Tomographic Reconstruction of 2-convex Polyominoes using Dual Horn Clauses

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Abstract

Among the very many research interests of Maurice Nivat, a special role, according to the produced literature, was played by the study of the algorithmic and combinatorial aspects of connected finite discrete sets of points called polyominoes. In particular, he addressed the problem of a faithful reconstruction of some subclasses of them by imposing convexity constraints. The present study fits in this research line, and relies on a well known algorithm that Maurice Nivat and co-authors defined 1996 for the reconstruction of *hv*-convex polyominoes by orthogonal projections in polynomial time. Here, we consider a recently defined hierarchy on this class of polyominoes and we continue a longstanding research on the reconstruction of its first levels by specializing the above mentioned algorithm on them. The achieved result bases on the possibility of expressing the notion of 2-convexity that characterizes the elements of the second level of the hierarchy, by a logic formula belonging to *Dual-Horn* and so polynomially solvable. Some related open problems are also presented.

1. Introduction

Maurice Nivat has often faced with algorithmic problems, arising from different areas of mathematics, computer science, and even biology. Many of these problems concerned the study of classes of finite discrete sets of points that are combinatorial objects commonly used as general-purpose tool to model phenomena and situations arising in these different contexts.

In particular, he investigated *polyominoes* that are connected finite sets of points without holes, usually represented as a finite union of cells on the square lattice $\mathbb{Z}^2$. In the half XX century since Solomon Golomb used the term in his seminal article [21], the study of polyominoes has proved a fertile topic of research.

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The attention of Maurice Nivat was attracted by numerous problems related to them, such as the problem of covering a polyomino by rectangles [13], tiling regions by polyominoes [3, 15], reconstructing or counting them and subclasses of interest or, in general, inspecting their geometrical properties. But, at the same time, there remain many challenging, open problems, starting with the general enumeration one: the number a_n of polyominoes with n cells has been determined only for small n (up to $n = 94$ in [24]); when n increases, only the asymptotic growth is known:

$$lim_n \{a_n\}^{1/n} = \mu, \quad 3.72 < \mu < 4.64.$$

Consequently, in order to probe further, several subclasses of polyominoes have been introduced on which to hone mathematical and algorithmic techniques.

One very natural subclass is that of the polyominoes that are horizontally and vertically convex, say *hv-convex*, with which we are concerned in this paper. The notion of convexity is commonly related to convexity along a set of discrete directions in the intent of transposing into the discrete framework the standard Euclidean convexity. In particular, concerning polyominoes, the horizontal and vertical directions are commonly preferred by some specific characteristics that relate both with the notion of four connectedness of the cells of a polyomino and with the natural correspondence with the directions of the orthogonal basis of the discrete plane.

It was Maurice Nivat's habit to keep his various research interests current by moving between different topics, often represented by different teams of co-authors. But certain themes related to polyominoes do emerge in retrospect over the course of his researches, beginning with discrete tomography, discrete geometry and tilings. One attractive result appears in [2], from 1996: reconstructability by means of orthogonal projections is shown to differentiate between horizontally [resp. vertically] convex polyominoes and hv-convex ones. Whereas the former reconstruction problem is NP-hard, the latter is solvable in polynomial time, and the algorithm achieving this can be put to other purposes in a matching problem.

As a representative example of Maurice Nivat's interests in (convex) polyominoes from a discrete geometry perspective, we also consider [17] and later [8], which, in fact, relates back to [2]. Median points in the integer square lattice are defined as those points whose sum of the distances from a finite set S of the lattice is minimal. The object of the research in [8] is to extend these results to another type of distance also defined in terms of two directions.

The result in [2] inspired the present research that aims at extending the reconstruction technique for hv-convex polyominoes to the elements of a hierarchy recently defined on them in [11]: it was observed that every pair of cells of an hv-convex polyomino can be connected by a monotone internal path (these notions will be properly defined in Section 2). The minimal number k of changes of direction among all the paths connecting every pairs of cells is the convexity degree of the polyomino. The class of k-convex polyominoes is made of all hv-

convex polyominoes whose convexity degree is less than or equal to k. Clearly, k-convex polyominoes are also $k+1$-convex ones, so the obtained classes form a hierarchy on hv-convex polyominoes.

