THE HADAMARD VARIATIONAL FORMULA AND THE MINKOWSKI PROBLEM FOR p-CAPACITY

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ABSTRACT. A Hadamard variational formula for p-capacity of convex bodies in $\mathbb{R}^n$ is established when $1 < p < n$. The formula is applied to solve the Minkowski problem for p-capacity which involves a degenerate Monge-Ampère type equation. Uniqueness for the Minkowski problem for p-capacity is established when $1 < p < n$ and existence and regularity when $1 < p < 2$. These results are (non-linear) extensions of the now classical solution of Jerison of the Minkowski problem for electrostatic capacity ($p = 2$).

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1. INTRODUCTION

The Brunn-Minkowski theory (or theory of mixed volumes) in convex geometric analysis begins with the variation of volume. When volume is viewed as a functional defined on the support functions of compact convex sets in $\mathbb{R}^n$, its variational functional turns out to be the classical surface area measure. The notion of mixed volume arises in the variation of volume. Specifically, if K and L are compact convex sets with non-empty interiors, i.e. convex bodies in $\mathbb{R}^n$, then there exists a finite Borel measure S_K, called the *surface area measure of* K, defined on the unit sphere $\mathbb{S}^{n-1}$, so that

$$\frac{d}{dt} V(K + tL)\Big|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(\xi)\, dS_K(\xi), \tag{1.1}$$

where V is the n-dimensional volume (i.e. Lebesgue measure in $\mathbb{R}^n$), h_L is the support function of L, and for $t > 0$, the body $K + tL$, is the Minkowski sum of K and tL, the dilate of L by a factor of t. The surface area measure S_K on $\mathbb{S}^{n-1}$ can be explicitly defined, for Borel $E \subset \mathbb{S}^{n-1}$, by

$$S_K(E) = \mathcal{H}^{n-1}\big(\mathsf{g}_K^{-1}(E)\big),$$

where $\mathsf{g}_K : \partial K \to \mathbb{S}^{n-1}$ is the (multi-valued) Gauss map and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. The integral in (1.1), divided by the ambient dimension n, is called the first mixed volume of K and L. This first mixed volume formula (1.1) is a generalization of the well-known volume formula,

$$V(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(\xi)\, dS_K(\xi). \tag{1.2}$$

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The Minkowski problem, which characterizes the surface area measure, is a fundamental problem in convex geometric analysis. Since for smooth convex bodies the reciprocal of the Gauss curvature (viewed as a function of the outer unit normals) is the density of the surface area measure with respect to the spherical Lebesgue measure, the Minkowski problem is the problem in differential geometry of characterizing the Gauss curvature of closed convex hypersurfaces. More precisely the Minkowski problem reads: *given a finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$, under what necessary and sufficient conditions does there exist a unique (up to translations) convex body K such that $S_K = \mu$?* This problem was first studied by Minkowski [81, 82], who demonstrated both existence and uniqueness of solutions when the given measure is either discrete or has a continuous density. Aleksandrov [2, 3] and Fenchel-Jessen [31] solved the problem in 1938 for arbitrary measures. Their methods are variational and use formulas (1.1) and (1.2). The solution of the Minkowski problem identified the conditions

(i) the measure μ is not concentrated on any great subsphere; that is,

$$\int_{\mathbb{S}^{n-1}} |\theta \cdot \xi| \, d\mu(\xi) > 0, \quad \text{for each } \theta \in \mathbb{S}^{n-1},$$

(ii) the centroid of the measure μ is at the origin; that is,

$$\int_{\mathbb{S}^{n-1}} \xi \, d\mu(\xi) = 0, \tag{1.3}$$

on the measure as *necessary and sufficient* conditions for existence and uniqueness. In the smooth case, the Minkowski problem can be formulated via a second order partial differential equation *of Monge-Ampère type* on the unit sphere and, for this reason, establishing the regularity of the solutions to the Minkowski problem is difficult and has led to a long series of influential works, see for example Lewy [67,68], Nirenberg [83], Cheng and Yau [21], Pogorelov [87], Caffarelli [15, 16], etc.

The Minkowski problem has inspired many other problems of a similar nature. Examples include the L_p-Minkowski problem which prescribes the p-surface area measure, see e.g., [5, 6, 22, 48, 74, 76], the logarithmic Minkowski problem which prescribes the cone-volume measure, see [5, 12, 96], the Christoffel-Minkowski problem which prescribes intermediate surface area measures, see [41], and Minkowski type problems which prescribe curvature measures, see [38, 40, 42]. The measures prescribed in these works are the variational functionals of volume and other quermassintegrals with respect to various operations on compact convex sets. These problems present central questions in geometric analysis. As a specific example, the Minkowski problem and its L^p generalization have been used to establish sharp affine Sobolev inequalities, see [23, 45, 46, 78, 79, 105]. Operators that arise as a consequence of the solution of the Minkowski problem (and its L^p generalization) have appeared in, e.g., [71–73, 101].

In his celebrated paper [52], Jerison solved the Minkowski problem that prescribes the capacitary measure, i.e. the measure that is the variational functional arising from the electrostatic (or Newtonian) capacity. The work of Jerison demonstrates a striking similarity between the Minkowski problem for the electrostatic capacitary measure and the Minkowski problem for the surface area measure. Recall that given $E \subset \mathbb{R}^n$, the classical electrostatic capacity $C_2(E)$ of E is defined by

$$C_2(E) = \inf \Big\{ \int_{\mathbb{R}^n} |\nabla u|^2 dx : \ u \in C_c^\infty(\mathbb{R}^n), \ u \geq 1 \text{ on } E \Big\}, \tag{1.4}$$

where $C_c^\infty(\mathbb{R}^n)$ is the set of C^∞ functions in $\mathbb{R}^n$ with compact support. Let Ω be a bounded convex domain, i.e. a bounded open convex set, in $\mathbb{R}^n$, and let $\bar{\Omega}$ be its closure. The equilibrium potential $U = U_\Omega$ of Ω, is the unique solution to the boundary value problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \\ U = 1 & \text{on } \partial\Omega \text{ and } \lim_{|x|\to\infty} U(x) = 0, \end{cases} \tag{1.5}$$

where Δ is the Laplace operator. Using the, by now, classical results on harmonic functions in Lipschitz domains due to Dahlberg [29], it follows that ∇U has non-tangential limits, almost everywhere on $\partial\Omega$, with respect to $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$, and that $|\nabla U| \in L^2(\partial\Omega, \mathcal{H}^{n-1})$. The *electrostatic capacitary measure* $\mu_2(\Omega, \cdot)$ of Ω is then the finite and well-defined Borel measure on the unit sphere $\mathbb{S}^{n-1}$ given, for Borel $E \subset \mathbb{S}^{n-1}$, by

$$\mu_2(\Omega, E) = \int_{\mathsf{g}^{-1}(E)} |\nabla U|^2 \, d\mathcal{H}^{n-1}, \tag{1.6}$$

where $\mathsf{g} : \partial\Omega \to \mathbb{S}^{n-1}$ is the Gauss map. The Minkowski problem for the electrostatic capacitary measure is: *given a finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$, under what necessary and sufficient conditions does there exist a unique (up to translations) bounded convex domain Ω for which $\mu_2(\Omega, \cdot) = \mu$?* In [52] Jerison solved the problem by giving the necessary and sufficient conditions for the existence of a solution and these conditions are identical to corresponding conditions in classical Minkowski problem for the surface area measure and stated as (1.3) (i) and (ii) above. Regularity was also obtained in [52]. Uniqueness was settled by Caffarelli, Jerison and Lieb in [18]. The general outline of Jerison's approach is quite similar to that for the Minkowski problem of surface area measure, but details are different and substantially more complicated compared to the classical Minkowski problem. The Hadamard variational formula,

$$\frac{d}{dt} C_2(\Omega + t\Omega_1)\Big|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_{\Omega_1}(\xi) \, d\mu_2(\Omega, \xi), \tag{1.7}$$

where Ω_1 is an arbitrary convex domain, and its special case, the Poincaré capacity formula,

$$C_2(\Omega) = \frac{1}{n-2} \int_{\mathbb{S}^{n-1}} h_\Omega(\xi) \, d\mu_2(\Omega, \xi), \tag{1.8}$$

play crucial roles in Jerison's proof and bear an amazing resemblance to the volume-formulas (1.1) and (1.2). The work of Jerison demonstrated a striking and unexpected similarity between the Minkowski problem for electrostatic capacity and the Minkowski problem for the surface area measure and the work of Jerison inspired subsequent research in this area. For example, similar problems, still involving a *linear* operator as the Laplace operator Δ, were studied in [53] and more recently in [25] where capacity is replaced by the first eigenvalue of $-\Delta$ and by the torsional rigidity, respectively.

In this paper we extend Jerison's work on electrostatic capacity to p-capacity, hence continuing Jerison's work in a *non-linear* setting. For p such that $1 < p < n$, recall that for $E \subset \mathbb{R}^n$, the p-capacity $C_p(E)$ of E is defined by

$$C_p(E) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla u|^p dx : \; u \in C_c^\infty(\mathbb{R}^n) \text{ and } u \geq 1 \text{ on } E \right\}. \tag{1.9}$$

In this context Jerison's work on the electrostatic capacity corresponds to the case $p = 2$. To extend Jerison's pioneering $p = 2$ results is demanding and highly nontrivial because the linear Laplace operator needs to be replaced with the nonlinear and degenerate p-Laplace operator. Many well-known facts for harmonic functions have not yet been established for p-harmonic functions. Neither of the formulas analogous to (1.7) and (1.8) for p-capacity is known. Fortunately, recent work of Lewis and Nyström on p-harmonic functions, see [59] – [66], makes it possible to define *p-capacitary measures* which generalize the notion of electrostatic capacitary measure. This opens the door to study the p-capacitary Minkowski problem. In this paper we establish extensions of Jerison's work to p-capacity and study the p-capacitary Minkowski problem. To do this we follow a similar but more direct approach than in the linear case $p = 2$ of Jerison [52]. We emphasize that, due to the non-linearity and degeneracy of the underlying partial differential equation, the cases where $p \neq 2$ are considerably more complicated, requiring both new ideas and techniques.

If Ω is a bounded convex domain in $\mathbb{R}^n$ and $1 < p < n$, then U, the p-equilibrium potential of Ω, is the unique solution to the boundary value problem

$$\begin{cases} \Delta_p U = 0 \quad \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \\ \\ U = 1 \qquad \text{on } \partial\Omega \text{ and } \lim_{|x|\to\infty} U(x) = 0\,, \end{cases} \tag{1.10}$$

where Δ_p is the p-Laplace operator defined in (2.1) and (2.2) below. A proof of the existence and uniqueness of U can be found in [57]. In [59] (see also [60]) Lewis and Nyström extended Dahlberg's [29] results for $p = 2$ to the general case $1 < p < \infty$ and, as a consequence, we can conclude that ∇U has non-tangential limits $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega$ and that $|\nabla U| \in L^p(\partial\Omega, \mathcal{H}^{n-1})$. Hence the *$p$-capacitary measure* $\mu_p(\Omega, \cdot)$ of Ω can be defined, for Borel $E \subset \mathbb{S}^{n-1}$, by

$$\mu_p(\Omega, E) = \int_{\mathbf{g}^{-1}(E)} |\nabla U|^p \, d\mathcal{H}^{n-1}. \tag{1.11}$$

In this paper we consider the following Minkowski problem for p-capacity.

Minkowski problem for p-capacity. *Suppose $1 < p < n$. Let μ be a finite Borel measure on $\mathbb{S}^{n-1}$. Under what necessary and sufficient conditions does there exist a (unique) bounded convex domain Ω such that $\mu_p(\Omega, \cdot) = \mu$?*

Our approach to the Minkowski problem for p-capacity is more direct than the approach used by Jerison [52] for the case of $p = 2$. However, it requires a more general variational formula for p-capacity – more general than (1.7). Note that the variation in (1.7) involves only support functions and a limit from above, however we need a variational formula with respect to a generic continuous function on $\mathbb{S}^{n-1}$ and also a two-sided limit. Our approach uses the notion of *Alexandrov domain*, or Wulff shape, associated with a function: if h is a positive continuous function on $\mathbb{S}^{n-1}$, then *the Alexandrov domain associated with h* is the convex domain given by

$$\bigcap_{\xi \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : x \cdot \xi < h(\xi)\}\,. \tag{1.12}$$

Our first result is the following Hadamard variational formula for p-capacity.

Theorem 1.1. *Suppose $1 < p < n$. Let Ω be a bounded convex domain in $\mathbb{R}^n$ whose support function is h_Ω and $f \in C(\mathbb{S}^{n-1})$. Denote by Ω_t the Alexandrov domain associated with the* *function $h_\Omega + tf$. Then*

$$\frac{d}{dt} C_p(\Omega_t)\bigg|_{t=0} = (p-1) \int_{\mathbb{S}^{n-1}} f(\xi)\, d\mu_p(\Omega, \xi)\,. \tag{1.13}$$

If f is also a support function then we recover variational formulas as (1.1) and (1.7) for p-capacity. Moreover, when $f = h_\Omega$, (1.13) gives the Poincaré p-capacity formula,

$$C_p(\Omega) = \frac{p-1}{n-p} \int_{\mathbb{S}^{n-1}} h_\Omega(\xi)\, d\mu_p(\Omega, \xi)\,.$$

The case $p = 2$ of Theorem 1.1 was treated by Jerison in [53]. Our proof is quite different compared to the proof of Jerison, although it follows the same general scheme, and it relies on the Brunn-Minkowski inequality for p-capacity established by Colesanti and Salani, see [28].

We use the Hadamard variational formula (1.13) and the Colesanti-Salani Brunn-Minkowski inequality to establish the following uniqueness result for the Minkowski problem for p-capacity. Note that the case $p = 2$ was proved by Caffarelli, Jerison and Lieb in [18].

Theorem 1.2. *Suppose $1 < p < n$. If Ω_0, Ω_1 are bounded convex domains in $\mathbb{R}^n$ which have the same p-capacitary measures, then Ω_0 is a translate of Ω_1 when $p \neq n-1$, and Ω_0, Ω_1 are homothetic when $p = n-1$.*

Concerning the existence for the Minkowski problem for p-capacity, we have the following result.

Theorem 1.3. *Suppose $1 < p < 2$. Let μ be a finite Borel measure on $\mathbb{S}^{n-1}$ which is not concentrated on any great subsphere and whose centroid is at the origin, i.e., (1.3) (i) and (ii) hold. If, in addition, μ does not have a pair of antipodal point masses, then there exists a bounded convex domain Ω in $\mathbb{R}^n$ such that $\mu_p(\Omega, \cdot) = \mu$.*

The conditions that μ is not concentrated on any great subsphere and that the centroid of μ is at the origin are, as discussed above, necessary and we emphasize that these conditions are exactly the same necessary and sufficient conditions as in Jerison's solution to the Minkowski problem for electrostatic capacity, as well as in the Alexandrov, Fenchel and Jessen's solution to the classical Minkowski problem for the surface area measure. The assumption that μ does not have a pair of antipodal point masses is instead not a necessary condition. It would be interesting if this assumption could be removed. Naturally the extension of Theorem 1.3 to the range $2 < p < n$ is a very interesting open problem.

Concerning the regularity of the solution of the Minkowski problem for p-capacity, we are able to establish the following.

Theorem 1.4. *Suppose $1 < p < 2$, $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Let Ω be a bounded convex domain in $\mathbb{R}^n$. If the p-capacitary measure $\mu_p(\Omega, \cdot)$ of Ω is absolutely continuous with respect to spherical Lebesgue measure, with a strictly positive density of class $C^{k,\alpha}(\mathbb{S}^{n-1})$, then the boundary of Ω is $C^{k+2,\alpha}$ smooth.*

The proof of Theorem 1.4 combines results and techniques contained in [59, 60, 62], with the generalization of the regularity theory for the Monge-Ampère equation, due to Caffarelli, see [14–17], and developed by Jerison [52].

The paper is organized as follows. In Section 2, which is partly of preliminary nature, we introduce notation, recall some basic results concerning the boundary behaviour of p-harmonic functions in Lipschitz domains, and state some facts we will need regarding bounded convex domains. We then derive integral formulas for p-capacity and some estimates for p-harmonic functions. In Section 3, we prove Theorems 1.1 and 1.2 for bodies with $C^{2,\alpha}$-smooth boundaries of positive Gauss curvature. In Section 4, we establish the weak convergence of the p-capacitary measure by using estimates for p-harmonic functions. Section 5 is devoted to the proof of Theorems 1.1 and 1.2 in the general case. The existence result stated in Theorem 1.3 is proved in Section 6. The regularity result, Theorem 1.4, is established in Section 7.

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2. Preliminaries and integral formulas for p-capacity

Throughout the paper we will work in Euclidean n-dimensional space $\mathbb{R}^n$, $n \geq 2$, endowed with the usual norm $|\cdot|$. Points in $\mathbb{R}^n$ are denoted by $x = (x_1, \dots, x_n)$ or sometimes by (x', x_n), where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. The scalar product of $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. The unit sphere of $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x, r)$ denotes the open ball centered at x with radius r. For a subset E of $\mathbb{R}^n$, we denote by $\bar{E}$, ∂E and $\operatorname{diam}(E)$ the closure, boundary and diameter of the set E, respectively. For a positive integer $k \leq n$, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^n$. Integration with respect to Lebesgue measure on $\mathbb{S}^{n-1}$ will often be abbreviated by simply writing $d\xi$. For $E, F \subset \mathbb{R}^n$, let $d(E, F)$ denote the Euclidean distance between E and F. In case $E = \{y\}$, we write $d(y, F)$ and let

$$h(E, F) = \max\{\sup_{y\in E} d(y, F), \sup_{y\in F} d(y, E)\}$$

denote the Hausdorff distance between E and F.

If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions $f \in L^q(O)$ with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$ which is in $L^q(O)$ as well. Let $\|f\|_{1,q} = \|f\|_q + \|\nabla f\|_q$ be the norm in $W^{1,q}(O)$ where $\|\cdot\|_q$ denotes the usual norm in $L^q(O)$. Next, let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O, and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$.

Given a bounded domain G, i.e. a bounded, connected open set, and $1 < p < \infty$, we say that u is p-harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int_G |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{2.1}$$

whenever $\theta \in W_0^{1,p}(G)$. Observe that if u is smooth and satisfies (2.1), and if $\nabla u \neq 0$ in G, then

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \quad \text{in } G, \tag{2.2}$$

and u is a classical solution in G to the p-Laplace partial differential equation. As usual, $\nabla \cdot$ denotes the divergence operator. We will often write $\Delta_p u = 0$ as abbreviated notation for condition (2.1), with a slight abuse of notation.

2.1. **p-harmonic functions in Lipschitz domains.** We say that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain if there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$, $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighborhood of $\partial\Omega$ and, for each i,

$$\Omega \cap B(x_i, 4r_i) \;\; = \;\; \{y = (y', y_n) \in \mathbb{R}^n : y_n > \phi_i(y')\} \cap B(x_i, 4r_i),$$

$$\partial\Omega\cap B(x_i,4r_i) \;=\; \{y=(y',y_n)\in\mathbb{R}^n: y_n=\phi_i(y')\}\cap B(x_i,4r_i), \tag{2.3}$$

in an appropriate coordinate system and for a Lipschitz function ϕ_i. The Lipschitz constant of Ω is defined to be $M=\max_i \left\|\nabla\phi_i\right\|_\infty$, and we let $r_0=\min_i r_i$. A bounded domain $\tilde\Omega\subset\mathbb{R}^n$ is said to be *starlike Lipschitz*, with respect to $\hat x\in\tilde\Omega$, provided

$$\partial\tilde\Omega \;=\; \{\hat x+R(\omega)\omega:\omega\in\partial B(0,1)\},$$

where the radial function R, defined on $\mathbb{S}^{n-1}$, is such that $\log R$ is Lipschitz on $\mathbb{S}^{n-1}$. We will refer to $\|\log \hat{R}\|_{\mathbb{S}^{n-1}}$ as the Lipschitz constant for $\tilde\Omega$. Observe that this constant is invariant under scaling about $\hat x$. By elementary geometric considerations it follows that if Ω is a Lipschitz domain with constants M, r_0, then there exist, for any $w\in\partial\Omega$ and $0<r<r_0$, points $a_r(w)\in\Omega$, $a_r'(w)\in\mathbb{R}^n\setminus\bar\Omega$, such that

$$\begin{cases} (i)\;\; M^{-1}r<|a_r(w)-w|<r,\;\; d(a_r(w),\partial\Omega)>M^{-1}r,\\ (ii)\;\; M^{-1}r<|a_r'(w)-w|<r,\;\; d(a_r'(w),\partial\Omega)>M^{-1}r.\end{cases} \tag{2.4}$$

In the following we state a number of estimates for non-negative p-harmonic functions defined in a Lipschitz domain Ω with constants M, r_0. Throughout this section and this paper, unless otherwise stated, and when we work in the context of Lipschitz domains with Lipschitz constants M and r_0, c will denote a positive constant ≥ 1, which is not necessarily the same at each occurrence, depending only on p, n and M. In general, $c(a_1,\dots,a_m)$ denotes a positive constant ≥ 1, which may depend only on p, n, M and $a_1,\dots,a_m$, and which is not necessarily the same at each occurrence. The notation $A\approx B$ means that A/B is bounded from above and below by strictly positive constants which, unless otherwise stated, only depend on p, n and M. Finally, given $w\in\partial\Omega$ and $r>0$, we let

$$\Delta(w,r)=\partial\Omega\cap B(w,r)\,.$$

For the proofs of the following Lemmas 2.1-2.5, we refer the reader to [59] and [60]. Lemma 2.1 was proved by Serrin [94].

Lemma 2.1. *Suppose $1<p<\infty$, and let u be a positive p-harmonic function in $B(w,2r)$. Then,*

$$(i)\qquad \max_{B(w,r)} u\leq c\min_{B(w,r)} u.$$

Furthermore, there exists $\alpha=\alpha(p,n)\in(0,1)$ such that if $x,y\in B(w,r)$, then

$$(ii)\qquad |u(x)-u(y)|\leq c\left(\frac{|x-y|}{r}\right)^\alpha \max_{B(w,2r)} u.$$

Lemma 2.2. *Suppose $1<p<\infty$, and let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain. Let $w\in\partial\Omega$, $0<r<r_0$, and suppose that $u>0$ is p-harmonic in $\Omega\cap B(w,2r)$, continuous in $\bar\Omega\cap B(w,2r)$, and $u=0$ on $\Delta(w,2r)$. Then,*

$$(i)\qquad r^{p-n}\int\limits_{\Omega\cap B(w,r/2)}|\nabla u|^p\,dx\;\leq\; c\left(\max_{\Omega\cap B(w,r)} u\right)^p.$$

Furthermore, there exists $\alpha=\alpha(p,n,M)\in(0,1)$ such that if $x,y\in\Omega\cap B(w,r)$, then

$$(ii)\qquad |u(x)-u(y)|\leq c\left(\frac{|x-y|}{r}\right)^\alpha \max_{\Omega\cap B(w,2r)} u.$$

Lemma 2.3. *Suppose that $1 < p < \infty$, and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that $u > 0$ is p-harmonic in $\Omega \cap B(w, 2r)$, continuous in $\bar{\Omega} \cap B(w, 2r)$, and $u = 0$ on $\Delta(w, 2r)$. Then there exists $c = c(p, n, M)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then*

$$\max_{\Omega \cap B(w,\tilde{r})} u \,\leq\, c\, u(a_{\tilde{r}}(w))\,.$$

Lemma 2.4. *Suppose that $1 < p < \infty$, and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that $u > 0$ is p-harmonic in $\Omega \cap B(w, 2r)$, continuous in $\bar{\Omega} \cap B(w, 2r)$, and $u = 0$ on $\Delta(w, 2r)$. Extend u to $B(w, 2r)$ by defining $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. Then u has a representative in $W^{1,p}(B(w, 2r))$ with Hölder continuous partial derivatives in $\Omega \cap B(w, 2r)$. In particular, there exists $\sigma \in (0, 1]$, depending only on p and n, such that if $x, y \in B(\hat{w}, \hat{r}/2)$, $B(\hat{w}, 4\hat{r}) \subset \Omega \cap B(w, 2r)$, then*

$$c^{-1}\, |\nabla u(x) - \nabla u(y)| \,\leq\, (|x - y|/\hat{r})^{\sigma} \max_{B(\hat{w},\hat{r})} |\nabla u| \,\leq\, c\,\hat{r}^{-1}\, (|x - y|/\hat{r})^{\sigma} \max_{B(\hat{w},2\hat{r})} u\,.$$

Moreover, if for some $\beta \in (1, \infty)$,

$$\frac{u(y)}{d(y, \partial\Omega)} \leq \beta\, |\nabla u(y)| \text{ for all } y \in B(\hat{w}, \hat{r}/2),$$

then $\hat{u} \in C^{\infty}(B(\hat{w}, \hat{r}/2))$ and given a positive integer k there exists $\bar{c} \geq 1$, depending only on p, n, β, k, such that

$$\max_{B(\hat{w},\frac{\hat{r}}{4})} \ |D^k u| \,\leq\, \bar{c}\, \frac{u(\hat{w})}{d(\hat{w}, \partial\Omega)^k}$$

where $D^k u$ denotes an arbitrary k-th order derivative of u. In particular, u is infinitely differentiable in $\Omega \cap B(w, 2r) \cap \{x : |\nabla u(x)| > 0\}$.

Lemma 2.5. *Suppose that $1 < p < \infty$, and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Given $w \in \partial\Omega$, $0 < r < r_0$, suppose that $u > 0$ is p-harmonic in $\Omega \cap B(w, 2r)$, continuous in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0$ on $\Delta(w, 2r)$. Extend u to $B(w, 2r)$ by defining $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. Then there exists a unique locally finite positive Borel measure ν on $\mathbb{R}^n$ with support in $\Delta(w, 2r)$ such that whenever $\theta \in C_0^{\infty}(B(w, 2r))$,*

$$(i) \qquad \int_{\mathbb{R}^n} |\nabla u|^{p-2}\, \langle \nabla u,\, \nabla\theta \rangle\, dx \,=\, -\int_{\mathbb{R}^n} \theta\, d\nu.$$

Moreover, there exists $c = c(p, n, M)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then

$$(ii) \qquad c^{-1} r^{p-n} \nu(\Delta(w, \tilde{r})) \leq (u(a_{\tilde{r}}(w)))^{p-1} \leq c\, r^{p-n}\, \nu(\Delta(w, \tilde{r}/2)).$$

We next quote a number of results proved in [59], [62], and [60]. In particular, the following two results are Lemma 4.28 and Theorem 2 in [62], respectively.

Theorem 2.6. *Suppose that $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with constants M, r_0. Given $w \in \partial\Omega$ and $0 < r < r_0$, suppose that u is a positive p-harmonic function in $\Omega \cap B(w, 4r)$, continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Suppose that (2.3) holds for some i and that $B(w, 4r) \subset B(x_i, 4r_i)$. There exists $c_2 = c_2(p, n, M) \geq 1$ and $\bar{\lambda} = \bar{\lambda}(p, n, M) \geq 1$ such that*

$$\bar{\lambda}^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq \langle \nabla u(y), e_n \rangle \leq |\nabla u(y)| \leq \bar{\lambda} \frac{u(y)}{d(y, \partial\Omega)}$$

whenever $y \in \Omega \cap B(w, r/c_2)$.

Theorem 2.7. *Suppose $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with constants M, r_0. Given $w \in \partial\Omega$ and $0 < r < r_0$, suppose that u and v are positive p-harmonic functions in $\Omega \cap B(w, 4r)$, continuous in $\bar{\Omega} \cap B(w, 4r)$, and $u = 0 = v$ on $\Delta(w, 4r)$. There exists $c_1 = c_1(p, n, M) \geq 1$ and $\alpha = \alpha(p, n, M)$, $\alpha \in (0, 1)$, such that*

$$\left| \log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)} \right| \leq c_1 \left(\frac{|y_1 - y_2|}{r} \right)^{\alpha}$$

whenever $y_1, y_2 \in \Omega \cap B(w, r/c_1)$.

