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ABSTRACT. Let (M, J, g, D) be a Norden manifold with the natural canonical connection D and let $\widehat{J}$ be the generalized complex structure on M defined by g and J. We prove that $\widehat{J}$ is $D-$integrable and we find conditions on the curvature of D under which the $\pm i-$eigenbundles of $\widehat{J}$, $E_{\widehat{J}}^{1,0}$, $E_{\widehat{J}}^{0,1}$, are complex Lie algebroids. Moreover we proove that $E_{\widehat{J}}^{1,0}$ and $\left(E_{\widehat{J}}^{1,0}\right)^*$ are canonically isomorphic and this allow us to define the concept of generalized $\overline{\partial}_{\widehat{J}} - operator$ of (M, J, g, D). Also we describe some generalized holomorphic sections. The class of Kähler-Norden manifolds plays an important role in this paper because for these manifolds $E_{\widehat{J}}^{1,0}$ and $E_{\widehat{J}}^{0,1}$ are complex Lie algebroids. [1] [2] [3]

1 Introduction

Generalized complex structures were introduced by N. Hitchin in [6], and further investigated by M. Gualtieri in [8], in order to unify symplectic and complex geometry. In this paper we consider a more general concept of generalized complex structure introduced in [15], [16] and also studied in [17], [18], [3]. Let (M, g) be a smooth pseudo-Riemannian manifold, let $T(M)$ be the tangent bundle, let $T^*(M)$ be the cotangent bundle and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of M. In the previous papers [15], [16], we defined a generalized complex structure of M as a complex structure on E and we studied some classes of such structures, in particular calibrated complex structures with respect to the canonical symplectic structure, $(\, , \,)$, of E. Using a linear connection, ∇, on M we introduced a bracket, $[\, , \,]_\nabla$, on sections of E, the corresponding concept of $\nabla-$integrability for generalized complex structures and we studied integrability conditions. In [18] we concentrated on the canonical generalized complex structure defined by g, $J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}$. We proved that in the case J^g is $\nabla-$integrable the $\pm i-$eigenbundles of J^g, $E_{J^g}^{1,0}$, $E_{J^g}^{0,1}$, are complex Lie

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algebroids and, by using the canonical isomorphism between $E_{J^g}^{0,1}$ and $\left(E_{J^g}^{1,0}\right)^*$ induced by the natural symplectic structure of $T(M) \oplus T^*(M)$, we defined the *generalized* $\overline{\partial}_{J^g}-$*operator* on M. We remark that this case is strictly related to the field of statistical manifolds introduced in [1]. In this paper we observe that Norden manifolds fit naturally in the context of our concept of generalized complex structures and we extend the results of [18] to the case of Norden manifolds. Precisely we prove that on a Norden manifold, (M, J, g), with the natural canonical connection D, the generalized complex structure defined by $\widehat{J} = \begin{pmatrix} J & O \\ g & -J^* \end{pmatrix}$ is $D-$integrable. Then we describe the $\pm i-$eigenbundles of $\widehat{J}$, $E_{\widehat{J}}^{1,0}$, $E_{\widehat{J}}^{0,1}$, we find conditions under which they are complex Lie algebroids and we prove that for Kähler-Norden manifolds these conditions are automatically satisfied, that is, for this class of manifolds, $E_{\widehat{J}}^{1,0}$ and $E_{\widehat{J}}^{0,1}$ are complex Lie algebroids. Then we define the *generalized* $\overline{\partial}_{\widehat{J}}-$*operator* on M, from the Jacobi identity on $E_{\widehat{J}}^{1,0}$ it follows that $\left(\overline{\partial}_{\widehat{J}}\right)^2 = 0$ and, as $\overline{\partial}_{\widehat{J}}$ is the exterior derivative of the Lie algebroid $E_{\widehat{J}}^{1,0}$, we get that $\left(C^\infty\left(\wedge^\bullet\left(E_{\widehat{J}}^{1,0}\right)\right), \wedge, \overline{\partial}_{\widehat{J}}, [\ ,\ ]_D\right)$ is a differential Gerstenhaber algebra, where $\wedge$ denotes the Schouten bracket, [12], [24]. The paper is organized as in the following. In section 2 we introduce preliminary material: first we describe the main geometrical properties of the generalized tangent bundle and of generalized complex structures, then we recall the basic definitions in the setting of Norden manifolds, Kähler-Norden manifolds and complex Lie algebroids. Original results are concentrated in section 3: the geometrical description of the generalized complex structure $\widehat{J}$ associated naturally to a Norden manifold, the definition of the generalized $\overline{\partial}_{\widehat{J}}-$*operator* and the description of some generalized holomorphic sections.

2 Preliminaries

2.1 Generalized geometry

Let M be a smooth manifold of real dimension n and let $E = T(M) \oplus T^*(M)$ be the *generalized tangent bundle* of M. Smooth sections of E are elements $X + \xi \in C^\infty(E)$ where $X \in C^\infty(T(M))$ is a vector field and $\xi \in C^\infty(T^*(M))$ is a $1-$ form.

E is equipped with a natural *symplectic structure* defined by:

$$(X + \xi, Y + \eta) = -\frac{1}{2}(\xi(Y) - \eta(X)) \tag{1}$$

and a natural *indefinite metric* defined by:

$$< X + \xi, Y + \eta > \ = -\frac{1}{2}(\xi(Y) + \eta(X)). \tag{2}$$

$< , >$ is non degenerate and of signature (n, n).