As a matter of fact, the idea of this hierarchy was, in nuce, in the research of A. Del Lungo and M. Nivat in [16] about the already mentioned median points of an hv-convex polyomino and the construction of its spine (the whole study can also be found in the comprehensive book [23] on discrete tomography edited by G. Hermann and A. Kuba). The median points lie inside the core of an hv-convex polyomino and can be computed by looking at the partial sums of the vectors of projection; the authors shown how to quickly detect them and use to compute some paths that connect the extremal points of the polyomino. These paths are constructed by considering the minimal numbers of changes of direction and lead naturally to the notion of k-convexity. We underline that in the case of 2-convex polyominoes, the spine is easier to reconstruct than for the general case as shown in [26].

In the last decades, 1-convex polyominoes have been analyzed from different points of view: the combinatorial aspects are investigated in [9, 7] providing the enumeration according to the semi-perimeter and the area. An initial tomographical study of the class is in [11, 12], that led to the successive definition of a polynomial reconstruction strategy from orthogonal projections and the solution of the related uniqueness problem in [10].

Substantially more difficult is the study of the higher classes of the hierarchy due to the lack of a tractable geometrical characterization of their elements, starting with the 2-convex ones. Duchi et al. enumerated them in [20] using a purely analytical approach, but their technique gives no clues for the tomographical reconstruction.

In this paper, we adapt the algorithm in [2] to obtain a different and still efficient one for reconstructing 2-convex polyominoes from orthogonal projections. The key point of the new algorithm consists in expressing by means of a subset of solvable *SAT* clauses a property that explicitly captures the geometrical structure of a 2-convex polyomino.

Section 2 introduces the reconstruction problem we are going to study: some basic definitions about polyominoes are provided and it is presented the above mentioned hierarchy on hv-convex ones. A new characterization of k-convex polyominoes relying on the notion of forbidden area is furnished: it plays a key role to reach a fast reconstruction strategy for 2-convex polyominoes.

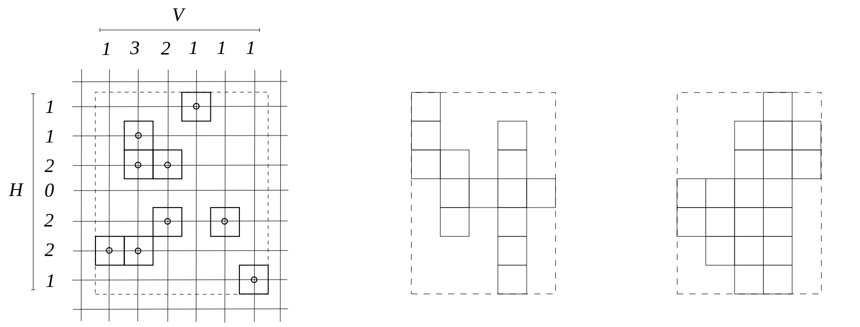
In Section 3, we finally recall the reconstruction algorithm in [2] and we show how to modify it in order to detect 2-convex solutions, if any. In particular, the strategy is integrated by a *Dual-Horn* formula that imposes the desired convexity; the satisfiability of HornSat in polynomial time assures the polynomiality of the whole reconstruction.

In Section 4 we provide some final comments and we propose new research lines that our study may open.

2. Definitions and preliminary results

A planar discrete set S of points is a finite subset of the lattice $\mathbb{Z}^2$, and it can be represented in a natural way by a finite union of cells of the square lattice so that each element of the discrete set is associated with the presence of a cell in the correspondent position, as shown in Fig. 1, (a). The *dimensions* of the set are those of its minimal bounding rectangle, whose rows and columns are enumerated from up to down and from left to right, by convention.

To each $m \times n$ discrete set S we associate two integer vectors $H = (h_1, ..., h_m)$ and $V = (v_1, ..., v_n)$ such that for each $1 \le i \le m$, $1 \le j \le n$, h_i and v_j are the number of cells of S which lie on row i and column j, respectively, as shown in Fig. 1, (a). The vectors H and V are called the *horizontal* and *vertical projections* of S, respectively.



