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Generalized geometry of Norden manifolds

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ABSTRACT. Let (M, J, g, D) be a Norden manifold with the natural canonical connection D and let \widehat{J} be the generalized complex structure on M defined by g and J. We prove that \widehat{J} is D-integrable and we find conditions on the curvature of D under which the $\pm i$ -eigenbundles of \widehat{J} , $E_{\widehat{J}}^{1,0}$, $E_{\widehat{J}}^{0,1}$, are complex Lie algebroids. Moreover we proove that $E_{\widehat{J}}^{1,0}$ and $\left(E_{\widehat{J}}^{1,0}\right)^*$ are canonically isomorphic and this allow us to define the concept of generalized $\overline{\partial}_{\widehat{J}} - operator$ of (M, J, g, D). Also we describe some generalized holomorphic sections. The class of Kähler-Norden manifolds plays an important role in this paper because for these manifolds $E_{\widehat{J}}^{1,0}$ and $E_{\widehat{J}}^{0,1}$ are complex Lie algebroids. 1 2 3

Introduction 1

Generalized complex structures were introduced by N. Hitchin in [6], and further investigated by M. Gualtieri in [8], in order to unify symplectic and complex geometry. In this paper we consider a more general concept of generalized complex structure introduced in [15], [16] and also studied in [17], [18], [3]. Let (M, g)be a smooth pseudo-Riemannian manifold, let T(M) be the tangent bundle, let $T^{*}(M)$ be the cotangent bundle and let $E = T(M) \oplus T^{*}(M)$ be the generalized tangent bundle of M. In the previous papers [15], [16], we defined a generalized complex structure of M as a complex structure on E and we studied some classes of such structures, in particular calibrated complex structures with respect to the canonical symplectic structure, (,), of E. Using a linear connection, ∇ , on M we introduced a bracket, $[,]_{\nabla}$, on sections of E, the corresponding concept of ∇ -integrability for generalized complex structures and we studied integrability conditions. In [18] we concentrated on the canonical generalized complex structure defined by g, $J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}$. We proved that in the case J^g is ∇ -integrable the $\pm i$ -eigenbundles of J^g , $E_{J^g}^{1,0}$, $E_{J^g}^{0,1}$, are complex Lie

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algebroids and, by using the canonical isomorphism between $E_{Jg}^{0,1}$ and $\left(E_{Jg}^{1,0}\right)^*$ induced by the natural symplectic structure of $T(M) \oplus T^{*}(M)$, we defined the generalized $\overline{\partial}_{J^g}$ -operator on M. We remark that this case is strictly related to the field of statistical manifolds introduced in [1]. In this paper we observe that Norden manifolds fit naturally in the context of our concept of generalized complex structures and we extend the results of [18] to the case of Norden manifolds. Precisely we prove that on a Norden manifold, (M, J, g), with the natural canonical connection D, the generalized complex structure defined by $\widehat{J} = \begin{pmatrix} J & O \\ g & -J^* \end{pmatrix}$ is *D*-integrable. Then we describe the $\pm i$ -eigenbundles of \widehat{J} , $E_{\widehat{J}}^{1,0}$, $E_{\widehat{J}}^{0,1}$, we find conditions under which they are complex Lie algebroids and we prove that for Kähler-Norden manifolds these conditions are automati-cally satisfied, that is, for this class of manifolds, $E_{\hat{j}}^{1,0}$ and $E_{\hat{j}}^{0,1}$ are complex Lie algebroids. Then we define the generalized $\overline{\partial}_{\hat{j}}$ -operator on M, from the Jacobi identity on $E_{\widehat{j}}^{1,0}$ it follows that $(\overline{\partial}_{\widehat{j}})^2 = 0$ and, as $\overline{\partial}_{\widehat{j}}$ is the exterior derivative of the Lie algebroid $E_{\widehat{j}}^{1,0}$, we get that $(C^{\infty}\left(\wedge^{\bullet}\left(E_{\widehat{j}}^{1,0}\right)\right), \wedge, \overline{\partial}_{\widehat{j}}, [,]_D)$ is a differential Gerstenhaber algebra, where \wedge denotes the Schouten bracket, [12], [24]. The paper is organized as in the following. In section 2 we introduce preliminary material: first we describe the main geometrical properties of the generalized tangent bundle and of generalized complex structures, then we recall the basic definitions in the setting of Norden manifolds, Kähler-Norden manifolds and complex Lie algebroids. Original results are concentrated in section 3: the geometrical description of the generalized complex structure J associated naturally to a Norden manifold, the definition of the generalized $\overline{\partial}_{\hat{j}}$ -operator and the description of some generalized holomorphic sections.

2 Preliminaries

2.1 Generalized geometry

Let M be a smooth manifold of real dimension n and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of M. Smooth sections of E are elements $X + \xi \in C^{\infty}(E)$ where $X \in C^{\infty}(T(M))$ is a vector field and $\xi \in C^{\infty}(T^*(M))$ is a 1- form.

E is equipped with a natural symplectic structure defined by:

$$(X + \xi, Y + \eta) = -\frac{1}{2}(\xi(Y) - \eta(X))$$
(1)

and a natural *indefinite metric* defined by:

$$\langle X + \xi, Y + \eta \rangle = -\frac{1}{2}(\xi(Y) + \eta(X)).$$
 (2)

< , > is non degenerate and of signature (n, n).

