Variants of theorems of Baer and Hall on finite-by-hypercentral groups

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dedicated to the memory of Guido Zappa

Abstract We show that if a group G has a finite normal subgroup L such that G/L is hypercentral, then the index of the hypercenter of G is bounded by a function of the order of L. This completes recent results generalizing classical theorems by R. Baer and P. Hall. Then we apply our results to groups of automorphisms of a group G acting in a restricted way on an ascending normal series of G.
[1]

1 Introduction

A classical theorem by R. Baer states that, if the m-th term $Z_m(G)$ of the upper central series a group G has finite index t in G for some positive integer m, then there is a finite normal subgroup L of G such that G/L is nilpotent of class at most m, that is $G/L = Z_m(G/L)$ (see 14.5.1 in [7], which shall be the reference for undefined notation). Recently, in [6] it has been shown that there is such an L with finite order d bounded by a function of t and m.

In the opposite direction, P. Hall showed that, if there is a normal subgroup L with finite order d such that G/L is nilpotent of class at most m, then $G/Z_{2m}(G)$ has finite order bounded by a function of d and m (see [7], page 118).

Recently, in [2] it has been shown that the hypercenter of G has finite index t if and only if there is a finite normal subgroup L with order d such that G/L is hypercentral, that is coincides with its hypercenter. Recall that the *hypercenter* of a group G is the last term of the upper central series of G (see details below). Then in [5] it has been shown that d may be bounded by a function of t, namely $t^{(1+\log_2 t)/2}$. Here we complete the picture by showing that t in turn may be bounded by a function of d.

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Theorem 1 *If a group G has a finite normal subgroup L such that G/L is hypercentral, then the hypercenter of G has index bounded by $|\mathrm{Aut}(L)| \cdot |Z(L)|$.*

Corollary 1 *If a group G has a finite normal subgroup L such that G/L is nilpotent of class m, then $|G/Z_{2m}(G)|$ is bounded by a function of $d := |L|$.*

There are many generalization and variants of Baer and Hall theorems. By applying Theorem 1 above, we improve the results in [3] which are concerned with possibly non-inner automorphisms.

Before stating our Theorem 2 we recall some definition. As usual, we say that the group A *acts* on a group G if and only if there is a homomorphism $\tilde{}: A \to \mathrm{Aut}(G)$ (called *action*). We will regard both G and $\tilde{A}$ as subgroups of the holomorph group $G \rtimes \mathrm{Aut}(G)$ of G. In particular, we will denote by a bar $\bar{}$ the action of a group G on itself by conjugation, that is the natural $\mathrm{Aut}(G)$-homomorphism $G \to \bar{G} \leq \mathrm{Aut}(G)$. If an action is such that its image $\tilde{A}$ is normalized by $\bar{G} = \mathrm{Inn}(G)$, we define by recursion an ascending G-series $Z_\alpha(G, A)$ (with α ordinal number) by $Z_0(G, A) := 1$, $Z_{\alpha+1}(G, A)/Z_\alpha(G, A) := C_{G/Z_\alpha(G,A)}(A)$ and $Z_\lambda(G, A) := \cup_{\alpha<\lambda} Z_\alpha(G, A)$ when λ is a limit ordinal. We call $Z_\alpha(G, A)$ the αth A-*center* of G. Recall that an ascending G-series is a well ordered (by inclusion) set of normal subgroups of G. Clearly the series $Z_\alpha(G, A)$ is stabilized by A, in the sense that A acts trivially on the factors beetwen consecutive terms. The last term $Z_\infty(G, A)$ of this series is called A-*hypercenter* of G.

We say that G *is A-hypercentral with (ordinal) type at most* α if and only if $G = Z_\alpha(G, A)$. Clearly $Z_\alpha(G) := Z_\alpha(G, \bar{G})$ is the usual αth center of G and if $G = Z_\alpha(G)$, then G is hypercentral of type at most α.

Now we are in a position to state our second result, which consists in two parts that refer to theorems of Baer and Hall, respectively. In fact, if $A = \mathrm{Inn}(G)$, then part (B) reduces to Theorem B in [5] and part (H) to Theorem 1 above.