(a) (b) (c)

Figure 1: (a) a finite set of $\mathbb{N} \times \mathbb{N}$, and its representation in terms of a set of cells. The vectors of projections H and V are also shown; (b) a vertically convex polyomino; (c) an hv-convex polyomino.

A planar discrete set whose cells are connected is called a *polyomino*; in addition, we embrace the commonly assumed requirement that a polyomino does not contain holes, i.e., its interior is simply connected. A polyomino P is *horizontally-convex* [resp. *vertically-convex*] if its cells lying on each column [resp. row] are connected, while it is *hv-convex* if it is both horizontally and vertically convex, see Fig. 1, (b) and (c).

Each hv-convex polyomino touches the borders of the minimal bounding rectangle in four bars called *N(orth)*, *S(outh)*, *E(ast)* and *W(est)* foot according to their positions.

The extremal points of two consecutive feet of P delimit four disjoint (possibly void) regions including the corners of the minimal bounding rectangle, and lying outside P; we indicate them as NW, NE, SW and SE corner regions, according to the delimiting feet as shown in Fig. 2, (a).

The mutual positions of the feet of an hv-convex polyomino determine its *orientation*, i.e., a well known geometrical characteristic that grabs the concept

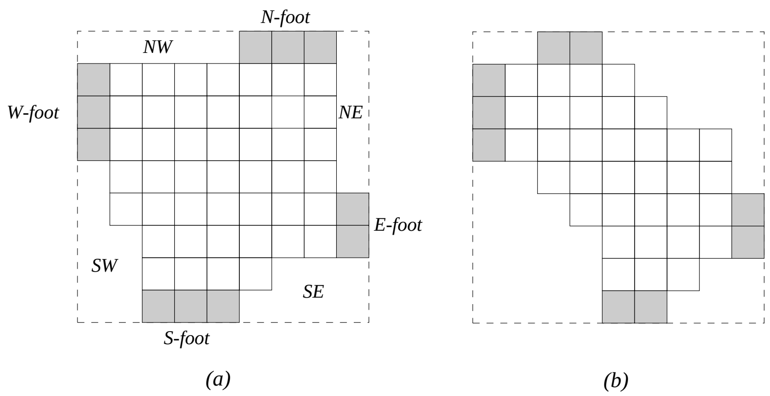


Figure 2: (a) a non oriented (hv-convex) polyomino whose feet and corner regions are highlighted; (b) a $NW \to SE$ oriented polyomino

of preferred direction of growth of a polyomino: if (all the cells of) the N-foot and the W-foot lie on the left and above the opposite S-foot and E-foot, respectively, then the polyomino is $NW \to SE$ oriented, as depicted in Fig. 2, (b). On the other hand, if the N-foot and the W-foot lie on the right and below the opposite S-foot and E-foot, respectively, then the polyomino is $SW \to NE$ oriented; the polyomino is non oriented otherwise, as shown in Fig. 2, (a).

The few notions above introduced will be involved in the problem we are going to study:

Reconstruction $(H, V, \mathcal{C})$

Input: two integer vectors H and V, and a class of discrete sets $\mathcal{C}$.

Task: reconstruct an element of $\mathcal{C}$ whose horizontal and vertical projections are H and V, respectively, if it exists, otherwise give FAILURE.

Maurice Nivat made a fine contribution to the whole field of Discrete Tomography and in particular he addressed this problem with $\mathcal{C}$ being the class of hv-convex polyominoes. The reconstruction algorithm defined in [2], here recalled in Section 3, was the result of a longstanding research that required a deep analysis of the convexity constraint on polyominoes littered with previous and successive combinatorial and geometrical side results.

Although non optimal in terms of the computational complexity, the proposed reconstruction technique deeply inspired researchers in the forthcoming years for its striking combining of simplicity and generalization perspectives: we make use of it to deal with the reconstruction problem on the classes of a recently defined hierarchy for hv-convex polyominoes.

A hierarchy on hv-convex polyominoes

For any two cells A and B in a polyomino P, a *path* from A to B, is a sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)$ of adjacent disjoint cells of P, with $A = (i_1, j_1)$, and

$B = (i_r, j_r)$. For each $1 \leq k \leq r-1$, we say that the two consecutive cells $(i_k, j_k), (i_{k+1}, j_{k+1})$ form

- an *east* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k + 1$;
- a *north* step if $i_{k+1} = i_k - 1$ and $j_{k+1} = j_k$;
- a *west* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k - 1$;
- a *south* step if $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k$.