A linear connection on M, ∇, defines, in a canonical way, a bracket on $C^{\infty}(E)$, $[\ ,\ ]_{\nabla}$, as follows:

$$[X+\xi, Y+\eta]_{\nabla} = [X, Y] + \nabla_X \eta - \nabla_Y \xi. \tag{3}$$

The following holds:

Lemma 1 ([*15*]) *For all $X, Y \in C^{\infty}(T(M))$, for all $\xi, \eta \in C^{\infty}(T^*(M))$ and for all $f \in C^{\infty}(M)$ we have:*

1. $[X+\xi, Y+\eta]_{\nabla} = -[Y+\eta, X+\xi]_{\nabla}$,
2. $[f(X+\xi), Y+\eta]_{\nabla} = f[X+\xi, Y+\eta]_{\nabla} - Y(f)(X+\xi)$,
3. *Jacobi's identity holds for $[\ ,\ ]_{\nabla}$ if and only if ∇ has zero curvature.*

We consider the following concept of generalized complex structure, introduced in [15], [16] and further investigated in [17], [18], [3] :

Definition 2 *A* generalized complex structure *on M is an endomorphism $\widehat{J}$, $\widehat{J} : E \to E$ such that $\widehat{J}^2 = -I$.*

A pseudo-Riemannian metric on M, g, defines, in a natural way, a complex structure J^g on E by:

$$J^g(X+\xi) = -g^{-1}(\xi) + g(X) \tag{4}$$

where $g : T(M) \to T^*(M)$ is identified to the bemolle musical isomorphism defined by:

$$g(X)(Y) = g(X, Y), \tag{5}$$

in block matrix form, is:

$$J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}. \tag{6}$$

Definition 3 *A generalized complex structure $\widehat{J}$ is called* pseudo calibrated *if is $(\ ,\ )-$invariant and if the bilinear symmetric form on $T(M)$ defined by $(\ , J\ )$ is non degenerate, moreover $\widehat{J}$ is called* calibrated *if $(\ , \widehat{J}\ )$ is positive definite,* [*15*].

A direct computation shows that J^g is pseudo calibrated.

Let ∇ be a linear connection on M and let $[\ ,\ ]_{\nabla}$ be the bracket on $C^{\infty}(E)$ defined by ∇, the following holds:

Lemma 4 ([16]) *Let $\widehat{J} : E \to E$ be a generalized complex structure on M and let*

$$N^{\nabla}(\widehat{J}) : C^{\infty}(E) \times C^{\infty}(E) \to C^{\infty}(E) \tag{7}$$

defined by:

$$N^{\nabla}(\widehat{J})(\sigma, \tau) = \left[\widehat{J}\sigma, \widehat{J}\tau\right]_{\nabla} - \widehat{J}\left[\widehat{J}\sigma, \tau\right]_{\nabla} - \widehat{J}\left[\sigma, J\tau\right]_{\nabla} - [\sigma, \tau]_{\nabla} \tag{8}$$

for all $\sigma, \tau \in C^{\infty}(E)$; $N^{\nabla}(\widehat{J})$ is a skew symmetric tensor.

Definition 5 *$N^{\nabla}(\widehat{J})$ is called the* Nijenhuis tensor *of $\widehat{J}$ with respect to ∇.*

Definition 6 *Let $\widehat{J} : E \to E$ be a generalized complex structure on M, $\widehat{J}$ is called $\nabla-$*integrable *if $N^{\nabla}(\widehat{J}) = 0$.*

Proposition 7 ([16]) *Let ∇ be a torsion free connection on M and let*

$$J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix} \tag{9}$$

be the generalized complex structure on M defined by a pseudo-Riemannian metric g, J^g is $\nabla-$integrable if and only if g is a Codazzi tensor, that is for all $X, Y \in C^{\infty}(T(M))$ we have:

$$(\nabla_X g)Y = (\nabla_Y g)X. \tag{10}$$

Definition 8 ([1]), ([4]), ([19]) *Let (M, g, ∇) be a pseudo-Riemannian manifold with a torsion free linear connection, if ∇g is symmetric then (M, g, ∇) is called a* statistical manifold.

Corollary 9 *Let ∇ be a torsion free connection on M and let J^g be the generalized complex structure on M defined by a pseudo-Riemannian metric g, J^g is $\nabla-$integrable if and only if (M, g, ∇) is a statistical manifold.*

2.2 Norden manifolds

Norden manifolds were introduced by A. P. Norden in [20] and then studied also under the names of almost complex manifolds with B-metric and anti-Kählerian manifolds, [2], [9]. They have applications in mathematics and in theoretical physics.

Definition 10 *Let (M, J) be an almost complex manifold of real dimension $2n$ and let g be a pseudo-Riemannian metric on M, if J is a g-symmetric operator then g is called* Norden metric *and (M, J, g) is called* Norden manifold.

Remark 11 *We can easily prove that a Norden metric g on a $2n-$dimensional almost complex manifold is of $(n, n)-$signature, that is g is a neutral metric.*

Let (M, J, g) be a complex Norden manifold, that is a Norden manifold with J integrable, then there exists a natural canonical connection on M, precisely the following holds:

Theorem 12 ([9]) *On a complex manifold with Norden metric (M, J, g) there exists a unique linear connection D with torsion T such that:*

$$(D_X g)(Y, Z) = 0 \tag{11}$$

$$T(JX, Y) = -T(X, JY) \tag{12}$$

$$g(T(X, Y), Z) + g(T(Y, Z), X) + g(T(Z, X), Y) = 0 \tag{13}$$

for all vector fields X, Y, Z on M. D is called the natural canonical connection *of the Norden manifold or $B-$*connection *and it is defined by:*

$$D_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J)Y \tag{14}$$

where ∇ is the Levi-Civita connection of g.

We remark that (14) is equivalent to:

$$D_X Y = \frac{1}{2}(\nabla_X Y - J\nabla_X JY) \tag{15}$$

then, by direct computation we get the following Proposition.