A linear connection on M, ∇ , defines, in a canonical way, a bracket on $C^{\infty}(E)$, $[,]_{\nabla}$, as follows:

$$[X + \xi, Y + \eta]_{\nabla} = [X, Y] + \nabla_X \eta - \nabla_Y \xi.$$
(3)

The following holds:

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Lemma 1 ([15]) For all $X, Y \in C^{\infty}(T(M))$, for all $\xi, \eta \in C^{\infty}(T^*(M))$ and for all $f \in C^{\infty}(M)$ we have:

1. $[X + \xi, Y + \eta]_{\nabla} = -[Y + \eta, X + \xi]_{\nabla}$,

2. $[f(X + \xi), Y + \eta]_{\nabla} = f[X + \xi, Y + \eta]_{\nabla} - Y(f)(X + \xi),$

3. Jacobi's identity holds for $[\ ,\]_{\nabla}$ if and only if ∇ has zero curvature.

We consider the following concept of generalized complex structure, introduced in [15], [16] and further investigated in [17], [18], [3]:

Definition 2 A generalized complex structure on M is an endomorphism \widehat{J} , $\widehat{J}: E \to E$ such that $\widehat{J}^2 = -I$.

A pseudo-Riemannian metric on M, g, defines, in a natural way, a complex structure J^g on E by:

$$J^{g}(X+\xi) = -g^{-1}(\xi) + g(X)$$
(4)

where $g: T(M) \to T^*(M)$ is identified to the bemolle musical isomorphism defined by:

$$g(X)(Y) = g(X, Y), \tag{5}$$

in block matrix form, is:

$$J^{g} = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}.$$
 (6)

Definition 3 A generalized complex structure \hat{J} is called pseudo calibrated if is (,)-invariant and if the bilinear symmetric form on T(M) defined by (, J) is non degenerate, moreover \hat{J} is called calibrated if $(, \hat{J})$ is positive definite, [15].

A direct computation shows that J^g is pseudo calibrated.

Let ∇ be a linear connection on M and let $[,]_{\nabla}$ be the bracket on $C^{\infty}(E)$ defined by ∇ , the following holds:

Lemma 4 ([16]) Let $\widehat{J} : E \to E$ be a generalized complex structure on M and let

$$N^{\nabla}(\widehat{J}): C^{\infty}(E) \times C^{\infty}(E) \to C^{\infty}(E)$$
(7)

defined by:

$$N^{\nabla}(\widehat{J})(\sigma,\tau) = \left[\widehat{J}\sigma,\widehat{J}\tau\right]_{\nabla} - \widehat{J}\left[\widehat{J}\sigma,\tau\right]_{\nabla} - \widehat{J}\left[\sigma,J\tau\right]_{\nabla} - [\sigma,\tau]_{\nabla}$$
(8)

for all $\sigma, \tau \in C^{\infty}(E)$; $N^{\nabla}(\widehat{J})$ is a skew symmetric tensor.

Definition 5 $N^{\nabla}(\widehat{J})$ is called the Nijenhuis tensor of \widehat{J} with respect to ∇ .

Definition 6 Let $\widehat{J} : E \to E$ be a generalized complex structure on M, \widehat{J} is called ∇ -integrable if $N^{\nabla}(\widehat{J}) = 0$.

Proposition 7 ([16]) Let ∇ be a torsion free connection on M and let

$$J^{g} = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}$$
(9)

be the generalized complex structure on M defined by a pseudo-Riemannian metric g, J^g is ∇ -integrable if and only if g is a Codazzi tensor, that is for all $X, Y \in \mathbb{C}^{\infty}(T(M))$ we have:

$$(\nabla_X g)Y = (\nabla_Y g)X. \tag{10}$$

Definition 8 ([1]), ([4]), ([19]) Let (M, g, ∇) be a pseudo-Riemannian manifold with a torsion free linear connection, if ∇g is symmetric then (M, g, ∇) is called a statistical manifold.

Corollary 9 Let ∇ be a torsion free connection on M and let J^g be the generalized complex structure on M defined by a pseudo-Riemannian metric g, J^g is ∇ -integrable if and only if (M, g, ∇) is a statistical manifold.

2.2 Norden manifolds

Norden manifolds were introduced by A. P. Norden in [20] and then studied also under the names of almost complex manifolds with B-metric and anti-Kählerian manifolds, [2], [9]. They have applications in mathematics and in theoretical physics.

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Definition 10 Let (M, J) be an almost complex manifold of real dimension 2n and let g be a pseudo-Riemannian metric on M, if J is a g-symmetric operator then g is called Norden metric and (M, J, g) is called Norden manifold.

Remark 11 We can easily prove that a Norden metric g on a 2n-dimensional almost complex manifold is of (n, n)-signature, that is g is a neutral metric.

Let (M, J, g) be a complex Norden manifold, that is a Norden manifold with J integrable, then there exists a natural canonical connection on M, precisely the following holds:

Theorem 12 ([9]) On a complex manifold with Norden metric (M, J, g) there exists a unique linear connection D with torsion T such that:

$$(D_X g)(Y, Z) = 0 \tag{11}$$

$$T(JX,Y) = -T(X,JY)$$
⁽¹²⁾

$$g(T(X,Y),Z) + g(T(Y,Z),X) + g(T(Z,X),Y) = 0$$
(13)

for all vector fields X, Y, Z on M. D is called the natural canonical connection of the Norden manifold or B-connection and it is defined by:

$$D_X Y = \nabla_X Y - \frac{1}{2} J (\nabla_X J) Y \tag{14}$$

where ∇ is the Levi-Civita connection of g.

We remark that (14) is equivalent to:

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$$D_X Y = \frac{1}{2} \left(\nabla_X Y - J \nabla_X J Y \right) \tag{15}$$

then, by direct computation we get the following Proposition.