Let Ω be a bounded Lipschitz domain; for $0 < b < 1$ and $y \in \partial\Omega$, let

$$\Gamma(y) = \Gamma_b(y) = \{x \in \Omega : \ d(x, \partial\Omega) > b|x - y|\}.$$

Fix $w \in \partial\Omega$ and $0 < r < r_0$. Given a measurable function k defined on

$$\cup_{y \in \Delta(w,2r)} \Gamma(y) \cap B(w, 4r),$$

we define the non-tangential maximal function of k as

$$N(k) : \Delta(w, 2r) \to \mathbb{R}, \quad N(k)(y) = \sup_{x \in \Gamma(y) \cap B(w,4r)} |k|(x).$$

Given a measurable function f on $\Delta(w, 2r)$ we say that f is of *bounded mean oscillation* on $\Delta(w, r)$, and we write $f \in \mathrm{BMO}(\Delta(w, r))$, if there exists A, $0 < A < \infty$, such that

$$\int\limits_{\Delta(y,s)} |f - f_\Delta|^2 d\mathcal{H}^{n-1} \leq A^2 \mathcal{H}^{n-1}(\Delta(y, s)) \tag{2.5}$$

whenever $y \in \Delta(w, r)$ and $0 < s \leq r$. Here f_Δ denotes the average of f on $\Delta = \Delta(y, s)$ with respect to the surface measure $\mathcal{H}^{n-1}$. The least A for which (2.5) holds is denoted by $\|f\|_{\mathrm{BMO}(\Delta(w,r))}$. If f is a vector-valued function, $f = (f_1, .., f_n)$, then $f_\Delta = (f_{1,\Delta}, .., f_{n,\Delta})$ and the BMO-norm of f is defined as in (2.5) with $|f - f_\Delta|^2 = \langle f - f_\Delta, f - f_\Delta \rangle$. For more details on BMO functions we refer the reader to chapter IV of [98]. Suppose now that u is a positive p-harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$, and $u = 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then there exists (see Lemma 2.5) a unique locally finite positive Borel measure ν on $\mathbb{R}^n$, with support in $\Delta(w, 4r)$, such that

$$\int\limits_{\mathbb{R}^n} |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle dx = - \int\limits_{\mathbb{R}^n} \theta d\nu \tag{2.6}$$

whenever $\theta \in C_0^\infty(B(w, 4r))$. Moreover, using Lemma 2.5 and Harnack's inequality for p-harmonic functions we can conclude that ν is a doubling measure in the following sense. There exists $c = c(p, n, M)$, $1 \leq c < \infty$, such that

$$\nu(\Delta(z, 2s)) \leq c\nu(\Delta(z, s)) \text{ whenever } z \in \Delta(w, 3r),\ s \leq r/c.$$

Here and henceforth, we say that ν is an A^∞-*measure* with respect to $\mathcal{H}^{n-1}$ on $\Delta(w, 2r)$, $d\nu \in A^\infty(\Delta(w, 2r), d\mathcal{H}^{n-1})$ for short, if for some $\gamma > 0$ there exists $\epsilon = \epsilon(\gamma) > 0$ with the property that if $z \in \Delta(w, 2r)$, $0 < s < r$ and if $E \subset \Delta(z, s)$, then

$$\frac{\mathcal{H}^{n-1}(E)}{\mathcal{H}^{n-1}(\Delta(z, s))} \geq \gamma \text{ implies that } \frac{\nu(E)}{\nu(\Delta(z, s))} \geq \epsilon.$$

The following result is a summary of Theorems 1 and 3 in [60].

Theorem 2.8. *Suppose that $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with constants M, r_0. Given $w \in \partial\Omega$, and $0 < r < r_0$, suppose that u is a positive p-harmonic function in $\Omega \cap B(w, 4r)$, continuous in $\bar{\Omega} \cap B(w, 4r)$, and $u = 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$ and let ν be as in (2.6). Then ν is absolutely continuous with respect to $\mathcal{H}^{n-1}$ on $\Delta(w, 4r)$ and $d\nu \in A^\infty(\Delta(w, 2r), d\mathcal{H}^{n-1})$. Moreover,*

$$\nabla u(y) := \lim_{x \in \Gamma(y) \cap B(w,4r), x \to y} \nabla u(x)$$

exists for $\mathcal{H}^{n-1}$-a.e. $y \in \Delta(w, 4r)$ and for some b, $0 < b < 1$, fixed in the definition of $\Gamma(y)$. Also, there exists $q > p$ and a constant $c \geq 1$, both depending only on p, n, M, such that

$$
\begin{aligned}
&(i) && N(|\nabla u|) \in L^q(\Delta(w, 2r)),\\
&(ii) && \int_{\Delta(w,2r)} |\nabla u|^q d\mathcal{H}^{n-1} \leq c r^{(n-1)(\frac{p-1-q}{p-1})} \Bigg(\int_{\Delta(w,2r)} |\nabla u|^{p-1} d\mathcal{H}^{n-1} \Bigg)^{q/(p-1)},\\
&(iii) && \log |\nabla u| \in BMO(\Delta(w, r)), \quad \| \log |\nabla u| \|_{BMO(\Delta(w,r))} \leq c,\\
&(iv) && d\nu = |\nabla u|^{p-1} d\mathcal{H}^{n-1}, \ \mathcal{H}^{n-1}\text{-a.e. on } \Delta(w, 2r).
\end{aligned}
$$

Finally, $\Delta(w, 4r)$ has a tangent plane at $y \in \Delta(w, r)$ for $\mathcal{H}^{n-1}$ almost every y. If $n(y)$ denotes the unit normal to this tangent plane pointing into $\Omega \cap B(w, 4r)$, then $\nabla u(y) = |\nabla u(y)| n(y)$.

2.2. **Basics of convex domains.** By definition, a domain in $\mathbb{R}^n$ is a (non-empty) open and connected subset of $\mathbb{R}^n$. In general we will work with bounded convex domains and will often simply refer to such a domain as a convex domain. The closure of a (bounded) convex domain is called a bounded convex body, or simply a convex body, and hence a convex body is a compact convex set with non-empty interior. In convex geometry, convex bodies are usually the objects of study. However, most notions and results for convex bodies carry over to convex domains without any difficulty and hence we will freely use many of these notions for open as well as closed domains. The book of Schneider [93] is a standard reference for convex bodies.

Let K be a convex body in $\mathbb{R}^n$. The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of K is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \sup_{y \in K} x \cdot y.$$

The support function is a convex function that is homogeneous of degree 1. Let K_1 and K_2 be convex bodies in $\mathbb{R}^n$ and let $\alpha, \beta \geq 0$. The Minkowski linear combination of K_1 and K_2, with coefficients α and β, is defined by

$$\alpha K_1 + \beta K_2 = \{\alpha x + \beta y \, : \, x \in K_1 \, , \, y \in K_2\} \, .$$

This is a convex body whose support function is given by

$$h_{\alpha K_1 + \beta K_2} = \alpha h_{K_1} + \beta h_{K_2} \, .$$

When $\alpha = 1 = \beta$ the result is often referred to as the Minkowski sum of K_1 and K_2. In what follows $\Omega \subset \mathbb{R}^n$ will be a bounded convex domain and $K \subset \mathbb{R}^n$ its closure. In particular, K is a convex body. Convexity guarantees that Ω is a Lipschitz domain, i.e. its boundary can be written locally as the graph of a Lipschitz function, see (2.3). Using this we see that the outer unit normal vector to ∂K at x, denoted by $\mathsf{g}(x)$, is well defined for $\mathcal{H}^{n-1}$ almost all $x \in \partial K$. The map $\mathsf{g} : \partial K \to \mathbb{S}^{n-1}$ is called the Gauss map of K. For $\omega \subset \mathbb{S}^{n-1}$, let

$$\mathsf{g}^{-1}(\omega) = \{x \in \partial K \,:\, \mathsf{g}(x) \text{ is defined and } \mathsf{g}(x) \in \omega\}\,.$$

If ω is a Borel subset of $\mathbb{S}^{n-1}$, then $\mathsf{g}^{-1}(\omega)$ is $\mathcal{H}^{n-1}$-measurable (see [93], Chapter 2). The Borel measure S_K, on $\mathbb{S}^{n-1}$, is defined for Borel $\omega \subset \mathbb{S}^{n-1}$ by

$$S_K(\omega) = \mathcal{H}^{n-1}(\mathsf{g}^{-1}(\omega)),$$

and is called the surface area measure of K. For every $f \in C(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} f(\xi)\, dS_K(\xi) = \int_{\partial K} f(\mathsf{g}(x))\, d\mathcal{H}^{n-1}(x)\,. \tag{2.7}$$

If K contains the origin, then the radial function $\rho_K : \mathbb{S}^{n-1} \to (0,\infty)$ of K is defined, for $\xi \in \mathbb{S}^{n-1}$, by

$$\rho_K(\xi) = \sup\{\rho \geq 0 \,:\, \rho\xi \in K\}.$$

The radial map $r_K : \mathbb{S}^{n-1} \to \partial K$ is

$$r_K(\xi) = \rho_K(\xi)\,\xi,$$

for $\xi \in \mathbb{S}^{n-1}$, i.e. $r_K(\xi)$ is the unique point on ∂K located on the ray parallel to ξ and emanating from the origin.

Remark 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and assume that $0 \in \Omega$. Let r_{int} be the largest radius such that $B(0, r_{\text{int}}) \subset \Omega$. Similarly, let r_{ext} be the smallest radius such that $\bar{\Omega} \subset B(0, r_{\text{ext}})$. Using the convexity of Ω, one can prove that Ω is a starlike Lipschitz domain with Lipschitz constant M bounded by $r_{\text{ext}}/r_{\text{int}}$.*

Given a bounded convex domain $\Omega \subset \mathbb{R}^n$, we have, using Remark 2.9, that there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$, $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighborhood of $\partial\Omega$ and, for each i, the representation in (2.3) in an appropriate coordinate system, for a convex Lipschitz function $\phi := \phi_i$. A bounded convex domain Ω, or body $K := \bar{\Omega}$, is said to be of class $C^{2,\alpha}$ if its boundary is $C^{2,\alpha}$-smooth, for some $\alpha \in (0,1)$, i.e., if each $\phi := \phi_i$ can be chosen to be $C^{2,\alpha}$-smooth. Ω, K, are said to be strongly convex, locally at $(y', \phi(y'))$ if the matrix $(n-1) \times (n-1)$-dimensional matrix $\nabla^2\phi(y')$ is positive definite. If this holds at all boundary points of Ω, K, then Ω, K, are said to be strongly convex. If $K := \bar{\Omega}$ is $C^{2,\alpha}$-smooth and strongly convex then the Gauss map $\mathsf{g}_K : \partial K \to \mathbb{S}^{n-1}$ is a diffeomorphism. Hence, for every $\xi \in \mathbb{S}^{n-1}$ there exists a unique $x \in \partial K$ such that $\mathsf{g}_K(x) = \xi$. Furthermore, locally the function $\phi := \phi_i$ satisfies

$$\det(\nabla^2\phi(y')) = (1 + |\nabla\phi(y')|^2)^{(n+1)/2}\kappa(\xi),\ \ \xi = (-1, \nabla\phi(y'))/(1 + |\nabla\phi(y')|^2)^{1/2} \tag{2.8}$$

where $\kappa(\cdot)$ denotes the Gauss curvature. In particular, if K is $C^{2,\alpha}$-smooth and strongly convex then the Gauss curvature is positive. In the following we say that Ω, K, are of class $C^{2,\alpha}_+$ if its boundary is $C^{2,\alpha}$-smooth, for some $\alpha \in (0,1)$, and of positive Gauss curvature. Finally, Ω, K, are said to be strictly convex if their boundary contain no line segments.

If a convex body K is of class $C^{2,\alpha}_+$ then, using the notation introduced above, the support function of K can be expressed as

$$h_K(\xi) = \xi \cdot \mathsf{g}_K^{-1}(\xi) = \mathsf{g}_K(x) \cdot x,\ \text{ where } \xi \in \mathbb{S}^{n-1},\, \mathsf{g}_K(x) = \xi,\, x \in \partial K. \tag{2.9}$$

Moreover, the gradient of h_K satisfies

$$\nabla h_K(\xi) = \mathsf{g}_K^{-1}(\xi), \tag{2.10}$$

and h_K is of class $C^{2,\alpha}$. Let $\{e_1, e_2, \ldots, e_{n-1}\}$ be an orthonormal frame on $\mathbb{S}^{n-1}$. Denote by h_i and h_{ij} the first and second order covariant derivatives of h_K on $\mathbb{S}^{n-1}$, and by ∇h_K and $\nabla^2 h_K$ the gradient and Hessian of h_K in $\mathbb{R}^n$. Then,

$$\nabla h_K(\xi) = h_i e_i + h\xi, \quad (\nabla^2 h_K(\xi))e_i = a_{ij}e_j, \tag{2.11}$$

where $a_{ij} = h_{ij} + h\delta_{ij}$, $h = h_K(\xi)$, δ_{ij} is the Kronecker delta, and we will use the usual convention that repeated indices means summation over all possible values of that index. Note that if K is of class $C^{2,\alpha}_+$, then the $(n-1) \times (n-1)$ matrix (a_{ij}) is symmetric and positive definite. The matrix (a_{ij}) is the inverse of the matrix associated with the Weingarten map with respect to the frame $\{e_1, e_2, \ldots, e_{n-1}\}$. In particular, the Gauss curvature of K, κ, is given by

$$\kappa(\mathsf{g}_K^{-1}(\xi)) = \frac{1}{\det(a_{ij}(\xi))} = \frac{1}{\det(h_{ij}(\xi) + h(\xi)\delta_{ij})}\,. \tag{2.12}$$

Denote by (c_{ij}) the *cofactor* matrix of the matrix (a_{ij}), and let c_{ijk} be the covariant derivative tensor of c_{ij}. Then

$$\sum_j c_{ijj} = 0\,, \tag{2.13}$$

see [21] for a proof. Let $F(\xi) = \mathsf{g}^{-1}(\xi)$ be the inverse Gauss map of ∂K. Then, using (2.11) we see that $F(\xi) = \nabla h_K(\xi)$ and

$$F(\xi) = h_i e_i + h\xi, \quad F_i = a_{ij}e_j, \quad F_{ij} = a_{ijk}e_k - a_{ij}\xi, \tag{2.14}$$

where a_{ijk} are the covariant derivatives of a_{ij}. As mentioned above, if K is of class $C^{2,\alpha}_+$, then the matrix $(h_{ij} + h\delta_{ij})$ is positive definite. Conversely, if $h \in C^{2,\alpha}(\mathbb{S}^{n-1})$ and $(h_{ij} + h\delta_{ij})$ is positive definite, then there exists a unique convex domain K, of class $C^{2,\alpha}_+$, such that $h = h_K$, see [16] and Proposition 1 in [50]. As a consequence, the set of functions

$$\mathcal{C} = \{h \in C^{2,\alpha}(\mathbb{S}^{n-1}) \,:\, (h_{ij} + h\delta_{ij}) \text{ is positive definite}\}\,, \tag{2.15}$$

consists precisely of support functions of convex domains of class $C^{2,\alpha}_+$. Furthermore, when K is of class $C^{2,\alpha}_+$, the surface area measure S_K is absolutely continuous with respect to the Lebesgue measure on $\mathbb{S}^{n-1}$ and

$$dS_K(\xi) = \det(h_{ij}(\xi) + h(\xi)\delta_{ij})\,d\xi\,. \tag{2.16}$$

The following lemma, see Alexandrov [3] and also [52], provides a change of variable formula based on the radial map, along with some related properties. We recall that r_{int} and r_{ext} were defined in Remark 2.9.

Lemma 2.10. *Let Ω be a bounded convex domain that contains the origin, let $K = \bar{\Omega}$ and let $f : \partial K \to \mathbb{R}$ be $\mathcal{H}^{n-1}$-integrable. Then,*

$$\int_{\partial K} f(x)\,d\mathcal{H}^{n-1}(x) = \int_{\mathbb{S}^{n-1}} f(r_K(\xi))J(\xi)\,d\xi\,,$$

where J is defined $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$ by

$$J(\xi) = \frac{(\rho_K(\xi))^n}{h_K(\mathsf{g}_K(r_K(\xi)))}.$$

Moreover, there exist constants $c_1, c_2 > 0$, depending only on $r_{\text{int}}(K)$ and $r_{\text{ext}}(K)$, such that $c_1 \le J(\xi) \le c_2$ for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$. Furthermore, assume that $\{K_i\}_{i\in\mathbb{N}}$ is a sequence of bounded convex bodies converging to K with respect to the Hausdorff metric. Define functions $J_i : \mathbb{S}^{n-1} \to (0, \infty)$,

$$J_i(\xi) = \frac{(\rho_{K_i}(\xi))^n}{h_{K_i}(\mathsf{g}_{K_i}(r_{K_i}(\xi)))}, \text{ for } i \in \mathbb{N}.$$

Then there exists $i_0 \ge 1$ such that if $i \ge i_0$, then $J_i(\xi)$ is bounded from below and above, uniformly with respect to ξ and i, and $\{J_i\}$ converge to J, $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$.

The following divergence formula for unbounded domains will also be needed.

Lemma 2.11. *Let Ω be a bounded convex domain of class $C^{2,\alpha}_+$ with Gauss map g, and let X be a C^1 vector field in $\mathbb{R}^n \setminus \bar{\Omega}$. Assume,*

(*i*) *The limit $X(x) := \lim_{t\to 0^+} X(x + t\mathsf{g}(x))$ exists for almost all $x \in \partial\Omega$, with respect to $\mathcal{H}^{n-1}$.*

(*ii*) *The integrals $\int_{\partial\Omega} |X|\, d\mathcal{H}^{n-1}(x)$ and $\int_{\mathbb{R}^n\setminus\bar{\Omega}} \operatorname{div} X\, dx$, exist.*

(*iii*) $|X| = o\big(|x|^{1-n}\big)$ *as $x \to \infty$.*

Then

$$\int_{\mathbb{R}^n\setminus\bar{\Omega}} \operatorname{div} X\, dx = -\int_{\partial\Omega} X(x)\cdot \mathsf{g}(x)\, d\mathcal{H}^{n-1}(x).$$

Proof. For $t > 0$, let

$$\Omega_t = \Omega \cup \{x + \tau\mathsf{g}(x) : x \in \partial\Omega,\ 0 \le \tau < t\}.$$

Let $R \gg 1$ be such that $\overline{\Omega_t} \subset B(0, R)$. By the divergence theorem, on the bounded domain $B(0, R) \setminus \overline{\Omega_t}$, we have that

$$\int_{B(0,R)\setminus\bar{\Omega}_t} \operatorname{div} X\, dx = -\int_{\partial\Omega_t} X(x)\cdot \mathsf{g}_t(x)\, d\mathcal{H}^{n-1}(x) + \int_{\partial B(0,R)} X\cdot \frac{x}{|x|}\, d\mathcal{H}^{n-1}(x),$$

where g_t is the Gauss map of $\partial\Omega_t$. For $x \in \partial\Omega$, let $x_t = x + t\mathsf{g}(x) \in \partial\Omega_t$. This is a diffeomorphism between $\partial\Omega$ and $\partial\Omega_t$. The surface area elements satisfy

$$d\mathcal{H}^{n-1}(x_t) = (1 + O(t))d\mathcal{H}^{n-1}(x),$$

and the Gauss maps satisfy

$$\mathsf{g}_t(x_t) = \mathsf{g}(x).$$

Thus,

$$\int_{\partial\Omega_t} X(x)\cdot \mathsf{g}_t(x)\, d\mathcal{H}^{n-1}(x) = \int_{\partial\Omega} X(x + t\mathsf{g}(x))\cdot \mathsf{g}(x)(1 + O(t))\, d\mathcal{H}^{n-1}(x).$$

From (i) and (ii) in the hypothesis of the lemma, and the Lebesgue dominating convergence theorem, we deduce that as $t \to 0^+$,

$$\int_{\partial\Omega} X(x + t\mathsf{g}(x)) \cdot \mathsf{g}(x)(1 + O(t))\, d\mathcal{H}^{n-1}(x) \to \int_{\partial\Omega} X(x) \cdot \mathsf{g}(x)\, d\mathcal{H}^{n-1}(x).$$

Finally, from (iii) in the hypothesis of the lemma we see that, as $R \to \infty$,

$$\int_{\partial B(0,R)} X \cdot \frac{x}{|x|}\, d\mathcal{H}^{n-1}(x) \to 0.$$

This completes the proof of the lemma. $\square$

2.3. **p-capacity of convex domains and its integral formulas.** Suppose $1 < p < \infty$, and let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. The p-capacity $C_p(\Omega)$ was defined in (1.9). Recall that the associated p-equilibrium potential is the function U which is defined and continuous on the closure of $\mathbb{R}^n \setminus \bar{\Omega}$, and which solves

$$\begin{cases} \Delta_p U = 0 & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \\[2ex] U = 1 & \text{on } \partial\Omega, \text{ and } \lim_{|x|\to\infty} U(x) = 0\,. \end{cases} \tag{2.17}$$

In particular, $U \in W_0^{1,p}(\mathbb{R}^n \setminus \bar{\Omega})$ is a weak solution to (2.17) in the sense of (2.1). As mentioned in the introduction, a proof of the existence and uniqueness of U can be found in Lewis [57], see also Theorem 2 in [28]. For the following theorem we refer to Lewis [57].

Theorem 2.12. *Suppose $1 < p < n$, and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded convex domain. Then there exists a unique weak solution U to (2.17) satisfying the following.*

- (a) $U \in C^\infty(\mathbb{R}^n \setminus \bar{\Omega}) \cap C(\mathbb{R}^n \setminus \Omega)$.
- (b) $0 < U < 1$ *and* $|\nabla U| \neq 0$ *in* $\mathbb{R}^n \setminus \bar{\Omega}$.
- (c) $C_p(\Omega) = \displaystyle\int_{\mathbb{R}^n\setminus\bar{\Omega}} |\nabla U|^p\, dx\,.$
- (d) *If U is defined to be 1 in Ω, then $\Omega_t = \{x \in \mathbb{R}^n : \; U(x) > t\}$ is convex for each $t \in [0, 1]$ and $\partial\Omega_t$ is a C^∞ manifold for $0 < t < 1$.*

Note that by the definition of p-capacity, and (c) of Theorem 2.12, we have

$$\int_{\mathbb{R}^n\setminus\bar{\Omega}} |\nabla U|^p\, dx = \inf\Big\{ \int_{\mathbb{R}^n} |\nabla u|^p\, dx,\; u \in C_0^\infty(\mathbb{R}^n),\; u \geq 1 \;\text{ on }\; \Omega \Big\}. \tag{2.18}$$

For $0 < b < 1$, $y \in \partial\Omega$ we let

$$\tilde{\Gamma}(y) = \tilde{\Gamma}_b(y) = \{x \in \mathbb{R}^n \setminus \bar{\Omega} : \; d(x, \partial\Omega) > b|x - y|\}\,. \tag{2.19}$$

The following lemma is a direct consequence of Theorem 2.8 stated above.

Lemma 2.13. *Suppose $1 < p < n$, and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded convex domain. Then*

$$\nabla U(y) := \lim_{x\to y,\; x\in\tilde{\Gamma}(y)} \nabla U(x),$$

exists for $\mathcal{H}^{n-1}$ almost all $y \in \partial\Omega$. Moreover, for $\mathcal{H}^{n-1}$ almost all $y \in \partial\Omega$,

$$\nabla U(y) = -|\nabla U(y)|\mathsf{g}(y),$$

and $|\nabla U| \in L^p(\partial\Omega, \mathcal{H}^{n-1})$.

Remark 2.14. Let Ω and U be as in the statement of Lemma 2.13. Then, as stated, ∇U has non-tangential limits at $\mathcal{H}^{n-1}$ almost all boundary point of Ω. Furthermore, the quantitative statement of Theorem 2.8 holds with constants q and c, where $q > p$, and $c \geq 1$, depending only on p, n and the eccentricity of Ω, i.e. the quotient between r_{ext} and r_{int}, see Remark 2.9. In particular, based on Lemma 2.13 we can conclude that the measure $\mu_p(\Omega, \cdot)$ in (1.11) is well-defined.

The following lemma concerns the behavior at infinity of the p-equilibrium potential U and its gradient. It was mentioned in a remark in [56, Remark 1.6] (see also [28]). For the sake of completeness we provide the proof of this result.

Lemma 2.15. *Suppose* $1 < p < n$, *and let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a bounded convex domain. If* U *is the solution of* (2.17), *then*

$$
\begin{aligned}
&(a) \qquad \lim_{|x|\to\infty} U(x)|x|^{\frac{n-p}{p-1}} = (n\omega_n)^{\frac{1}{1-p}} \left(\frac{p-1}{n-p}\right) C_p(\Omega)^{\frac{1}{p-1}},\\
&(b) \qquad \lim_{|x|\to\infty} |x|^{\frac{n-1}{p-1}} |\nabla U(x)| = (n\omega_n)^{\frac{1}{1-p}} C_p(\Omega)^{\frac{1}{p-1}},
\end{aligned}
$$

where ω_n *is the volume of the unit ball in* $\mathbb{R}^n$.

Proof. Let in the following $\varsigma : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ denote the fundamental solution of the p-Laplace equation, $\varsigma(x) = |x|^{\frac{p-n}{p-1}}$. Then $\Delta_p \varsigma = 0$ in $\mathbb{R}^n \setminus \{0\}$. Let R_1, $R_2 > 0$ be such that

$$B(0, R_1) \subset \Omega, \quad \overline{\Omega} \subset B(0, R_2).$$

Step 1. There exist positive constants C_1 and C_2 such that

$$C_1 \varsigma(x) \leq U(x) \leq C_2 \varsigma(x), \tag{2.20}$$

for all x such that $|x| \geq R_2$. This is a straightforward consequence of the comparison principle for p-harmonic functions. Indeed, consider the functions $U_1, U_2 : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined as

$$U_1(x) = R_1^{\frac{n-p}{p-1}} \varsigma(x), \quad U_2(x) = R_2^{\frac{n-p}{p-1}} \varsigma(x).$$

Then by the comparison principle we see that

$$U_1(x) \leq U(x) \leq U_2(x)$$

for all x such that $|x| \geq R_2$ and this proves (2.20).

Step 2. There exist C, $R > 0$, $\sigma \in (0, 1)$ such that

$$|x|\,|\nabla U(x)| \leq C\varsigma(x), \quad |\nabla U(x) - \nabla U(x')| \leq C\, \frac{\varsigma(x)|x - x'|^\sigma}{|x|^{1+\sigma}}, \tag{2.21}$$

for all $|x|, |x'| \geq R$. To see this, let $R_0 > 3R_2$ and let

$$V(y) := U(R_0\, y)\, R_0^{\frac{n-p}{p-1}}.$$

Then V is a p-harmonic function in

$$O := \{y \in \mathbb{R}^n \,:\, 3 < |y| < 6\}.$$

Moreover, by the previous step, in O, V is bounded by constants depending on Ω only. Now, using Theorem 1 in [58], see also Lemma 2.4 stated above, there exist $A>0$ and $\sigma\in(0,1)$, both depending on Ω, n and p, such that

$$|\nabla V(y)|\leq A,\quad |\nabla V(y)-\nabla V(y')|\leq A|y-y'|^\sigma, \tag{2.22}$$

whenever $y,y'\in D$ and where $D=\{y\,:\,4\leq|y|\leq 5\}$. Hence

$$R_0^{\frac{n-p}{p-1}}\,R_0\,|\nabla U(x)|\leq A,\ \text{ whenever } 4R_0\leq|x|\leq 5R_0. \tag{2.23}$$

Using the restriction $R_0>3R_2$ and (2.23) we can conclude, in particular, that there exists $A'>0$, depending on Ω, n and p, such that

$$|x|\,|\nabla U(x)|\leq A'\varsigma(x),\ \text{ whenever } |x|\geq R:=12R_2. \tag{2.24}$$

Furthermore, again using (2.22) we see that

$$R_0^{\frac{n-p}{p-1}}\,R_0\,|\nabla U(x)-\nabla U(x')|\leq A'\frac{|x-x'|^\sigma}{R_0^\sigma},$$

for all x,x' such that $4R_0\leq|x|,|x'|\leq 5R_0$ and by arguing as above we deduce that

$$|\nabla U(x)-\nabla U(x')|\leq A'\frac{\varsigma(x)}{|x|^{\sigma+1}}\,|x-x'|^\sigma,$$

for every x,x' such that $|x|,|x'|\geq R$. This concludes the proof of (2.21).