We define a path to be *monotone* if it uses only two of the four types of steps above defined.

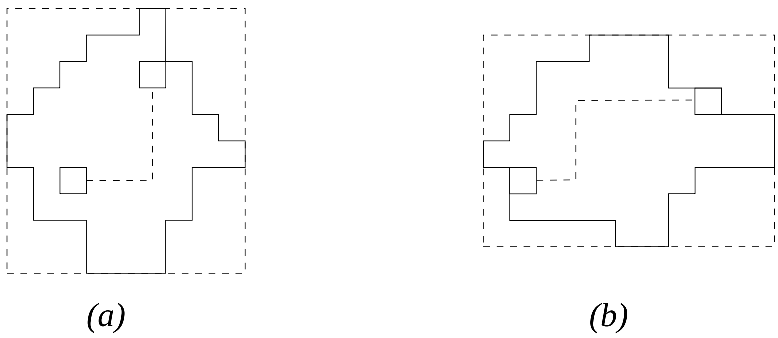


Figure 3: (a) a 1-convex polyomino: each couple of cells admits a (monotone) path joining them with at most one change of direction. (b) a 2-convex polyomino: the two highlighted cells does not admit a path with less than two changes of directions.

Proposition 1 (Castiglione, Restivo [11]). *A polyomino P is hv-convex if and only if for every pair of cells there exists a monotone path connecting them and contained in P.*

For $k \in \mathbb{N}$, we call k-path a monotone path with *at most* k changes of direction. An *hv*-convex polyomino P is said to be a k-convex polyomino if every pair of cells in P can be connected by an internal k-path. By definition, it follows that the classes of k-convex polyominoes form a hierarchy on *hv*-convex ones.

If $k = 1$, we have the well known class of 1-convex polyominoes, sometimes addressed as L-convex polyominoes for the shape of a path having at most one change of direction: tomographical and combinatorial aspects of 1-convex polyominoes have been investigated in these last years. In particular, they were characterized both by horizontal and vertical projections [10], and by maximal L shapes [11, 12], whose knowledge, in both cases, led to a fast reconstruction algorithm. These remarkable properties make 1-convex polyominoes to be one of the very few classes of discrete sets where the possibility of a faithful reconstruction from projections realizes; they also related to their enumeration according to the perimeter [9] and to the area [7].

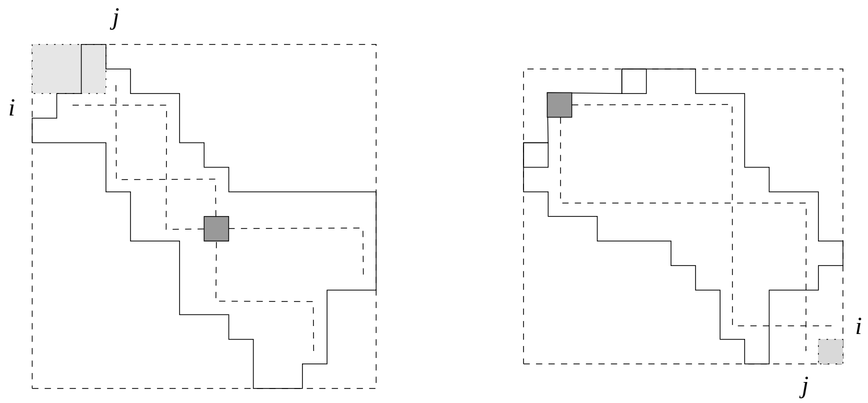
The class of 2-convex polyominoes we are interested in, exhibits geometrical and combinatorial properties substantially harder and more interesting than those of 1-convex polyominoes.

A useful characterization of k-convexity

The k-convex property can be rephrased in a more efficiently testable form using the notion of *forbidden areas* of a polyomino: let us define *maximal Nk-path* a (monotone) k-path starting with a North step and having the maximum number of consecutive steps of the same type, i.e., each change of direction occurs on a border cell of the polyomino, till reaching a last one where no further steps along the two directions are possible (see Fig. 4).