Proposition 13 If D is the natural canonical connection of the complex Norden manifold (M, J, g) then

$$DJ = 0. \tag{16}$$

Definition 14 Let (M, J, g) be a Norden manifold and let

$$\widetilde{g}(X,Y) = g(JX,Y). \tag{17}$$

for all X and Y vector fields on M. \tilde{g} is a pseudo-Riemannian metric on M with (n, n)-signature and (M, J, \tilde{g}) is a Norden manifold. \tilde{g} is called the associated metric to g. \tilde{g} is also called the twin or the dual metric of g.

2.3 Kähler-Norden manifolds

Kähler-Norden manifolds are strictly related with complex analysis and they will be the main object of our theory. We recall here the definition and the main properties of Kähler-Norden manifolds, for details see [2],[11], [23].

Definition 15 Let (M, J, g) be a Norden manifold and let ∇ be the Levi-Civita connection of g, if $\nabla J = 0$ then (M, J, g) is called Kähler-Norden manifold.

We remark that for a Kähler-Norden manifold (M, J, g) the structure J is integrable and the natural canonical connection is the Levi-Civita connection.

Moreover the following holds:

Theorem 16 ([22]) Let (M, J, g) be a Kähler-Norden manifold, the Levi-Civita connection of g coincides with the Levi-Civita connection of the associated metric \tilde{g} , in particular the Riemann curvature tensors of g and \tilde{g} coincide.

A large class of Kähler-Norden manifolds is given by complex parallelisable manifolds, ([2]).

An interesting property of Kähler-Norden manifolds is the following:

Proposition 17 ([2]) Let (M, J, g) be a Kähler-Norden manifold then, extending g by \mathbb{C} -linearity to the complexified tangent bundle $T(M) \otimes \mathbb{C}$, the components of the complex extended metric, \hat{g} , are holomorphic functions.

We recall that on a complex manifold (M, J) an element $X \in C^{\infty}(TM)$ is an *infinitesimal automorphism* of the complex structure J on M if and only if X satisfies the following condition:

$$[X, JY] = J[X, Y] \tag{18}$$

for all $Y \in C^{\infty}(TM)$.

On Kähler-Norden manifolds, from the condition $\nabla J = 0$, (18) can be written as:

$$\nabla_{JY}X = \nabla_Y JX. \tag{19}$$

The Riemannian curvature tensor of a Kähler-Norden manifold has interesting properties, precisely we have the following:

Theorem 18 ([11]), ([22]) In a Kähler-Norden manifold the Riemannian curvature tensor, \mathbb{R}^{∇} , of the Norden metric g is pure in all arguments, that is, for all $X, Y, Z, W \in C^{\infty}(T(M))$:

$$g(R^{\nabla}(JX,Y)Z,W) = g(R^{\nabla}(X,JY)Z,W)$$

= $g(R^{\nabla}(X,Y)JZ,W)$
= $g(R^{\nabla}(X,Y)Z,JW).$ (20)

2.4 Complex Lie algebroids

Lie algebroids were introduced by J. Pradines in [21]; we recall here the definition and the main properties.

Definition 19 A complex Lie algebroid is a complex vector bundle L over a smooth real manifold M such that: a Lie bracket [,] is defined on $C^{\infty}(L)$, a smooth bundle map $\rho : L \to T(M)$, called anchor, is defined and, for all $\sigma, \tau \in C^{\infty}(L)$, for all $f \in C^{\infty}(M)$ the following conditions hold:

1.
$$\rho([\sigma, \tau]) = [\rho(\sigma), \rho(\tau)]$$

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2. $[f\sigma, \tau] = f([\sigma, \tau]) - (\rho(\tau)(f))\sigma.$

Let L and its dual vector bundle L^* be Lie algebroids; on sections of $\wedge L$, respectively $\wedge L^*$, the Schouten bracket is defined by:

$$[,]_{L}: C^{\infty}(\wedge^{p}L) \times C^{\infty}(\wedge^{q}L) \longrightarrow C^{\infty}(\wedge^{p+q-1}L)$$
(21)

$$[X_1 \wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_q]_L =$$

and, for $f \in C^{\infty}(M)$, $X \in C^{\infty}(L)$

$$[X, f]_L = -[f, X]_L = \rho(X)(f);$$
(23)

respectively, by:

$$[,]_{L^*}: C^{\infty}(\wedge^p L^*) \times C^{\infty}(\wedge^q L^*) \longrightarrow C^{\infty}(\wedge^{p+q-1}L^*)$$
(24)
$$[\mathbf{X}^* \wedge \dots \wedge \mathbf{X}^* \mathbf{Y}^* \wedge \dots \wedge \mathbf{Y}^*] = -$$

$$[X_{1}^{*} \land ... \land X_{p}, I_{1}^{*} \land ... \land I_{q}]_{L^{*}} = \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [X_{i}^{*}, Y_{j}^{*}]_{L^{*}} \land X_{1}^{*} \land ..^{\hat{i}} ... \land X_{p}^{*} \Lambda Y_{1}^{*} \land ..^{\hat{j}} ... \land Y_{q}^{*}$$

$$(25)$$

and, for $f \in C^{\infty}(M)$, $X \in C^{\infty}(L^*)$

$$[X, f]_{L^{\star}} = -[f, X]_{L^{\star}} = \rho(X)(f).$$
(26)