Step 3. There exists a constant γ such that

$$\lim_{|x|\to\infty}\frac{U(x)}{\varsigma(x)}=\gamma. \tag{2.25}$$

Let

$$\gamma:=\limsup_{|x|\to\infty}\frac{U(x)}{\varsigma(x)}.$$

Then, using [99, Proposition 3.3.2], see also [56, Corollary 1.1], it follows that

$$\sup_{R_0<|x|\leq R}\frac{U(x)}{\varsigma(x)}=\sup_{|x|=R}\frac{U(x)}{\varsigma(x)}$$

and hence

$$\gamma=\lim_{R\to\infty}\left(\sup_{|x|=R}\frac{U(x)}{\varsigma(x)}\right). \tag{2.26}$$

Consequently, from (2.26) and by the continuity of U it follows that there exists, whenever $r\geq 2R_2$, $x_r\in\mathbb{R}^n$, such that $|x_r|=r$ and such that

$$\lim_{r\to\infty}\frac{U(x_r)}{\varsigma(x_r)}=\gamma.$$

Now, for $r\geq 2R_2$, we consider

$$U_r(\xi)=U(r\xi)\;r^{\frac{n-p}{p-1}},\ \text{ whenever } |\xi|\geq\frac{1}{2}.$$

In particular, $\{U_r\}_{r\geq 2R_2}$, is a family of functions defined for $|\xi|\geq\frac12$. Using (2.21) and the Ascoli-Arzelá theorem, we can conclude that there exists a sequence $\{r_k\}_{k\in\mathbb{N}}$, with $r_k\to+\infty$

as $k \to +\infty$, and a function $W = W(\xi)$, defined for $|\xi| \geq \frac{1}{2}$, such that U_{r_k} converges to W in the norm C^1, on compact sets. In particular, W is p-harmonic on $|\xi| \geq \frac{1}{2}$. Since

$$\frac{U_r(\xi)}{\varsigma(\xi)} = \frac{U(r\xi)}{\varsigma(r\xi)},$$

(2.26) implies that

$$\frac{W(\xi)}{\varsigma(\xi)} \leq \gamma, \text{ whenever } |\xi| \geq \tfrac{1}{2}.$$

For $k \in \mathbb{N}$, we let $\xi_{r_k} = \frac{1}{r_k}\, x_{r_k}$. Note that $\{\xi_k\}_{k\in\mathbb{N}}$ is a compact family of points and hence $\{\xi_k\}$ converges to some point ξ_0, with $|\xi_0| = 1$, as $k \to +\infty$. Using the definition of x_r, and the uniform convergence, we see that

$$\frac{W(\xi_0)}{\varsigma(\xi_0)} = \lim_{k\to+\infty} \frac{U_{r_k}(\xi_{r_k})}{\varsigma(\xi_{r_k})} = \lim_{k\to+\infty} \frac{U(x_{r_k})}{\varsigma(x_{r_k})} = \gamma.$$

Next, again using Proposition 3.3.2 in [99] we deduce that

$$\frac{W(\xi)}{\varsigma(\xi)} = \gamma, \text{ whenever } |\xi| \geq \tfrac{1}{2},$$

and, in particular, it follows that the family U_r (and not just a subsequence of it) converges to W as $r \to +\infty$. Using this we see that

$$\lim_{r\to+\infty} \frac{U(r\xi)}{\varsigma(r\xi)} = \gamma,$$

uniformly on the unit sphere, and consequently (2.25) holds.

The final step. By Step 3 we have

$$\lim_{r\to+\infty} U_r(\xi) = W(\xi) = \gamma\varsigma(\xi), \text{ whenever } |\xi| \geq \tfrac{1}{2},$$

and the convergence is C^1 on compact subsets. Using this it follows that

$$\lim_{r\to+\infty} \nabla U_r(\xi) = \lim_{r\to+\infty} \nabla U(r\xi)\, r^{\frac{n-1}{p-1}} = \nabla W(\xi) = \gamma \frac{p-n}{p-1} |\xi|^{-\frac{n-1}{p-1}} \frac{\xi}{|\xi|},$$

and hence

$$\lim_{r\to+\infty} |r\xi|^{\frac{n-1}{p-1}} |\nabla U(r\xi)| = \frac{n-p}{p-1}\gamma, \tag{2.27}$$

whenever $|\xi| \geq \frac{1}{2}$. Note that the convergence in the last display is uniform on compact sets. Finally, to deduce the value of the constant γ we can argue as in [28]. Indeed, we first note that

$$\begin{aligned}\int\limits_{\Omega_t\setminus\bar\Omega} |\nabla U|^p dx &= \int\limits_{\Omega_t\setminus\bar\Omega} \nabla\cdot((U-1)|\nabla U|^{p-2}\nabla U) dx \\ &= (1-t)\int\limits_{\partial\Omega_t} |\nabla U|^{p-1} d\mathcal{H}^{n-1},\end{aligned}$$

where $\Omega_t = \{x \in \mathbb{R}^n,\ U(x) > t\}$. Taking the limit as $t \to 0^+$ in the last display, using Theorem 2.12 (c) and (2.27), we can conclude that

$$C_p(\Omega) = n\omega_n(\frac{n-p}{p-1}\gamma)^{p-1} \lim_{r\to+\infty} r^{n-1} \cdot (r^{\frac{1-n}{p-1}})^{p-1} = n\omega_n \cdot (\frac{n-p}{p-1}\gamma)^{p-1}$$

This completes the proofs of (a) and (b) in the statement of the lemma. □

Lemma 2.16. *Suppose $1 < p < n$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded convex domain of class $C^{2,\alpha}_+$. Let $\Omega_t = \{x \in \mathbb{R}^n : U(x) > t\}$. Then,*

$$(a) \qquad C_p(\Omega) = \int_{\partial\Omega_t} |\nabla U|^{p-1}\, d\mathcal{H}^{n-1} \text{ for every } t \in (0,1).$$

$$(b) \qquad C_p(\Omega) = \frac{p-1}{n-p} \int_{\partial\Omega} |\nabla U(x)|^p \big(x \cdot \mathrm{g}(x)\big)\, d\mathcal{H}^{n-1}(x)\,.$$

Proof. To prove the statement in (a), let Φ denote for the class of all non-decreasing C^∞ functions $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \phi(t) = 0 \quad \text{as } t \in (-\infty, 0], \\ \mathrm{supp}(\phi') \ \subset \ (0,1), \\ \phi(t) = 1 \quad \text{as } t \in [1, \infty). \end{cases}$$

Let

$$f(t) = \int_{\partial\Omega_t} |\nabla U|^{p-1}\, d\mathcal{H}^{n-1}.$$

Then, using (c) and (d) of Theorem 2.12, and the co-area formula, we conclude that

$$C_p(\Omega) = \int_0^1 \Big(\int_{\partial\Omega_t} |\nabla U|^{p-1}\, d\mathcal{H}^{n-1} \Big) dt = \int_0^1 f(t)\, dt. \tag{2.28}$$

Next, using (2.18), the co-area formula, and (2.2.3) of [80, Chapter 2], we see that

$$\begin{aligned} C_p(\Omega) &= \inf_{\phi\in\Phi} \int_{\mathbb{R}^n\setminus\bar\Omega} |\nabla\phi(U)|^p\, dx = \inf_{\phi\in\Phi} \int_{\mathbb{R}^n\setminus\bar\Omega} |\phi'(U)|^p |\nabla U|^p\, dx \\ &= \inf_{\phi\in\Phi} \int_0^1 |\phi'(t)|^p f(t)\, dt = \Big(\int_0^1 f^{\frac{1}{1-p}}(t)\, dt \Big)^{1-p}. \end{aligned} \tag{2.29}$$

Combining (2.28), (2.29), and using the Hölder inequality, we conclude that

$$1 \leq \Big(\int_0^1 f(t)\, dt \Big)^{\frac{1}{p}} \Big(\int_0^1 f^{\frac{1}{1-p}}(t)\, dt \Big)^{\frac{p-1}{p}} = 1.$$

From this, it follows that $f(t) = cf(t)^{1-p}$ for a constant $c > 0$ and that $f(t)$ must be identical to $C_p(\Omega)$ for all $t \in (0,1)$. This completes the proof of the statement in (a).

The statement in (b) is proved by repeated integration by parts. As mentioned above, we adopt the Einstein convention for summation over repeated indices. We denote, for $j = 1, \ldots, n$, by U_i and U_{ij} the first and second partial derivatives of U, respectively. Note that by statement (a) of Theorem 2.12 we have $U \in C^\infty(\mathbb{R}^n \setminus \bar\Omega) \cap C(\mathbb{R}^n \setminus \Omega)$ and hence we can differentiate U freely in $\mathbb{R}^n \setminus \bar\Omega$. To proceed, recall that $\nabla\cdot$ denotes the divergence operator, we first observe the identities

$$\nabla \cdot (|\nabla U(x)|^p x) = n|\nabla U(x)|^p + p|\nabla U(x)|^{p-2} U_{ij}(x) x_i U_j(x)$$

and

$$\begin{aligned}(2.30)\qquad &\nabla\cdot\big(\nabla U(x)|\nabla U(x)|^{p-2}(\nabla U(x)\cdot x)\big)\\ &=\nabla\cdot\big(\nabla U(x)|\nabla U(x)|^{p-2}\big)\big(\nabla U(x)\cdot x\big)+|\nabla U(x)|^{p-2}\big(\nabla U(x)\cdot\nabla(\nabla U(x)\cdot x)\big)\,.\end{aligned}$$

Using (2.17) we see that (2.30) implies that

$$\nabla\cdot\big(\nabla U(x)|\nabla U(x)|^{p-2}(\nabla U(x)\cdot x)\big)=|\nabla U(x)|^{p}+|\nabla U(x)|^{p-2}U_{ij}(x)x_iU_j(x), \tag{2.31}$$

and, consequently, we can conclude that

$$(n-p)|\nabla U(x)|^p=\nabla\cdot\big(x|\nabla U(x)|^p\big)-p\nabla\cdot\Big(\nabla U(x)|\nabla U(x)|^{p-2}\big(\nabla U(x)\cdot x\big)\Big) \tag{2.32}$$

holds, whenever $x\in\mathbb{R}^n\setminus\bar\Omega$. Integrating both sides of (2.32) over $\mathbb{R}^n\setminus\bar\Omega$, and using (c) of Theorem 2.12, we see that

$$\begin{aligned}(n-p)C_p(\Omega)\;&=\;\int_{\mathbb{R}^n\setminus\bar\Omega}\nabla\cdot\big(x|\nabla U(x)|^p\big)\,dx\\ &\quad -p\int_{\mathbb{R}^n\setminus\bar\Omega}\nabla\cdot\Big(\nabla U(x)|\nabla U(x)|^{p-2}\big(\nabla U(x)\cdot x\big)\Big)\,dx.\end{aligned}$$

Next, using Lemma 2.15 and Lemma 2.13, we can apply Lemma 2.11 and conclude that

$$\begin{aligned}(n-p)C_p(\Omega)\;&=\;-\int_{\partial\Omega}|\nabla U(x)|^p\big(x\cdot\mathsf{g}(x)\big)\,d\mathcal{H}^{n-1}(x)\\ &\quad -p\int_{\partial\Omega}\Big(\mathsf{g}(x)|\nabla U(x)|^{p-1}\big(\nabla U(x)\cdot x\big)\Big)\cdot\mathsf{g}(x)\,d\mathcal{H}^{n-1}(x)\\ &=\;(p-1)\int_{\partial\Omega}|\nabla U(x)|^p\big(x\cdot\mathsf{g}(x)\big)\,d\mathcal{H}^{n-1}(x)\,.\end{aligned}$$

This completes the proof of statement in (b) and hence the proof of Lemma 2.16. □

Remark 2.17. Using (2.7) and (2.9) we see that the statement in Lemma 2.16 (b) can be expressed as

$$C_p(\Omega)=\frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}}h_\Omega(\xi)|\nabla U(\mathsf{g}^{-1}(\xi))|^p\,dS_\Omega(\xi)\,. \tag{2.33}$$

Hence, from the definition of the measure $\mu_p(\Omega,\cdot)$, see (1.11), we have

$$C_p(\Omega)=\frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}}h_\Omega(\xi)\,d\mu_p(\Omega,\xi). \tag{2.34}$$

Lemma 2.18. *Suppose $1<p<n$. Let $\Omega\subset\mathbb{R}^n$, $n\ge 2$, be a bounded convex domain, assume $0\in\Omega$ and let $R>0$ be such $\bar\Omega$ is contained in $B(0,R)$. Let U be the unique solution to (2.17). Then there exists $c=c(n,p,R)$, with $1\le c<\infty$, such that*

$$|\nabla U|\ge c^{-1},\ \mathcal{H}^{n-1}\text{-a.e. on }\partial\Omega.$$

Proof. Using the continuity of U in $\mathbb{R}^n\setminus\Omega$, and (b) of Theorem 2.12, we see that there exists $\hat t\in(0,1)$ such that

$$\Omega_t:=\{x\in\mathbb{R}^n\setminus\Omega:\ U(x)>t\}\subseteq B(0,R), \tag{2.35}$$

for all $t \in (\hat{t}, 1)$. We fix $t \in (\hat{t}, 1)$ and consider

$$\hat{U}(x) = \frac{U(x)}{t}, \quad \text{for all } x \in \mathbb{R}^n \setminus \Omega_t. \tag{2.36}$$

Then $\hat{U}$ is the p-equilibrium potential of Ω_t. Using (d) of Theorem 2.12 we conclude that the closure $\overline{\Omega_t}$ of Ω_t is a $C^{2,\alpha}$-smooth (even C^∞-smooth) convex body. Let $\theta_t = -\langle x, \nabla \hat{U} \rangle$. Then, essentially using (a) of Theorem 2.12 and [69] we can conclude that θ_t is at least Hölder continuous on the closure of Ω_t. Furthermore, using barrier arguments, see [58] or Lemma 2.4, Lemma 2.5 in [59], we can conclude, for some $\varepsilon > 0$, that $\theta_t \geq \varepsilon$ at every point of $\partial\Omega_t$. This implies that that closure $\overline{\Omega_t}$ of Ω_t is a convex body of class $C^{2,\alpha}_+$. In particular, $\hat{U} \in C^2(\mathbb{R}^n \setminus \Omega_t)$ and the closure of

$$\tilde{\Omega}_s = \{x \in \mathbb{R}^n \setminus \Omega_t : \ \hat{U}(x) > s\}, \tag{2.37}$$

is a convex body of class $C^{2,\alpha}_+$ for all $s \in (0, 1)$. Based on this, we now consider the function

$$h = h(x, s), \quad (x, s) \in \mathbb{S}^{n-1} \times (0, 1] \tag{2.38}$$

where, for every $s \in (0, 1]$, the function $h(\cdot, s) : \mathbb{S}^{n-1} \to \mathbb{R}^1$, is defined as the support function of the closure of $\tilde{\Omega}_s$. Note that the function h, which was studied in [28], is well-defined since, for each $s \in (0, 1)$, the closure of $\tilde{\Omega}_s$ is a convex body of class $C^{2,\alpha}_+$. Since $\tilde{\Omega}_1 = \Omega_t \subseteq B(0, R)$ we can use the maximum principle to conclude that $\hat{U}$ is dominated, in $\mathbb{R}^n \setminus \overline{B(0, R)}$, by the p-equilibrium potential of $B(0, R)$. This implies, in particular, that

$$\operatorname{diam}(\tilde{\Omega}_{1/2}) \leq D,$$

where $D > 0$ is a constant depending on n, p, R. As a consequence,

$$0 \leq h(x, 1/2) \leq D, \ \text{ for each } x \in \mathbb{S}^{n-1}. \tag{2.39}$$

Using [28, Proposition 1] we have that $\frac{\partial h}{\partial s}(x, \cdot)$ is non-decreasing for every fixed x, and

$$h(x, 1) = h(x, 1/2) + \int_{1/2}^{1} \frac{\partial h}{\partial s}(x, s)\, ds, \tag{2.40}$$

for each $x \in \mathbb{S}^{n-1}$. Thus, for $x \in \mathbb{S}^{n-1}$, we deduce that

$$\begin{aligned} \frac{\partial h}{\partial s}(x, 1) \ &\geq \ 2\big[h(x, 1) - h(x, 1/2)\big] \\ &\geq \ -2h(x, 1/2) \geq -2D. \end{aligned} \tag{2.41}$$

On the other hand, using [28, Theorem 4], we see that

$$\frac{\partial h}{\partial s}(x, 1) = -1/|\nabla \hat{U}(x)|, \quad \text{at } x = \nabla \hat{U}(x)/|\nabla \hat{U}(x)|. \tag{2.42}$$

Hence, we can first conclude that there is a constant $c' = c'(n, p, R)$, such that $1 \leq c' < \infty$, and such that

$$|\nabla \hat{U}(x)| \geq (c')^{-1} \text{ for all } x \in \partial\Omega_t. \tag{2.43}$$

Then, using the definition of $\hat{U}$, we see that (2.43) implies that

$$|\nabla U(x)| \geq c^{-1} \text{ for all } x \in \partial\Omega_t, \tag{2.44}$$

for yet another constant $1 \leq c = c(n, p, R) < \infty$. To complete the proof, we let $t \to 1^-$ and use Lemma 2.13. □

We end this section by recalling an important Brunn-Minkowski type inequality for p-capacity.

Theorem 2.19. *Suppose $n \geq 2$, and that $\Omega_0, \Omega_1 \subset \mathbb{R}^n$ are bounded convex domains. If $1 < p < n$, then*

$$C_p(\Omega_0 + \Omega_1)^{\frac{1}{n-p}} \geq C_p(\Omega_0)^{\frac{1}{n-p}} + C_p(\Omega_1)^{\frac{1}{n-p}},$$

with equality if and only if Ω_0 and Ω_1 are homothetic.

Remark 2.20. The classical Brunn-Minkowski inequality states that the volume (i.e. Lebesgue measure) raised to power $1/n$ is concave with respect to Minkowski addition in the class of convex bodies in $\mathbb{R}^n$, i.e. $V^{1/n}(A+B) \geq V(A)^{1/n} + V(B)^{1/n}$, where $A, B \subset \mathbb{R}^n$ are convex bodies. It is at the core of the Brunn-Minkowski theory of convex bodies and it is strongly related to many other important inequalities of analysis, see [93] and the beautiful paper by Gardner [33]. The Brunn-Minkowski inequality in fact holds for measurable sets (provided their Minkowsky sum is measurable as well) and suitable versions hold for the other quermassintegrals, see [93, Theorem 6.4.3]. Recently, Brunn-Minkowski type inequalities have been proved also for several functionals from calculus of variations, among which there are of course the Newton capacity [8, 18] and the p-capacity [28]; other examples are the first Dirichlet eigenvalue of the Laplacian [11,13,24], the torsional rigidity [10], the logarithmic capacity (or transfinite diameter) in the plane [9] and its n-dimensional counterpart [26] (which is the natural extension of the p-capacity when $p = n$), the Monge-Ampère eigenvalue [91], the Dirichlet eigenvalue of the p-Laplacian and the p-torsional rigidity [27], the Bernoulli constant [7], eigenvalues of Hessian equations [70,92], the first Dirichlet eigenvalue for the Finsler laplacian [103]. In general, Brunn-Minkowski type inequalities include the characterization of equality conditions, and this often plays an important role in Minkowski type problems, especially for the uniqueness part. In the case of p-capacity, this characterization has been obtained in [18] for $p = 2$ and in [28] for $1 < p < n$.

Remark 2.21. As mentioned in the introduction there are many extensions of the classical Minkowski problem. One of them is the so-called L_p-Minkowski problem considered in Lutwak [74], [75] (extending Firey [32]), Lutwak-Oliker [76], Lutwak-Yang-Zhang [77], Chou-Wang [22], Hug-Lutwak-Yang-Zhang [48], Stancu [96], [97], Umanskiy [100], and Haberl-Lutwak-Yang-Zhang [44]. This problem arises from the notion of L_p *surface area measure* of a convex body K, introduced in [74], whose total mass is the p-surface area $S_p(K)$ given by

$$S_p(K) = \int_{\partial K} |x \cdot \mathsf{g}(x)|^{1-p} \, d\mathcal{H}^{n-1}(x).$$

In connection with p-capacity, studied here, we mention the following sharp inequality proved by Ludwig-Xiao-Zhang [73]:

$$S_p(K) \;\geq\; \Big(\frac{p-1}{n-p}\Big)^{p-1} C_p(K).$$

See also Pólya-Szegö [88] for $(n, p) = (3, 2)$ as well as Maz'ya [80] and Xiao [104] for $p = 1$.

3. Variational formula for the p-capacity of smooth convex domains

In this section we prove Theorem 1.1 and Theorem 1.2 for bounded convex domains of class $C^{2,\alpha}_+$. In the following we let Ω, $\tilde{\Omega}$ be two such domains, and we let h and v be the support functions of Ω and $\tilde{\Omega}$, respectively. We let g denote the Gauss map of $\partial\Omega$. Recall the definition of the class $\mathcal{C}$ in (2.15) and note that $h, v \in \mathcal{C}$. Then, also $h + tv \in \mathcal{C}$ for $|t|$ sufficiently small, and hence there exists a convex body K_t of class $C^{2,\alpha}_+$ with support function $h_t := h + tv$. We let Ω_t denote the interior of K_t and we note, for $t \geq 0$, that $\Omega_t = \Omega + t\tilde{\Omega}$. In the following we we first develop an explicit expression for $dC_p(\Omega_t)/dt$ at $t = 0$.

3.1. **A self-adjoint operator.** Using the notation above, we let $U_t = U(x,t)$ be the solution to the problem in (1.10) in $\mathbb{R}^n \setminus \bar{\Omega}_t$. We are interested in the functional $\mathcal{F} : \mathcal{C} \to C(\mathbb{S}^{n-1})$, defined, for $\xi \in \mathbb{S}^{n-1}$, by

$$\mathcal{F}(h)(\xi) = \frac{|\nabla U(\mathsf{g}^{-1}(\xi))|^p}{\kappa(\mathsf{g}^{-1}(\xi))} = |\nabla U(\mathsf{g}^{-1}(\xi))|^p \det(h_{ij}(\xi) + h(\xi)\delta_{ij}).$$

Given $v \in C^\infty(\mathbb{S}^{n-1})$, let

$$\mathcal{L}(v) = \frac{d}{dt}\mathcal{F}(h + tv)\Big|_{t=0}$$

denote the directional derivative of $\mathcal{F}$ at h along v. As we will see, $\mathcal{L}$ is a linear functional acting on $C^\infty(\mathbb{S}^{n-1})$. One of the key steps in computing the first variation of $C_p(\Omega_t)$ is to prove that $\mathcal{L}$ is self-adjoint on $L^2(\mathbb{S}^{n-1})$, viewing $L^2(\mathbb{S}^{n-1})$ as equipped with the standard scalar product.

Lemma 3.1. *Suppose* $1 < p < n$. *Let* Ω, $\tilde{\Omega}$, Ω_t, h, v, g, h_t, U, U_t, *be as above. Then, for each fixed* $x \in \mathbb{R}^n \setminus \bar{\Omega}$, *the function* $t \to U(x,t)$ *is differentiable with respect to* t *at* $(x, 0)$. *Let* $\dot{U}(x) = \frac{\partial U}{\partial t}(x, 0)$. *The function* $\dot{U} : \mathbb{R}^n \setminus \bar{\Omega} \to \mathbb{R}$ *can be extended to* $\partial\Omega$ *so that* $\dot{U} \in C^2(\mathbb{R}^n \setminus \Omega)$. *Moreover,*

$$(a) \qquad \dot{U}(x) = |\nabla U(x)| v(\mathsf{g}(x)) \ \textit{ for all } x \in \partial\Omega,$$

and there exists $c = c(n,p)$ *such that*

$$(b) \qquad 0 \leq \dot{U}(x) \leq c|x|^{\frac{p-n}{p-1}} \ \textit{ as } |x| \to \infty,$$

$$(c) \qquad 0 \leq |\nabla \dot{U}(x)| \leq c|x|^{\frac{1-n}{p-1}} \ \textit{ as } |x| \to \infty.$$

Furthermore,

$$(d) \qquad \nabla \dot{U}(y) := \lim_{x \to y,\ x \in \tilde{\Gamma}(y)} \nabla \dot{U}(x) \ \textit{ exists for every } y \in \partial\Omega,$$

with $\tilde{\Gamma}(y)$ *defined as in* (2.19), *and*

$$(e) \qquad \int_{\partial\Omega} |\nabla U|^{p-1} |\nabla \dot{U}(x)| \, d\mathcal{H}^{n-1} < \infty.$$

Proof. Assume, without loss of generality, that $0 \in \tilde{\Omega}$, which implies $v \geq 0$ on $\mathbb{S}^{n-1}$. Then Ω_t contains Ω for $t > 0$ and Ω_t is contained in Ω for $t < 0$. Recall that, for $|t|$ small, Ω, $\tilde{\Omega}$, and Ω_t are all bounded convex domains of class $C^{2,\alpha}_+$.

Step 1. We first prove that there exist ε, such that $0 < \varepsilon \ll 1$, and a constant c, such that $1 \leq c < \infty$, for which we have

$$|\nabla U(x,t)| \leq c \ \text{ for all } (x,t) \in (\mathbb{R}^n \setminus \Omega_t) \times [-\varepsilon, \varepsilon]. \tag{3.1}$$

Let $\varepsilon > 0$ be such that Ω_t is of class $C^{2,\alpha}_+$ for $|t| \leq \varepsilon$, and fix t restricted to $|t| \leq \varepsilon$. Using [28, Proposition 1], we get that $\frac{\partial h}{\partial s}(x, \cdot)$ is non-decreasing for every fixed x. Thus [28, Theorem 4] implies that $|\nabla U(x,t)|$ attains its maximum on $\partial\Omega_t$. Now let $x \in \partial\Omega_t$ and note that there exists a ball B, included in Ω_t and internally tangent to $\partial\Omega_t$ at x, with radius r which can be chosen to be independent of t and x. Let $\tilde{U}$ be the p-equilibrium potential of B. By the comparison principle $\tilde{U}(\cdot) \le U(\cdot, t)$ in $\mathbb{R}^n \setminus \Omega_t$, and, since $U(x) = \tilde{U}(x,t)$,we have $|\nabla U(x,t)| \le |\nabla \tilde{U}(x)|$. On the other hand the value $|\nabla \tilde{U}(x)|$ can be explicitly computed and is a positive constant depending on r and n only. Hence (3.1) is proved.

Step 2. We let, for $(x,t) \in (\mathbb{R}^n \setminus (\Omega \cup \Omega_t)) \times [-\varepsilon, \varepsilon]$,

$$V(x,t) = U(x,t) - U(x,0)\,, \tag{3.2}$$

and, for $t \neq 0$,

$$W(x,t) = \frac{V(x,t)}{t}\,. \tag{3.3}$$

Consider $t > 0$ and recall that this implies that Ω_t contains Ω. Then, using (b) of Theorem 2.12, we see that $U(x) = U(x,0) \le 1 = U(x,t)$ when $x \in \partial\Omega_t$. Moreover, since $U(x), U(x,t) \to 0$ as $|x| \to \infty$ it follows directly from the comparison principle for the p-Laplacian, see [89, Theorem 5.4], that whenever $x \in \mathbb{R}^n \setminus \Omega_t$, we have $U(x,0) \le U(x,t)$. Let

$$\psi(t) := \frac{1}{\min_{x \in \Omega_t} U(x)} = \frac{1}{\min_{x \in \partial\Omega_t} U(x)} \tag{3.4}$$

where the equality in (3.4) follows from the comparison principle on the domain $\Omega_t \setminus \Omega$. Moreover, we have $\psi(t) > 0$ by the strong maximum principle and ψ is an increasing function of t. Furthermore, using (3.1) it is easy to see that there exists a constant c, depending on Ω but independent of t, such that for all $t \in (0, \varepsilon]$,

$$\frac{\psi(t) - \psi(0)}{t} \le c. \tag{3.5}$$

In particular, for all $x \in \partial\Omega_t$ and $t \in (0, \varepsilon]$,

$$U(x,t) \le \psi(t) U(x,0). \tag{3.6}$$

By the maximum principle we see that this inequality also holds in $\mathbb{R}^n \setminus \Omega_t$, and hence we have proved that

$$0 \le W(x,t) \le c\,U(x,0) \;\text{ whenever }\; (x,t) \in (\mathbb{R}^n \setminus \Omega_t) \times (0, \varepsilon].$$

An analogous estimate can be found for $t < 0$ and all in all we can conclude that there exists a constant $c > 0$, depending on Ω but independent of t, such that

$$|W(x,t)| \le c\,U(x,0) \le c \;\text{ whenever }\; (x,t) \in (\mathbb{R}^n \setminus (\Omega \cup \Omega_t) \times ([-\varepsilon, \varepsilon] \setminus \{0\})\,. \tag{3.7}$$

From this inequality it follows, in particular, that $U(\cdot, t)$ converges to $U(\cdot, 0)$ as t tends to 0, uniformly on compact subsets of $\mathbb{R}^n \setminus \bar{\Omega}$.