Theorem 2 *Let G be a group and A be a subgroup of $\mathrm{Aut}(G)$ such that $A^{\mathrm{Inn}(G)} = A$ and the hypercenter of $A/(A \cap \mathrm{Inn}(G))$ has finite index k.*

(B) If the A-hypercenter of G has finite index t, then there is a finite normal A-subgroup L with order bounded by a function of (t, k) such that G/L is A-hypercentral.

(H) If there is a finite normal A-subgroup L with order d such that G/L is A-hypercentral, then the A-hypercenter of G has finite index bounded by a funcion of (d, k).

Remark that this theorem generalizes Theorems 4 and 3 of [3] where the same picture is considered, but with more restrictive conditions, that is A contains $\mathrm{Inn}(G)$, the factor $A/\mathrm{Inn}(G)$ is finite and the involved series which are stabilized by A are finite. Clearly, our bounding functions do not depend on the length of the considered series.

Finally note that the hypothesis that A is normalized by $\mathrm{Inn}(G)$ is necessary, as shown by Example in Sect. 2 below.

2 Proof of Theorem 1

To prove Theorem 1 we use a key lemma. Recall that we denote the hypercenter of a group G by $Z_\infty(G)$.

Lemma 1 *Let $A \le H$ be normal subgroups of a group G with A finite and $A \le Z(H)$. If $G/C_G(H)$ is locally nilpotent and $H/A \le Z_\infty(G/A)$, then $H \le Z_\infty(G)A$.*

Proof. Arguing by induction on the order of A, we may assume that A is minimal normal in G. Then A is an elementary abelian p-group for some prime p. If $A \cap Z(G) \neq 1$, then $A \le Z(G)$ by minimality of A and so we have $H \le Z_\infty(G)A$.

Suppose then $A \cap Z(G) = 1$ (and so $A \cap Z_\infty(G) = 1$) and let $N := Z_\infty(G) \cap H$. Note that the hypotheses hold for the subgroups $\bar{A} := AN/N$, $\bar{H} := H/N$ of the group $\bar{G} := G/N$. Since from $\bar{H} \le Z_\infty(\bar{G})\bar{A}$ it follows $H \le Z_\infty(G)A$, we may assume $Z_\infty(G) \cap H = 1$.

We claim that $H = A$ (note that $H \le Z_\infty(G)A$ if and only if $H = H \cap Z_\infty(G)A = (H \cap Z_\infty(G))A = A$). Suppose, by contradiction, $H > A$ and let $X/A \neq 1$ be either infinite cyclic or of prime order r and contained in $(H/A) \cap Z(G/A)$. Since by hypotheses $A \le Z(H)$, then X is abelian and $X \lhd G$, clearly.

Let us show now that X is a p-group. If, by contradiction, X/A is infinite or $r \neq p$, then $X^p \neq 1$ and $X^p \cap A = 1$. Thus X^p is G-isomorphic to $X^pA/A \le Z_\infty(G/A)$. Hence $X^p \le H \cap Z_\infty(G) = 1$, a contradiction. So X/A has order p.

Assume, again by contradiction, $X^p \neq 1$. By minimality of A, we have $X^p = A = [G, X]$ and so $[G, A] = [G, X^p] = [G, X]^p = A^p = 1$, a contradiction.

Then X is a finite elementary abelian p-group. Since $[G, A] = A \le X$, the subgroup $X \rtimes (G/C_G(X))$ of the holomorph of X is not nilpotent, and so

$G/C_G(X)$ is not a p-group. Hence there are a prime $q \neq p$ and a normal non-trivial q-subgroup $Q/C_G(X)$ of $G/C_G(X)$. Since $Q \not\leq C_G(X)$, then $[X,Q] \neq 1$. Thus $[X,Q] = A$, as $[X,Q] \leq A$ and by minimality of A.