Moreover the exterior derivatives d and d_* associated with the Lie algebroid structure of L and L^* are defined respectively by:

$$d: C^{\infty}(\wedge^{p}L^{*}) \longrightarrow C^{\infty}(\wedge^{p+1}L^{*})$$
(27)

$$(d\alpha) (\sigma_0, ..., \sigma_p) =$$

$$= \sum_{\substack{i=0 \\ i \neq j}}^{p} (-1)^i \rho(\sigma_i) \alpha \left(\sigma_0, ..^{\widehat{i}} .., \sigma_p\right) + \sum_{i < j} (-1)^{i+j} \alpha \left([\sigma_i, \sigma_j]_L, \sigma_0, ..^{\widehat{i}} ..^{\widehat{j}} .., \sigma_p \right)$$
(28)

for $\alpha \in C^{\infty}(\wedge^{p}L^{*}), \sigma_{0}, ..., \sigma_{p} \in C^{\infty}(L)$, and:

a:

$$d_*: C^{\infty}(\wedge^p L) \longrightarrow C^{\infty}(\wedge^{p+1}L) \qquad (29)$$

$$(d_*\alpha)(\sigma_0, ..., \sigma_p) =$$

$$=\sum_{i=0}^{p}(-1)^{i}\rho\left(\sigma_{i}\right)\alpha\left(\sigma_{0},\ldots^{\widehat{i}},\sigma_{p}\right)+\sum_{i\leqslant j}(-1)^{i+j}\alpha\left(\left[\sigma_{i},\sigma_{j}\right]_{L^{\bullet}},\sigma_{0},\ldots^{\widehat{i}},\widehat{j},\sigma_{p}\right)$$
(30)

for $\alpha \in C^{\infty}(\wedge^{p}L), \sigma_{0}, ..., \sigma_{p} \in C^{\infty}(L^{*})$.

3 Generalized geometry of Norden manifolds

3.1 Generalized complex structures

Let (M, J, g) be a Norden manifold, the almost complex structure J and the pseudo Riemannian metric g define, in a natural way, a complex structure \widehat{J} on E by:

$$\widehat{J}(X+\xi) = J(X) + g(X) - J^{*}(\xi)$$
(31)

where $J^*: T^*(M) \to T^*(M)$ is the dual operator of J defined by:

$$J^{*}(\xi)(X) = \xi(J(X)).$$
(32)

In block matrix form, is:

$$\widehat{J} = \begin{pmatrix} J & O \\ g & -J^* \end{pmatrix}.$$
(33)

Remark 20 From the g-symmetry of J it follows immediately that \hat{J} is a pseudo calibrated generalized complex structure on M, see also [16].

A direct computation gives the following:

Proposition 21 Let (M, J, g) be a Norden manifold and let ∇ be a linear connection on M with torsion T, let \widehat{J} be the generalized complex structure defined by J and g, we have:

$$N^{\nabla}(\widehat{J})(X,Y) = (\nabla_{JX}J)Y - J(\nabla_XJ)Y - (\nabla_{JY}J)X + J(\nabla_YJ)X + -T(JX,JY) + JT(X,JY) + JT(JX,Y) + T(X,Y) + +g((\nabla_YJ)X - (\nabla_XJ)Y) + g(T(X,JY) + T(JX,Y)) + + (\nabla_{JX}g)Y - (\nabla_{JY}g)X + (\nabla_Xg)JY - (\nabla_Yg)JX$$
(34)

$$N^{\nabla}(\widehat{J})(X,\xi) = -J^*(\nabla_X J^*)\xi - (\nabla_J J^*)\xi$$
(35)

$$N^{\nabla}(\widehat{J})(\xi,\eta) = 0 \tag{36}$$

for all $X, Y \in C^{\infty}(T(M))$ and for all $\xi, \eta \in C^{\infty}(T^{*}(M))$.

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Corollary 22 \hat{J} is ∇ -integrable if and only if the following conditions hold:

$$(37)$$

$$T(JX, JY) - JT(X, JY) - JT(JX, Y) - T(X, Y) = O$$
(38)

$$g((\nabla_Y J)X - (\nabla_X J)Y) + g(T(X, JY) + T(JX, Y)) + (\nabla_J Xg)Y - (\nabla_J Yg)X + (\nabla_X g)JY - (\nabla_Y g)JX = 0$$
(39)

for all $X, Y \in C^{\infty}(T(M))$.

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Corollary 23 If \widehat{J} is ∇ -integrable then J is integrable.

Proof. Let N(J) be the Nijenhuis tensor of the almost complex structure J, we have:

$$N(J)(X,Y) = (\nabla_{JX}J)Y - J(\nabla_XJ)Y - (\nabla_{JY}J)X + J(\nabla_YJ)X + -T(JX,JY) + JT(X,JY) + JT(JX,Y) + T(X,Y)$$
(40)

for all $X, Y \in C^{\infty}(T(M))$, then the statement follows from Corollary 22.

As we are interested in integrable generalized complex structures in the following we will assume that (M, J, g) is a complex Norden manifold. In particular we get:

Proposition 24 Let (M, J, g) be a complex Norden manifold and let D be the natural canonical connection on M, let \hat{J} be the generalized complex structure defined by J and g, then \hat{J} is D-integrable.

Proof. It follows from the properties of D described in Theorem 12 and in Proposition 13.

Analogous statement can be given for the associated metric, precisely the following holds:

Proposition 25 Let (M, J, g) be a complex Norden manifold and let \tilde{D} be the natural canonical connection of the associated metric \tilde{g} , let \tilde{J} be the generalized complex structure defined by J and \tilde{g} , then \tilde{J} is \tilde{D} -integrable.