Analogously, we can define the Sk, Ek and Wk (maximal) paths. In case of ambiguity, the second direction of the path will be explicitly stated.

For each cell A of an hv-convex polyomino, we can define its NWk forbidden area as the rectangular area of the maximal bounding rectangle whose lower rightmost corner is in position $(i-1, j-1)$, with (i, j) being the k-th intersection, if it exists, of two maximal Nk and Wk paths (or their prolongations) starting from A and using only N and W steps. In case the two paths ends with parallel steps, then the forbidden area is considered void. In a similar way, we define the NE, SW, and SE forbidden areas of A. Fig. 4 tries to clarify these definitions.



(a) (b)

Figure 4: (a) the $NW2$ and $SE2$ forbidden areas of the highlighted cell of the polyomino. The first one is in light grey in the NW region, while the latter is void, being the final steps of the maximal $S2$ and $E2$ path parallel. (b) the $SE2$ forbidden area of the NW border cell. The area lies outside the polyomino, meaning that the cell can be connected by a 2-path to all the cells of the polyomino, as suggested by Proposition 3.

The following proposition gives a characterization of k-convex polyominoes in terms of forbidden areas:

Proposition 2. *A polyomino P is k-convex if and only if the four forbidden areas of every cell obtained with maximal k-paths do not intersect P.*

The proof is a direct consequence of the definition of k-convexity.

We point out that for each cell of a non oriented hv-convex polyomino, at least one among the two maximal paths determining each forbidden area has at most one change of direction (see Fig. 4, (a)), so it follows:

Property 1. *A non oriented hv-polyomino is* 2*-convex.*

The algorithm defined in [2] can be used to detect and reconstruct non oriented hv-convex polyominoes compatible with a given couple of orthogonal projections, if any, since the solutions are computed in parallel for each possible feet placement. Relying on that, in the remaining part of the paper we focus on oriented polyominoes only, and in particular, w.l.g. on $NW \to SE$ oriented ones.

In [4], the authors noticed (Properties 2 and 3) that the degree of convexity of an hv-convex polyomino can be detected by considering only the maximal paths joining the extremal cells that lie on opposite corner regions, where an *extremal cell* is a cell of the polyomino having two only adjacent cells of the polyomino. They named those cells as NW, NE, SW, and SE corner cells, according to the belonging corner region (see Fig. 4, (b)). Furthermore, if the polyomino is $NW \to SE$ oriented, then only the paths of the NW and SE extremal cells need to be checked. This property was previously observed for 1-convex polyominoes in [5].

All the previous observations lead to rephrase Proposition 2 in the following useful form:

Proposition 3. *A $NW \to SE$ oriented polyomino P is k-convex if and only if the SEk-forbidden areas of every NW corner cell (obtained with maximal k-paths) do not intersect P.*

3. Guiding a reconstruction strategy toward 2-convex polyominoes

It is well known that the reconstruction of a generic discrete set and, in particular, an hv-convex polyomino, from projections is an ill-posed problem, i.e., there are many solutions that may totally differ one from the other. Maurice Nivat shown a special interest in the configurations of cells, say *switchings*, that underpin this ambiguity and studied the algebra that characterizes their interaction with the intent of enumerate and manage classes of solutions [18].

In [2], it was observed that hv-convex polyominoes allow only specific switchings, called *cycles*, that have a preferred orientation according to that of the polyomino itself: as detailed in the sequel, the authors provided an efficient strategy to detect and manage them by means of a (polynomially solvable)

2-SAT logic formula whose valuations correspond to the solutions of the reconstruction problem.

So, the mere presence of cycles in *hv*-convex polyominoes, not assured in case of horizontal [resp. vertical] convexity only, lower the computational complexity of the reconstruction process from the non polynomiality of the latter class to $O(n^4m^4)$ of *hv*-convex polyominoes, where m and n are the dimensions of the unknown polyomino. Let us briefly recall the mentioned reconstruction algorithm:

RecConv (H, V)

Input: a couple of integer vectors H and V.

Procedure: For each placement of the feet of the polyomino

Step 1: detect the cells that are in common with all the *hv*-convex polyominoes having H and V as horizontal and vertical projections, and the cells that are external to them, say the *kernel* and the *shell*, respectively by means of two operations. If the kernel and the shell intersect, then gives failure as output.