Proposition 13 *If D is the natural canonical connection of the complex Norden manifold (M, J, g) then*

$$DJ = 0. \tag{16}$$

Definition 14 *Let (M, J, g) be a Norden manifold and let*

$$\tilde{g}(X, Y) = g(JX, Y). \tag{17}$$

for all X and Y vector fields on M. $\tilde{g}$ is a pseudo-Riemannian metric on M with $(n, n)-$signature and $(M, J, \tilde{g})$ is a Norden manifold. $\tilde{g}$ is called the associated metric *to g. $\tilde{g}$ is also called the* twin *or the* dual metric *of g.*

2.3 Kähler-Norden manifolds

Kähler-Norden manifolds are strictly related with complex analysis and they will be the main object of our theory. We recall here the definition and the main properties of Kähler-Norden manifolds, for details see [2],[11], [23].

Definition 15 *Let (M, J, g) be a Norden manifold and let ∇ be the Levi-Civita connection of g, if $\nabla J = 0$ then (M, J, g) is called* Kähler-Norden manifold.

We remark that for a Kähler-Norden manifold (M, J, g) the structure J is integrable and the natural canonical connection is the Levi-Civita connection.

Moreover the following holds:

Theorem 16 ([22]) *Let (M, J, g) be a Kähler-Norden manifold, the Levi-Civita connection of g coincides with the Levi-Civita connection of the associated metric $\widetilde{g}$, in particular the Riemann curvature tensors of g and $\widetilde{g}$ coincide.*

A large class of Kähler-Norden manifolds is given by complex parallelisable manifolds, ([2]).

An interesting property of Kähler-Norden manifolds is the following:

Proposition 17 ([2]) *Let (M, J, g) be a Kähler-Norden manifold then, extending g by $\mathbb{C}-$linearity to the complexified tangent bundle $T(M) \otimes \mathbb{C}$, the components of the complex extended metric, $\widehat{g}$, are holomorphic functions.*

We recall that on a complex manifold (M, J) an element $X \in C^\infty(TM)$ is an *infinitesimal automorphism* of the complex structure J on M if and only if X satisfies the following condition:

$$[X, JY] = J[X, Y] \tag{18}$$

for all $Y \in C^\infty(TM)$.

On Kähler-Norden manifolds, from the condition $\nabla J = 0$, (18) can be written as:

$$\nabla_{JY} X = \nabla_Y JX. \tag{19}$$

The Riemannian curvature tensor of a Kähler-Norden manifold has interesting properties, precisely we have the following:

Theorem 18 ([11]), ([22]) *In a Kähler-Norden manifold the Riemannian curvature tensor, R^∇, of the Norden metric g is pure in all arguments, that is, for all $X, Y, Z, W \in C^\infty(T(M))$:*

$$\begin{aligned} g(R^\nabla(JX, Y)Z, W) &= g(R^\nabla(X, JY)Z, W) \\ &= g(R^\nabla(X, Y)JZ, W) \\ &= g(R^\nabla(X, Y)Z, JW). \end{aligned} \tag{20}$$

2.4 Complex Lie algebroids

Lie algebroids were introduced by J. Pradines in [21]; we recall here the definition and the main properties.

Definition 19 *A* complex Lie algebroid *is a complex vector bundle L over a smooth real manifold M such that: a Lie bracket $[\ ,\ ]$ is defined on $C^\infty(L)$, a smooth bundle map $\rho : L \to T(M)$, called* anchor, *is defined and, for all $\sigma, \tau \in C^\infty(L)$, for all $f \in C^\infty(M)$ the following conditions hold:*

1. $\rho([\sigma,\tau]) = [\rho(\sigma), \rho(\tau)]$
2. $[f\sigma,\tau] = f([\sigma,\tau]) - (\rho(\tau)(f))\sigma$.

Let L and its dual vector bundle L^* be Lie algebroids; on sections of $\wedge L$, respectively $\wedge L^*$, the *Schouten bracket* is defined by:

$$[\ ,\ ]_L : C^\infty(\wedge^p L) \times C^\infty(\wedge^q L) \longrightarrow C^\infty(\wedge^{p+q-1} L) \tag{21}$$

$$[X_1 \wedge ... \wedge X_p, Y_1 \wedge ... \wedge Y_q]_L =$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{i+j}[X_i, Y_j]_L \wedge X_1 \wedge ..\widehat{i}.. \wedge X_p \wedge Y_1 \wedge ..\widehat{j}.. \wedge Y_q \tag{22}$$

and, for $f \in C^\infty(M)$, $X \in C^\infty(L)$

$$[X, f]_L = -[f, X]_L = \rho(X)(f); \tag{23}$$

respectively, by:

$$[\ ,\ ]_{L^*} : C^\infty(\wedge^p L^*) \times C^\infty(\wedge^q L^*) \longrightarrow C^\infty(\wedge^{p+q-1} L^*) \tag{24}$$

$$[X_1^* \wedge ... \wedge X_p^*, Y_1^* \wedge ... \wedge Y_q^*]_{L^*} =$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{i+j}[X_i^*, Y_j^*]_{L^*} \wedge X_1^* \wedge ..\widehat{i}.. \wedge X_p^* \wedge Y_1^* \wedge ..\widehat{j}.. \wedge Y_q^* \tag{25}$$

and, for $f \in C^\infty(M)$, $X \in C^\infty(L^*)$

$$[X, f]_{L^*} = -[f, X]_{L^*} = \rho(X)(f). \tag{26}$$

Moreover the *exterior derivatives* d and d_* associated with the Lie algebroid structure of L and L^* are defined respectively by:

$$d : C^\infty(\wedge^p L^*) \longrightarrow C^\infty(\wedge^{p+1} L^*) \tag{27}$$