Step 3. In the following, let D be a compact subset of $\mathbb{R}^n \setminus \bar{\Omega}$. Then, using (3.1) and Lemma 2.4, concerning interior Hölder continuity of partial derivatives, see also Theorem 1 in [58], we can conclude that there exists $\sigma \in (0,1)$ and a constant $\tilde{c}$, with $1 \le \tilde{c} < \infty$, both (σ and $\tilde{c}$) independent of t such that $\|U(\cdot, t)\|_{C^{1,\sigma}(D)} \le \tilde{c}$, whenever $t \in [-\varepsilon, \varepsilon] \setminus \{0\}$. Consequently, using the Ascoli-Arzelá Theorem, we can conclude that $U(\cdot, t)$ converges to $U(\cdot, 0)$ in $C^1(D)$, as $t \to 0$. Now using (b) of Theorem 2.12 we see that there exists $\varepsilon > 0$ and a constant $\hat{c} > 0$, independent of t, such that

$$|\nabla U(x,t)| \geq \hat{c} \text{ whenever } (x,t) \in D \times [-\varepsilon, \varepsilon]\,. \tag{3.8}$$

Note that

$$\begin{aligned} 0 &= \Delta_p U(x,t) - \Delta_p U(x,0) \\ &= \int_0^1 \frac{d}{ds} \Delta_p (sU(x,t) + (1-s)U(x,0))\, ds \\ &= \frac{\partial}{\partial x_i}\big(b_{ij}(x,t) \frac{\partial}{\partial x_j} V(x,t)\big)\,, \end{aligned} \tag{3.9}$$

where, for $i, j = 1 \ldots, n$,

$$b_{ij}(x,t) = \int_0^1 \Big(|\nabla U_s(x,t)|^{p-4}\big((p-2)(U_s(x,t))_{x_i}(U_s(x,t))_{x_j} + \delta_{ij}|\nabla U_s(x,t)|^2\big)\Big)\, ds\,,$$

$U_s(x,t) = sU(x,t) + (1-s)U(x,0)$, and δ_{ij} is the Kronecker delta. As $t \to 0$, we see that $U_s(\cdot,t) \to U(\cdot,0)$, uniformly and, by the argument above, we see that $b_{ij}(\cdot,t)$ converges uniformly to

$$b_{ij}(x) = |\nabla U(x,0)|^{p-4}\big((p-2)(U(x,0))_{x_i}(U(x,0))_{x_j} + \delta_{ij}|\nabla U(x,0)|^2\big)\,.$$

Dividing in (3.9) by t we can conclude that W solves

$$\frac{\partial}{\partial x_i}\big(b_{ij}(x,t) W_j(x,t)\big) = 0\,, \tag{3.10}$$

in D. Now, using (3.1), (3.8), and (3.10), we see that equation in (3.10) is uniformly elliptic in D with ellipticity constants independent of t. Moreover, the $C^{0,\sigma}(D)$ norms of the coefficients b_{ij} are uniformly bounded. Hence, using Schauder estimates (see, for instance, [34, Theorem 6.2]), we deduce that there exists a constant c such that

$$\|W(\cdot,t)\|_{C^{2,\alpha}(D)} \leq c \text{ whenever } t \in [-\varepsilon,\varepsilon] \setminus \{0\}.$$

Applying again the Ascoli-Arzelá Theorem and a standard diagonalization procedure, we obtain a sequence $\{t_k\}_{k\in\mathbb{N}}$, tending to 0 as k tends to infinity, and a function $\dot{U} : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$, such that, as $k \to \infty$, we have $W(\cdot,t_k)$ converging to $\dot{U}(\cdot)$ uniformly on compact sets of $\mathbb{R}^n \setminus \bar{\Omega}$. Using (3.10) we then have

$$\frac{\partial}{\partial x_i}\big(b_{ij}(x) \frac{\partial}{\partial x_j} \dot{U}(x)\big) = 0 \text{ whenever } x \in \mathbb{R}^n \setminus \bar{\Omega}\,. \tag{3.11}$$

We have thus proved the existence of the limit of $W(\cdot,t)$, as t tends to 0, at least up to choosing a suitable sequence of t's in the interior of $\mathbb{R}^n \setminus \bar{\Omega}$. In particular, for each fixed $x \in \mathbb{R}^n \setminus \bar{\Omega}$, the function $t \to U(x,t)$ is differentiable with respect to t at $(x,0)$.

Step 4. We now focus on the boundary behavior of $\dot{U}$. Let $R > 0$ be such that $\Omega_t \subset B_R$ for every $t \in [-\varepsilon,\varepsilon]$. In what follows we let $K := \bar{\Omega}$ and $K_t = \overline{\Omega_t}$. From the assumption that K_t is of class $C^{2,\alpha}_+$ and using a standard compactness argument, it follows that there exists $\rho > 0$ such that for every $t \in [-\varepsilon,\varepsilon]$ and for every $x \in \partial K_t$, there exists a closed ball B, of radius ρ, with $B \supset K_t$ and $x \in \partial B$. Let $\hat{U}$ be the p-equilibrium potential of B. From the comparison principle we see that $\hat{U}(\cdot) \geq U(\cdot,t)$ in $\mathbb{R}^n \setminus B$ and, since they coincide at x, that

$|\nabla U(x,t)| \geq |\nabla \hat{U}(x)|$. Observe that $|\nabla \hat{U}(x)|$ can be explicitly computed and it is a positive constant depending on ρ and n. Hence there exists a constant $c > 0$ such that

$$|\nabla U(x,t)| \geq c, \qquad \text{for all } (x,t) \in B_R \setminus (\Omega \cup \Omega_t)\,. \tag{3.12}$$

This, together with (3.1), implies that the p-Laplace operator is uniformly elliptic on $U(\cdot,t)$ in $B_R \setminus K_t$, and that the ellipticity constants can be chosen to be independent of t. As a consequence, using also the smoothness assumptions on ∂K_t, the boundary condition verified by $U(\cdot,t)$ and (3.1), we may apply the boundary Hölder estimates for the gradient of solutions of quasi-linear elliptic equations, see [34, Theorem 13.2], to infer that there exist $\sigma \in (0,1)$ and a constant $c > 0$ such that

$$\|\nabla U(\cdot,t)\|_{C^{0,\sigma}(B_R\setminus\Omega_t)} \leq c\,, \qquad \text{for all } t \in [-\varepsilon,\varepsilon]\,.$$

This in turn implies that the coefficients of equation (3.10) are uniformly bounded in the norm of $C^{0,\sigma}(B_r \setminus \Omega_t)$. Hence, by Theorem 6.6 in [34], there exists $c > 0$, independent of t, such that

$$\|W(\cdot,t)\|_{C^{2,\sigma}(B_r\setminus(\Omega\cup\Omega_t))} \leq c, \qquad \text{for all } t \in [-\varepsilon,\varepsilon] \setminus \{0\}. \tag{3.13}$$

Clearly the same estimate extends to the function $\dot{U}$:

$$\|\dot{U}\|_{C^{2,\sigma}(B_R\setminus\Omega)} \leq c\,. \tag{3.14}$$

In particular, the function $\dot{U} : \mathbb{R}^n \setminus \bar{\Omega} \to \mathbb{R}$ can be extended to $\partial\Omega$ so that $\dot{U} \in C^2(\mathbb{R}^n \setminus \Omega)$.

Step 5. Proof of (a) and (b). Let $x \in \partial\Omega$ and let $\mathsf{g}(x)$ be the outer unit normal to $\partial\Omega$ at x. For $k \in \mathbb{N}$, let $x_k \in \partial\Omega_{t_k}$ be the point whose outer unit normal to $\partial\Omega_{t_k}$ is $\mathsf{g}(x)$. Since K_t is of class $C^{2,\alpha}_+$, the point x_k is uniquely determined and the sequence x_k tends to x as k tends to infinity. We have

$$x_k = \nabla h_{\Omega_{t_k}}(\xi) = \nabla h_\Omega(\xi) + t_k \nabla v(\xi) = x + t_k \nabla v(\xi),$$

and thus,

$$\frac{x_k - x}{t_k} = \nabla v(\xi),$$

where $\xi = \mathsf{g}(x)$. From (3.13),

$$\dot{U}(x) = \lim_{k\to\infty} W_k(x_k)\,,$$

and hence

$$\begin{aligned} \dot{U}(x) &= \lim_{k\to\infty} \frac{U(x_k,t_k) - U(x_k,0)}{t_k} = \lim_{k\to\infty} \frac{1 - U(x_k,0)}{t_k} \\ &= \lim_{k\to\infty} \frac{U(x,0) - U(x_k,0)}{t_k} = -\nabla U(x) \cdot \nabla v(\xi) \\ &= |\nabla U|\xi \cdot \nabla v(\xi) = |\nabla U(x)| v(\mathsf{g}(x))\,, \end{aligned} \tag{3.15}$$

and this completes the proof of part (a). Note also that by (3.7) we have, as $|x| \to \infty$, that $\dot{U}(x) \leq c|x|^{\frac{p-n}{p-1}}$. Hence, from (3.11) and the maximum principle, $\dot{U}$ is uniquely determined. As a consequence, as t tends to zero, the family of functions $W(\cdot,t)$ tends to $\dot{U}$ (not just some sequence of it). In particular, the proofs of statements (a) and (b) are complete.

Step 6. Proof of (c), (d), (e). let $R_1, R_2 > 0$ be such that $R_1 < R_2$ and $K \subset B(0, R_1)$. Furthermore, for $r > 0$ we define $\varsigma(r) = r^{\frac{p-n}{p-1}}$, and consider,

$$\dot{X}(y) = \frac{\dot{U}(ry)}{\varsigma(r)},\ Z(y) = \frac{U(ry)}{\varsigma(r)} \text{ for } y \in D = \{y \in \mathbb{R}^n \,:\, R_1 < |y| < R_2\}. \tag{3.16}$$

Now, arguing as above we see that $\dot{X}(y)$ is a bounded solution of

$$\frac{\partial}{\partial y_i}\big(c_{ij}(y)\frac{\partial}{\partial y_j}\dot{X}(y)\big) = 0 \quad \text{in} \quad D, \tag{3.17}$$

where

$$c_{ij}(y) \;=\; |\nabla Z(y)|^{p-4}\big((p-2)(Z(y))_{y_i}(Z(y))_{y_j} + \delta_{ij}|\nabla Z(y)|^2\big)\,. \tag{3.18}$$

Using the asymptotic behavior of $\nabla U(x)$ as $|x| \to \infty$, see Lemma 2.15 (b), we can conclude that the equation in (3.17) is uniformly elliptic in D with ellipticity constants independent of r. Furthermore, using uniform Hölder continuity of the coefficients $\{c_{ij}(y)\}$ in D, it follows from statement (b) of the lemma as well as well-known a priori estimates for such equations, see, for instance, [37, Lemma 3.1], that

$$|\nabla \dot{X}(y)| \leq c \text{ whenever } y \in D = \{y \in \mathbb{R}^n \,:\, R_1 < |y| < R_2\}, \tag{3.19}$$

and for some constant c independent of r. Hence, letting $r \to \infty$ the proof of (c) is complete. Furthermore, also (d), (e), are easily seen to hold and we omit further details. $\square$

Lemma 3.2. *Suppose $1 < p < n$. Let Ω be a bounded convex domains of class $C^{2,\alpha}_+$, with support function h, and let g denote the Gauss map of $\partial\Omega$. For $i = 1, 2$, let Ω_i be a convex domain of class $C^{2,\alpha}_+$ with support function v_i. Let $\varepsilon > 0$ be such that, for both $i = 1, 2$, the function $h + tv_i \in \mathcal{C}$ whenever $|t| \leq \varepsilon$. For $|t| \leq \varepsilon$, and $i = 1, 2$, let $\Omega_{i,t}$ be the uniquely determined convex domain of class $C^{2,\alpha}_+$ such that $h + tv_i$ is the support functions of $\Omega_{i,t}$. Let $U_i(x,t)$ be the solution to problem (1.10) in $\Omega_{i,t}$. Using Lemma 3.1, let*

$$\dot{U}_i(x) = \frac{\partial}{\partial t}U_i(x,t)\bigg|_{t=0} \qquad \textit{for all } x \in \mathbb{R}^n \setminus \Omega.$$

Then,

$$\begin{aligned}\int_{\partial\Omega} & v_1(\mathsf{g}(x))|\nabla U(x)|^{p-1}(\mathsf{g}(x)\cdot\nabla\dot{U}_2(x))\,d\mathcal{H}^{n-1}(x)\\ &= \int_{\partial\Omega} v_2(\mathsf{g}(x))|\nabla U(x)|^{p-1}(\mathsf{g}(x)\cdot\nabla\dot{U}_1(x))\,d\mathcal{H}^{n-1}(x)\,.\end{aligned}$$

Proof. Arguing as in the proof of Lemma 3.1, see (3.11), we have

$$\begin{aligned}&\nabla\cdot\big((p-2)|\nabla U|^{p-4}(\nabla U\cdot\nabla\dot{U}_1)\nabla U + |\nabla U|^{p-2}\nabla\dot{U}_1\big) = 0\,,\\ &\nabla\cdot\big((p-2)|\nabla U|^{p-4}(\nabla U\cdot\nabla\dot{U}_2)\nabla U + |\nabla U|^{p-2}\nabla\dot{U}_2\big) = 0\,.\end{aligned} \tag{3.20}$$

Therefore,

$$\begin{aligned}0 =& \dot{U}_2\,\nabla\cdot((p-2)|\nabla U|^{p-4}(\nabla U\cdot\nabla\dot{U}_1)\nabla U + |\nabla U|^{p-2}\nabla\dot{U}_1)\\ &- \dot{U}_1\,\nabla\cdot((p-2)|\nabla U|^{p-4}(\nabla U\cdot\nabla\dot{U}_2)\nabla U + |\nabla U|^{p-2}\nabla\dot{U}_2)\\ =& \nabla\cdot(\dot{U}_2((p-2)|\nabla U|^{p-4}(\nabla U\cdot\nabla\dot{U}_1)\nabla U + |\nabla U|^{p-2}\nabla\dot{U}_1))\end{aligned}$$

$$
- \quad \nabla\cdot(\dot U_1((p-2)|\nabla U|^{p-4}(\nabla U\cdot\nabla\dot U_2)\nabla U+|\nabla U|^{p-2}\nabla\dot U_2))\,. \tag{3.21}
$$

Now, applying Lemma 2.11 to (3.21), using Lemma 2.13, Lemma 2.15, and Lemma 3.1, we can conclude that

$$
\begin{aligned}
&\int_{\partial\Omega}\dot U_2\big((p-2)|\nabla U|^{p-4}(\nabla U\cdot\nabla\dot U_1)(\mathsf{g}\cdot\nabla U)+|\nabla U|^{p-2}(\mathsf{g}\cdot\nabla\dot U_1)\big)\,d\mathcal{H}^{n-1}\\
&=\int_{\partial\Omega}\dot U_1\big((p-2)|\nabla U|^{p-4}(\nabla U\cdot\nabla\dot U_2)(\mathsf{g}\cdot\nabla U)+|\nabla U|^{p-2}(\mathsf{g}\cdot\nabla\dot U_2)\big)\,d\mathcal{H}^{n-1}.
\end{aligned}
$$

Hence, a final application of Lemma 2.13, together with (a) of Lemma 3.1, completes the proof. □

Lemma 3.3. *Suppose $1<p<n$. Let $e_1,\dots,e_{n-1}$ be an orthonormal frame on $\mathbb{S}^{n-1}$. Let Ω be a bounded convex domain of class $C^{2,\alpha}_+$ in $\mathbb{R}^n$ and let h be the support function of Ω. Let U be the solution to (2.17). Let $\xi\in\mathbb{S}^{n-1}$ and $x\in\partial\Omega$ be such that $\mathsf{g}(x)=\xi$. Then,*

$$
\begin{aligned}
&(a)\qquad \big(\nabla^2U(x)e_i\big)\cdot e_j=-\kappa(x)\,|\nabla U(x)|\,c_{ij}(\xi),\\
&(b)\qquad \big(\nabla^2U(x)e_i\big)\cdot\xi=-\kappa(x)\textstyle\sum_j c_{ij}(\xi)|\nabla U(x)|_j,\\
&(c)\qquad \big(\nabla^2U(x)\xi\big)\cdot\xi=(p-1)^{-1}\kappa(x)\,|\nabla U(x)|\textstyle\sum_i c_{ii}(\xi),
\end{aligned}
$$

where $\kappa(x)$ denotes the Gauss curvature of $\partial\Omega$ at $x\in\partial\Omega$, $|\nabla U(\cdot)|_j$ denotes first covariant derivatives of $|\nabla U(\cdot)|$ on $\mathbb{S}^{n-1}$, and (c_{ij}) is the cofactor matrix introduced below (2.12).

Proof. Lemma 3.3 is a generalization of Lemma A in [50], see also [52], to the non-linear setting. Compared to [50] though, we here give a different and simpler proof. We first note, since Ω is a bounded convex domain of class $C^{2,\alpha}_+$ in $\mathbb{R}^n$, that it follows, as in Step 4 of the proof of Lemma 3.1, from an elementary barrier argument, and boundary Schauder estimates for quasi-linear elliptic equations, see [34], that partial derivatives of U up to, and including, order two are pointwise well-defined on $\partial\Omega$. Using this we start the proof of Lemma 3.3 and we first note, using Lemma 2.13 and (2.14), that $\nabla U=-|\nabla U|\xi$ and $\nabla U\cdot F_i=0$, where $F(\xi)=\mathsf{g}^{-1}(\xi)$ is the inverse Gauss map of $\partial\Omega$. Differentiating the second relation gives

$$
((\nabla^2U)F_j)\cdot F_i+\nabla U\cdot F_{ij}=0.
$$

Recalling (2.14) we have that

$$
F_i=a_{ij}e_j,\quad F_{ij}=a_{ijk}e_k-a_{ij}\xi,
$$

where a_{ijk} are the covariant derivatives of a_{ij}. In particular, combining the last two displays we see that

$$
a_{ik}a_{jl}((\nabla^2U)e_l)\cdot e_k+a_{ij}|\nabla U|=0. \tag{3.22}
$$

Multiplying (3.22) by the cofactor matrix c_{ij} of a_{ij} we see that (a) holds. Similarly, taking the covariant derivative of $|\nabla U|=-\xi\cdot\nabla U$, and using (2.14), we see that

$$
|\nabla U|_i=-e_i\cdot\nabla U-\xi\cdot((\nabla^2U)F_i)=-a_{ij}((\nabla^2U)e_j)\cdot\xi.
$$

Next, again multiplying by the cofactor matrix c_{ij} of a_{ij} we see that (b) holds. Finally, using the p-Laplace equation for U, we have

$$
(p-2)((\nabla^2U)\xi)\cdot\xi+\Delta U=0.
$$

This and (a) allows us to conclude that

$$\begin{aligned}(p-1)((\nabla^2 U)\xi)\cdot\xi &= ((\nabla^2 U)\xi)\cdot\xi - \Delta U\\ &= -((\nabla^2 U)e_i)\cdot e_i = \kappa|\nabla U|\sum_i c_{ii},\end{aligned}$$

which completes our proof. ☐

Lemma 3.4. *Suppose $1<p<n$. Let Ω be a bounded convex domain of class $C_+^{2,\alpha}$ and let U be the unique solution to (1.10). Define the operator $\mathcal{L} : \mathcal{C} \to C(\mathbb{S}^{n-1})$ by*

$$\begin{aligned}\mathcal{L}(v)(\xi) &= -\frac{p|\nabla U(\mathsf{g}^{-1}(\xi))|^{p-1}}{\kappa(\mathsf{g}^{-1}(\xi))}\,(\xi\cdot\nabla\dot U(\mathsf{g}^{-1}(\xi)))\\ &\quad+\sum_j(|\nabla U(\mathsf{g}^{-1}(\xi))|^p\sum_i c_{ij}(\xi)v_i(\xi))_j \\ &\quad-(p-1)^{-1}|\nabla U(\mathsf{g}^{-1}(\xi))|^p\,v(\xi)\sum_i c_{ii}(\xi)\,,\end{aligned}\tag{3.23}$$

where $\dot U$ is the function defined in Lemma 3.1 corresponding to v and where $\kappa(x)$ denotes the Gauss curvature of $\partial\Omega$ at x. Then $\mathcal{L}$ is self-adjoint on $L^2(\mathbb{S}^{n-1})$, i.e.

$$\int_{\mathbb{S}^{n-1}} v_1\,\mathcal{L}(v_2)\,d\xi = \int_{\mathbb{S}^{n-1}} v_2\,\mathcal{L}(v_1)\,d\xi, \qquad \text{for all } v_1, v_2\in\mathcal{C}\,.$$

Proof. For $i=1,2,3$, let $\mathcal{L}_i : \mathcal{C}\to C(\mathbb{S}^{n-1})$ be defined by

$$\begin{aligned}\mathcal{L}_1(v)(\xi) &= -\frac{p|\nabla U(\mathsf{g}^{-1}(\xi))|^{p-1}}{\kappa(\mathsf{g}^{-1}(\xi))}\left(\xi\cdot\nabla\dot U(\mathsf{g}^{-1}(\xi))\right),\\ \mathcal{L}_2(v)(\xi) &= \sum_j(|\nabla U(\mathsf{g}^{-1}(\xi))|^p\sum_i c_{ij}(\xi)v_i(\xi))_j\,,\\ \mathcal{L}_3(v)(\xi) &= -(p-1)^{-1}|\nabla U(\mathsf{g}^{-1}(\xi))|^p v(\xi)\sum_i c_{ii}(\xi)\,,\end{aligned}$$

Note that $\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3$. Using this decomposition we see immediately that $\mathcal{L}_2$ and $\mathcal{L}_3$ are self-adjoint. To prove that $\mathcal{L}_1$ is self-adjoint, use the change of variable formula (2.7), together with (2.16), and Lemma 3.2. ☐

3.2. **Variational formula and uniqueness for smooth domains.** We are now ready to state and prove Theorem 1.1 in the context of bounded convex domains of class $C_+^{2,\alpha}$.

Theorem 3.5. *Suppose $1<p<n$. Let Ω and L be bounded convex domains of class $C_+^{2,\alpha}$, with support functions h and v, respectively, and let g denote the Gauss map of Ω. For $t\in\mathbb{R}$ with $|t|$ sufficiently small, let Ω_t be the bounded convex domain having $h+tv$ as its support function. Then,*

$$\begin{aligned}\frac{d}{dt}C_p(\Omega_t)\Big|_{t=0} &= (p-1)\int_{\partial\Omega} v\big(\mathsf{g}(x)\big)|\nabla U(x)|^p\,d\mathcal{H}^{n-1}(x)\\ &= (p-1)\int_{\mathbb{S}^{n-1}} v(\xi)\,d\mu_p(\Omega,\xi)\,,\end{aligned}\tag{3.24}$$

where μ_p is the measure defined in (1.11).

Proof. Note that from the definition of the measure μ_p and (2.7) it suffices to prove that

$$\frac{d}{dt}C_p(\Omega_t)\Big|_{t=0} = (p-1)\int_{\mathbb{S}^{n-1}} v(\xi)\, d\mu_p(\Omega,\xi). \tag{3.25}$$

Let $U(\cdot,t)$ be the unique solution to (1.10) in Ω_t, and let $h_t(\cdot)$, $\mathsf{g}(\cdot,t)$, and $F(\cdot,t)=\mathsf{g}_t^{-1}(\cdot,t)$ denote the support function of Ω_t, its Gauss and its inverse, respectively. Let $\dot U(\cdot)$, $\dot h(\cdot)$, $\dot{\mathsf{g}}(\cdot)$ and $\dot F(\cdot)$ denote the partial derivatives of these functions, with respect to t, at $t=0$. Since $h_t = h+tv$ we observe that $\dot h = v$. Moreover, assuming that an orthonormal coordinate frame $e_1,\dots,e_{n-1}$ has been chosen on $\mathbb{S}^{n-1}$, from (2.14) it follows, for $\xi\in\mathbb{S}^{n-1}$, that

$$\dot F(\xi) = v_i(\xi)e_i + v(\xi)\xi. \tag{3.26}$$

Using (1.11), (2.7), and (2.16), we see that

$$d\mu_p(\Omega_t,\cdot) = |\nabla U(F(\xi,t),t)|^p\, \det((h_t)_{ij}(\xi)+h_t(\xi)\delta_{ij})\, d\xi\,.$$

For $\xi\in\mathbb{S}^{n-1}$, let

$$\mathcal{F}(h_t)(\xi) = |\nabla U(F(\xi,t),t)|^p\, \det((h_t)_{ij}(\xi)+h_t(\xi)\delta_{ij}).$$

Then, using the representation formula (2.34) we see that

$$\begin{aligned}\frac{d}{dt}C_p(\Omega_t)\Big|_{t=0} &= \frac{p-1}{n-p}\frac{d}{dt}\int_{\mathbb{S}^{n-1}} h_t\mathcal{F}(h_t)\,d\xi\Big|_{t=0}\\ &= \frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}}\left(v\mathcal{F}(h)+h\frac{d}{dt}\mathcal{F}(h_t)\Big|_{t=0}\right)d\xi.\end{aligned} \tag{3.27}$$

Let (c_{ij}) be the cofactor matrix of $(h_{ij}+h\delta_{ij})$. By standard properties of the cofactor matrix we have

$$\frac{d}{dt}(\det((h_t)_{ij}+h_t\delta_{ij}))\Big|_{t=0} = c_{ij}(v_{ij}+v\delta_{ij}),$$

and hence

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(h_t)\Big|_{t=0}(\xi) &= |\nabla U(F(\xi))|^p c_{ij}(\xi)(v_{ij}(\xi)+v(\xi)\delta_{ij})\\ &+ p|\nabla U(F(\xi))|^{p-1}\det(h_{ij}(\xi)+h(\xi)\delta_{ij})\,\frac{d}{dt}|\nabla U(F(\xi,t),t)|\Big|_{t=0},\end{aligned} \tag{3.28}$$

where for simplicity, we have let $F(\cdot)=F(\cdot,0)$. Using the boundary condition in problem (1.10) we see that

$$|\nabla U(F(\xi,t),t)| = -\nabla U(F(\xi,t),t)\cdot\xi\,.$$

and hence

$$\begin{aligned}&\frac{d}{dt}|\nabla U(F(\xi,t),t)|\Big|_{t=0}\\ &\quad= -\nabla^2(U(F(\xi)))\xi\cdot\dot F(\xi) - \nabla\dot U(F(\xi))\cdot\xi\\ &\quad= -\nabla^2(U(F(\xi)))\xi\cdot\big(v_i(\xi)e_i+v(\xi)\xi\big) - \nabla\dot U(F(\xi))\cdot\xi\\ &\quad= \kappa(F(\xi))c_{ij}(\xi)(|\nabla U(F(\xi))|)_j v_i(\xi)\\ &\qquad -(p-1)^{-1}\kappa(F(\xi))c_{ii}(\xi)|\nabla U(F(\xi))|v(\xi) - \nabla\dot U(F(\xi))\cdot\xi\,,\end{aligned}$$

where we have used (3.26) and Lemma 3.3. In particular, using (2.12), (3.28), and (2.13), we see that

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(h_t)\Big|_{t=0}(\xi) &= -p|\nabla U(F(\xi))|^{p-1}\det(h_{ij}(\xi)+h(\xi)\delta_{ij})(\nabla\dot{U}(F(\xi))\cdot\xi) \\ &\quad -(p-1)^{-1}|\nabla U(F(\xi))|^p c_{ii}(\xi)v(\xi)+\left(c_{ij}(\xi)|\nabla U(F(\xi))|^p v_j\right)_i . \end{aligned}$$

All put together, we can conclude that

$$\frac{d}{dt}\mathcal{F}(h_t)\Big|_{t=0}=\mathcal{L}(v)\,, \tag{3.29}$$

where $\mathcal{L}$ is the operator defined in Lemma 3.4. Next, by a standard homogeneity argument we note that

$$\mathcal{F}((1+t)h)=(1+t)^{n-p-1}\mathcal{F}(h)\,,$$

and hence by taking $v=h$, we see that

$$\mathcal{L}(h)=(n-p-1)\mathcal{F}(h). \tag{3.30}$$

Using (3.27), (3.29), Lemma 3.4, and (3.30) we have

$$\begin{aligned} \frac{d}{dt}C_p(\Omega_t)\Big|_{t=0} &= \frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}}(v\mathcal{F}(h)+h\mathcal{L}(v))\,d\xi \\ &= \frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}}(v\mathcal{F}(h)+v\mathcal{L}(h))\,d\xi \\ &= (p-1)\int_{\mathbb{S}^{n-1}}v\mathcal{F}(h)d\xi. \end{aligned}$$

Hence the proof of the lemma is complete. □

Let Ω_0, Ω_1 be bounded convex domains in $\mathbb{R}^n$. Define the mixed p-capacity of Ω_0 and Ω_1 by

$$C_p(\Omega_0,\Omega_1)=\frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}}h_{\Omega_1}(\xi)\,d\mu_p(\Omega_0,\xi).$$

Obviously, $C_p(\Omega_0,\Omega_0)=C_p(\Omega_0)$. The following theorem gives the Minkowski inequality for p-capacity in the context of bounded convex domains of class $C_+^{2,\alpha}$.

Theorem 3.6. *Suppose $1<p<n$. Let Ω_0, Ω_1 be bounded convex domains in $\mathbb{R}^n$ of class $C_+^{2,\alpha}$. Then*

$$C_p(\Omega_0,\Omega_1)^{n-p}\geq C_p(\Omega_0)^{n-p-1}C_p(\Omega_1),$$

with equality if and only if Ω_0,Ω_1 are homothetic.