By a standard argument on coprime actions (see for example Exercise 4.1 in [1]), we have

$$X = [X,Q] \times C_X(Q) = A \times C_X(Q),$$

therefore $C_X(Q) \neq 1$. On the other hand, $C_X(Q)$ is a normal subgroup of G and so $C_X(Q) \leq Z_\infty(G) \cap H = 1$, a contradition which gives the claim $H = A$. □

Proof of Theorem 1. Let us apply Lemma 1 with $A := Z(L)$ and $H := C_G(L)$. In fact on one hand $H/A = H/(H \cap L) \simeq_G LH/L$, then $H/A \leq Z_\infty(G/A)$. On the other hand $L \leq C_G(H)$ and so $G/C_G(H)$ is hypercentral, since it is an image of G/L. Therefore $H \leq Z_\infty(G)A$. Hence

$$|H/(H \cap Z_\infty(G))| = |A(Z_\infty(G) \cap H)/(Z_\infty(G) \cap H)| \leq |A| = |Z(L)|.$$

Since $H = C_G(L)$, then $|G/H| \leq |\mathrm{Aut}(L)|$. Thus

$$|G/Z_\infty(G)| \leq |G/H| \cdot |H/(H \cap Z_\infty(G))| \leq |\mathrm{Aut}(L)| \cdot |Z(L)|.$$ □

Proof of Corollary 1. Note that $Z_{d+m}(G) = Z_\infty(G)$ has finite index. Thus if $d \leq m$, the statement follows directly from Theorem 1. Otherwise, $|G/Z_{2m}(G)|$ is bounded by the maximum of the $h(d,i)$ with $i = 1, \ldots, d$, where $h(d,m)$ is the bounding function in Hall Theorem. □

From Theorem 1 and the above quoted result from [5] we deduce a corollary which gives a rather complete picture of finite-by-hypercentral groups.

Corollary 2 *If G is a group with a (finite) normal series*

$$G = G_0 \geq F_1 \geq G_1 \geq \ldots \geq F_n \geq G_n = 1$$

where
- each factor F_i/G_i is finite with order $t_i > 1$,
- each factor G_{i-1}/F_i is contained in the hypercenter of G/F_i,
then there is a normal subgroup L with finite order bounded by a function of $t = t_1 \cdot \ldots \cdot t_n$ such that G/L is hypercentral.

Moreover the hypercenter of G has finite index bounded by a function of t.

Proof. Define recursively a function $f : \mathbb{N} \to \mathbb{N}$ by means of $f(1) = 1$ and $f(t+1) = (t+1)g(g(f(t)))$ for each $t \in \mathbb{N}$, where $g(t) := t^{1+\log_2 t}$.

We show that there is $L \lhd G$ such that $|L| \leq f(t)$ and $G/L = Z_\alpha(G/L)$ for $\alpha := \alpha_n + \ldots + \alpha_1 + m'$, where the α_i's are ordinal numbers such that $G_{i-1}/F_i \leq Z_{\alpha_i}(G/F_i)$ for each i and $m' \in \mathbb{N}$ may be bounded by a function of t and of the α_i's which are finite. Since $f(t) \geq t$ for each t, the statement is trivial if $n = 1$.

Assume then by induction on n that there is a normal series

$$G \geq F_{n-1} \geq G_{n-1} \geq F_n \geq G_n = 1$$

such that G/F_{n-1} is hypercentral of type $\alpha' = \alpha_{n-1} + \ldots + \alpha_1 + m''$, with $m'' \in \mathbb{N}$ and $|F_{n-1}/G_{n-1}| \leq f(t_*)$ with $t_* = t_1 \cdot \ldots \cdot t_{n-1}$. Applying Theorem 1 to G/G_{n-1}, if $Z/G_{n-1} := Z_{(\lceil \log_2 f(t_*)\rceil + \alpha')}(G/G_{n-1})$, then $|G/Z| \leq g(f(t_*))$. Thus, applying Theorem B of [KOS] to G/F_n, we have that there is a normal subgroup L such that G/L is hypercentral with ordinal type at most $\alpha_n + \lceil \log_2 f(t_*)\rceil + \alpha' + \lceil \log_2 g(f(t_*))\rceil$ and $|L/F_n| \leq g(g(f(t_*)))$. We have: $|L| \leq t_1 g(g(f(t_*))) \leq tg(g(f(t-1))) = f(t)$, as wished. □

Remark: In the above proof, if α is infinite, then clearly $G/Z_\alpha(G)$ is finite. Otherwise, if $G_{i-1}/F_i \leq Z_{m_i}(G/F_i)$ for each i with $m_i \in \mathbb{N}$, then there is a finite normal subgroup L such that $G/L = Z_m(G/L)$ with $m := m_1 + m_2 + \ldots + m_n$, by Theorem B in [4]. Hence, in this case, $G/Z_{2m}(G)$ is finite.