$$(d\alpha)(\sigma_0, ..., \sigma_p) =$$

$$= \sum_{i=0}^{p}(-1)^i \rho(\sigma_i)\alpha\left(\sigma_0, ..\widehat{i}.., \sigma_p\right) + \sum_{i<j}(-1)^{i+j}\alpha\left([\sigma_i, \sigma_j]_L, \sigma_0, ..\widehat{i}..\widehat{j}.., \sigma_p\right) \tag{28}$$

for $\alpha \in C^{\infty}(\wedge^p L^*)$, $\sigma_0, ..., \sigma_p \in C^{\infty}(L)$,
and:

$$d_* : C^{\infty}(\wedge^p L) \longrightarrow C^{\infty}(\wedge^{p+1} L) \tag{29}$$

$$(d_*\alpha)(\sigma_0, ..., \sigma_p) =$$

$$= \sum_{i=0}^{p} (-1)^i \rho(\sigma_i)\, \alpha\left(\sigma_0, ..\overset{\widehat{i}}{.}.., \sigma_p\right) + \sum_{i<j} (-1)^{i+j} \alpha\left([\sigma_i, \sigma_j]_{L^*}, \sigma_0, ..\overset{\widehat{i}}{.}.\overset{\widehat{j}}{.}.., \sigma_p\right) \tag{30}$$

for $\alpha \in C^{\infty}(\wedge^p L)$, $\sigma_0, ..., \sigma_p \in C^{\infty}(L^*)$.

3 Generalized geometry of Norden manifolds

3.1 Generalized complex structures

Let (M, J, g) be a Norden manifold, the almost complex structure J and the pseudo Riemannian metric g define, in a natural way, a complex structure $\widehat{J}$ on E by:

$$\widehat{J}(X + \xi) = J(X) + g(X) - J^*(\xi) \tag{31}$$

where $J^* : T^*(M) \to T^*(M)$ is the dual operator of J defined by:

$$J^*(\xi)(X) = \xi(J(X)). \tag{32}$$

In block matrix form, is:

$$\widehat{J} = \begin{pmatrix} J & O \\ g & -J^* \end{pmatrix}. \tag{33}$$

Remark 20 *From the $g-$symmetry of J it follows immediately that $\widehat{J}$ is a pseudo calibrated generalized complex structure on M, see also [16].*

A direct computation gives the following:

Proposition 21 *Let (M, J, g) be a Norden manifold and let ∇ be a linear connection on M with torsion T, let $\widehat{J}$ be the generalized complex structure defined by J and g, we have:*

$$\begin{aligned} N^{\nabla}(\widehat{J})(X, Y) &= (\nabla_{JX} J)Y - J(\nabla_X J)Y - (\nabla_{JY} J)X + J(\nabla_Y J)X + \\ &-T(JX, JY) + JT(X, JY) + JT(JX, Y) + T(X, Y) + \\ &+g((\nabla_Y J)X - (\nabla_X J)Y) + g(T(X, JY) + T(JX, Y)) + \\ &+ (\nabla_{JX} g) Y - (\nabla_{JY} g) X + (\nabla_X g) JY - (\nabla_Y g) JX \end{aligned} \tag{34}$$

$$N^{\nabla}(\widehat{J})(X, \xi) = -J^*(\nabla_X J^*)\xi - (\nabla_{JX} J^*)\xi \tag{35}$$

$$N^{\nabla}(\widehat{J})(\xi, \eta) = 0 \tag{36}$$

for all $X, Y \in C^{\infty}(T(M))$ and for all $\xi, \eta \in C^{\infty}(T^*(M))$.

Corollary 22 *$\widehat{J}$ is $\nabla-$integrable if and only if the following conditions hold:*

$$(\nabla_{JX}J) = J(\nabla_X J) \tag{37}$$

$$T(JX, JY) - JT(X, JY) - JT(JX, Y) - T(X, Y) = O \tag{38}$$

$$\begin{aligned} &g((\nabla_Y J)X - (\nabla_X J)Y) + g(T(X, JY) + T(JX, Y)) + \\ &+ (\nabla_{JX} g)\, Y - (\nabla_{JY} g)\, X + (\nabla_X g)\, JY - (\nabla_Y g)\, JX = O \end{aligned} \tag{39}$$

for all $X, Y \in C^\infty(T(M))$.

Corollary 23 *If $\widehat{J}$ is $\nabla-$integrable then J is integrable.*

Proof. Let $N(J)$ be the Nijenhuis tensor of the almost complex structure J, we have:

$$\begin{aligned} N(J)(X,Y) &= (\nabla_{JX}J)Y - J(\nabla_X J)Y - (\nabla_{JY}J)X + J(\nabla_Y J)X + \\ &\quad -T(JX, JY) + JT(X, JY) + JT(JX, Y) + T(X, Y) \end{aligned} \tag{40}$$

for all $X, Y \in C^\infty(T(M))$, then the statement follows from Corollary 22. ■

As we are interested in integrable generalized complex structures in the following we will assume that (M, J, g) is a complex Norden manifold. In particular we get:

Proposition 24 *Let (M, J, g) be a complex Norden manifold and let D be the natural canonical connection on M, let $\widehat{J}$ be the generalized complex structure defined by J and g, then $\widehat{J}$ is $D-$integrable.*

Proof. It follows from the properties of D described in Theorem 12 and in Proposition 13. ■

Analogous statement can be given for the associated metric, precisely the following holds:

Proposition 25 *Let (M, J, g) be a complex Norden manifold and let $\widetilde{D}$ be the natural canonical connection of the associated metric $\widetilde{g}$, let $\widetilde{J}$ be the generalized complex structure defined by J and $\widetilde{g}$, then $\widetilde{J}$ is $\widetilde{D}-$integrable.*