3.2 Generalized $\overline{\partial}_{\widehat{j}}$ -operator

Let (M, J, g) be a complex Norden manifold and let \widehat{J} be the generalized complex structure on M defined by J and g, let

$$E^{\mathbb{C}} = (T(M) \oplus T^{*}(M)) \otimes \mathbb{C}$$
(41)

be the complexified generalized tangent bundle. The splitting in $\pm i$ eigenspaces of \hat{J} is denoted by:

$$E^{\mathbb{C}} = E_{\hat{j}}^{1,0} \oplus E_{\hat{j}}^{0,1}$$
(42)

with

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$$E_{\hat{j}}^{0,1} = \overline{E_{\hat{j}}^{1,0}}.$$
 (43)

A direct computation gives:

$$E_{\widehat{J}}^{1,0} = \{ Z - iJZ + g(W + iJW - iZ) \mid Z, W \in T(M) \otimes \mathbb{C} \}, \qquad (44)$$

equivalently $E_{\widehat{J}}^{1,0}$ is generated by elements of the following type:

$$X - iJX - ig(X) \text{ with } X \in C^{\infty}(TM),$$
(45)

$$g(Y+iJY)$$
 with $Y \in C^{\infty}(TM)$. (46)

Analogously we have:

$$\overset{\delta}{=} E_{\widehat{J}}^{0,1} = \{ Z + iJZ + g(W - iJW + iZ) \mid Z, W \in T(M) \otimes \mathbb{C} \}$$

$$(47)$$

and $E_{\widehat{I}}^{0,1}$ is generated by elements of the following type:

$$X + iJX + ig(X) \text{ with } X \in C^{\infty}(TM),$$
(48)

$$g(Y - iJY)$$
 with $Y \in C^{\infty}(TM)$. (49)

Moreover, for any linear connection ∇ , the following holds:

Lemma 26 $E_{\widehat{J}}^{1,0}$ and $E_{\widehat{J}}^{0,1}$ are $[,]_{\nabla}$ -involutive if and only if $N^{\nabla}(\widehat{J}) = 0$.

Proof. Let $P_+ : E^{\mathbb{C}} \to E_{\widehat{J}}^{1,0}$ and $P_- : E^{\mathbb{C}} \to E_{\widehat{J}}^{0,1}$ be the projection operators:

$$P_{\pm} = \frac{1}{2} (I \mp i \widehat{J}), \tag{50}$$

for all $\sigma, \tau \in C^{\infty}(E^{\mathbb{C}})$ we have:

$$P_{\mp} \left[P_{\pm}(\sigma), P_{\pm}(\tau) \right]_{\nabla} = P_{\mp} \left[\frac{1}{2} \left(\sigma \mp i \widehat{J} \sigma \right), \frac{1}{2} \left(\tau \mp i \widehat{J} \tau \right) \right]_{\nabla}$$

$$= -\frac{1}{8} (N^{\nabla}(\widehat{J}) \left(\sigma, \tau \right) \pm i \widehat{J} N^{\nabla}(\widehat{J}) \left(\sigma, \tau \right)) = -\frac{1}{4} P_{\mp} \left(N^{\nabla}(\widehat{J}) \left(\sigma, \tau \right) \right).$$
(51)

From now on we suppose that (M, J, g, D) is a complex Norden manifold with the natural canonical connection. A direct computation of the bracket associated to D on $E_{\widehat{J}}^{1,0}$ and $E_{\widehat{J}}^{0,1}$ gives the following:

$$\sigma = X - iJX - ig(X)$$

$$\tau = Y - iJY - ig(Y)$$

$$v = Z - iJZ - ig(Z).$$
(66)

Let us compute

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$$Jac[[g(X+iJX), Y-iJY-ig(Y)]_D, Z-iJZ-ig(Z)]_D.$$
 (67)

We have:

$$[[g(X+iJX), Y-iJY-ig(Y)]_D, Z-iJZ-ig(Z)]_D = g(K+iJK)$$
(68)

$$[[Y - iJY - ig(Y), Z - iJZ - ig(Z)]_D, g(X + iJX)]_D = g(L + iJL)$$
(69)

$$g[[Z - iJZ - ig(Z), g(X + iJX)]_D Y - iJY - ig(Y)]_D = g(H + iJH)$$
(70)

where

$$K = D_Z D_Y X + D_Z J D_{JY} X + J D_{JZ} D_Y X + J D_{JZ} J D_{JY} X$$
(71)

$$L = D_{[Y,Z]}X + JD_{J[Y,Z]}X - D_{[JY,JZ]}X - JD_{J[JY,JZ]}X$$
(72)

$$H = -D_Y D_Z X - J D_Y D_{JZ} X - J D_{JY} D_Z X + D_{JY} D_{JZ} X.$$
(73)

Then we get

$$Jac\left[\left[\sigma,\tau\right]_{D},\upsilon\right]_{D}=O\tag{74}$$

 $\begin{array}{c} \text{if and only if} \\ \overset{\bullet}{} \end{array}$

$$K + L + H = O \tag{75}$$

or, by direct computation, if and only if:

$$R^{D}(JY, JZ) - JR^{D}(JY, Z) - JR^{D}(Y, JZ) - R^{D}(Y, Z) - JD_{JN(J)(Y,Z)} = O$$
 (76)

where N(J) is the Nijenhuis tensor of J. By using the integrability of J, we have the first condition.