3 Proof of Theorem 2

Proof of Theorem 2. Let α' such that $B/(A \cap \mathrm{Inn}(G)) := Z_{\alpha'}(A/(A \cap \mathrm{Inn}(G))$ has finite index in $A/(A \cap \mathrm{Inn}(G))$. Consider the subgroup $S := G \rtimes A$ of the holomorph group of G.

Assume first $A \geq \mathrm{Inn}(G)$. Let $G_\delta := Z_\delta(G, A)$ for any ordinal δ. We claim:

$$(*) \qquad \forall \delta \quad S_\delta := G_\delta \bar{G}_\delta \leq Z_\delta(S).$$

By induction, suppose true for δ. Note that $\bar{G} \leq A$ acts by conjugation on G the same way as G. We have $[S_{\delta+1}, S] = [G_{\delta+1}\bar{G}_{\delta+1}, GA]$. On one hand, we have $[G_{\delta+1}, GA] \leq [G_{\delta+1}, A] \cdot [G_{\delta+1}, G]^A \leq G_\delta$. On the other hand, $[\bar{G}_{\delta+1}, GA] \leq [\bar{G}_{\delta+1}, A] \cdot [\bar{G}_{\delta+1}, G]^A \leq \bar{G}_\delta G_\delta = S_\delta$. It follows $S_{\delta+1} \leq Z_{\delta+1}(S)$ and the claim is proved since the limit ordinal step is trivial.

To prove (B) in the case $A \geq \mathrm{Inn}(G)$, let α be such that $Z_\alpha(G, A)$ has finite index in G and note that in the normal series

$$S = GA \geq GB \geq G\bar{G} \geq G_\alpha \bar{G}_\alpha \geq 1$$

the factors GA/GB and $G\bar{G}/G_\alpha\bar{G}_\alpha$ are finite with order k and t^2, respectively. Moreover, by $(*)$, factors $GB/G\bar{G}$ and $G_\alpha\bar{G}_\alpha$ are contained in the α'th and αth center of $S/G\bar{G}$ and S, respectively. Thus we apply Corollary 2 to the group $S = GA$. Then the statement (for the group G) follows easily.

Concerning part (H) in the case $A \geq \mathrm{Inn}(G)$, consider the normal series

$$S = GA \geq GB \geq G\bar{G} \geq L\bar{L} \geq 1.$$

Note that GA/GB and $L\bar{L}$ are finite with order k and d^2, respectively. Moreover, if α_1 is such that $Z_{\alpha_1}(G/L, A)$ has finite index in G/L, then by $(*)$ we have that $GB/L\bar{L}$ is contained in the $(\alpha_1 + \alpha')$th A-center of $S/L\bar{L}$. We may apply Corollary 2 and get the statement.

To deal with the more general case, let $\bar{N} := A \cap \mathrm{Inn}(G)$ such that $Z(G) \leq N \leq G$. Note that $[G, A] \leq N$, as $\overline{[g, \gamma]} = [\bar{g}, \gamma] \in A \cap \mathrm{Inn}(G)$ $\forall \gamma \in A$ since $A^{\mathrm{Inn}(G)} = A$. Thus A acts trivially on G/N. Moreover the group $\tilde{A} := A/C_A(N)$ may be considered as a group of automorphisms on N containing $\mathrm{Inn}(N)$. Thus, to prove (H), one may apply the above case to N and $\tilde{A} := A/C_A(N)$.