3.2 Generalized $\overline{\partial}_{\widehat{J}}$-operator

Let (M, J, g) be a complex Norden manifold and let $\widehat{J}$ be the generalized complex structure on M defined by J and g, let

$$E^{\mathbb{C}} = (T(M) \oplus T^*(M)) \otimes \mathbb{C} \tag{41}$$

be the complexified generalized tangent bundle. The splitting in $\pm i$ eigenspaces of $\widehat{J}$ is denoted by:

$$E^{\mathbb{C}} = E^{1,0}_{\widehat{J}} \oplus E^{0,1}_{\widehat{J}} \tag{42}$$

with

$$E^{0,1}_{\widehat{J}} = \overline{E^{1,0}_{\widehat{J}}}. \tag{43}$$

A direct computation gives:

$$E^{1,0}_{\widehat{J}} = \{Z - iJZ + g(W + iJW - iZ) \mid Z, W \in T(M) \otimes \mathbb{C}\}, \tag{44}$$

equivalently $E^{1,0}_{\widehat{J}}$ is generated by elements of the following type:

$$X - iJX - ig(X) \text{ with } X \in C^\infty(TM), \tag{45}$$

$$g(Y + iJY) \text{ with } Y \in C^\infty(TM). \tag{46}$$

Analogously we have:

$$E^{0,1}_{\widehat{J}} = \{Z + iJZ + g(W - iJW + iZ) \mid Z, W \in T(M) \otimes \mathbb{C}\} \tag{47}$$

and $E^{0,1}_{\widehat{J}}$ is generated by elements of the following type:

$$X + iJX + ig(X) \text{ with } X \in C^\infty(TM), \tag{48}$$

$$g(Y - iJY) \text{ with } Y \in C^\infty(TM). \tag{49}$$

Moreover, for any linear connection ∇, the following holds:

Lemma 26 *$E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$ are $[\ ,\ ]_\nabla$ -involutive if and only if $N^\nabla(\widehat{J}) = 0$.*

Proof. Let $P_+ : E^{\mathbb{C}} \to E^{1,0}_{\widehat{J}}$ and $P_- : E^{\mathbb{C}} \to E^{0,1}_{\widehat{J}}$ be the projection operators:

$$P_\pm = \frac{1}{2}(I \mp i\widehat{J}), \tag{50}$$

for all $\sigma, \tau \in C^\infty(E^{\mathbb{C}})$ we have:

$$\begin{aligned} P_\mp \left[P_\pm(\sigma), P_\pm(\tau)\right]_\nabla &= P_\mp \left[\tfrac{1}{2}\left(\sigma \mp i\widehat{J}\sigma\right), \tfrac{1}{2}\left(\tau \mp i\widehat{J}\tau\right)\right]_\nabla \\ &= -\tfrac{1}{8}(N^\nabla(\widehat{J})(\sigma,\tau) \pm i\widehat{J}N^\nabla(\widehat{J})(\sigma,\tau)) = -\tfrac{1}{4}P_\mp\left(N^\nabla(\widehat{J})(\sigma,\tau)\right). \end{aligned} \tag{51}$$

■

From now on we suppose that (M, J, g, D) is a complex Norden manifold with the natural canonical connection. A direct computation of the bracket associated to D on $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$ gives the following:

or

$$\begin{aligned}\sigma &= X - iJX - ig(X)\\ \tau &= Y - iJY - ig(Y)\\ \upsilon &= Z - iJZ - ig(Z).\end{aligned} \tag{66}$$

Let us compute

$$Jac\,[[g(X+iJX), Y - iJY - ig(Y)]_D\,, Z - iJZ - ig(Z)]_D\,. \tag{67}$$

We have:

$$[[g(X+iJX), Y - iJY - ig(Y)]_D\,, Z - iJZ - ig(Z)]_D = g(K + iJK) \tag{68}$$

$$[[Y - iJY - ig(Y), Z - iJZ - ig(Z)]_D\,, g(X+iJX)]_D = g(L + iJL) \tag{69}$$

$$g\,[[Z - iJZ - ig(Z), g(X+iJX)]_D\, Y - iJY - ig(Y)]_D = g(H + iJH) \tag{70}$$

where

$$K = D_Z D_Y X + D_Z J D_{JY} X + J D_{JZ} D_Y X + J D_{JZ} J D_{JY} X \tag{71}$$

$$L = D_{[Y,Z]} X + J D_{J[Y,Z]} X - D_{[JY,JZ]} X - J D_{J[JY,JZ]} X \tag{72}$$

$$H = -D_Y D_Z X - J D_Y D_{JZ} X - J D_{JY} D_Z X + D_{JY} D_{JZ} X. \tag{73}$$

Then we get

$$Jac\,[[\sigma, \tau]_D\,, \upsilon]_D = O \tag{74}$$

if and only if

$$K + L + H = O \tag{75}$$

or, by direct computation, if and only if:

$$R^D(JY, JZ) - JR^D(JY, Z) - JR^D(Y, JZ) - R^D(Y, Z) - JD_{JN(J)(Y,Z)} = O \tag{76}$$

where $N(J)$ is the Nijenhuis tensor of J. By using the integrability of J, we have the first condition.
Let us compute

$$Jac\,[[X - iJX - ig(X), Y - iJY - ig(Y)]_D\,, Z - iJZ - ig(Z)]_D\,. \tag{77}$$

We have:

$$\begin{aligned}&[[X - iJX - ig(X), Y - iJY - ig(Y)]_D\,, Z - iJZ - ig(Z)]_D =\\ &\qquad = A - iJA - ig(A) + g(B + iJB)\end{aligned} \tag{78}$$

where

$$A = [[X,Y] - [JX,JY], Z] - [J[X,Y] - J[JX,JY], JZ] \tag{79}$$

and

$$B = D_{JZ}[X,Y] + D_{JZ}T^D(JX,JY) - D_{J[X,Y]}Z+ \\ +D_{J[JX,JY]}Z - D_Z D_{JY}X + D_Z D_{JX}Y \tag{80}$$

where T^D denotes the torsion tensor of the connection D.
Fom the Jacobi identity of $[\ ,\ ]$ we have that $Jac(A) = O$, then it is enough to compute $Jac(B)$.
From the properties of the torsion tensor T^D we get:

$$Jac(B) = \left(R^D(JX,Y) + R^D(X,JY)\right) Z+ \\ + \left(R^D(JZ,X) + R^D(Z,JX)\right) Y + \left(R^D(Y,JZ) + R^D(JY,Z)\right) X. \tag{81}$$