Let us compute

$$Jac[[X - iJX - ig(X), Y - iJY - ig(Y)]_{D}, Z - iJZ - ig(Z)]_{D}.$$
 (77)

We have:

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$$[[X - iJX - ig(X), Y - iJY - ig(Y)]_D, Z - iJZ - ig(Z)]_D = = A - iJA - ig(A) + g(B + iJB)$$
(78)

or

where

\$

$$A = [[X, Y] - [JX, JY], Z] - [J[X, Y] - J[JX, JY], JZ]$$
(79)

 and

$$B = D_{JZ} [X, Y] + D_{JZ} T^{D} (JX, JY) - D_{J} [X, Y] Z + + D_{J} [JX, JY] Z - D_{Z} D_{JY} X + D_{Z} D_{JX} Y$$
(80)

where T^D denotes the torsion tensor of the connection D. Fom the Jacobi identity of [,] we have that Jac(A) = O, then it is enough to

compute Jac(B). From the properties of the torsion tensor T^{D} we get:

$$Jac(B) = (R^{D}(JX, Y) + R^{D}(X, JY))Z + (R^{D}(JZ, X) + R^{D}(Z, JX))Y + (R^{D}(Y, JZ) + R^{D}(JY, Z))X.$$
(81)

Analogous computations for $E_{\widehat{J}}^{0,1}$ gives exactly the same conditions, then the Proof is complete.

Remark 29 We observe that (61) is equivalent to:

$$(R^D)^{(0,2)} = O (82)$$

where $(R^D)^{(0,2)}$ denotes the (0,2)-part of the curvature with respect to the complex structure J on M. Moreover, if the torsion is zero, from the first Bianchi identity with zero torsion, we get that (62) is automatically satisfied; instead, from the first Bianchi identity with torsion:

$$R^{D}(X,Y)Z + R^{D}(Y,Z)X + R^{D}(Z,X)Y + -T^{D}(X,[Y,Z]) - T^{D}(Y,[Z,X]) - T^{D}(Z,[X,Y]) + -D_{X}T(Y,Z) - D_{Y}T(Z,X) - D_{Z}T(X,Y) = O,$$
(83)

we obtain that (62) is equivalent to the following:

$$(R^{D}(JX, JY) - R^{D}(X, Y)) Z + (R^{D}(JZ, JX) - R^{D}(Z, X)) Y + + (R^{D}(JY, JZ) - R^{D}(Y, Z)) X = O.$$
(84)

From Proposition 26 we get in particular the following:

Proposition 30 If
$$R^D = O$$
 then $E_{\hat{j}}^{1,0}$ and $E_{\hat{j}}^{0,1}$ are complex Lie algebroids.

In this sense the following result provides a class of examples, ([10]), ([13]).

Theorem 31 ([10]), ([13]) Each hyper-Kaehler NH-manifold is a flat pseudo-Riemannian manifold of signature (2n, 2n). More generally we have the following:

Theorem 32 Let (M, J, g) be a Kähler-Norden manifold then $E_{\hat{J}}^{1,0}$ and $E_{\hat{J}}^{0,1}$ are complex Lie algebroids.

Proof. In this case the natural canonical connection D is the Levi-Civita connection ∇ and, as its torsion is zero, (62) is automatically satisfied. Moreover from (20) we get that (61) is equivalent to:

$$R^{\nabla}(Y,Z) + R^{\nabla}(JY,Z)J = O \tag{85}$$

and, by using again the fact that R^{∇} is a pure tensor, we have that, for all $Y, Z, W \in C^{\infty}(T(M))$, (85) becomes:

$$R^{\nabla}(Y,Z)W + R^{\nabla}(Y,Z)JJW = O$$
(86)

which is automatically satisfied. Thus the proof is complete. \blacksquare

Remark 33 Analogous statement can be given for $E_{\tilde{J}}^{1,0}$ and $E_{\tilde{J}}^{0,1}$. In the following we will consider only \hat{J} .

The following holds:

Proposition 34 The natural symplectic structure on E defines a canonical isomorphism between $E_{\widehat{J}}^{0,1}$ and the dual bundle of $E_{\widehat{J}}^{1,0}$, $\left(E_{\widehat{J}}^{1,0}\right)^*$.

Proof. We define

$$\varphi: E_{\widehat{J}}^{0,1} \to \left(E_{\widehat{J}}^{1,0}\right)^* \tag{87}$$

by:

$$(\varphi \left(Z + iJZ + g(W - iJW + iZ)\right)) \left(X - iJX + g(Y + iJY - iX)\right) =$$

= $(Z + iJZ + g(W - iJW + iZ), X - iJX + g(Y + iJY - iX))$
(88)

for all $X, Y, Z, W \in T(M) \otimes \mathbb{C}$. We get:

$$(\varphi (Z + iJZ + g(W - iJW + iZ))) (X - iJX + g(Y + iJY - iX)) = = g(Y, Z) - g(W, X) + i (g(W, JX) + g(Y, JZ) - g(X, Z))$$
(89)

and we extend by linearity. We have immediately that φ is injective and furthermore φ is an isomorphism.

The canonical isomorphism φ between $E_{\widehat{J}}^{0,1}$ and the dual bundle $\left(E_{\widehat{J}}^{1,0}\right)^*$ allows us to define the $\overline{\partial}_{\widehat{J}}$ – operator associated to the complex structure \widehat{J} as in the following:

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let $f \in C^{\infty}(M)$ and let $df \in C^{\infty}(T^*(M)) \hookrightarrow C^{\infty}(T(M) \oplus T^*(M))$, we pose

$$\overline{\partial}_{\widehat{J}}f = 2\left(df\right)^{0,1} = df + i\widehat{J}df \tag{90}$$

or:

$$\overline{\partial}_{\widehat{J}}f = df - iJ^* (df)$$

$$= df - i (df) J;$$
(91)

moreover we define:

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$$\overline{\partial}_{\widehat{j}}: C^{\infty}\left(E^{0,1}_{\widehat{j}}\right) \to C^{\infty}\left(\wedge^{2}\left(E^{0,1}_{\widehat{j}}\right)\right)$$
(92)

via the natural isomorphism

$$E_{\widehat{j}}^{0,1} \stackrel{\varphi}{\simeq} \left(E_{\widehat{j}}^{1,0}\right)^* \tag{93}$$

as:

$$\overline{\partial}_{\widehat{J}}: C^{\infty}\left(\left(E_{\widehat{J}}^{1,0}\right)^{*}\right) \to C^{\infty}\left(\wedge^{2}\left(E_{\widehat{J}}^{1,0}\right)^{*}\right)$$
(94)

$$\left(\overline{\partial}_{\tilde{J}}\alpha\right)(\sigma,\tau) = \rho\left(\sigma\right)\alpha\left(\tau\right) - \rho\left(\tau\right)\alpha\left(\sigma\right) - \alpha\left(\left[\sigma,\tau\right]_{D}\right)$$
(95)

for $\alpha \in C^{\infty}\left(\left(E_{\widehat{J}}^{1,0}\right)^{*}\right), \, \sigma, \tau \in C^{\infty}\left(E_{\widehat{J}}^{1,0}\right).$

In general:

$$\overline{\partial}_{\widehat{J}}: C^{\infty}\left(\wedge^{p}\left(E_{\widehat{J}}^{1,0}\right)^{*}\right) \to C^{\infty}\left(\wedge^{p+1}\left(E_{\widehat{J}}^{1,0}\right)^{*}\right)$$
(96)

is defined by:

$$(\overline{\partial}_{\widehat{j}}\alpha) (\sigma_{0},...,\sigma_{p}) =$$

$$= \sum_{i=0}^{p} (-1)^{i} \rho (\sigma_{i}) \alpha \left(\sigma_{0},..^{\widehat{i}}..,\sigma_{p}\right) + \sum_{i < j} (-1)^{i+j} \alpha \left(\left[\sigma_{i},\sigma_{j}\right]_{D},\sigma_{0},..^{\widehat{i}}..^{\widehat{j}}..,\sigma_{p}\right)$$
for $\alpha \in C^{\infty} \left(\wedge^{p} \left(E_{\widehat{j}}^{1,0}\right)^{*}\right), \sigma_{0},...,\sigma_{p} \in C^{\infty} \left(E_{\widehat{j}}^{1,0}\right).$

$$(97)$$

Definition 35 $\overline{\partial}_{\hat{j}}$ is called generalized $\overline{\partial}$ – operator of (M, J, g, D) or generalized $\overline{\partial}_{\hat{j}}$ – operator.

We get the following:

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Proposition 36 If (61) and (62) hold then $(\overline{\partial}_{j})^{2} = 0$ and $(\partial_{j})^{2} = 0$.

Proof. It follows from the fact that Jacobi identity holds on $E_{\tilde{j}}^{1,0}$ and $\left(E_{\tilde{j}}^{1,0}\right)^*$.

From now on we suppose that (61) and (62) hold. We have immediately that $\overline{\partial}_{\widehat{J}}$ is the exterior derivative, d_L , of the Lie algebroid $L = E_{\widehat{J}}^{1,0}$. Moreover the exterior derivative d_{L^*} of $L^* = \left(E_{\widehat{J}}^{1,0}\right)^*$ is given by the operator $\partial_{\widehat{J}}$ defined by:

$$\partial_{\widehat{J}^{g}}: C^{\infty}\left(\wedge^{p}\left(E_{\widehat{J}}^{1,0}\right)\right) \to C^{\infty}\left(\wedge^{p+1}\left(E_{\widehat{J}}^{1,0}\right)\right)$$
(98)

$$(\partial_{\widehat{j}}\sigma)\left(\alpha_{0}^{*},...,\alpha_{p}^{*}\right) =$$

$$= \sum_{i=0}^{j} (-1)^{i} \rho\left(\alpha_{i}^{*}\right) \sigma\left(\alpha_{0}^{*},..^{\widehat{i}}..,\alpha_{p}^{*}\right) + \sum_{i < j} (-1)^{i+j} \sigma\left(\left[\alpha_{i}^{*},\alpha_{j}^{*}\right]_{D},\alpha_{0}^{*},..^{\widehat{i}}..^{\widehat{j}}..,\alpha_{p}^{*}\right)$$

$$(99)$$

for $\sigma \in C^{\infty}\left(\wedge^{p}\left(E_{\widehat{J}}^{1,0}\right)\right), \alpha_{0}^{*}, ..., \alpha_{p}^{*} \in C^{\infty}\left(\left(E_{\widehat{J}}^{1,0}\right)^{*}\right).$

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3.3 Generalized holomorphic sections

Definition 37 Let $\alpha \in C^{\infty}\left(\wedge^{p}\left(E_{\hat{j}}^{1,0}\right)^{*}\right)$, α is called generalized holomorphic section if

$$\overline{\partial}_{\widehat{J}}\alpha = 0. \tag{100}$$

We remark that for all $f \in C^{\infty}(M)$ we have $\overline{\partial}_{\widehat{J}}f = 0$ if and only if df = 0, so the generalized holomorphic condition for functions gives only constant functions on connected components of M.