Proof. From the Brunn-Minkowski inequality for p-capacity, see Theorem 2.19, we see that the function

$$f(t)=C_p(\Omega_0+t\Omega_1)^{\frac{1}{n-p}}-C_p(\Omega_0)^{\frac{1}{n-p}}-tC_p(\Omega_1)^{\frac{1}{n-p}},\quad t\geq 0,$$

is non-negative and concave. Now, using the variational formula for p-capacity established in (3.24) we have

$$\lim_{t\to 0^+}\frac{f(t)-f(0)}{t}=C_p(\Omega_0)^{\frac{1}{n-p}-1}C_p(\Omega_0,\Omega_1)-C_p(\Omega_1)^{\frac{1}{n-p}}\geq 0.$$

Hence, if equality holds, then f must be linear and Ω_0,Ω_1 must be homothetic. □

We are now ready to state and prove Theorem 1.2 in the context of bounded convex domains of class $C^{2,\alpha}_+$.

Theorem 3.7. *Suppose* $1 < p < n$. *Let* Ω_0, Ω_1 *be bounded convex domains in* $\mathbb{R}^n$ *of class* $C^{2,\alpha}_+$. *If* Ω_0, Ω_1 *have the same* p*-capacitary measure, then* Ω_0 *is a translate of* Ω_1 *when* $p \neq n-1$, *and* Ω_0, Ω_1 *are homothetic when* $p = n-1$.

Proof. By the assumption and using Theorem 3.6 we see that

$$C_p(\Omega_0) = C_p(\Omega_0, \Omega_0) = C_p(\Omega_1, \Omega_0) \geq C_p(\Omega_1)^{1-\frac{1}{n-p}} C_p(\Omega_0)^{\frac{1}{n-p}}. \tag{3.31}$$

Hence, $C_p(\Omega_0)^{1-\frac{1}{n-p}} \geq C_p(\Omega_1)^{1-\frac{1}{n-p}}$ and by reversing the roles of Ω_0 and Ω_1 we can conclude that $C_p(\Omega_0) = C_p(\Omega_1)$ when $p \neq n-1$. In particular, this implies that there is equality in the Minkowski inequality for p-capacity and hence Ω_0 must be a translation of Ω_1 when $p \neq n-1$, and Ω_0, Ω_1 are homothetic when $p = n-1$. $\square$

4. Weak convergence of p-capacitary measures

The purpose of the section is to prove the following important lemma.

Lemma 4.1. *Suppose* $1 < p < n$. *Let* Ω *be a bounded convex domain and let* $\{\Omega_i\}_{i=1}^\infty$ *be a sequence of bounded convex domains in* $\mathbb{R}^n$. *If* $\{\Omega_i\}$ *converges to* Ω *in the Hausdorff distance sense, then the sequence of measures* $\{\mu_p(\Omega_i, \cdot)\}$ *converges weakly to* $\mu_p(\Omega, \cdot)$.

4.1. A refined integral estimate for p-harmonic functions.

The following Theorem 4.2 is a refinement of Theorem 2.8 and it will be important in the proof of Lemma 4.1. The proof of Theorem 4.2, which we provide for completeness, is implicitly contained in Theorem 2 of [60], while for $p = 2$ it is Corollary 5.2 of [55].

Theorem 4.2. *Suppose* $1 < p < \infty$, *let* $1 < M < \infty$, $r_0 > 0$, *and consider* $0 < r < r_0$. *For every* $\varepsilon > 0$, $\varepsilon \ll 1$, *there exists* $\eta = \eta(\varepsilon)$ *such that the following is true. Assume that* $\Omega \subset \mathbb{R}^n$ *is a Lipschitz domain with constants* M, r_0, $0 \in \partial\Omega$, *and that*

$$\begin{aligned} &\partial\Omega \cap B(0, 4r) = \{(x', \phi(x')) : \ x' \in \mathbb{R}^{n-1}\} \cap B(0, 4r), \\ &\phi(0) = 0, \ \sup_{|x'|<r} |\nabla_{x'} \phi(x')| < \eta. \end{aligned}$$

Then, for every positive p*-harmonic function* u *in* $\Omega \cap B(0, 4r)$, *continuous on* $\bar{\Omega} \cap B(0, 4r)$ *with* $u = 0$ *on* $\Delta(0, 4r)$, *and for every* $y \in \Delta(0, \eta r)$ *and* $s < \eta r$, *there exists a constant* $c_{\Delta(y,s)} = c_{\Delta(y,s)}(u)$ *such that*

$$\frac{1}{\mathcal{H}^{n-1}(\Delta(y, s))} \int\limits_{\Delta(y,s)} \big| \log |\nabla u| - c_{\Delta(y,s)} \big| \, d\mathcal{H}^{n-1} < \varepsilon \, .$$

Proof. Let q be as in the statement of Theorem 2.8 and let $\tilde{q} = \min\{(q + p - 1)/2, p\}$. Note that we may, without loss of generality, assume that $r = 1$. To prove the theorem it suffices, by way of a lemma of Sarason (see [55]), to prove that there exists $\tilde{\epsilon}_0 > 0$ and $\tilde{\eta} = \tilde{\eta}(\tilde{\epsilon})$, defined for $\tilde{\epsilon} \in (0, \tilde{\epsilon}_0)$, such that if $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with constants M, r_0, $0 \in \partial\Omega$,

$$\begin{aligned} &\partial\Omega \cap B(0, 4) = \{(x', \phi(x')) : \ x' \in \mathbb{R}^{n-1}\} \cap B(0, 4), \\ &\phi(0) = 0, \ \sup_{|x'|<1} |\nabla_{x'} \phi(x')| < \tilde{\eta}, \end{aligned}$$

and $y \in \Delta(0,\tilde\eta)$ and $0<s<\tilde\eta$, then

$$\text{(4.1)}\quad \frac{1}{\mathcal{H}^{n-1}(\Delta(y,s))}\int\limits_{\Delta(y,s)} |\nabla u|^{\tilde q}\, d\mathcal{H}^{n-1} \le (1+\tilde\epsilon)\left(\frac{1}{\mathcal{H}^{n-1}(\Delta(y,s))}\int\limits_{\Delta(y,s)} |\nabla u|^{p-1}\, d\mathcal{H}^{n-1}\right)^{\frac{\tilde q}{p-1}},$$

whenever u is as in the statement of the present theorem. Indeed, assuming (4.1) for now, consider $y\in\Delta(0,\tilde\eta)$, $0<s<\tilde\eta$, let $\Delta=\Delta(y,s)$, $k=|\nabla u|^{p-1}$, and introduce the measure

$$d\lambda = \Big(\int_\Delta k d\mathcal{H}^{n-1}\Big)^{-1} k d\mathcal{H}^{n-1}.$$

Let β be defined through the relation $(p-1)(1+\beta)=\tilde q$. The, applying the Hölder inequality we see that

$$\begin{aligned}\int_\Delta k^\beta d\lambda \int_\Delta k^{-\beta} d\lambda &= \Big(\int_\Delta k d\mathcal{H}^{n-1}\Big)^{-2}\int_\Delta k^{1+\beta} d\mathcal{H}^{n-1}\int_\Delta k^{1-\beta} d\mathcal{H}^{n-1}\\ &\le \Big(\frac{1}{\mathcal{H}^{n-1}(\Delta)}\int_\Delta k d\mathcal{H}^{n-1}\Big)^{-2}\Big(\frac{1}{\mathcal{H}^{n-1}(\Delta)}\int_\Delta k^{1+\beta} d\mathcal{H}^{n-1}\Big)^{\frac{2}{1+\beta}}.\end{aligned}$$

In particular, using (4.1) we can conclude that

$$\int_\Delta k^\beta d\lambda \int_\Delta k^{-\beta} d\lambda \le (1+\tilde\varepsilon)^{\frac{2(p-1)}{\tilde q}} \le 1 + C\tilde\varepsilon.$$

Applying the lemma of Sarason, see (5.26) of [55], the inequality in above display implies that

$$\int_\Delta \left|\log k^\beta - \Big(\int_\Delta \log k^\beta d\lambda\Big)\right| d\lambda \le C\tilde\varepsilon,$$

which is the desired conclusion once we recall that $kd\mathcal{H}^{n-1}\in A^\infty(2\Delta, d\mathcal{H}^{n-1})$, see Theorem 2.8. Hence, to prove the theorem it suffices to prove (4.1). For this, we follow the proof of Theorem 2 of [60] and we argue by contradiction. Indeed, if (4.1) is not true, then there exist a sequence of Lipschitz domains $\{\Omega_m\}_{m=1}^\infty$, a sequence of constants $\{\tilde\eta_m\}_{m=1}^\infty$, a sequence of functions $\{u_m\}_{m=1}^\infty$ and sequences $\{s_m\}_{m=1}^\infty, \{y_m\}_{m=1}^\infty$, for which the following facts hold. For every m, Ω_m is a Lipschitz domain with constants M, r_0, such that $0\in\partial\Omega_m$,

$$\begin{aligned}&\partial\Omega_m\cap B(0,4) = \{(x',\phi_m(x')) :\ x'\in\mathbb{R}^{n-1}\}\cap B(0,4),\\ &\phi_m(0)=0,\ \sup_{|x'|<r}|\nabla_{x'}\phi_m(x')| < \tilde\eta_m,\end{aligned}$$

with $\tilde\eta_m\to 0$ as $m\to\infty$, u_m is p-harmonic in $\Omega_m\cap B(0,4)$, u_m is continuous in $\bar\Omega_m\cap B(0,4)$, and $u_m=0$ on $\partial\Omega_m\cap B(0,4)$. Furthermore, $y_m\in\partial\Omega_m\cap B(0,\tilde\eta_m)$, $s_m\le\tilde\eta_m$, and

$$\text{(4.2)}\quad \frac{1}{\mathcal{H}^{n-1}(\Delta_m)}\int\limits_{\Delta_m} |\nabla u_m|^{\tilde q}\, d\mathcal{H}^{n-1} > (1+\tilde\epsilon)\left(\frac{1}{\mathcal{H}^{n-1}(\Delta_m)}\int\limits_{\Delta_m} |\nabla u_m|^{p-1}\, d\mathcal{H}^{n-1}\right)^{\tilde q/(p-1)}.$$

Here and in the following, $\Delta_m=\Delta_m(y_m,s_m)=\partial\Omega_m\cap B(y_m,s_m)$. For fixed $0<\tilde\varepsilon\ll 1$, let

$$\text{(4.3)}\qquad A = e^{1/\tilde\varepsilon},\ \tilde\eta = \tilde\varepsilon^2.$$

Since $\tilde\eta_m\to 0$ as $m\to\infty$ we may assume that

$$0<\tilde\eta_m\le\tilde\eta,\quad \text{for every } m.$$

Let $n_m(y_m)$ be the inner unit normal to $\partial\Omega_m$ at $y_m \in \partial\Omega_m$ and $P_m(y_m)$ be the hyperplane which is orthogonal to n_m and which contains y_m. Using coordinates $x = (x', x_n)$, with $x' = x - x_n n_m(y_m) \in P_m(y_m)$, we introduce, for $s > 0$, the cylinders

$$C_m(y_m, s) = \{x = (x', x_n) = x' + x_n n_m(y_m) \text{ such that } x' \in P_m(y_m) \cap B(y_m, s),\ |x_n| \leq s\}.$$

For brevity we let

$$P_m = P(y_m),\ n_m = n(y_m, As_m) \text{ and } C_m = C(y_m, As_m)\,.$$

Extend u_m to $B(0,4)$ by letting $u_m \equiv 0$ in $B(0,4) \setminus \bar{\Omega}_m$ and let ν_m be the measure associated with u_m in the sense of Lemma 2.5. In view of (iv) of Theorem 2.8,

$$\nu_m(\Delta_m) = \int_{\Delta_m} |\nabla u_m|^{p-1}\, d\mathcal{H}^{n-1}.$$

Let

$$A_m = \frac{1}{\mathcal{H}^{n-1}(\Delta_m)} \int\limits_{\Delta_m} |\nabla u_m|^{\tilde{q}}\, d\mathcal{H}^{n-1}\,.$$

Below we prove that there exists $c \geq 1$, independent of m, such that

$$\limsup_{m\to\infty} A_m \left(\frac{\mathcal{H}^{n-1}(\Delta_m)}{\nu_m(\Delta_m)} \right)^{\tilde{q}/(p-1)} \leq (1 + e^{-1/(c\tilde{\varepsilon})}). \tag{4.4}$$

To see that this is sufficient to reach a contradiction, note that (4.2) and (4.4) give

$$1 + \tilde{\varepsilon} \leq \limsup_{m\to\infty} A_m \left(\frac{\mathcal{H}^{n-1}(\Delta_m)}{\nu(\Delta_m)} \right)^{\tilde{q}/(p-1)} \leq 1 + e^{-1/(c\tilde{\epsilon})} \tag{4.5}$$

for some $c \geq 1$ independent of m, provided $\tilde{\epsilon}_0$ is sufficiently small. Choosing $\tilde{\epsilon}_0$ still smaller, if necessary, we see that (4.5) can not hold if $0 < \tilde{\varepsilon} \leq \tilde{\epsilon}_0$. Hence the statement above and (4.1) must hold.

To start the proof of (4.4), we let $\hat{y}_m = y_m + \frac{1}{10} A s_m n_m$. Note that if $\tilde{\varepsilon}$ is sufficiently small, then the domain D_m, obtained by drawing all line segments from points in $B(\hat{y}_m, \frac{As_m}{100})$ to points in $\partial\Omega_m \cap B(y_m, \frac{As_m}{10})$, is starlike Lipschitz with respect to $\hat{y}_m$. Let $\hat{D}_m = D_m \setminus \bar{B}(\hat{y}_m, \frac{As_m}{1000})$ and note that the Lipschitz constant of $\hat{D}_m$ is smaller or equal to $c = c(n, M)$. Let $\hat{u}_m$ be the p-capacitary function for $\hat{D}_m$, i.e. $\hat{u}_m$ is non-negative, $\hat{u}_m = 0$, and $\hat{u}_m = 1$ continuously on ∂D_m and $\partial B(\hat{y}_m, \frac{As_m}{1000})$, respectively, and $\hat{u}_m$ is p-harmonic in $\hat{D}_m$. Extend $\hat{u}_m$ to $\mathbb{R}^n \setminus D_m$ by setting $\hat{u}_m \equiv 0$ on $\mathbb{R}^n \setminus D_m$ and let $\hat{\nu}_m$ be the measure, with support on ∂D_m, corresponding to $\hat{u}_m$ as in Lemma 2.5. Next suppose $\tilde{\epsilon}_0$ is so small that $A/100 \geq 2c_1$, where c_1 is as in Theorem 2.7. Then, using Theorem 2.7 with r, w, u_1, u_2 replaced by $As_m/100, y_m, u_m, \hat{u}_m$, we deduce, for $\tilde{\varepsilon}$ sufficiently small, that if $w_1, w_2 \in B(y_m, 2s_m) \cap \hat{D}_m$, then

$$\left| \log\left(\frac{\hat{u}_m(w_1)}{u_m(w_1)} \right) - \log\left(\frac{\hat{u}_m(w_2)}{u_m(w_2)} \right) \right| \leq cA^{-\alpha}, \tag{4.6}$$

where c, α are the constants in Theorem 2.7 and hence independent of m. Letting $w_1, w_2 \to z_1, z_2 \in \partial\hat{D}_m \cap B(y_m^-, 2s_m)$ in (4.6) and using Theorem 2.8 we see that

$$\left| \log\left(\frac{|\nabla \hat{u}_m(z_1)|}{|\nabla u_m(z_1)|} \right) - \log\left(\frac{|\nabla \hat{u}_m(z_2)|}{|\nabla u_m(z_2)|} \right) \right| \leq cA^{-\alpha} \tag{4.7}$$

for $\mathcal{H}^{n-1}$-almost all $z_1, z_2 \in \partial\hat{D}_m \cap B(y_m, 2s_m)$. From the inequality in (4.7) we deduce

$$(1 - c\,A^{-\alpha})\,\frac{|\nabla \hat{u}_m(z_1)|}{|\nabla \hat{u}_m(z_2)|} \le \frac{|\nabla u_m(z_1)|}{|\nabla u_m(z_2)|} \le (1 + c\,A^{-\alpha})\,\frac{|\nabla \hat{u}_m(z_1)|}{|\nabla \hat{u}_m(z_2)|}, \tag{4.8}$$

where $c = c(p, n, M)$. Let

$$\hat{A}_m := \frac{1}{\mathcal{H}^{n-1}(\Delta_m)} \int\limits_{\Delta_m} |\nabla \hat{u}_m|^{\tilde{q}}\, d\mathcal{H}^{n-1}. \tag{4.9}$$

From (4.8) and Theorem 2.8, we see that

$$\frac{\hat{A}_m}{(\hat{\nu}_m(\Delta_m))^{\tilde{q}/(p-1)}} \ge (1 - cA^{-\alpha})^{\tilde{q}} \frac{A_m}{(\nu_m(\Delta_m))^{\tilde{q}/(p-1)}}\,. \tag{4.10}$$

One concludes from (4.9)-(4.10) and simple estimates that it suffices to prove (4.4) with ν_m, u_m, replaced by $\hat{\nu}_m, \hat{u}_m$. Thus one desires to prove that, for $c = c(p, n, M)$ suitably large and $\tilde{\epsilon}_0$ sufficiently small,

$$\limsup_{m \to \infty} \hat{A}_m \left(\frac{\mathcal{H}^{n-1}(\Delta_m)}{\hat{\nu}_m(\Delta_m)} \right)^{\tilde{q}/(p-1)} \le 1 + e^{-1/(c\tilde{\varepsilon})}. \tag{4.11}$$

To prove (4.11), let T_m be a conformal affine mapping of $\mathbb{R}^n$ which maps the plane W containing the origin and with normal e_n onto $\hat{P}_m$ with $T_m(0) = y_m$ and $T_m(e_n) = \hat{y}_m$. Let $\tilde{D}_m, \tilde{u}_m$ be such that $T_m(\tilde{D}_m) = \hat{D}_m$ and $\hat{u}_m(T_m x) = \tilde{u}_m(x)$ whenever $x \in \tilde{D}_m$. Then, since the p-Laplace equation is invariant under translations, rotations, and dilations, we see that $\tilde{u}_m$ is the p-capacitary function for $\tilde{D}_m$. Moreover, if $\tilde{\nu}_m$ corresponds to $\tilde{u}_m$ as in Lemma 2.5, then

$$\hat{A}_m \left(\frac{\mathcal{H}^{n-1}(\Delta_m)}{\hat{\nu}_m(\Delta_m)} \right)^{\tilde{q}/(p-1)} = \tilde{A}_m \left(\frac{\mathcal{H}^{n-1}(\partial\tilde{D}_m \cap B(0, 10/A))}{\tilde{\nu}_m(\partial\tilde{D}_m \cap B(0, 10/A))} \right)^{\tilde{q}/(p-1)}, \tag{4.12}$$

where

$$\tilde{A}_m := \frac{1}{\mathcal{H}^{n-1}(\partial\tilde{D}_m \cap B(0, 10/A))} \int\limits_{\partial\tilde{D}_m \cap B(0,10/A)} |\nabla \tilde{u}_m|^{\tilde{q}}\, d\mathcal{H}^{n-1}.$$

Letting $m \to \infty$, one deduces from Lemma 2.1 and Lemma 2.2 that $\tilde{u}_m$ converges uniformly on $\mathbb{R}^n$ to $\tilde{u}$ where $\tilde{u}$ is the p-capacitary function for the starlike Lipschitz ring domain, $\tilde{D} = \hat{D} \setminus B(e_n, 1/100)$. Now, $\hat{D}$ is obtained by drawing all line segments connecting points in $B(0,1) \cap W$ to points in $B(e_n, 1/10)$. Using a Rellich type inequality and arguing as in (5.27)-(5.41) in [59], it follows that

$$\left\{ \begin{array}{l} \limsup_{m \to \infty} (\tilde{A}_m)^{1/\tilde{q}} \le \left(\frac{1}{\Gamma_A} \int\limits_{W \cap B(0,10/A)} |\nabla \tilde{u}|^p \, d\mathcal{H}^{n-1} \right)^{1/p} \\ \liminf_{m \to \infty} \frac{\tilde{\nu}_m(\partial\tilde{D}_m \cap B(0,10/A))}{\mathcal{H}^{n-1}(\partial\tilde{D}_m \cap B(0,10/A))} \ge \frac{1}{\Gamma_A} \int\limits_{W \cap B(0,10/A)} |\nabla \tilde{u}|^{p-1} \, d\mathcal{H}^{n-1}, \end{array} \right. \tag{4.13}$$

where $\Gamma_A = \mathcal{H}^{n-1}(W \cap B(0, 10/A))$ and $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Lebesgue or Hausdorff measure on W. Assuming (4.13), by using Schwarz reflection, we note that $\tilde{u}$ has a p-harmonic extension to $B(0, 1/2)$ with $\tilde{u} \equiv 0$ on $W \cap B(0, 1/2)$. From barrier estimates, we have $c^{-1} \leq |\nabla \tilde{u}| \leq c$ on $B(0,1/4)$, where c depends only on $p, n,$. From Lemma 2.4 we find that $|\nabla \tilde{u}|$ is Hölder continuous with exponent σ on $\bar{B}(0,1/4) \cap W$. Using these facts, we conclude first that there exist $\hat{z} \in \bar{B}(0,10/A) \cap W$ and a constant c such that

$$(1 - cA^{-\sigma})\, |\nabla \tilde{u}(\hat{z})| \;\leq\; |\nabla \tilde{u}(z)| \;\leq\; (1 + cA^{-\sigma})\, |\nabla \tilde{u}(\hat{z})|, \tag{4.14}$$

whenever $z \in B(0,10/A) \cap W$. Combining (4.12), (4.13), and (4.14), in light of (4.3), we deduce that, for c and $\tilde{\epsilon}_0^{-1}$ sufficiently large,

$$\limsup_{m \to \infty} \hat{A}_m \left(\frac{\mathcal{H}^{n-1}(\Delta_m)}{\hat{\nu}_m(\Delta_m)} \right)^{\tilde{q}/(p-1)} \leq (1 + cA^{-\sigma})^{\tilde{q}} \leq 1 + e^{-1/(c\tilde{\varepsilon})},$$

which is (4.11). This completes the proof of Theorem 4.2. □

4.2. L^1 **convergence of the p-capacitary density.** To prove Lemma 4.1, we first prove the L^1 convergence of p-capacitary densities. To begin the argument, we recall the following purely geometric lemma due to Jerison, see [52, Lemma 3.3]. The lemma essentially says that $\partial\Omega$ and $\partial\Omega'$ are flat on a set of large measure.

Lemma 4.3. *Let Ω and Ω' be bounded convex domains in $\mathbb{R}^n$, and let $\varepsilon_1 > 0$ be given. Then there exist $\delta_1 > 0$ and a finite collection of balls $B(x_j, r_j)$, $j = 1, \ldots, N$, such that $x_j \in \partial\Omega$ for every j,*

$$(a) \qquad \mathcal{H}^{n-1}(\partial\Omega \setminus \cup_{j=1}^{N} B(x_j, r_j)) \leq \varepsilon_1,$$

and such that, if the Hausdorff distance between Ω and Ω' is less than δ_1, then

(b) *for every $j \in \{1, \ldots, N\}$, both $\partial\Omega \cap B(x_j, r_j/\varepsilon_1)$ and $\partial\Omega' \cap B(x_j, r_j/\varepsilon_1)$ can, after a suitable translation and rotation of coordinates depending on j, be expressed as the graphs of two functions ϕ and ϕ', respectively, such that $|\nabla\phi|, |\nabla\phi'| \leq \varepsilon_1$, for all $x \in \mathbb{R}^n$ such that $|x| < r_j/\varepsilon_1$.*

Given Ω we in the following let r_{int} and r_{ext} be defined with respect to Ω. Let now $\{\Omega_i\}_{i=1}^{\infty}$ be a sequence of bounded convex domains in $\mathbb{R}^n$ which converges to Ω in the Hausdorff distance sense. We let, for $i \in \mathbb{N}$, $U_i = U_{\Omega_i}$ denote the p-equilibrium potential of Ω_i and we let g_i denote the Gauss map of $K_i = \overline{\Omega_i}$. Let $U = U_\Omega$ and g be the corresponding objects defined with respect to Ω. In the following we can assume, without loss of generality, that $0 \in \Omega$ and that $0 \in \Omega_i$ for every $i \in \mathbb{N}$. This implies that the radial maps, r_K and r_{K_i}, of K and K_i, respectively, are well defined, see Section 2. Let J and J_i, $i \in \mathbb{N}$, be the Jacobian functions introduced in Lemma 2.10 associated with K and K_i respectively, and let

$$q_i(\xi) = \frac{J_i(\xi)}{J(\xi)} \text{ whenever } \xi \in \mathbb{S}^{n-1}. \tag{4.15}$$

Using Lemma 2.10 we see that this quotient is bounded above and below, for i large enough, by positive constants, uniformly with respect to i, and

$$q_i(\xi) \to 1,\ \mathcal{H}^{n-1}\text{-a.e. on } \mathbb{S}^{n-1}, \text{ as } i \to \infty. \tag{4.16}$$

Finally, we let, for $\mathcal{H}^{n-1}$-a.a $\xi \in \mathbb{S}^{n-1}$

$$H_i(\xi) = |\nabla U_i(r_{K_i}(\xi))|\, (J_i(\xi))^{\frac{1}{p}}, \tag{4.17}$$

and

$$H(\xi) = |\nabla U(r_K(\xi))| \, (J(\xi))^{\frac{1}{p}} \,. \tag{4.18}$$

These are the p-th roots of the densities of the pullback measures, with respect to the corresponding radial maps, of $\mu_p(\Omega_i,\cdot)$ and $\mu_p(\Omega,\cdot)$, respectively. Recall that we denote the Hausdorff distance between Ω_i and Ω by $h(\Omega_i,\Omega)$. Using that $\{\Omega_i\}_{i=1}^{\infty}$ is a sequence of bounded convex domains in $\mathbb{R}^n$ which converges to Ω in the Hausdorff distance sense we can in the following assume, with out loss of generality, by using Lemma 2.10 and Lemma 2.18, that there exists a constant c, $1 \le c < \infty$ such that

$$\min\{H_i(\xi), H(\xi)\} \ge c^{-1}, \text{ for } \mathcal{H}^{n-1}\text{-a.a } \xi \in \mathbb{S}^{n-1} \text{ and for all } i. \tag{4.19}$$

In fact, c can be chosen to depend only on p, n and the Euclidian diameter of Ω. The following lemma is a non-linear version of Lemma 3.7 in [52].