Analogous computations for $E_{\tilde{J}}^{0,1}$ gives exactly the same conditions, then the Proof is complete. ■

Remark 29 *We observe that* (61) *is equivalent to:*

$$\left(R^D\right)^{(0,2)} = O \tag{82}$$

where $\left(R^D\right)^{(0,2)}$ *denotes the* $(0,2)-$*part of the curvature with respect to the complex structure* J *on* M. *Moreover, if the torsion is zero, from the first Bianchi identity with zero torsion, we get that* (62) *is automatically satisfied; instead, from the first Bianchi identity with torsion:*

$$R^D(X,Y)Z + R^D(Y,Z)X + R^D(Z,X)Y+ \\ -T^D(X,[Y,Z]) - T^D(Y,[Z,X]) - T^D(Z,[X,Y])+ \\ -D_X T(Y,Z) - D_Y T(Z,X) - D_Z T(X,Y) = O, \tag{83}$$

we obtain that (62) *is equivalent to the following:*

$$\left(R^D(JX,JY) - R^D(X,Y)\right) Z + \left(R^D(JZ,JX) - R^D(Z,X)\right) Y+ \\ + \left(R^D(JY,JZ) - R^D(Y,Z)\right) X = O. \tag{84}$$

From Proposition 26 we get in particular the following:

Proposition 30 *If* $R^D = O$ *then* $E_{\tilde{J}}^{1,0}$ *and* $E_{\tilde{J}}^{0,1}$ *are complex Lie algebroids.*

In this sense the following result provides a class of examples, ([10]), ([13]).

Theorem 31 ([*10*]), ([*13*]) *Each hyper-Kaehler NH-manifold is a flat pseudo-Riemannian manifold of signature* $(2n, 2n)$.

More generally we have the following:

Theorem 32 *Let (M, J, g) be a Kähler-Norden manifold then $E_{\widehat{J}}^{1,0}$ and $E_{\widehat{J}}^{0,1}$ are complex Lie algebroids.*

Proof. In this case the natural canonical connection D is the Levi-Civita connection ∇ and, as its torsion is zero, (62) is automatically satisfied. Moreover from (20) we get that (61) is equivalent to:

$$R^{\nabla}(Y, Z) + R^{\nabla}(JY, Z)J = O \tag{85}$$

and, by using again the fact that R^{∇} is a pure tensor, we have that, for all $Y, Z, W \in C^{\infty}(T(M))$, (85) becomes:

$$R^{\nabla}(Y, Z)W + R^{\nabla}(Y, Z)JJW = O \tag{86}$$

which is automatically satisfied. Thus the proof is complete. ∎

Remark 33 *Analogous statement can be given for $E_{\overline{J}}^{1,0}$ and $E_{\overline{J}}^{0,1}$. In the following we will consider only $\widehat{J}$.*

The following holds:

Proposition 34 *The natural symplectic structure on E defines a canonical isomorphism between $E_{\widehat{J}}^{0,1}$ and the dual bundle of $E_{\widehat{J}}^{1,0}$, $\left(E_{\widehat{J}}^{1,0}\right)^{*}$.*

Proof. We define

$$\varphi : E_{\widehat{J}}^{0,1} \to \left(E_{\widehat{J}}^{1,0}\right)^{*} \tag{87}$$

by:

$$\begin{aligned} &(\varphi\,(Z + iJZ + g(W - iJW + iZ)))\,(X - iJX + g(Y + iJY - iX)) = \\ &\quad = (Z + iJZ + g(W - iJW + iZ), X - iJX + g(Y + iJY - iX)) \end{aligned} \tag{88}$$

for all $X, Y, Z, W \in T\,(M) \otimes \mathbb{C}$.
We get:

$$\begin{aligned} &(\varphi\,(Z + iJZ + g(W - iJW + iZ)))\,(X - iJX + g(Y + iJY - iX)) = \\ &\quad = g(Y, Z) - g(W, X) + i\,(g(W, JX) + g(Y, JZ) - g(X, Z)) \end{aligned} \tag{89}$$

and we extend by linearity. We have immediately that φ is injective and furthermore φ is an isomorphism. ∎