Proposition 38 Let $W \in C^{\infty}(T(M))$ and let $\sigma = g(W - iJW) \in E_{\widehat{J}}^{0,1}$ then $\overline{\partial}_{\widehat{J}}\sigma = 0$ if and only if for all $X, Y \in C^{\infty}(T(M))$ holds:

$$g(D_X W - D_{JX} J W, Y) = g(D_Y W - D_{JY} J W, X).$$
(101)

Proof. Let $X, Y \in C^{\infty}(T(M))$, from (95), direct computations give:

$$\delta_{j} \qquad \left(\overline{\partial}_{\hat{j}}\sigma\right)\left(g(X+iJX),g(Y+iJY)\right) = 0 \tag{102}$$

$$\left(\overline{\partial}_{\widehat{J}}\sigma\right)\left(g(X+iJX),Y-iJY-ig(Y)\right) = 0 \tag{103}$$

$$\left(\overline{\partial}_{\widehat{J}}\sigma
ight)\left(X-iJX-ig(X),Y-iJY-ig(Y)
ight)=$$

$$= g(-D_XW + D_{JX}JW + i(D_{JX}W + iJD_XW, Y) + (104) + g(D_YW - D_{JY}JW - i(D_{JY}W + JD_YW), X).$$

In particular we have $(\overline{\partial}_{\widehat{\jmath}}\sigma) = 0$ if and only if:

$$g(-D_X W + D_{JX} JW + i(D_{JX} W + iJD_X W, Y) + g(D_Y W - D_{JY} JW - i(D_{JY} W + JD_Y W), X) = 0$$
(105)

and then, by separating real and imaginary parts, we get the statement. \blacksquare

Equivalently we can state Proposition 36 as follows:

Proposition 39 Let $W \in C^{\infty}(T(M))$ and let $\sigma = g(W - iJW) \in E_{\widehat{J}}^{0,1}$ then $\overline{\partial}_{\widehat{J}}\sigma = 0$ if and only if for all $X, Y \in C^{\infty}(T(M))$ holds:

$$(d(g(W)))(X,Y) = (d(g(W)))(JX,JY).$$
(106)

Proof. We have:

$$(d(g(W)))(X,Y) = Xg(W,Y) - Yg(W,X) - g(W,[X,Y])$$

= $g(D_XW,Y) - g(D_YW,X) - g(W,T^D(X,Y)).$ (107)

On the other hand: λ

$$(d^{\bullet}(g(W))) (JX, JY) = JXg(W, JY) - JYg(W, JX) - g(W, [JX, JY])$$

= $g(D_{JX}W, JY) - g(D_{JY}W, JX) - g(W, T^{D}(JX, JY))$
= $g(D_{JX}JW, Y) - g(D_{JY}JW, X) - g(W, T^{D}(JX, JY)).$
(108)

From the property (12) of the torsion T^D of the natural canonical connection we get the conclusion.

Moreover:

Proposition 40 Let $Z \in C^{\infty}(T(M))$ and let $\sigma = Z + iJZ + ig(Z) \in E_{\widehat{J}}^{0,1}$ then $\overline{\partial}_{\widehat{J}}\sigma = 0$ if and only if for all $X, Y \in C^{\infty}(T(M))$ the following conditions hold:

$$D_{JY}JZ = -D_YZ \tag{109}$$

$$g(D_X Z, Y) = g(D_Y Z, X).$$
(110)

Proof. Let $X, Y \in C^{\infty}(T(M))$, direct computations give:

$$\left(\overline{\partial}_{\widehat{J}}\sigma\right)\left(g(X+iJX),g(Y+iJY)\right) = 0 \tag{111}$$

$$\left(\bar{\partial}_{j}\sigma\right)\left(g(X+iJX),Y-iJY-ig(Y)\right) =$$
(112)

$$= -g(D_YZ + D_{JY}JZ, X) + ig(D_{JY}Z - D_YJZ, X)$$
⁽¹¹²⁾

$$\left(\overline{\partial}_{\overline{J}}\sigma\right)\left(X - iJX - ig(X), Y - iJY - ig(Y)\right) =$$

$$= -g(iD_X Z + D_{JX} Z, Y) + g(iD_Y Z + D_{JY} Z, X)$$
(113)

$$= g(D_{JY}Z, X) - g(D_{JX}Z, Y) + i(g(D_YZ, X) - g(D_XZ, Y)).$$

.

and, by separating real and imaginary parts, we get the following conditions:

$$D_{JY}JZ + D_YZ = O \tag{114}$$

$$g(D_{JY}Z, X) - g(D_{JX}Z, Y) = O;$$
 (115)

From (114) we get

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$$D_{JY}Z = JD_YZ \tag{116}$$

and, substituting in (115), we have

$$g(D_Y Z, JX) - g(D_{JX} Z, Y) = O$$
(117)

for all $X, Y \in C^{\infty}(T(M))$, then we get the statement.

Corollary 41 Given $Z \in C^{\infty}(T(M))$, infinitesimal automorphism of J, Z defines the following generalized holomorphic sections of $E_{\tilde{J}}^{0,1}$:

$$\sigma = g(Z - iJZ) \tag{118}$$

$$\tau = Z + iJZ + ig(Z) \tag{119}$$

if and only if for all $X, Y \in C^{\infty}(T(M))$ the following condition hold:

$$g(D_X Z, Y) = g(D_Y Z, X).$$
(120)

In particular for Kähler-Norden manifolds, as D is the Levi-Civita connection and then torsion free, condition (120) is equivalent to the d-closure of g(Z), and, by using a classical result in symplectic geometry, [14], we have:

Proposition 42 Let M be a Kähler-Norden manifold and let $Z \in C^{\infty}(T(M))$ be an infinitesimal automorphism of J then g(Z - iJZ) and Z + iJZ + ig(Z)are generalized holomorphic sections of $E_{\widehat{J}}^{0,1}$ if and only if g(Z) is a Lagrangian submanifold of $T^{*}(M)$ with respect to the standard symplectic structure.

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