Lemma 4.4. *Suppose $1 < p < n$. Then, for every $\varepsilon_2 > 0$ and $\gamma > 0$, there exist $s_0 > 0$, $\delta_2 > 0$, and a family of balls $\mathcal{B}$ on $\mathbb{S}^{n-1}$, such that the following holds.*

- (a) *Every member in $\mathcal{B}$ has radius s_0.*
- (b) *There is a constant $c > 0$, depending only on r_{int} and r_{ext}, such that any point of $\mathbb{S}^{n-1}$ lies in at most c balls of $\mathcal{B}$.*
- (c) *$\mathcal{H}^{n-1}(\mathbb{S}^{n-1} \setminus F) < \varepsilon_2$, where $F = \cup_{B\in\mathcal{B}} B$.*
- (d) *If $h(\Omega_i,\Omega) < \delta_2$, then, for any $B \in \mathcal{B}$,*
$$s_0^{1-n}\left(\int_B \left|\left(\frac{H_i}{H}\right)^{p-1} - 1\right|^{\gamma} d\xi + \int_B \left|\left(\frac{H}{H_i}\right)^{p-1} - 1\right|^{\gamma} d\xi\right) < \varepsilon_2 \,.$$

Proof. Our proof of Lemma 4.4 proceeds along the lines of Lemma 3.7 in [52], with Theorem 4.2 replacing Lemma 3.6 of [52]. Let $\varepsilon_2 > 0$ and $\gamma > 0$ be given and let in the following $\varepsilon > 0$ be a degree of freedom to be determined based on $\varepsilon_2 > 0$ and $\gamma > 0$. Given $\varepsilon > 0$ we let $\eta = \min\{\eta(\varepsilon), \varepsilon/10\}$ where $\eta(\varepsilon)$ is as stated in Theorem 4.2. Using this η we let $\varepsilon_1 = \eta/10$ and apply Lemma 4.3. Doing this we get $\delta_1 = \delta_1(\varepsilon_1) = \delta_1(\varepsilon)$, $\{B(x_j, r_j)\}$, as in Lemma 4.3 and we in the following only consider i large enough to ensure $h(\Omega_i,\Omega) < \delta_1$. Next, following the proof of Lemma 3.7 in [52] we let $s_0 < \min\{r_j : j = 1,\dots,N\}$ be small enough to ensure that the oscillation of J and J_i is less than ε_1 when the oscillation of ξ is bounded by s_0 and $r_K(\xi) \in B(x_j, r_j/\varepsilon_1)$ for some index j. One can then choose a family $\mathcal{B}$ of balls for which properties (a), (b), hold and for which

$$\mathcal{H}^{n-1}(\mathbb{S}^{n-1} \setminus F) < \varepsilon_1 \le \varepsilon, \text{ where } F = \cup_{B\in\mathcal{B}} B. \tag{4.20}$$

Now, using Theorem 4.2, and the choice for η made above, we see that if $\tilde{B} \in \mathcal{B}$, and if B is a ball of radius $s \le s_0$, contained in the concentric copy of $\tilde{B}$ rescaled by $1/\varepsilon_1$, then

$$\frac{1}{\mathcal{H}^{n-1}(B)} \int_B |\log H - c_B| d\xi < \varepsilon, \; c_B = \frac{1}{\mathcal{H}^{n-1}(B)} \int_B \log H \, d\xi. \tag{4.21}$$

That c_B can be chosen as stated follows from the proof of Theorem 4.2. Furthermore, using Lemma 4.3 and the choice for δ_1 made above, and arguing similarly, we see that if $h(\Omega_i,\Omega) < \delta_1$, then also

$$\frac{1}{\mathcal{H}^{n-1}(B)} \int_B |\log H_i - c_{B,i}| d\xi < \varepsilon, \; c_{B,i} = \frac{1}{\mathcal{H}^{n-1}(B)} \int_B \log H_i \, d\xi. \tag{4.22}$$

Let

$$I_i(B) := \frac{\int_B |H_i|^{p-1} d\xi}{\int_B |H|^{p-1} d\xi} \tag{4.23}$$

Then, using the construction above, using Theorem 4.2, Theorem 2.8, (4.21), (4.22), (4.19) and the John-Nirenberg inequality, see [54], we can conclude, given $\gamma > 0$, that if $h(\Omega_i, \Omega) < \delta_1$ then

$$s^{1-n} \int_B \left| I_i(B) \left| \frac{H}{H_i} \right|^{p-1} - 1 \right|^{\gamma} d\xi \leq \hat{c}\varepsilon \tag{4.24}$$

for a harmless constant $\hat{c}$ which may depend on p, n, γ but which is independent of ε. Furthermore, using Lemma 4.5, stated and proved below, we see that given ε there exists $\delta_3 = \delta_3(\varepsilon)$ such that if $h(\Omega_i, \Omega) < \delta_3$, then

$$|I_i(B) - 1| = \left| \frac{\int_B |H_i|^{p-1} d\xi}{\int_B |H|^{p-1} d\xi} - 1 \right| \leq 2\hat{c}\varepsilon \tag{4.25}$$

Combining (4.24) and (4.25), and using (4.19), we deduce that

$$\begin{aligned} s^{1-n} \int_B \left| \left(\frac{H}{H_i} \right)^{p-1} - 1 \right|^{\gamma} d\xi &\leq c\varepsilon \left(1 + s^{1-n} \int_B \left| \frac{H}{H_i} \right|^{(p-1)\gamma} d\xi \right) \\ &\leq \hat{c}\varepsilon \left(1 + c s^{1-n} \int_B H^{(p-1)\gamma} d\xi \right), \end{aligned} \tag{4.26}$$

where all constants c may depend on p, n, γ but are independent of ε. Finally, using Theorem 4.2, Lemma 2.5, choosing $\varepsilon = \varepsilon(p, n, \gamma)$ sufficiently small, and elementary estimates we can conclude that

$$s^{1-n} \int_B \left| \left(\frac{H}{H_i} \right)^{p-1} - 1 \right|^{\gamma} d\xi \leq \bar{c}\varepsilon \tag{4.27}$$

for yet an other constant $\bar{c}$ may depend on p, n, γ but which is independent of ε. Given $\varepsilon_2 > 0$ we now choose ε so small that $\bar{c}\varepsilon \leq \varepsilon_2$, $\delta_2 := \min\{\delta_1, \delta_3\} = \min\{\delta_1(\varepsilon), \delta_3(\varepsilon)\}$. Then (c) through (4.20) holds and the proof of one half of (d) is complete. The other part of (d) is proved similarly. □

Lemma 4.5. *Suppose $1 < p < n$. Given ε, B, as in the proof of Lemma 4.4 there exists $\delta_3 = \delta_3(\varepsilon)$ such that if $h(\Omega_i, \Omega) < \delta_3$, then*

$$|I_i(B) - 1| = \left| \frac{\int_B |H_i|^{p-1} d\xi}{\int_B |H|^{p-1} d\xi} - 1 \right| \leq 2\hat{c}\varepsilon \tag{4.28}$$

Proof. Using (4.15) we see that that is suffice to prove that there exists $\delta_3 = \delta_3(\varepsilon)$ such that if $h(\Omega_i, \Omega) < \delta_3$, then

$$\left| \frac{\int_{\mathsf{g}_i^{-1}(B)} |\nabla U_i|^{p-1} d\sigma_j}{\int_{\mathsf{g}^{-1}(B)} |\nabla U|^{p-1} d\sigma} - 1 \right| \leq 2\hat{c}\varepsilon \tag{4.29}$$

where $d\sigma_i$ and $d\sigma$ here denote the surface measure on $\partial\Omega_i$ and $\partial\Omega$, respectively. Let $\Delta_i = \mathsf{g}_i^{-1}(B)$, $\Delta = \mathsf{g}^{-1}(B)$ and note, using Theorem 2.8, that (4.29) can be rewritten as

$$\left| \frac{\nu_i(\Delta_i)}{\nu(\Delta)} - 1 \right| \leq 2\hat{c}\varepsilon \tag{4.30}$$

where ν_i and ν are, respectively, the measures associated to U_i and U in the sense of Theorem 2.8. (4.30) can now be proved by essentially arguing by contradiction as in the proof of Theorem 4.2, see the proof of Lemma 4.4 in [66] for example. We here omit further details. $\square$

Lemma 4.6. *Suppose* $1 < p < n$. *Then,*

$$\lim_{i\to\infty}\int_{\mathbb{S}^{n-1}} |H_i^p - H^p|\, d\xi = 0\,.$$

Proof. To prove Lemma 4.6 we use an argument similar to [52, Proposition 3.14]. In the following the notation $A \lesssim B$ simply means that A/B is bounded from above by a positive constant. Let $q > p$ be as in Theorem 2.8 (i), (ii). Using this q, let $\gamma := pq/((p-1)(q-p))$. Given $\varepsilon_2 > 0$ and γ, we can use Lemma 4.4 to conclude that there exist $s_0 > 0$ and $\delta_2 > 0$, and a family of balls $\mathcal{B}$ on $\mathbb{S}^{n-1}$ such that statements $(a) - (d)$ of Lemma 4.4 hold. In the following let F be as in the statement of Lemma 4.4. Now, using the elementary fact that

$$|a^p - b^p| \le p(p-1)^{-1}(a+b)|a^{p-1} - b^{p-1}| \text{ for all } a, b \ge 0,$$

we see that

$$\int_F |H_i^p - H^p|\, d\xi \lesssim \int_F |H_i^{p-1} - H^{p-1}|(H_i + H)\, d\xi. \tag{4.31}$$

Next, using the Hölder inequality we see that

$$\begin{aligned} \int_F |H_i^p - H^p|\, d\xi &\lesssim \Big(\int_F |H_i^{p-1} - H^{p-1}|^{\frac{p}{p-1}}\, d\xi\Big)^{\frac{p-1}{p}} \Big(\int_F (H_i + H)^p\, d\xi\Big)^{\frac{1}{p}} \\ &= \Big(\int_F \Big|\Big(\frac{H_i}{H}\Big)^{p-1} - 1\Big|^{\frac{p}{p-1}} H^p\, d\xi\Big)^{\frac{p-1}{p}} \Big(\int_F (H_i + H)^p\, d\xi\Big)^{\frac{1}{p}}. \end{aligned} \tag{4.32}$$

Using the Hölder inequality once again we get

$$\begin{aligned} &\Big(\int_F \Big|\Big(\frac{H_i}{H}\Big)^{p-1} - 1\Big|^{\frac{p}{p-1}} H^p\, d\xi\Big)^{\frac{p-1}{p}} \\ &\lesssim \Big(\int_F \Big|\Big(\frac{H_i}{H}\Big)^{p-1} - 1\Big|^{\gamma} d\xi\Big)^{1/\gamma} \Big(\int_F H^q\, d\xi\Big)^{\frac{p-1}{q}}. \end{aligned} \tag{4.33}$$

Furthermore, using Theorem 2.8, Lemma 2.5 and elementary estimate we see that there exists a constant c, independent of ε and well as i, so that

$$\Big(\int_F (H_i + H)^p\, d\xi\Big)^{\frac{1}{p}} + \Big(\int_F H^q\, d\xi\Big)^{\frac{p-1}{q}} \le c. \tag{4.34}$$

In particular, using this and the above displays, we can conclude that

$$\int_F |H_i^p - H^p|\, d\xi \lesssim c\Big(\int_F \Big|\Big(\frac{H_i}{H}\Big)^{p-1} - 1\Big|^{\gamma} d\xi\Big)^{1/\gamma}. \tag{4.35}$$

Hence, using statement (d) of Lemma 4.4, and the last display, we get

$$\int_F |H_i^p - H^p|\, d\xi \lesssim \varepsilon_2 \quad \text{as} \quad i \to \infty\,. \tag{4.36}$$

Furthermore, again using the Hölder inequality, we obtain

$$\begin{aligned}\int_{\mathbb{S}^{n-1}\setminus F} |H_i^p - H^p|\,d\xi &\le \int_{\mathbb{S}^{n-1}\setminus F} (H_i^p + H^p)\,d\xi \\ &\le \left(\mathcal{H}^{n-1}(\mathbb{S}^{n-1}\setminus F)\right)^{\frac{q}{q-p}} \left(\int_{\mathbb{S}^{n-1}\setminus F} (H_i^q + H^q)\,d\xi\right)^{\frac{p}{q}}.\end{aligned}$$

Now, this, (c) of Lemma 4.4 and Theorem 2.8 (ii) give us the desired result. □

4.3. **The final proof of Lemma 4.3: weak convergence of p-capacitary measures.** Let ρ_i, ρ, and g_i, g be the radial functions and Gauss maps of Ω_i, Ω, respectively. Define

$$\alpha_i(\xi) = \mathsf{g}_i(\rho_i(\xi)\xi), \quad \alpha(\xi) = \mathsf{g}(\rho(\xi)\xi), \text{ whenever } \xi \in \mathbb{S}^{n-1}.$$

Using that g_i converges to g almost everywhere on $\mathbb{S}^{n-1}$, and that ρ_i converges to ρ uniformly, we see that α_i converges to α almost everywhere on $\mathbb{S}^{n-1}$ as $i \to \infty$. To prove that $\mu_p(\Omega_i, \cdot)$ converges to $\mu_p(\Omega, \cdot)$ weakly, we need to prove that

$$\int_{\mathbb{S}^{n-1}} f(\xi)\,d\mu_p(\Omega_i, \xi) - \int_{\mathbb{S}^{n-1}} f(\xi)\,d\mu_p(\Omega, \xi) \to 0,$$

or equivalently, that

$$\int_{\mathbb{S}^{n-1}} f(\alpha_i) H_i^p\,d\xi - \int_{\mathbb{S}^{n-1}} f(\alpha) H^p\,d\xi \to 0, \tag{4.37}$$

for each continuous function f on $\mathbb{S}^{n-1}$. However, (4.37) follows immediately from Lemma 4.6, Theorem 2.8, the a.e. convergence of α_i, and the inequality,

$$\begin{aligned}&\left|\int_{\mathbb{S}^{n-1}} f(\alpha_i) H_i^p\,d\xi - \int_{\mathbb{S}^{n-1}} f(\alpha) H^p\,d\xi\right| \\ &\le \left|\int_{\mathbb{S}^{n-1}} f(\alpha_i)(H_i^p - H^p)\,d\xi\right| + \left|\int_{\mathbb{S}^{n-1}} (f(\alpha_i) - f(\alpha)) H^p\,d\xi\right|.\end{aligned}$$

This completes the proof of Lemma 4.3. □

5. Variational formula for p-capacity of general convex domains

In this section we present the proof of Theorem 1.1 and Theorem 1.2. Given a bounded convex domain, $\Omega \subset \mathbb{R}^n$, let h_Ω denote the support function of Ω, let S_Ω denote the surface area measure on $\partial\Omega$, and let g be the Gauss map of $\partial\Omega$. Let $\mathcal{C}_o^n$ denote the class of all bounded convex domains in $\mathbb{R}^n$ that contain the origin, and $C_+(\mathbb{S}^{n-1})$ the class of positive continuous function on $\mathbb{S}^{n-1}$.

5.1. **Aleksandrov domains.** Given a function $h \in C_+(\mathbb{S}^{n-1})$, let $\Omega \subset \mathbb{R}^n$ be such that

$$\bar{\Omega} := \bigcap_{\xi \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \ x \cdot \xi \le h(\xi)\}. \tag{5.1}$$

Note that since h is both positive and continuous, Ω must be an element of $\mathcal{C}_o^n$. The convex domain $\Omega \subset \mathbb{R}^n$ is often called the *Aleksandrov domain associated with h*. For the Aleksandrov domain Ω associated with h, we see that

$$h_\Omega \le h.$$

Let

$$\omega_h = \{\xi \in \mathbb{S}^{n-1} : \ h_\Omega(\xi) < h(\xi)\}.$$

A basic fact established by Aleksandrov is that

$$S_\Omega(\omega_h) = 0.$$

Consequently,

$$h_\Omega = h, \text{ a.e. with respect to } S_\Omega. \tag{5.2}$$

By (2.34) and Lemma 4.1, we have for the p-capacity of any bounded convex domain $\Omega \subset \mathbb{R}^n$,

$$C_p(\Omega) = \frac{p-1}{n-p} \int_{\mathbb{S}^{n-1}} h_\Omega(\xi)\, d\mu_p(\Omega, \xi). \tag{5.3}$$

For $\omega \subset \mathbb{S}^{n-1}$, if $S_\Omega(\omega) = 0$, then $\mu_p(\Omega, \omega) = 0$. This follows from the definition of $\mu_p(\Omega, \cdot)$ since $|\nabla U|^p$ is integrable on $\partial\Omega$. Thus, $\mu_p(\Omega, \cdot)$ is absolutely continuous with respect to S_Ω. Using Theorem 3.6 and Lemma 4.1, we obtain

$$C_p(\Omega, L)^{n-p} \geq C_p(\Omega)^{n-p-1} C_p(L), \tag{5.4}$$

for all convex domains Ω and L in $\mathbb{R}^n$ whenever $1 < p < n$. Obviously, if h is the support function of a convex domain $\Omega \in \mathcal{C}_0^n$, then Ω itself is the Aleksandrov domain associated with h. We need *Aleksandrov's Convergence Lemma*: if the functions $h_i \in C_+(\mathbb{S}^{n-1})$ have associated Aleksandrov domains $\Omega_i \in \mathcal{C}_0^n$, then

$$h_i \to h \in C_+(\mathbb{S}^{n-1}) \text{ uniformly} \quad \Longrightarrow \quad \Omega_i \to \Omega \text{ in the Hausdorff metric,} \tag{5.5}$$

where Ω is the Aleksandrov domain associated with h. For $h \in C_+(\mathbb{S}^{n-1})$, denote by $C_p(h)$ the p-capacity of the Aleksandrov domain associated with h. Since the Aleksandrov domain associated with the support function h_Ω of a convex domain $\Omega \in \mathcal{C}_0^n$ is the domain Ω itself, we have

$$C_p(h_\Omega) = C_p(\Omega). \tag{5.6}$$

From Aleksandrov's convergence lemma and the continuity of p-capacity on $\mathcal{C}_o^n$ we see that

$$C_p : C_+(\mathbb{S}^{n-1}) \to \mathbb{R} \quad \text{is continuous.}$$

Let $I \subset \mathbb{R}$ be an interval containing 0 and suppose that

$$h_t(\xi) = h(t, \xi) : I \times \mathbb{S}^{n-1} \to (0, \infty)$$

is continuous. For fixed $t \in I$, let $\Omega_t \subset \mathbb{R}^n$ be such that

$$\overline{\Omega_t} = \bigcap_{\xi \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \ x \cdot \xi \leq h(t, \xi)\}.$$

This is the Aleksandrov domain associated with h_t. The family of convex domains $\{\Omega_t\}_{t \in I}$ will be called *the family of Aleksandrov domains associated with* h_t. Obviously, from (5.2) we have, for each $t \in I$,

$$h_{\Omega_t} \leq h_t \text{ and } h_{\Omega_t} = h_t, \text{ a.e. with respect to } S_{\Omega_t}. \tag{5.7}$$

5.2. **Variation of p-capacity for Alexandrov domains.** The proof of the following lemma regarding the variation of p-capacity is similar to that of its analogue for volume (see [93, Lemma 6.5.3]).

Lemma 5.1. *Let $I \subset \mathbb{R}$ be an interval containing both 0 and some positive number and let*

$$h(t,\xi) : I \times \mathbb{S}^{n-1} \to (0,\infty)$$

be continuous and such that the convergence in

$$h'_+(0,\xi) = \lim_{t\to 0^+} \frac{h(t,\xi) - h(0,\xi)}{t} \tag{5.8}$$

is uniform on $\mathbb{S}^{n-1}$. If $\{\Omega_t\}_{t\in I}$ is the family of Aleksandrov domains associated with h_t, then

$$\lim_{t\to 0^+} \frac{C_p(\Omega_t) - C_p(\Omega_0)}{t} = (p-1)\int_{\mathbb{S}^{n-1}} h'_+(0,\xi)\, d\mu_p(\Omega_0,\xi).$$

Proof. The uniform convergence of (5.8) implies that $h_t \to h_0$, uniformly on $\mathbb{S}^{n-1}$. Therefore, the Aleksandrov convergence lemma, see (5.5), yields

$$\lim_{t\to 0^+} \Omega_t = \Omega_0. \tag{5.9}$$

Thus we conclude that $\mu_p(\Omega_t,\cdot)$ converges weakly to $\mu_p(\Omega_0,\cdot)$ as $t\to 0$. Since the measures $\mu_p(\Omega_t,\cdot)$ are finite, converge weakly to $\mu_p(\Omega_0,\cdot)$ and since the convergence in

$$\lim_{t\to 0^+} \frac{h(t,\xi) - h(0,\xi)}{t}$$

is uniform on $\mathbb{S}^{n-1}$, we obtain

$$\lim_{t\to 0^+} \int_{\mathbb{S}^{n-1}} \frac{h_t(\xi) - h_0(\xi)}{t}\, d\mu_p(\Omega_t,\xi) = \int_{\mathbb{S}^{n-1}} h'_+(\xi,0)\, d\mu(\Omega_0,\xi). \tag{5.10}$$

Now, (5.3), (5.6), (5.7) and the fact that $\mu_p(\Omega,\cdot)$ is absolutely continuous with respect to S_Ω, imply that

$$C_p(\Omega_t) = \frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}} h_{\Omega_t}(\xi)\, d\mu_p(\Omega_t,\xi) = \frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}} h_t(\xi)\, d\mu_p(\Omega_t,\xi). \tag{5.11}$$

From (5.11), the definition of mixed p-capacity, and the inequality in (5.7) at $t=0$, we have

$$\begin{aligned}\liminf_{t\to 0^+} \frac{C_p(\Omega_t) - C_p(\Omega_t,\Omega_0)}{t} &= \frac{p-1}{n-p}\liminf_{t\to 0^+}\int_{\mathbb{S}^{n-1}} \frac{h_t(\xi) - h_{\Omega_0}(\xi)}{t}\, d\mu_p(\Omega_t,\xi)\\ &\geq \frac{p-1}{n-p}\liminf_{t\to 0^+}\int_{\mathbb{S}^{n-1}} \frac{h_t(\xi) - h_0(\xi)}{t}\, d\mu_p(\Omega_t,\xi),\end{aligned}$$

which, when combined with (5.10), gives

$$\liminf_{t\to 0^+} \frac{C_p(\Omega_t) - C_p(\Omega_t,\Omega_0)}{t} \geq \frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}} h'_+(\xi,0)\, d\mu_p(\Omega_0,\xi). \tag{5.12}$$

For the sake of brevity, set

$$l = \frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}} h'_+(\xi,0)\, d\mu_p(\Omega_0,\xi).$$

Inequality (5.12) and the mixed capacity inequality (5.4) show that

$$l \le \liminf_{t\to 0^+} \frac{C_p(\Omega_t) - C_p(\Omega_t,\Omega_0)}{t} \le \liminf_{t\to 0^+} \frac{C_p(\Omega_t) - C_p(\Omega_t)^{1-\frac{1}{n-p}} C_p(\Omega_0)^{\frac{1}{n-p}}}{t}.$$

However, (5.9) gives $\lim_{t\to 0^+} C_p(\Omega_t) = C_p(\Omega_0)$ and hence,

$$l \le C_p(\Omega_0)^{1-\frac{1}{n-p}} \liminf_{t\to 0^+} \frac{C_p(\Omega_t)^{\frac{1}{n-p}} - C_p(\Omega_0)^{\frac{1}{n-p}}}{t}. \tag{5.13}$$

Now the definition of mixed p-capacity, the inequality in (5.7) and the uniform convergence in (5.8) give

$$\begin{aligned}
\limsup_{t\to 0^+} \frac{C_p(\Omega_0,\Omega_t) - C_p(\Omega_0)}{t} &= \frac{p-1}{n-p} \limsup_{t\to 0^+} \int_{\mathbb{S}^{n-1}} \frac{h_{\Omega_t}(\xi) - h_0(\xi)}{t}\, d\mu_p(\Omega_0,\xi) \\
&\le \frac{p-1}{n-p} \limsup_{t\to 0^+} \int_{\mathbb{S}^{n-1}} \frac{h_t(\xi) - h_0(\xi)}{t}\, d\mu_p(\Omega_0,\xi) \\
&= \frac{p-1}{n-p} \int_{\mathbb{S}^{n-1}} h'_+(\xi,0)\, d\mu_p(\Omega_0,\xi) \\
&= l.
\end{aligned}$$

This, together with the mixed capacity inequality (5.4), yields

$$l \ge \limsup_{t\to 0^+} \frac{C_p(\Omega_0,\Omega_t) - C_p(\Omega_0)}{t} \ge \limsup_{t\to 0^+} \frac{C_p(\Omega_0)^{1-\frac{1}{n-p}} C_p(\Omega_t)^{\frac{1}{n-p}} - C_p(\Omega_0)}{t},$$

and hence,

$$l \ge C_p(\Omega_0)^{1-\frac{1}{n-p}} \limsup_{t\to 0^+} \frac{C_p(\Omega_t)^{\frac{1}{n-p}} - C_p(\Omega_0)^{\frac{1}{n-p}}}{t}. \tag{5.14}$$

Combining (5.13) and (5.14) we see that

$$l = C_p(\Omega_0)^{1-\frac{1}{n-p}} \lim_{t\to 0^+} \frac{C_p(\Omega_t)^{\frac{1}{n-p}} - C_p(\Omega_0)^{\frac{1}{n-p}}}{t}. \tag{5.15}$$

Define a function $f : I \to \mathbb{R}$ by $f(t) = C_p(\Omega_t)^{\frac{1}{n-p}}$. Identity (5.15) shows that the right derivative of f exists at 0. But this implies that the right derivative of f^n exists at 0 and that

$$\lim_{t\to 0^+} \frac{f(t)^{n-p} - f(0)^{n-p}}{t} = (n-p) f(0)^{n-p-1} \lim_{t\to 0^+} \frac{f(t) - f(0)}{t}.$$

Thus the definition of f and (5.15) prove that

$$\lim_{t\to 0^+} \frac{C_p(\Omega_t) - C_p(\Omega_0)}{t} = (n-p) l.$$

□

Theorem 5.2. *Let $I \subset \mathbb{R}$ be an interval containing 0 in its interior, and let*

$$h(t,\xi) : I \times \mathbb{S}^{n-1} \to (0,\infty)$$

be continuous, such that the convergence in

$$h'(0,\xi) = \lim_{t\to 0} \frac{h(t,\xi) - h(0,\xi)}{t}$$

is uniform on $\mathbb{S}^{n-1}$. *If* $\{\Omega_t\}_{t\in I}$ *is the family of Aleksandrov domains associated with* h, *then*

$$\left.\frac{dC_p(\Omega_t)}{dt}\right|_{t=0} = (p-1)\int_{\mathbb{S}^{n-1}} h'(0,\xi)\, d\mu_p(\Omega_0,\xi). \tag{5.16}$$

Proof. From Lemma 5.1 we see that we only need to show that

$$\lim_{t\to 0^-} \frac{C_p(\Omega_t) - C_p(\Omega_0)}{t} = (p-1)\int_{\mathbb{S}^{n-1}} h'(0,\xi)\, d\mu_p(\Omega_0,\xi). \tag{5.17}$$

To that end, define $\tilde{h}(t,\xi) : -I \times \mathbb{S}^{n-1} \to (0,\infty)$ by $\tilde{h}(t,\xi) = h(-t,\xi)$. For the corresponding family $\{\tilde{\Omega}_{-t}\}_{t\in I}$ of Aleksandrov domains associated with $\tilde{h}$ we have $\tilde{\Omega}_{-t} = \Omega_t$ and $\tilde{\Omega}_0 = \Omega_0$. Thus, by Lemma 5.1,

$$\lim_{t\to 0^-} \frac{C_p(\Omega_t) - C_p(\Omega_0)}{-t} = \lim_{t\to 0^+} \frac{C_p(\tilde{\Omega}_t) - C_p(\tilde{\Omega}_0)}{t} = (p-1)\int_{\mathbb{S}^{n-1}} \tilde{h}'(0,\xi)\, d\mu_p(\Omega_0,\xi).$$

Obviously, $\tilde{h}'(0,\xi) = -h'(0,\xi)$, which immediately implies (5.17). □

Remark 5.3. The Hadamard formula contained in Theorem 3.5 can be seen as a special case of Theorem 5.2. Indeed, if Ω and L are bounded convex domain of class $C^{2,\alpha}_+$, with support functions h_Ω and h_L respectively, applying Theorem 5.2 to the function $h(t,\xi) = h_\Omega(\xi) + t h_L(\xi)$ (for t in a sufficiently small neighborhood of the origin) we immediately get $h' = h_L$ and consequently (5.16) coincides with (3.24).

5.3. **Final proof of the variational formula and uniqueness.** Let Ω be a bounded convex domain containing the origin and let h_Ω be its support function. Then $h_\Omega > 0$ on $\mathbb{S}^{n-1}$. Let f be an arbitrary continuous function on $\mathbb{S}^{n-1}$. For $|t|$ sufficiently small we have $h_t(\xi) := h_\Omega(\xi) + t f(\xi) > 0$ for every $\xi \in \mathbb{S}^{n-1}$. Let Ω_t be the Aleksandrov domain associated with h_t. To complete the proof of Theorem 1.1 we simply note that Theorem 1.1 now follows immediately from Theorem 5.2 applied to h_t. To prove Theorem 1.2, we can state, in view of the general variational formula of p-capacity for general bounded convex domains, the p-capacitary Minkowski inequality for general bounded convex domains and its important consequences:

Theorem 5.4. *Suppose* $1 < p < n$. *Let* Ω_0, Ω_1 *be convex domains in* $\mathbb{R}^n$. *Then*

$$C_p(\Omega_0,\Omega_1)^{n-p} \geq C_p(\Omega_0)^{n-p-1} C_p(\Omega_1), \tag{5.18}$$

with equality if and only if Ω_0, Ω_1 *are homothetic.*

Theorem 5.5. *Let* Ω_0, Ω_1 *be convex domains in* $\mathbb{R}^n$ *and* $1 < p < n$. *If* Ω_0, Ω_1 *have the same p-capacitary measure, then* Ω_0 *is a translate of* Ω_1 *when* $p \neq n-1$, *and* Ω_0, Ω_1 *are homothetic when* $p = n-1$.

The proofs of Theorems 5.4 and 5.5 are exactly the same as those of Theorems 3.6 and 3.7 when the general variational formula (5.16) is used. Hence the proofs of Theorem 1.2 and Theorem 1.1 are complete.

Remark 5.6. *Suppose* $1 < p < n$ *and* Ω *is a bounded convex domain in* $\mathbb{R}^n$. *Let*

$$C_p(\Omega,\mathbb{S}^{n-1}) = \int_{\mathbb{S}^{n-1}} d\mu_p(\Omega,\xi)$$

denote the total p-capacitary measure of Ω. Then, using Theorem 5.4 we have the following p-capacitary isoperimetric inequality,

$$C_p(\Omega, \mathbb{S}^{n-1})^{n-p} \geq n\omega_n \left(\frac{n-p}{p-1}\right)^{p-1} C_p(\Omega)^{n-p-1}, \tag{5.19}$$

with equality if and only if Ω is a ball. To prove this isoperimetric inequality for p-capacity, simply let $\Omega_0 = \Omega$ and $\Omega_1 = B$, where B is the unit ball, and use the fact that $C_p(B) = n\omega_n \left(\frac{n-p}{p-1}\right)^{p-1}$. Then inequality (5.19) *follows from the Minkowski inequality for p-capacity* (5.18).