The canonical isomorphism φ between $E_{\widehat{J}}^{0,1}$ and the dual bundle $\left(E_{\widehat{J}}^{1,0}\right)^{*}$ allows us to define the $\overline{\partial}_{\widehat{J}}$ − *operator* associated to the complex structure $\widehat{J}$ as in the following:

let $f \in C^{\infty}(M)$ and let $df \in C^{\infty}(T^*(M)) \hookrightarrow C^{\infty}(T(M) \oplus T^*(M))$, we pose

$$\overline{\partial}_{\widehat{J}} f = 2\,(df)^{0,1} = df + i\widehat{J} df \tag{90}$$

or:

$$\begin{aligned} \overline{\partial}_{\widehat{J}} f &= df - iJ^*(df) \\ &= df - i\,(df)\,J; \end{aligned} \tag{91}$$

moreover we define:

$$\overline{\partial}_{\widehat{J}} : C^{\infty}\left(E_{\widehat{J}}^{0,1}\right) \to C^{\infty}\left(\wedge^2\left(E_{\widehat{J}}^{0,1}\right)\right) \tag{92}$$

via the natural isomorphism

$$E_{\widehat{J}}^{0,1} \overset{\varphi}{\simeq} \left(E_{\widehat{J}}^{1,0}\right)^* \tag{93}$$

as:

$$\overline{\partial}_{\widehat{J}} : C^{\infty}\left(\left(E_{\widehat{J}}^{1,0}\right)^*\right) \to C^{\infty}\left(\wedge^2\left(E_{\widehat{J}}^{1,0}\right)^*\right) \tag{94}$$

$$\left(\overline{\partial}_{\widehat{J}}\alpha\right)(\sigma,\tau) = \rho(\sigma)\,\alpha(\tau) - \rho(\tau)\,\alpha(\sigma) - \alpha([\sigma,\tau]_D) \tag{95}$$

for $\alpha \in C^{\infty}\left(\left(E_{\widehat{J}}^{1,0}\right)^*\right)$, $\sigma,\tau \in C^{\infty}\left(E_{\widehat{J}}^{1,0}\right)$.

In general:

$$\overline{\partial}_{\widehat{J}} : C^{\infty}\left(\wedge^p\left(E_{\widehat{J}}^{1,0}\right)^*\right) \to C^{\infty}\left(\wedge^{p+1}\left(E_{\widehat{J}}^{1,0}\right)^*\right) \tag{96}$$

is defined by:

$$\begin{aligned} &\left(\overline{\partial}_{\widehat{J}}\alpha\right)(\sigma_0,\dots,\sigma_p) = \\ &= \sum_{i=0}^{p}(-1)^i \rho(\sigma_i)\,\alpha\left(\sigma_0,..\overset{\widehat{i}}{.}..,\sigma_p\right) + \sum_{i<j}(-1)^{i+j}\alpha\left([\sigma_i,\sigma_j]_D,\sigma_0,..\overset{\widehat{i}}{.}.\overset{\widehat{j}}{.}..,\sigma_p\right) \end{aligned} \tag{97}$$

for $\alpha \in C^{\infty}\left(\wedge^p\left(E_{\widehat{J}}^{1,0}\right)^*\right)$, $\sigma_0,\dots,\sigma_p \in C^{\infty}\left(E_{\widehat{J}}^{1,0}\right)$.

Definition 35 $\overline{\partial}_{\widehat{J}}$ *is called* generalized $\overline{\partial}$ – operator *of* (M,J,g,D) *or* generalized $\overline{\partial}_{\widehat{J}}$ – operator.

We get the following:

Proposition 36 *If* (61) *and* (62) *hold then* $\left(\overline{\partial}_{\widehat{J}}\right)^2 = 0$ *and* $\left(\partial_{\widehat{J}}\right)^2 = 0$.

Proof. It follows from the fact that Jacobi identity holds on $E^{1,0}_{\mathcal{J}}$ and $\left(E^{1,0}_{\mathcal{J}}\right)^*$. ■

From now on we suppose that (61) and (62) hold. We have immediately that $\overline{\partial}_{\mathcal{J}}$ is the exterior derivative, d_L, of the Lie algebroid $L = E^{1,0}_{\mathcal{J}}$. Moreover the exterior derivative d_{L^*} of $L^* = \left(E^{1,0}_{\mathcal{J}}\right)^*$ is given by the operator $\partial_{\mathcal{J}}$ defined by:

$$\partial_{\mathcal{J}_g} : C^\infty\left(\wedge^p\left(E^{1,0}_{\mathcal{J}}\right)\right) \to C^\infty\left(\wedge^{p+1}\left(E^{1,0}_{\mathcal{J}}\right)\right) \tag{98}$$

$$\left(\partial_{\mathcal{J}}\sigma\right)\left(\alpha_0^*, ..., \alpha_p^*\right) =$$

$$= \sum_{i=0}^{p}(-1)^i \rho\left(\alpha_i^*\right)\sigma\left(\alpha_0^*, ..\overset{\widehat{i}}{}.., \alpha_p^*\right) + \sum_{i<j}(-1)^{i+j}\sigma\left(\left[\alpha_i^*, \alpha_j^*\right]_D, \alpha_0^*, ..\overset{\widehat{i}}{}..\overset{\widehat{j}}{}.., \alpha_p^*\right) \tag{99}$$

for $\sigma \in C^\infty\left(\wedge^p\left(E^{1,0}_{\mathcal{J}}\right)\right)$, $\alpha_0^*, ..., \alpha_p^* \in C^\infty\left(\left(E^{1,0}_{\mathcal{J}}\right)^*\right)$.

3.3 Generalized holomorphic sections

Definition 37 *Let* $\alpha \in C^\infty\left(\wedge^p\left(E^{1,0}_{\mathcal{J}}\right)^*\right)$, α *is called* generalized holomorphic section *if*

$$\overline{\partial}_{\mathcal{J}}\alpha = 0. \tag{100}$$

We remark that for all $f \in C^\infty(M)$ we have $\overline{\partial}_{\mathcal{J}}f = 0$ if and only if $df = 0$, so the generalized holomorphic condition for functions gives only constant functions on connected components of M.