To prove (B) in the general case note that, by the above, the subgroup $Z := Z_\infty(N, A)$ has finite index in N, bounded by a function of $|L \cap N| \leq |L|$. Let K/Z be the A-hypercenter of G/Z. Clearly, $K \cap N = Z$. Moreover $K/Z = Z(G/Z, A)$. Consider then $C/Z := C_{G/Z}([G, A]Z/Z)$ and note that C has finite index in G, since $[G, A] \leq N$. By applying the Three Subgroup Lemma to A, C/Z, C/Z, we have that A acts trivially on the derived subgroup of C/Z. Thus $C'Z/Z \leq C_{G/Z}(A) \leq K/Z$. Therefore CK/K is abelian. We consider the series

$$G \geq CK \geq K \geq Z \geq 1.$$

The index of CK in G is finite and bounded by a function of $d = |L|$, as $|N/Z|$ is. Then consider the action of A on the abelian group $\hat{G} := CK/K$. Since $K \cap N = Z$, we have that $|NK/K|$ is bounded by a function of d. Thus the image of $A \cap \mathrm{Inn}(G)$ in $\hat{A} := A/C_A(\hat{G})$ is finite with order bounded by a function of d. By Corollary 2, $Z_{\alpha'}(\hat{A})$ has finite index q in $\hat{A}$, bounded by a function of d and k. Recall that $\hat{G}$ is abelian and $[\hat{G}, \hat{A}]$ is finite, as $[G, A]$ is finite modulo K. Let $\hat{S} := \hat{G} \rtimes \hat{A}$. Then $Z_{1+\alpha'}(\hat{S}/[\hat{G}, \hat{A}])$ has finite index at most q. By Theorem 1, the index of $Z_{1+\alpha'}(\hat{S})$ in $\hat{S}$ is finite and bounded by a function of d and q. Thus the A-hypercenter of $\hat{G} := CK/K$ has finite index and bounded by a function of d and k, as wished. □

Remark: in the case $A \geq \mathrm{Inn}(G)$ of the above proof, if α, α_1 and α' are finite, we have that:
- in case (B), the $2(\alpha + \alpha')$th A-center has finite index in G, by the above quoted result in [4]. In particular, for $\alpha' = 0$ we have Theorem 3 of [3].
- in case (H), there is a boundedly finite normal A-subgroup L such that G/L coincides with its $(\alpha_1 + \alpha')$th A-center. This follows by applying the remarks after Corollary 2 to the group S. In particular, for $\alpha' = 0$ we have Theorem 2 and 4 of [3].

Let us see that the condition that A is normalized by $\mathrm{Inn}(G)$ is necessary.

Example *There is an elementary abelian group G and a bounded abelian group $A \leq \mathrm{Aut}(G)$ such that $G/Z_\omega(G, A)$ is finite (of prime order), while G/L is not A-hypercentral, for any finite A-subgroup $L \leq G$.*

Proof. Let $G := Dr_{i<\omega}\langle a_i\rangle$ be an elementary abelian p-group, where p is an odd prime and let $Z := Dr_{0<i<\omega}\langle a_i\rangle$. For any $i > 0$, consider $\gamma_i \in \mathrm{Aut}(G)$ centralizing Z, and such that $a_0^{\gamma_i} := a_0 a_i$. Let $\tau \in \mathrm{Aut}(G)$ centralizing Z and such that $a_0^\tau := a_0^2$. Let A be the subgroup of $\mathrm{Aut}(G)$ generated by τ and all the γ_i's. Then $Z = Z_1(G, A)$ has index p in G, while if K is a proper A-subgroup of G, then $a_0 \notin K$, as $a_0^A = G$. Clearly τ does not centralizes a_0 mod K. Thus G/K is not A-hypercentral, for any proper A-subgroup K of G and in particular for any finite A-subgroup $L \leq G$. □

We finish by noticing that Theorem 2 may be formulated in a different way. Recall that the factor of two consecutive terms of a series is called just factor.

Corollary 3 *Let A be a finite-by-hypercentral group of automorphisms of a group G such that $A^{\mathrm{Inn}(G)} = A$.*

If there is an ascending normal series in G with a finite number of finite factors and such that A acts trivially on all other factors, then:
i) there is a finite index normal A-subgroup G_0 of G such that A stabilizes an ascending G-series of G_0;
ii) there is is a finite normal A-subgroup L such that A stabilizes an ascending G-series of G/L.

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