6. Minkowski problem for p-capacity

6.1. **Existence in the discrete case.** In this part we prove a version of Theorem 1.3 for purely atomic measures. Let μ be a finite Borel measure on $\mathbb{S}^{n-1}$. Consider the following conditions.

(A_1) The measure μ is not concentrated on any great subsphere; that is,
$$\int_{\mathbb{S}^{n-1}} |\theta \cdot \xi| \, d\mu(\xi) > 0, \quad \text{for each } \theta \in \mathbb{S}^{n-1}.$$

(A_2) The centroid of the measure μ is at the origin; that is,
$$\int_{\mathbb{S}^{n-1}} \xi \, d\mu(\xi) = 0.$$

(A_3) The measure μ does not have a pair of antipodal point masses; that is, if $\mu(\{x\}) > 0$, then $\mu(\{-x\}) = 0$ for $x \in \mathbb{S}^{n-1}$.

For a function $h \in C_+(\mathbb{S}^{n-1})$, denote by $\Omega(h)$ the Aleksandrov domain associated with h.

Lemma 6.1. *Suppose $1 < p < 2$ and μ is a discrete measure on $\mathbb{S}^{n-1}$ satisfying conditions (A_1)-(A_3). Let*

$$b = \inf\left\{ \int_{\mathbb{S}^{n-1}} h \, d\mu \, : h \in C_+(\mathbb{S}^{n-1}) \, , \, C_p(\Omega(h)) \geq 1 \right\}. \tag{6.1}$$

Then there exists a polytope P_0, with positive support function h_{P_0}, such that

$$\int_{\mathbb{S}^{n-1}} h_{P_0} \, d\mu = b \, , \quad C_p(P_0) = 1 \, , \tag{6.2}$$

and

$$\mu = \frac{b(p-1)}{n-p} \mu_p(P_0, \cdot) \, . \tag{6.3}$$

Furthermore, there exists b_0 depending only on n and p such that

$$0 < b < b_0 \int_{\mathbb{S}^{n-1}} d\mu. \tag{6.4}$$

Proof. We first show that the minimization in (6.1) can be reduced to minimizing only over positive support functions of polytopes. Assume that the discrete measure μ has support $\{\xi_1, \dots, \xi_m\}$ and weights c_i, $i = 1, \dots, m$, that is,

$$\mu = \sum_{i=1}^{m} c_i \, \delta_{\xi_i}.$$

For $h \in C_+(\mathbb{S}^{n-1})$, define

$$P = \{x \in \mathbb{R}^n : x \cdot \xi_i < h(\xi_i), \ i = 1, \dots, m\}. \tag{6.5}$$

Hence P is bounded by hyperplanes orthogonal to the vectors ξ_i with distance $h(\xi_i)$ from the origin. By condition (A_1), P is bounded and thus P is an open convex polytope. It follows that $\Omega(h) \subset P$ and $h_P(\xi_i) \leq h(\xi_i)$, $i = 1, \dots, m$. Thus,

$$C_p(P) \geq C_p(\Omega(h)) \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} h_P \, d\mu \leq \int_{\mathbb{S}^{n-1}} h \, d\mu.$$

Therefore, we can take a minimizing sequence $\{h_j\}$ for problem (6.1) so that h_j is the support function of polytope P_j with faces orthogonal to ξ_i, $i = 1, \dots, m$. Since it is a minimizing sequence, there exists $M > 0$ such that, for all j,

$$\sum_{i=1}^{m} c_i h_{P_j}(\xi_i) = \int_{\mathbb{S}^{n-1}} h_{P_j} \, d\mu \leq M.$$

Since all the c_i's are positive and P_j contains the origin, we have for all j,

$$h_{P_j}(\xi_i) \leq M' := \frac{M}{\min\{c_i \, : \, i = 1, \dots, m\}}.$$

This implies that the sequence P_j is bounded. By the Blaschke selection theorem (see [93]), the sequence $\bar{P}_j$ has a convergent subsequence with limit P', with respect to the Hausdorff metric. Let P_0 be the interior of P'. By the continuity of p-capacity, h_{P_0} is a minimizer for problem (6.1). Next we prove that P_0 is not empty. Note that P' is a polytope whose facets have outer unit normals that belong to $\{\xi_1, \dots, \xi_m\}$. Assume that its interior is empty and let $k < n$ be its dimension. By condition (A_3), no two vectors in $\{\xi_1, \dots, \xi_m\}$ are antipodal. This implies that $k \neq n-1$, and thus $k \leq n-2$ is left as the only possibility. But $p < 2$, and this implies that $C_p(P') = 0$ (see [30, p.154, Theorem 3]), contradicting $C_p(P') \geq 1$.

By condition (A_2), the support function of a translate of P_0 is again a minimizer of (6.1). Hence, we may assume that $h_{P_0} > 0$ on $\mathbb{S}^{n-1}$. Let $f \in C(\mathbb{S}^{n-1})$. For $t \in \mathbb{R}$, with $|t|$ sufficiently small, $h_t = h_{P_0} + tf \in C_+(\mathbb{S}^{n-1})$. But h_{P_0} being a minimizer implies the existence of a constant b' such that

$$\frac{d}{dt} \left(\int_{\mathbb{S}^{n-1}} h_t \, d\mu \right) \bigg|_{t=0} = b' \, \frac{d}{dt} C_p(\Omega_t) \bigg|_{t=0},$$

where Ω_t is the Aleksandrov domain associated with h_t. Applying Theorem 1.1 yields

$$\int_{\mathbb{S}^{n-1}} f(\xi) \, d\mu(\xi) = b'(p-1) \int_{\mathbb{S}^{n-1}} f(\xi) \, d\mu_p(P_0, \xi).$$

But f being arbitrary allows us to conclude that $\mu = b'(p-1)\mu_p(P_0, \cdot)$. But $C_p(P_0) = 1$, now yields $b = b'(n-p)$.

To prove (6.4), we just let $r = r(n,p)$ be such that the ball centered at the origin with radius r has p-capacity equal to 1. Then,

$$b \le \int_{\mathbb{S}^{n-1}} r(n,p)\, d\mu.$$

$\square$

Lemma 6.2. *Suppose* $1 < p < 2$. *Let* μ *be a discrete measure on* $\mathbb{S}^{n-1}$ *satisfying conditions* (A_1) - (A_3). *Then there exists a polytope* P *such that*

$$\mu_p(P,\cdot) = \mu\,.$$

Proof. This follows by appropriately rescaling the polytope P_0 of Lemma 6.1. $\square$

6.2. **Existence in the general case.** We prove the following result, which is Theorem 1.3.

Theorem 6.3. *Suppose* $1 < p < 2$. *If* μ *is a finite Borel measure on* $\mathbb{S}^{n-1}$ *satisfying conditions* (A_1)-(A_3). *Then there exists a bounded convex domain* Ω *such that*

$$\mu_p(\Omega,\cdot) = \mu\,.$$

Proof. Consider a sequence $\{\mu_j\}_{j=1}^{\infty}$ of discrete measures, satisfying conditions (A_1)-(A_3) and converging to μ weakly. From condition (A_1), we see that

$$\inf_{\theta\in\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\theta\cdot\xi|\, d\mu(\xi) > 0. \tag{6.6}$$

Hence from the weak convergence, we may infer the existence of a constant $c > 0$ such that, for all j,

$$\inf_{\theta\in\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\theta\cdot\xi|\, d\mu_j(\xi) \ge c. \tag{6.7}$$

There are constants c', c'' so that

$$c' \le \int_{\mathbb{S}^{n-1}} d\mu_j(\xi) \le c''\,. \tag{6.8}$$

As in Lemma 6.1, let P_j be a polytope that solves the p-capacity Minkowski problem for the discrete measure μ_j. Let

$$b_j = \int_{\mathbb{S}^{n-1}} h_{P_j}\, d\mu_j\,,$$

and let d_j be the diameter of P_j. By condition (A_1), we may assume that P_j has been translated so that $\frac{d_j}{2}\,w_j$ and $-\frac{d_j}{2}\,w_j$ belong to $\bar{P}_j$ for some unit vector w_j. This, and the definition of a support function, implies that

$$h_{P_j}(\xi) \ge \frac{d_j}{2}|\xi\cdot w_j|\,, \quad \text{for all } \xi\in\mathbb{S}^{n-1}.$$

Thus, using (6.7), we see that

$$b_j = \int_{\mathbb{S}^{n-1}} h_{P_j}\, d\mu_j \ge \frac{d_j}{2}\int_{\mathbb{S}^{n-1}} |\xi\cdot w_j|\, d\mu_j(\xi) \ge \frac{d_j}{2}c\,.$$

Using this, (6.4) and (6.8), we obtain

$$d_j \le 2b_0c''c^{-1}.$$

Therefore, the sequence P_j is bounded. A subsequence of $\bar{P}_j$ converges to a compact convex set K with interior Ω.

Case I: Ω is non-empty. In this case the proof is complete. Indeed, using (6.3) and the weak convergence of p-capacitary measures (Lemma 4.1), we have

$$\mu = \frac{b(p-1)}{n-p}\mu_p(\Omega, \cdot), \quad \text{where } b = \int_{\mathbb{S}^{n-1}} h_\Omega \, d\mu.$$

The solution of the p-capacity Minkowski problem for μ is now obtained by suitably rescaling Ω.

Case II: Ω is empty. Then $\dim(K) < n$. Since $1 < p < 2$, the continuity of C_p and the fact that $C_p(P_j) = 1$, for every j, ensures $\dim(K) = n-1$. But this is only possible if the surface area measure, S_K, is concentrated at two antipodal point masses in $\mathbb{S}^{n-1}$ with equal weight, i.e. there is a real $\alpha > 0$ and a point $x \in \mathbb{S}^{n-1}$ such that

$$S_K = \alpha(\delta_x + \delta_{-x}).$$

By Lemma 2.18, there is a constant c that depends on n, p and the radius of a ball containing all P_j so that

$$|\nabla U_j| \geq c^{-1} \quad \text{a.e. on} \quad \partial\Omega_j,$$

where U_j is the p-equilibrium potential of Ω_j. Suppose $f \in C(\mathbb{S}^{n-1})$ is a non-negative function. Then, by (1.11) and (6.3), as $j \to \infty$,

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} f \, d\mu_j &\geq \frac{p-1}{n-p} b_j c^{-p} \int_{\mathbb{S}^{n-1}} f \, dS_{P_j} \\
&\to \frac{p-1}{n-p} b c^{-p} \int_{\mathbb{S}^{n-1}} f \, dS_K = \frac{p-1}{n-p} b c^{-p} \alpha\big(f(x) + f(-x)\big).
\end{aligned}$$

Using the weak convergence we have

$$\lim_{j\to\infty} \int_{\mathbb{S}^{n-1}} f \, d\mu_j = \int_{\mathbb{S}^{n-1}} f \, d\mu$$

and consequently,

$$\int_{\mathbb{S}^{n-1}} f \, d\mu \geq \big(f(x) + f(-x)\big)/c',$$

where c' is a positive constant independent of f. However, this is possible, for all continuous non-negative f, only if μ has equal point masses at x and $-x$, see [30, p.42, Theorem 3]. The latter contradicts the assumption that Ω is empty, hence Case II can not occur and the proof of Theorem 6.3 is complete. □

7. Regularity for the p-capacity Minkowski problem

In this section we prove Theorem 1.4 by developing a p-harmonic version of the corresponding regularity theorem valid for $p = 2$ and developed in [52].

7.1. **More convex geometric facts.** To proceed we need to introduce some additional notions and notation from convex geometric analysis. Throughout the section $\Omega \subset \mathbb{R}^n$ will denote a bounded convex domain, and we first note that, after a translation, we can without loss of generality, assume that

$$B(0, r_{\rm int}) \subset \Omega \subset B(0, r_{\rm ext}) \tag{7.1}$$

where $r_{\rm int}$ and $r_{\rm ext}$ are the ones defined in Remark 2.9. Recall also (following Remark 2.9) that the Lipschitz constant of Ω is bounded by a constant depending only on $r_{\rm ext}/r_{\rm int}$, the eccentricity of Ω. In the following we set $r_0 = r_{\rm int}/10$ and we let M denote the Lipschitz constant of Ω. In the following, we will also assume, after a dilation and without loss of generality, that

$$r_{\rm int} = 1 \text{ and hence that the eccentricity of } \Omega \text{ equals } r_{\rm ext}. \tag{7.2}$$

Let $x \in \Omega$ and let $\Gamma(x, \Omega)$ denote the family of all pairs of points (x_1, x_2) in $\partial\Omega$ for which x is on the line segment joining x_1 to x_2. The normalized distance, $\delta(x, \Omega)$, of x to $\partial\Omega$, is defined by

$$\delta(x, \Omega) = \min_{(x_1,x_2)\in\Gamma(x,\Omega)} \frac{|x - x_1|}{|x - x_2|}. \tag{7.3}$$

Note that since x, x_1 and x_2 are collinear in the definition of the set $\Gamma(x, \Omega)$, the distance $\delta(x, \Omega)$ is invariant under linear transformations. The distance $\delta(x, \Omega)$ is referred to as the normalized distance of x to $\partial\Omega$.

Consider a half-space H in $\mathbb{R}^n$ and assume that $H \cap B(0, r_{\rm int}) = \varnothing$. Let $\Pi = \partial H$ and let π denote the operation of radial projection onto Π, that is, if $y \in \Pi$ and $\pi(x) = y$, then there exists $a(x) \in \mathbb{R}$ such that $x = a(x)y$. We let

$$\gamma_H := H \cap \partial\Omega,\ \tilde{\Omega}_\Pi := \pi(\gamma_H). \tag{7.4}$$

Then $\tilde{\Omega}_\Pi$, the radial projection of γ_H onto Π, is a convex subset contained in the hyperplane Π. Let x be such that $\pi(x) \in \tilde{\Omega}_\Pi$. We then define a normalized distance from x to the boundary of γ_H through the relation

$$\delta(x, \gamma_H) = \delta(\pi(x), \tilde{\Omega}_\Pi).$$

Here $\delta(\pi(x), \tilde{\Omega}_\Pi)$ is defined in the same way as in (7.3), i.e.

$$\delta(\pi(x), \tilde{\Omega}_\Pi) = \min_{(\tilde{x}_1,\tilde{x}_2)\in\tilde{\Gamma}(\pi(x),\tilde{\Omega}_\Pi)} \frac{|\pi(x) - \tilde{x}_1|}{|\pi(x) - \tilde{x}_2|},$$

where $\tilde{\Gamma}(\pi(x), \tilde{\Omega}_\Pi)$ denotes the family of all pairs of points $(\tilde{x}_1, \tilde{x}_2)$ in $\partial\tilde{\Omega}_\Pi$ for which $\pi(x)$ is on the line segment joining $\tilde{x}_1$ to $\tilde{x}_2$. This distance is changed by at most a bounded factor for different choices of π depending on the location of the origin, provided the distance from the origin to $\partial\Omega$ is bounded below by a fixed constant times the inner radius. Note also that if we let n_Π denote the unit normal to Π pointing into H, then we have, using (7.2) and the fact that $H \cap B(0, 1) = \varnothing$, that

$$1 \le x \cdot n_\Pi \le r_{\rm ext}, \quad \text{ for all } x \in \tilde{\Omega}_H.$$

7.2. **Technical lemmas.** Recall that the p-equilibrium potential associated to Ω is the function U, which is defined and continuous on the closure of $\mathbb{R}^n \setminus \bar{\Omega}$ and satisfies

$$\begin{cases} \Delta_p U = 0 & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \\ \\ U = 1 & \text{on } \partial\Omega, \text{ and } \lim_{|x|\to\infty} U(x) = 0\,. \end{cases}$$

Let $u = 1 - U$ and extend u to be identically equal to zero in Ω. Then, using Lemma 2.5, we see that there exists a unique locally finite positive Borel measure ν on $\partial\Omega$ such that whenever $\theta \in C_0^\infty(\mathbb{R}^n)$, such that $(\text{supp } \theta) \cap B(0,1) = \varnothing$, then

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx \;=\; -\int \theta \, d\nu.$$

Moreover, there exists $c = c(p,n,M)$, $1 \le c < \infty$, such that if $w \in \partial\Omega$, $r < r_0$, then

$$c^{-1} r^{p-n} \nu(\Delta(w,r)) \le (u(a_r'(w)))^{p-1} \le c\, r^{p-n}\, \nu(\Delta(w,r/2)),$$

where $a_r'(w) \in \mathbb{R}^n \setminus \bar{\Omega}$ was introduced in (2.4) (ii). Furthermore, combining these facts with the Hölder inequality and Theorem 2.8 (ii), we see that if $w \in \partial\Omega$, $r < r_0$, then

$$\left(\frac{1}{\mathcal{H}^{n-1}(\Delta(w,r))} \int\limits_{\Delta(w,r)} |\nabla U|^p d\mathcal{H}^{n-1} \right)^{1/p} \approx \left(\frac{\nu(\Delta(w,r))}{\mathcal{H}^{n-1}(\Delta(w,r))} \right)^{1/(p-1)} \approx \frac{u(a_r'(w))}{r}.$$

Using Lemma 2.13 we have that

$$\lim_{x \in \tilde{\Gamma}(y), x \to y} \nabla u(x) = \nabla u(y) \tag{7.5}$$

exists for $\mathcal{H}^{n-1}$ almost all $y \in \Delta(w,r)$. Assume that the limit in (7.5) exists at $y \in \Delta(w,r)$, let $\tilde{\Pi}_y$ denote the supporting hyperplane to Ω at y, and let $\tilde{H}_y$ be the associated half-space which is contained in $\mathbb{R}^n \setminus \bar{\Omega}$. Without loss of generality, we can assume that $y = 0$, $\tilde{\Pi}_y = \{(x', x_n) : x_n = 0\}$ and $\tilde{H}_y = \{(x', x_n) : x_n > 0\}$. Let $D = \tilde{H}_y \cap B(0,r)$ and let $\hat{u}$ be the non-negative p-harmonic function in D that satisfies $\hat{u} = 1$ on $\partial D \cap \{(x', x_n) : x_n > r/2\}$, $\hat{u} = 0$ on $\partial D \cap \{(x', x_n) : x_n < r/4\}$, which is continuous on $\bar{D}$ and takes boundary values between 0 and 1 on $\partial D \cap \{(x', x_n) : r/4 \le x_n \le r/2\}$. Then, using the Harnack inequality and the maximum principle, we see that

$$u(x) \ge c^{-1} u(a_r'(w)) \hat{u}(x), \quad \text{ whenever } x \in D \cap B(0, r/10). \tag{7.6}$$

Hence, using the facts that $u(0) = 0 = \hat{u}(0)$, that (7.5) exists at $y = 0$, we see that

$$\langle \nabla u, e_n \rangle(0) \ge c^{-1} u(a_r'(w)) \langle \nabla \hat{u}, e_n \rangle(0). \tag{7.7}$$

Note that $\langle \nabla \hat{u}, e_n \rangle(0)$ exists by continuity of $\nabla \hat{u}$ up to the boundary $\partial D \cap \{(x', x_n) : x_n = 0\}$, see [69]. Furthermore, using Theorem 2.7 and Theorem 2.6 applied to the pair of functions $\hat{u}$, x_n/r, we deduce that

$$\langle \nabla u, e_n \rangle(0) \ge c^{-1} u(a_r'(w)) \langle \nabla \hat{u}, e_n \rangle(0) \ge c^{-2} \frac{u(a_r'(w))}{r}. \tag{7.8}$$

Combining the estimates above, we see that

$$\left(\frac{1}{\mathcal{H}^{n-1}(\Delta(w,r))} \int\limits_{\Delta(w,r)} |\nabla U|^p d\mathcal{H}^{n-1} \right)^{1/p} \approx \frac{u(a_r'(w))}{r} \lesssim \langle \nabla u, e_n \rangle(0). \tag{7.9}$$

Since this argument can be repeated for $\mathcal{H}^{n-1}$ almost all $y \in \Delta(w,r)$, we can conclude that

$$\left(\frac{1}{\mathcal{H}^{n-1}(\Delta(w,r))}\int\limits_{\Delta(w,r)}|\nabla U|^p d\mathcal{H}^{n-1}\right)^{1/p} \approx \frac{u(a_r'(w))}{r} \approx \inf_{\Delta(w,r)}|\nabla U|, \tag{7.10}$$

whenever $w \in \partial\Omega$, $r < r_0$. We emphasize that all constants in these estimates only depend on n, p and M. In particular, using (7.10), a simple covering argument and the Harnack inequality, we can conclude that there exists a constant $c = c(n,p,M)$ such that

$$\int\limits_{\partial\Omega}|\nabla U|^p d\mathcal{H}^{n-1} \leq c,\ c^{-1} \leq \inf_{\partial\Omega}|\nabla U|. \tag{7.11}$$

Following [52], the bulk of the argument below is devoted to the extension of estimates like (7.10) and (7.11) to certain cross sections of $\partial\Omega$ which may not be comparable to balls.

Lemma 7.1. *Let the sets γ_H, $\tilde{\Omega}_\Pi$ be as defined in* (7.4)*. Let $\hat{r}$ be the inner radius of the set $\tilde{\Omega}_\Pi$. Assume that $x_1, \hat{x}_1 \in \gamma_H$, $\delta(x_1,\gamma_H) \approx 1$. Then there exists $c = c(p,n,M)$, $1 \leq c < \infty$, such that*

$$\inf_{\Delta(x_1,\hat{r})}|\nabla U| \leq c \inf_{\Delta(\hat{x}_1,\hat{r})}|\nabla U|.$$

Proof. This lemma is proved by using (7.10) and essentially copying, verbatim, the corresponding proof, valid in the case $p = 2$, in [52] (see the proof of lemma 6.8 in [52]). In particular, the proof uses only a few elementary facts about convex sets, the maximum principle, Harnack's inequality, the fact that the p-Laplace operator is independent of translations and dilations and (7.10). Details are omitted. □

Lemma 7.2. *Let the sets γ_H, $\tilde{\Omega}_\Pi$ be as defined in* (7.4)*. Let $\hat{r}$ be the inner radius of the set $\tilde{\Omega}_\Pi$. Then there exist $c = c(p,n,M)$, $1 \leq c < \infty$, and $\varepsilon = \varepsilon(p,n,M) > 0$, such that if $x_2 \in \gamma_H$, then*

$$\frac{1}{\mathcal{H}^{n-1}(\Delta(x_2,\hat{r}))}\int\limits_{\Delta(x_2,\hat{r})}|\nabla U(x)|^{p-1}d\mathcal{H}^{n-1}(x) \leq c\delta(x_2,\gamma_H)^{(-1+\varepsilon)(p-1)}(\inf_{\gamma_H}|\nabla U|)^{p-1}.$$

Proof. Following [52, Theorem 6.13], we let $x_1 \in \gamma_H$ be a central point of γ_H, i.e. $\delta(x_1,\gamma_H) \approx 1$. We then choose $x_3 \in \partial\gamma_H \cap \tilde{\Omega}_\Pi$ such that $\pi(x_2)$ is on the line segment with endpoints determined by x_3 and $\pi(x_1)$. If $\delta(x_2,\gamma_H) \approx 1$, Lemma 7.1 implies that

$$\inf_{\Delta(x_2,\hat{r})}|\nabla U| \leq C \inf_{\Delta(x_1,\hat{r})}|\nabla U| \leq C \inf_{\gamma_H}|\nabla U|.$$

This and (7.10) yield the conclusion of Lemma 7.2. Hence, we assume in the following that $\delta(x_2,\gamma_H) \ll 1$, and we note that we may assume, in particular, that

$$|x_2 - x_3| < r/10, \text{ where } r = |x_1 - x_3|.$$

To proceed we introduce a number of auxiliary domains and functions. We let $D = \Omega \cap B(x_1, r/2)$ and $\tilde{\Omega}$ be the convex hull of D and x_3. Note that $\tilde{\Omega} \subset \Omega$. We also let

$$\mathcal{C} = \{x_3 + t(x - x_3) :\ x \in D,\ t > 0\}$$

and we note that $\mathcal{C}$ is an unbounded cone defined through x_3 and D. Recall the function $u = 1 - U$, where U is the p-capacitary potential for Ω, defined in $\mathbb{R}^n \setminus \bar{\Omega}$. Similarly there exists a positive p-harmonic function $u_{\tilde\Omega}$ in the complement of the closure of $\tilde\Omega$ which is identical to zero on $\partial\tilde\Omega$. In our argument we will also make use of the following lemma.

Lemma 7.3. *Let D, x_3 and $\mathcal{C}$ be as stated above. Let $e = (x_1-x_3)/|x_1-x_3|$ and let $\mathbb{S}^{n-1}(x_3) \subset \mathbb{R}^n$ denote the unit sphere centered at x_3. Given $x \in D$, let $x' = x_3 + (x-x_3)/|x-x_3|$, and let $D' = D_{\mathbb{S}^{n-1}}$ denote the set of all such points x'. Then D' is a bounded convex domain on $\mathbb{S}^{n-1}(x_3)$ with eccentricity bounded by a constant $c = c(n, M)$, $1 \leq c < \infty$, and $\mathbb{R}^n \setminus \bar{\mathcal{C}}$ is an unbounded Lipschitz domains with Lipschitz constant depending only on n, M. Furthermore, there exists a unique positive p-harmonic function $u_{\mathcal{C}}$ in $\mathbb{R}^n \setminus \bar{\mathcal{C}}$ which vanishes continuously on $\partial\mathcal{C}$ and satisfies $u_{\mathcal{C}}(x_3 - e) = 1$. Finally, there exists $\varepsilon > 0$, which is bounded from below by a positive constant which only depends on n, p, M, such that*

$$u_{\mathcal{C}}(x) = |x-x_3|^{\varepsilon} f((x-x_3)/|x-x_3|),$$

whenever $x \in \mathbb{R}^n \setminus \bar{\mathcal{C}}$.