Proposition 38 *Let* $W \in C^\infty(T(M))$ *and let* $\sigma = g(W - iJW) \in E^{0,1}_{\mathcal{J}}$ *then* $\overline{\partial}_{\mathcal{J}}\sigma = 0$ *if and only if for all* $X, Y \in C^\infty(T(M))$ *holds:*

$$g(D_X W - D_{JX}JW, Y) = g(D_Y W - D_{JY}JW, X). \tag{101}$$

Proof. Let $X, Y \in C^\infty(T(M))$, from (95), direct computations give:

$$\left(\overline{\partial}_{\mathcal{J}}\sigma\right)(g(X + iJX), g(Y + iJY)) = 0 \tag{102}$$

$$\left(\overline{\partial}_{\mathcal{J}}\sigma\right)(g(X + iJX), Y - iJY - ig(Y)) = 0 \tag{103}$$

$$\begin{aligned}&\left(\overline{\partial}_{\mathcal{J}}\sigma\right)(X - iJX - ig(X), Y - iJY - ig(Y)) = \\ &= g(-D_X W + D_{JX}JW + i(D_{JX}W + iJD_X W, Y) + \\ &+ g(D_Y W - D_{JY}JW - i(D_{JY}W + JD_Y W), X).\end{aligned} \tag{104}$$

In particular we have $(\bar{\partial}_{\bar{J}}\sigma) = 0$ if and only if:

$$g(-D_X W + D_{JX} JW + i(D_{JX} W + iJD_X W, Y) + \\ +g(D_Y W - D_{JY} JW - i(D_{JY} W + JD_Y W), X) = 0 \tag{105}$$

and then, by separating real and imaginary parts, we get the statement. ■

Equivalently we can state Proposition 36 as follows:

Proposition 39 *Let $W \in C^\infty(T(M))$ and let $\sigma = g(W - iJW) \in E^{0,1}_{\bar{J}}$ then $\bar{\partial}_{\bar{J}}\sigma = 0$ if and only if for all $X, Y \in C^\infty(T(M))$ holds:*

$$(d(g(W)))(X, Y) = (d(g(W)))(JX, JY). \tag{106}$$

Proof. We have:

$$\begin{aligned}(d(g(W)))(X, Y) &= Xg(W, Y) - Yg(W, X) - g(W, [X, Y]) \\ &= g(D_X W, Y) - g(D_Y W, X) - g(W, T^D(X, Y)).\end{aligned} \tag{107}$$

On the other hand:

$$\begin{aligned}(d(g(W)))(JX, JY) &= JXg(W, JY) - JYg(W, JX) - g(W, [JX, JY]) \\ &= g(D_{JX} W, JY) - g(D_{JY} W, JX) - g(W, T^D(JX, JY)) \\ &= g(D_{JX} JW, Y) - g(D_{JY} JW, X) - g(W, T^D(JX, JY)).\end{aligned} \tag{108}$$

From the property (12) of the torsion T^D of the natural canonical connection we get the conclusion. ■

Moreover:

Proposition 40 *Let $Z \in C^\infty(T(M))$ and let $\sigma = Z + iJZ + ig(Z) \in E^{0,1}_{\bar{J}}$ then $\bar{\partial}_{\bar{J}}\sigma = 0$ if and only if for all $X, Y \in C^\infty(T(M))$ the following conditions hold:*

$$D_{JY} JZ = -D_Y Z \tag{109}$$

$$g(D_X Z, Y) = g(D_Y Z, X). \tag{110}$$

Proof. Let $X, Y \in C^\infty(T(M))$, direct computations give:

$$(\bar{\partial}_{\bar{J}}\sigma)(g(X + iJX), g(Y + iJY)) = 0 \tag{111}$$

$$\begin{aligned}&(\bar{\partial}_{\bar{J}}\sigma)(g(X + iJX), Y - iJY - ig(Y)) = \\ &= -g(D_Y Z + D_{JY} JZ, X) + ig(D_{JY} Z - D_Y JZ, X)\end{aligned} \tag{112}$$

$$\begin{aligned}&(\bar{\partial}_{\bar{J}}\sigma)(X - iJX - ig(X), Y - iJY - ig(Y)) = \\ &= -g(iD_X Z + D_{JX} Z, Y) + g(iD_Y Z + D_{JY} Z, X) \\ &= g(D_{JY} Z, X) - g(D_{JX} Z, Y) + i(g(D_Y Z, X) - g(D_X Z, Y)).\end{aligned} \tag{113}$$

and, by separating real and imaginary parts, we get the following conditions:

$$D_{JY}JZ + D_Y Z = O \tag{114}$$

$$g(D_{JY}Z, X) - g(D_{JX}Z, Y) = O; \tag{115}$$

From (114) we get

$$D_{JY}Z = JD_Y Z \tag{116}$$

and, substituting in (115), we have

$$g(D_Y Z, JX) - g(D_{JX}Z, Y) = O \tag{117}$$

for all $X, Y \in C^\infty(T(M))$, then we get the statement. ■

Corollary 41 *Given $Z \in C^\infty(T(M))$, infinitesimal automorphism of J, Z defines the following generalized holomorphic sections of $E_{\hat{J}}^{0,1}$:*

$$\sigma = g(Z - iJZ) \tag{118}$$

$$\tau = Z + iJZ + ig(Z) \tag{119}$$

if and only if for all $X, Y \in C^\infty(T(M))$ the following condition hold:

$$g(D_X Z, Y) = g(D_Y Z, X). \tag{120}$$

In particular for Kähler-Norden manifolds, as D is the Levi-Civita connection and then torsion free, condition (120) is equivalent to the d−closure of $g(Z)$, and, by using a classical result in symplectic geometry, [14], we have:

Proposition 42 *Let M be a Kähler-Norden manifold and let $Z \in C^\infty(T(M))$ be an infinitesimal automorphism of J then $g(Z - iJZ)$ and $Z + iJZ + ig(Z)$ are generalized holomorphic sections of $E_{\hat{J}}^{0,1}$ if and only if $g(Z)$ is a Lagrangian submanifold of $T^*(M)$ with respect to the standard symplectic structure.*

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