We postpone the proof of Lemma 7.3 for now and proceed with the proof of Lemma 7.2. Let $A_{2r}(x_3) \in \partial B(x_3, 2r)$ be such that the Euclidean distance from $A_{2r}(x_3)$ to $\mathcal{C}$ is r. Based on $u_{\tilde\Omega}$, $u_{\mathcal{C}}$, we also introduce the normalized functions

$$\begin{aligned}\tilde u_{\tilde\Omega}(x) &= u_{\tilde\Omega}(x)\frac{u(A_{2r}(x_3))}{u_{\tilde\Omega}(A_{2r}(x_3))},\\ \tilde u_{\mathcal{C}}(x) &= u_{\mathcal{C}}(x)\frac{u(A_{2r}(x_3))}{u_{\mathcal{C}}(A_{2r}(x_3))}.\end{aligned}$$

We now first note, simply using the maximum principle, that

$$u(x)\frac{u(A_{2r}(x_3))}{u_{\tilde\Omega}(A_{2r}(x_3))} \leq \tilde u_{\tilde\Omega}(x), \tag{7.12}$$

whenever $x \in \mathbb{R}^n \setminus \Omega$. By construction, $\tilde\Omega \cap B(x_3, r/2) = \mathcal{C} \cap B(x_3, r/2)$ and using that Ω is convex we see that the interior of the domain $B(x_3, r/2) \setminus \tilde\Omega = B(x_3, r/2) \setminus \mathcal{C}$ is a Lipschitz domain with Lipschitz constant determined by M. In particular, using Theorem 2.7 and the Harnack inequality, we see that

$$\frac{\tilde u_{\mathcal{C}}(x)}{\tilde u_{\tilde\Omega}(x)} \approx \frac{\tilde u_{\mathcal{C}}(A_{2r}(x_3))}{\tilde u_{\tilde\Omega}(A_{2r}(x_3))} \approx 1,$$

whenever $x \in B(x_3, r/4) \setminus \tilde\Omega$. Furthermore, using the Harnack inequality, we also see that

$$\tilde u_{\tilde\Omega}(x) \approx \tilde u_{\tilde\Omega}(A_{2r}(x_3)) \approx u(A_{2r}(x_3)), \tag{7.13}$$

whenever $x \in \partial B(x_3, 3r)$. We now apply the maximum principle in the domain $B(x_3, 3r) \setminus \bar\Omega$, using (7.12) and (7.13), to conclude that

$$u(x) \lesssim \tilde u_{\tilde\Omega}(x), \qquad \text{whenever } x \in B(x_3, 3r) \setminus \bar\Omega.$$

Combining this estimate with (7.13), noting that $B(x_3, r/4) \setminus \Omega \subset B(x_3, r/4) \setminus \tilde\Omega$, we can conclude that

$$u(x) \leq \tilde c \tilde u_{\mathcal{C}}(x) = \tilde c u_{\mathcal{C}}(x)\frac{u(A_{2r}(x_3))}{u_{\mathcal{C}}(A_{2r}(x_3))}, \tag{7.14}$$

for all $x \in B(x_3, r/4) \setminus \Omega$, where $\tilde{c}$, $1 \leq \tilde{c} < \infty$, depends only on n, p and the eccentricity of Ω, and hence M. By a similar argument one can also prove (see [52]) that

$$\tilde{c}^{-1} u_{\mathcal{C}}(x) \frac{u(A_{2r}(x_3))}{u_{\mathcal{C}}(A_{2r}(x_3))} = \tilde{c}^{-1} \tilde{u}_{\mathcal{C}}(x) \leq u(x), \tag{7.15}$$

for all $x \in B(x_1, r/4) \setminus \mathcal{C}$ and with $\tilde{c}$ as above. Recall that $\hat{r}$ is the inner radius of the set $\tilde{\Omega}_{\Pi}$. Next, following [52] it can be seen that if c is sufficiently large, depending only on n, p and the eccentricity of Ω, then the segment $S = \{(1+c\hat{r})x : \ x = t\pi(x_1) + (1-t)x_3, \ 0 \leq t \leq 1\}$ (recall that $0 \in \Omega$) has the property that every point $x \in S$ is at a distance comparable to $\hat{r}$ from $\mathcal{C}$ and Ω. Using this, the facts that $(1 + c\hat{r})x_2 \in B(x_3, r/4) \setminus \Omega$, $(1 + c\hat{r})x_1 \in B(x_1, r/4) \setminus \mathcal{C}$, (7.14) and (7.15), it follows that

$$\begin{aligned} u((1+c\hat{r})x_2) &\leq \tilde{c} u_{\mathcal{C}}((1+c\hat{r})x_2) \frac{u(A_{2r}(x_3))}{u_{\mathcal{C}}(A_{2r}(x_3))}, \\ u_{\mathcal{C}}((1+c\hat{r})x_1) \frac{u(A_{2r}(x_3))}{u_{\mathcal{C}}(A_{2r}(x_3))} &\leq \tilde{c} u((1+c\hat{r})x_1). \end{aligned}$$

Next, let z be located on the line segment between x_3 and $(1 + c\hat{r})x_1$ and at distance $\delta\hat{r}$ from x_3 where $0 < \delta \ll 1$. In particular, z is at height $\delta\hat{r}$ above $\mathcal{C}$ and close to $\pi(x_2)$. Arguing similarly as in the proof of (7.6)-(7.8), we deduce that

$$\frac{u_{\mathcal{C}}(z)}{\delta\hat{r}} \geq \tilde{c}^{-1} \frac{u_{\mathcal{C}}((1+c\hat{r})x_2)}{\hat{r}}.$$

Hence, combining the estimates above, we see that

$$u((1+c\hat{r})x_2) \leq \tilde{c}^2 \frac{u_{\mathcal{C}}(z)}{\delta} \frac{u(A_{2r}(x_3))}{u_{\mathcal{C}}(A_{2r}(x_3))} \leq \tilde{c}^3 \frac{u_{\mathcal{C}}(z)}{\delta} \frac{u((1+c\hat{r})x_1)}{u_{\mathcal{C}}((1+c\hat{r})x_1)}. \tag{7.16}$$

Next, using Lemma 7.3 we see that there exists $0 < \varepsilon \ll 1$, depending only on n, p and the eccentricity of Ω, such that

$$u_{\mathfrak{C}}(z) = \delta^{\varepsilon} u_{\mathfrak{C}}((1+c\hat{r})x_1). \tag{7.17}$$

In particular, combining (7.16) and (7.17) we see that

$$u((1+c\hat{r})x_2) \leq \tilde{c}^3 \delta^{-1+\varepsilon} u((1+c\hat{r})x_1). \tag{7.18}$$

By (7.10), we see that (7.18) implies that

$$\left(\frac{1}{\mathcal{H}^{n-1}(\Delta(x_2, \hat{r}))} \int\limits_{\Delta(x_2,\hat{r})} |\nabla U(x)|^{p-1} d\mathcal{H}^{n-1} \right)^{1/(p-1)} \lesssim \delta^{-1+\varepsilon} \inf_{\Delta(x_1,\hat{r})} |\nabla U|.$$

Since $\delta(x_1, \gamma_H) \approx 1$, an application of Lemma 7.1 now completes the proof of the lemma. $\square$

Proof of Lemma 7.3. Recall the definition of D' stated in Lemma 7.3. Then D' is a bounded convex domain on $\mathbb{S}^{n-1}(x_3)$, with eccentricity bounded by a constant $c = c(n, M)$, $1 \leq c < \infty$, and $\mathbb{R}^n \setminus \bar{\mathcal{C}}$ is an unbounded Lipschitz domains with Lipschitz constant depending only on n, M. To prove the existence and uniqueness of $u_{\mathcal{C}}$ we can assume, without loss of generality, that $x_3 = 0$ and that $-e = e_n$ in an appropriate coordinate system. Given $p, 1 < p < \infty$, we say that $\hat{u}$ is a minimal positive p-harmonic function in $\mathbb{R}^n \setminus \bar{\mathcal{C}}$, relative to ∞, provided $\hat{u}$ is a positive p-harmonic function in $\mathbb{R}^n \setminus \bar{\mathcal{C}}$ with continuous boundary value zero on $\partial\mathcal{C}$. Using this notion we see that to prove the existence and uniqueness of $u_{\mathcal{C}}$, as stated in Lemma

7.3, it is sufficient to establish the existence and uniqueness of a minimal positive p-harmonic function $\hat u$ relative to ∞ in $\mathbb{R}^n \setminus \bar{\mathcal{C}}$. To begin the existence part of the proof we note that the existence of a minimal positive p-harmonic function $\hat u$ relative to ∞ in $\mathbb{R}^n \setminus \bar{\mathcal{C}}$ follows from standard arguments. For instance, one can take $\hat u$ to be the limit of a subsequence of $\{u_m\}_{m=1}^\infty$ where u_m is a positive p-harmonic function in $(\mathbb{R}^n \setminus \bar{\mathcal{C}}) \cap B(0,m)$ with continuous boundary value 0 on $\partial\mathcal{C} \cap B(0,m)$ and $u_m(e_n) = 1$. Existence of $u_m, m = 1, 2, \dots,$ follows from a calculus of variations argument. Applying Lemma 2.1 – Lemma 2.4 to $u_m, m = 1, 2, \dots,$ and using Ascoli-Arzelá theorem we can deduce the existence of $\hat u$ such that $\hat u(e_n) = 1$. To prove uniqueness of this $\hat u$, let $\hat v$ be another minimal positive p-harmonic function in $\mathbb{R}^n \setminus \bar{\mathcal{C}}$ with $\hat v(e_n) = 1$. Using Theorem 2.7 with $\Omega = (\mathbb{R}^n \setminus \bar{\mathcal{C}}) \cap B(0, 2r), w = 0$, we get, upon letting $r \to \infty$, that $\hat v = \hat u$. Thus $\hat u$ is the unique minimal positive p-harmonic function in $\mathbb{R}^n \setminus \bar{\mathcal{C}}$ with $\hat u(e_n) = 1$. Note also, using Theorem 2.6 and Lemma 2.4, that $\hat u$ is infinitely differentiable in $\mathbb{R}^n \setminus \bar{\mathcal{C}}$. Finally, to obtain the desired form for $u_{\mathcal{C}} = \hat u$ we first note that uniqueness of $\hat u$ and invariance of the p-Laplace equation under dilations immediately implies that

$$\hat u(\lambda x) = \hat u(\lambda e_n)\hat u(x), \tag{7.19}$$

whenever $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \bar{\mathcal{C}}$. We now want to prove that there exist $\varepsilon > 0$ such that

$$\hat u(\lambda e_n) = \lambda^\varepsilon, \quad \text{for all } \lambda > 0. \tag{7.20}$$

To do this we first consider $0 < \lambda < \lambda' < 1$ and we define ε and ε' through

$$\hat u(\lambda e_n) = \lambda^\varepsilon,\ \hat u(\lambda' e_n) = (\lambda')^{\varepsilon'}. \tag{7.21}$$

Let $\sigma^* \in \mathbb{S}^{n-1} \setminus \bar{\mathcal{C}}$ be such that

$$\hat u(\sigma^*) = \sup_{\sigma \in \mathbb{S}^{n-1}\setminus\bar{\mathcal{C}}} \hat u(\sigma). \tag{7.22}$$

Furthermore, let $\nu \in \mathbb{N}$ and let μ be a non-negative integer in the interval

$$[(\ln\lambda/\ln\lambda')\nu, (\ln\lambda/\ln\lambda')(\nu+1)].$$

Then

$$\lambda^{\nu+1} \le (\lambda')^\mu \le \lambda^\nu. \tag{7.23}$$

Using the maximum principle we see that

$$\hat u(\lambda^{\nu+1}\sigma^*) \le \hat u((\lambda')^\mu\sigma^*) \le \hat u(\lambda^\nu\sigma^*). \tag{7.24}$$

Now, (7.19), (7.21), (7.23) and (7.24) imply that

$$\begin{aligned} \lambda^{(\nu+1)\varepsilon} &\le (\lambda')^{\mu\varepsilon'} \le \lambda^{\nu\varepsilon}, \\ \lambda^{(\nu+1)\varepsilon'} &\le (\lambda')^{\mu\varepsilon'} \le \lambda^{\nu\varepsilon'}. \end{aligned} \tag{7.25}$$

These inequalities imply that

$$\lambda^{\nu(\varepsilon-\varepsilon')+\varepsilon} \le 1 \le \lambda^{\nu(\varepsilon-\varepsilon')-\varepsilon'}. \tag{7.26}$$

Hence if $\varepsilon - \varepsilon' > 0$, then since $\nu \in \mathbb{N}$ is arbitrary, we can derive a contradiction based on the right hand inequality in (7.26). Similarly when $\varepsilon - \varepsilon' < 0$, we see that $\varepsilon' = \varepsilon$ and hence (7.20) follows in this case. Furthermore, ε must be positive. The case $1 < \lambda < \lambda' < \infty$ can be treated similarly and by continuity we can then conclude the validity of (7.20), for all $0 < \lambda < \infty$, and hence

$$u_{\mathcal{C}}(x) = \hat u(x) = \hat u(|x|e_n)\hat u(x/|x|) = |x|^\varepsilon f(x/|x|). \tag{7.27}$$

Finally, we see that it now only remains to establish the bound on $\varepsilon > 0$ from below by a positive constant which only depends on n, p, M. To do this, we let $\theta(e, e_n)$ be the angle between $e \in \mathbb{R}^n$, $|e| = 1$, and e_n. Given $\theta_0 \in (0, \pi]$, we set

$$C(\theta_0) \;:=\; \{\lambda e : \; \theta_0 < \theta(e, e_n) \leq \pi \text{ and } 0 < \lambda < \infty\}. \tag{7.28}$$

We note that we can now, as a special case of the above construction, construct a unique minimal positive p-harmonic function $\tilde{u}$ in $\mathbb{R}^n \setminus \overline{C(\theta_0)}$, relative to ∞, which satisfies $\tilde{u}(e_n) = 1$. In this case we can specify the form of $\tilde{u}$ further by using the fact that uniqueness and the invariance of the p-Laplace equation under rotations imply that $\tilde{u}$ is symmetric about the x_n axis. Thus we write $\tilde{u}(r, \theta)$ for $\tilde{u}(x)$ when $x \in \mathbb{R}^n \setminus C(\theta_0)$ and $r = |x|, x_n = r\cos\theta, 0 \leq \theta \leq \theta_0$. Furthermore, differentiating (7.19) with respect to λ and evaluating at $\lambda = 1$ we find that

$$r\tilde{u}_r(r, \theta) = \langle x, \nabla\tilde{u}(x)\rangle = \langle \nabla\tilde{u}(e_n), e_n\rangle \hat{u}(r, \theta).$$

Dividing this by $r\tilde{u}(r, \theta)$, integrating, and then exponentiating, we get $\tilde{u}(r, \theta) = r^\gamma \psi(\cos\theta)$ where $\gamma = \langle e_n, \nabla\tilde{u}(e_n)\rangle$. Continuity of γ once again follows from uniqueness of $\tilde{u}(\cdot, \theta_0)$ and Lemmas 2.1 – 2.4. Also, $\gamma(\theta_0)$ is an increasing function of θ_0 for $\theta_0 \in (0, \pi)$, as follows easily from comparing solutions in different cones and using the maximum principle for p-harmonic functions. Finally $\tilde{u}(x) = x_n = r\cos\theta$ when $\theta_0 = \pi/2$, and hence $\gamma(\pi/2) = 1$. Now let $\tilde{\theta}_0$ be the smallest $\tilde{\theta}_0 \in (0, \pi)$ such

$$\mathbb{R}^n \setminus \overline{C(\tilde{\theta}_0)} \subset \mathbb{R}^n \setminus \bar{\mathcal{C}}.$$

Using this and the maximum principle we see that the ε in (7.20), (7.27), must satisfy $\varepsilon \geq \tilde{\theta}_0$ and this completes the proof of Lemma 7.3. $\square$

If Q is a cube in $\mathbb{R}^{n-1}$ with sides of length s, then in the following we denote by Q^* the concentric cube with sides of length $b_n s$ where $b_n = 10((n-1)!)^2$.

Lemma 7.4. *Let γ_H, $\tilde{\Omega}_\Pi$ be as defined in* (7.4)*. Let $E := \tilde{\Omega}_\Pi$ and let $\hat{r}$ be the inner radius of E. Choose coordinate axes parallel to the axes of an optimal inscribed ellipsoid in E. Let $\mathcal{Q}$ be a tiling of E by cubes with sides of length s parallel to the coordinate axes. Assume that $s < \hat{r}$. For each cube $Q \in \mathcal{Q}$, let*

$$\delta^*(Q) = \max_{x \in Q^* \cap E} \delta(x, E).$$

Then,

$$\sum_{\{Q:\; \delta^*(Q) < \delta\}} |Q| \leq c_n \delta |E|.$$

Proof. This purely geometric lemma is a formulation of Lemma 6.16 in [52]. $\square$

Lemma 7.5. *Suppose $1 < p < 2$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded convex domain. Then there exist $c = c(p, n, M)$, $1 \leq c < \infty$, and $\varepsilon = \varepsilon(p, n, M) > 0$, such that*

$$\int_{\gamma_H} \delta(x, \gamma_H)^{1-\varepsilon} |\nabla U(x)|^p d\mathcal{H}^{n-1}(x) \leq c\mathcal{H}^{n-1}(\gamma_H)(\inf_{\gamma_H} |\nabla U|)^p$$

for every set of the form $\gamma_H := H \cap \partial\Omega$ where H is a half-space in $\mathbb{R}^n$ such that $H \cap B(0, r_{int}) = \varnothing$.

Proof. Let $E := \tilde{\Omega}_\Pi$ be as introduced in (7.4) and let $\hat{r}$ be the inner radius of E. Following Lemma 7.4, let $\mathcal{Q}$ be a tiling of E by cubes with sides of length $s < \hat{r}$. Using Lemma 7.4 we see, for any $\varepsilon > 0$, that

$$\begin{aligned}\sum_{Q\in\mathcal{Q}} \delta^*(Q)^{(-1+\varepsilon)(p-1)}|Q| &= \sum_{k=1}^{\infty} \sum_{\{Q\in\mathcal{Q}\,:\,\delta^*(Q)\approx 2^{-k}\}} \delta^*(Q)^{(-1+\varepsilon)(p-1)}|Q| \\ &\leq c\sum_{k=1}^{\infty}(2^k)^{(1-\varepsilon)(p-1)}2^{-k}|E| \leq c_\varepsilon|E|, \end{aligned} \tag{7.29}$$

provided that $1 \leq p \leq 2$. Let $\hat{Q} = \pi^{-1}(Q)$. Then $\mathcal{H}^{n-1}(\hat{Q}) \approx |Q|$ and $\mathcal{H}^{n-1}(\gamma_H) \approx |E|$. Using this we deduce

$$\begin{aligned}\int_{\gamma_H} \delta(x,\gamma_H)^{1-\varepsilon}|\nabla U(x)|^p d\mathcal{H}^{n-1} &\leq \sum_{Q\in\mathcal{Q}} \int_{\hat{Q}} \delta(x,\gamma_H)^{1-\varepsilon}|\nabla U(x)|^p d\mathcal{H}^{n-1} \\ &\leq \sum_{Q\in\mathcal{Q}} \delta^*(Q)^{1-\varepsilon} \int_{\hat{Q}} |\nabla U(x)|^p d\mathcal{H}^{n-1}. \end{aligned} \tag{7.30}$$

Hence, using (7.30) and Theorem 2.8 we see that

$$\int_{\gamma_H} \delta(x,\gamma_H)^{1-\varepsilon}|\nabla U(x)|^p d\mathcal{H}^{n-1} \leq \sum_{Q\in\mathcal{Q}} \delta^*(Q)^{1-\varepsilon}\mathcal{H}^{n-1}(\hat{Q})\left(\frac{1}{\mathcal{H}^{n-1}(\hat{Q})}\int_{\hat{Q}} |\nabla U(x)|^{p-1} d\mathcal{H}^{n-1}\right)^{p/(p-1)}.$$

Combining this estimate, (7.29) and Lemma 7.2 we see that

$$\begin{aligned}\int_{\gamma_H} \delta(x,\gamma_H)^{1-\varepsilon}|\nabla U(x)|^p d\mathcal{H}^{n-1} &\leq \sum_{Q\in\mathcal{Q}} \delta^*(Q)^{(1-\varepsilon)}\mathcal{H}^{n-1}(\hat{Q})\delta^*(Q)^{(-1+\varepsilon)p}(\inf_{\gamma_H}|\nabla U|)^p \\ &\leq c\mathcal{H}^{n-1}(\gamma_H)(\inf_{\gamma_H}|\nabla U|)^p, \end{aligned}$$

and this completes the proof of the lemma. $\square$

7.3. **The final proof of regularity.** Let $O \subset \mathbb{R}^{n-1}$ be a bounded convex domain and let ν be a given Radon measure on O. We say that a convex function $\phi : O \to \mathbb{R}$ is an Alexandrov solution to the Monge-Ampére equation

$$\det(\nabla^2\phi) = d\nu \tag{7.31}$$

if for any Borel set $E \subset O$ it holds that

$$\left|\bigcup_{x\in E} \partial\phi(x)\right| = \nu(E) \tag{7.32}$$

where $\partial\phi(x)$ is the subdifferential of ϕ at x and $|E|$ here is the $(n-1)$-dimensional Lebesgue measure of $E \subset \mathbb{R}^{n-1}$. If this is the case, we also say that ϕ solves (7.31) in the sense of Alexandrov. Recall that $\partial\phi(x)$, $x \in O$, is the set of all $y \in \mathbb{R}^{n-1}$ such that the plane

$$\{(z, z_n) \in \mathbb{R}^n : z_n = \phi(x) + y\cdot(z - x)\}$$

is tangent to the graph of ϕ at $(x, \phi(x))$. For a subset $E \subset O$, we define

$$\partial\phi(E) = \cup\{\partial\phi(x) :\ x \in E\}.$$

The set-valued mapping $\partial\phi$ is related to the set-valued Gauss mapping

$$\mathtt{g}((x,\phi(x))) = \Big\{\xi = (y,-1)/\sqrt{1+|y|^2} \in \mathbb{S}^{n-1} : y \in \partial\phi(x)\Big\}.$$

The coordinate $\xi_n = -1/\sqrt{1+|y|^2}$ gives the Jacobian of the change of variable $dy = |\xi_n|^{-n} d\xi$. Using this notation and (7.32), we see that the function ϕ is an Alexandrov solution to (7.31) if

$$\nu(E) = \int\limits_{\mathtt{g}(\bar{E})} |\xi_n|^{-n} d\xi, \tag{7.33}$$

for every Borel set $E \subset O$ and where $\bar{E} = \{(x,\phi(x)) : x \in E\}$.

Theorem 7.6. *Let $O \subset \mathbb{R}^{n-1}$ be a bounded convex domain, let ν be a given Radon measure on O and let $\varepsilon > 0$. Suppose that ϕ is a convex function which solves*

$$\det(\nabla^2\phi) = d\nu$$

on O in the sense of Alexandrov. Suppose that for every set F of the form $F = \{x \in O : \phi(x) < a \cdot x + b\}$, we have

$$\int\limits_F \delta(x,F)^{1-\varepsilon} d\nu(x) \le c\nu(F/2)$$

for some constant c which is independent of a and b. Suppose that l_{x_0} is a supporting linear function to ϕ at $x_0 \in O$. Then either

$$\{\phi = l_{x_0}\} = \{x_0\} \tag{7.34}$$

or

(7.35) *the extremal points of the set $\{\phi = l_{x_0}\}$ are contained in ∂O.*

Furthermore, if ϕ is strictly convex, then $\phi \in C^{1,t}$ for some $t > 0$.

Proof. In the case $\varepsilon = 1$, Theorem 7.6 is a summary of Theorem 1 and Theorem 2 in [17], see also [15]. In Theorem 7.1 in [52], Jerison extended this theorem to allow for $\varepsilon > 0$ and the contribution in [52] is the proof of Theorem 7.6 for $\varepsilon > 0$ arbitrarily small. Furthermore, in [52], see p. 44-45, it is proved that a power $\delta(x,F)^{1-\varepsilon}$, for some $\varepsilon > 0$, is the best one can hope for in Theorem 7.6 and that this power is at the borderline of the regularity theory for the Monge-Ampére equation. □

Lemma 7.7. *If Ω is a bounded convex domain in $\mathbb{R}^n$, $1 < p < n$, and let U be the p-equilibrium potential of Ω. Let μ be a positive measure on $\mathbb{S}^{n-1}$ satisfying*

$$\mu_p(\Omega, E) = \int\limits_{\mathtt{g}^{-1}(E)} |\nabla U|^p \, d\mathcal{H}^{n-1} = \int\limits_E d\mu$$

for every Borel set $E \subset \mathbb{S}^{n-1}$. Suppose that $d\mu = \psi(\xi)d\xi$ for some integrable function ψ and $\psi(\xi) \ge c > 0$ for all $\xi \in \mathbb{S}^{n-1}$. Let ϕ denote the convex, Lipschitz function defined on a bounded convex domain $O \subset \mathbb{R}^{n-1}$ whose graph $\{(x',\phi(x')) : x' \in O\}$ is a portion of $\partial\Omega$. Then ϕ satisfies the Monge-Ampére equation

$$\det(\nabla^2\phi(x')) \;=\; (1+|\nabla\phi(x')|^2)^{(n+1)/2} |\nabla U(x',\phi(x'))|^p (\psi(\xi))^{-1} \text{ with}$$

$$\xi = \frac{(-1,\nabla\phi(x'))}{\sqrt{1+|\nabla\phi(x')|^2}}, \tag{7.36}$$

in the sense of Alexandrov.

Proof. The proof follows along the lines of the proof of Proposition 7.5 in [52]. □

Proof of Theorem 1.4. Let p, $1<p<2$, be fixed and consider $n\geq 2$. Let $\mu=\mu_p(\Omega,\cdot)$ and Ω be as in the statement of Theorem 1.4. Assume that $k=0$ and that $\alpha\in(0,1)$. By assumption $d\mu=\psi d\mathcal{H}^{n-1}$ with strictly positive density ψ, hence $\psi(\xi)\geq c>0$ for all $\xi\in\mathbb{S}^{n-1}$ and for some $c>0$, and if $\psi\in C^{0,\alpha}(\mathbb{S}^{n-1})$ then ψ is also bounded from above. Using (2.3) and a translation and rotation, we can without loss of generality assume, locally, that

$$\begin{aligned}\Omega\cap B(0,4r_0) &= \{x=(x',x_n)\in\mathbb{R}^n : x_n>\phi(x')\}\cap B(0,4r_0),\\ \partial\Omega\cap B(0,4r_0) &= \{x=(x',x_n)\in\mathbb{R}^n : x_n=\phi(x')\}\cap B(0,4r_0),\end{aligned} \tag{7.37}$$

$\phi(0)=0$, in an appropriate coordinate system, for a convex Lipschitz function ϕ and for some $r_0>0$. Let $B'(0,4r_0)$ be the orthogonal projection of the ball $B(0,4r_0)$ onto the plane $x_n=0$ in the local coordinate system. Then the graph $\{(x',\phi(x')) : x'\in B'(0,4r_0)\}$ describes $\partial\Omega\cap B(0,4r_0)$. Let $\eta_0=\sup_{x'\in B'(0,4r_0)}\phi(x')$, consider $\eta\in(0,\eta_0)$ and let $O_\eta=\{x'\in B'(0,4r_0) : \phi(x')<\eta\}$. Then O_η is a convex set in $\mathbb{R}^{n-1}$. Let $\phi_\eta=\phi-\eta$ in O_η. Now, by construction ϕ_η is convex in O_η, $\phi_\eta=0$ on ∂O_η, and using Lemma 7.7, we can conclude that

$$\det(\nabla^2\phi_\eta(x'))=d\nu(x'), \quad \text{ in } O_\eta \tag{7.38}$$

in the sense of Alexandrov, where

$$d\nu=\frac{|\nabla U(x',\phi(x'))|^p}{\psi(\xi)(1+|\nabla\phi(x')|^2)^{-(n+1)/2}} \ \text{ with } \ \xi=\frac{(-1,\nabla\phi(x'))}{\sqrt{1+|\nabla\phi(x')|^2}}, \tag{7.39}$$

and where U is the p-equilibrium potential associated to Ω. Using Lemma 7.5 we see there exist $c=c(p,n,M)$, $1\leq c<\infty$, and $\varepsilon=\varepsilon(p,n,M)$, $0<\varepsilon\ll 1$, such that for every set F of the form $F=\{x'\in O_\eta :\ \phi_\eta(x')<a\cdot x'+b\}$, we have

$$\int_F \delta(x',F)^{1-\varepsilon}d\nu(x')\leq c\int_{\frac12 F}d\nu(x'). \tag{7.40}$$

We now want to verify that ϕ_η is strictly convex. To do this suppose that l_{x_0} is a supporting linear function to ϕ_η at $x_0'\in O_\eta$. Using Theorem 7.6 we see that either (7.34) or (7.35) holds. If (7.34) holds then ϕ_η is strictly convex at $x_0'\in O_\eta$ and hence we can assume that (7.35) holds. In this case the extremal points of the set $\{\phi_\eta=l_{x_0'}\}$ are contained in ∂O_η and, as there must be at least two extremal points of the set $\{\phi_\eta=l_{x_0'}\}$ and since $\phi_\eta=0$ on ∂O_η, we can conclude that $l_{x_0'}\equiv 0$. However, since $x_0'\in O_\eta$ and $\{\phi_\eta(x_0)=l_{x_0'}(x_0')\equiv 0\}$ we see that this implies the existence of extremal points of $\{\phi_\eta=l_{x_0'}\}$ in the interior of O_η which is a contradiction. Hence (7.34) must hold and we can conclude that ϕ_η is strictly convex in O_η. Next, using the fact that ϕ_η is strictly convex we can apply Theorem 7.6 and conclude that $\partial\Omega$ is locally (and then globally, by a covering argument) $C^{1,t}$-regular for some $t>0$. Having concluded that Ω is $C^{1,t}$-regular for some $t>0$, it follows from [69] that ∇U is continuous on $\mathbb{R}\setminus\Omega$, i.e. ∇U is continuous up to the boundary. In fact, ∇U is even $C^{0,\tilde t}$-regular, for some $\tilde t>0$, up to the boundary. Using this and the fact that U is bounded we can conclude that $|\nabla U|$ is bounded from above on $\partial\Omega$. Furthermore, essentially repeating the barrier type argument in (7.6)-(7.8) and using the strong maximum principle, we can also conclude that $|\nabla U|$ is positively bounded from below on $\partial\Omega$. The same conclusion also follows by an application of Lemma 2.18. Hence, put together, using this, the assumption on ψ, (7.38), (7.39), we see that there exists $0 < \lambda_1 < \lambda_2 < \infty$ such that

$$0 < \lambda_1 \leq \det(\nabla^2 \phi_\eta(x')) \leq \lambda_2 < \infty, \quad \text{in } O_\eta, \tag{7.41}$$

and that the right hand side in (7.39) is $C^{0,\tilde{t}}$-regular for some $\tilde{t} > 0$. Using this and the results in [15] and [16], we can conclude that ϕ_η, and hence ϕ, is $C^{2,\tilde{t}}$-regular. This implies that ∇U is $C^{1,\hat{t}}$-regular up to the boundary of Ω, for some $\hat{t} > 0$, and hence, in particular, that the right hand side of the Monge-Ampére equation is $C^{0,\alpha}$-regular. Now again using results in [15] and [16] we can conclude that ϕ is in fact $C^{2,\alpha}$-regular. The reminder of the regularity statements of Theorem 1.4 now follow from the seminal and by now well-known work of Caffarelli, see [14], [15] and [16]. We omit additional details. □

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