Step 2: define a *2-SAT* formula φ involving the cells that do not lie either in the kernel or in the shell, if any, and whose valuations represent all the possible *hv*-convex solutions.

Output: an *hv*-convex polyomino obtained from one of the valuations of φ, if it exists, otherwise gives failure as output.

Detailing Step 1, the kernel expansion is performed by alternating a *connecting operation* that add cells to the polyomino to guarantee its horizontal and vertical convexity (see Fig.5, (a)), and the *coherence operation* that add cells according to the horizontal and vertical projections (see Fig. 5, (b)). The shell reduction acts similarly on the cells that lie outside each solution with the same couple of operations. Kernel expansion and shell reduction continue until a fixed configuration is reached, then three cases arise:

i) the kernel and the shell intersect, so no solution of the reconstruction problem exists according to the chosen feet placement;

ii) the kernel and the shell are a partition of the minimal bounding rectangle, so the kernel itself is a solution of the reconstruction problem and it is the unique one (according to the feet placement);

iii) there are some cells of the minimal bounding rectangle not yet included in the kernel or in the shell (see Fig.5, (c)). Step 2 is performed to assign them to one of the two sets, maintaining the *hv*-convexity and the orthogonal projections, if possible.

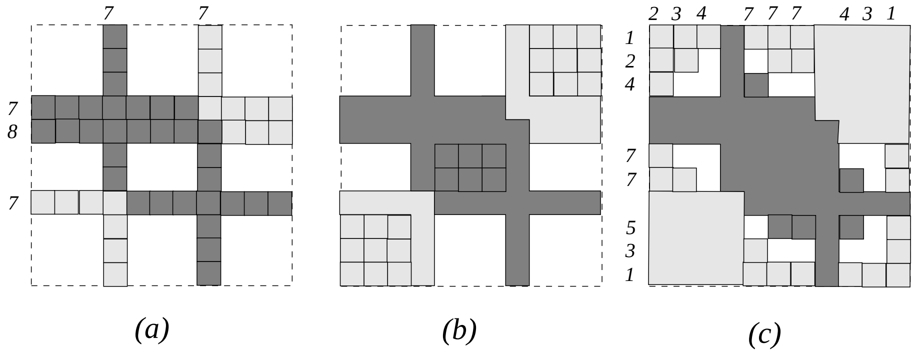


Figure 5: The run of Step 1 on the the instance $H = (1, 2, 4, 7, 8, 7, 7, 7, 5, 3, 1)$ and $V = (2, 3, 4, 7, 7, 7, 7, 7, 4, 3, 1)$ for the feet placement in (a). In (b) the connecting operations guarantee the hv-convexity both of the kernel (dark grey cells), and the shell (light grey cells). In (c) the coherence operations check the coherence with H and V. Both the connecting and the coherence operations are iterated until no more cells can be added. The projections involved by the operations and the added cells are highlighted.

Step 2 starts by labeling each unassigned cell of the NW corner with a binary variable in the set $x_1, x_2, \ldots, x_n$. As a matter of fact, this same set of variables, in positive or negative form, can be used to label all the unassigned cells of the remaining three corner regions, by using the constraint on the horizontal and vertical projections. So, the labeling process creates a mutual dependance on those cells, called *cycles*. Finally, the hv-convexity is imposed on the variables by means of a *2-SAT* formula φ.

The following example and the related Fig. 6 try to clarify the role of Step 2 in the reconstruction process:

Example 1. *Let us consider again the instance of Reconstruction* $(H, V, \mathcal{C})$ *as in Fig. 5.*

After Step 1*, three cells are left unassigned in the NW corner. Each of them is labeled with a variable and the related cycle is detected: Fig. 6,* (a)*, shows how the boolean value of* x_1 *that determines the presence or the absence of the cell in the polyomino, propagates in the corner regions according to the projections' values.*

In Fig. 6, (b)*, the three variables have been set and the cycles detected. Now, it is defined a 2-SAT formula* φ *to impose the hv-convexity of the final valuation in the four corner regions: each clause of* φ *is of the form* $(x_i \rightarrow x_j)$ *meaning that if the variable* x_i *has value* 1*, i.e. the labeled cell belongs to the polyomino, then the same value is imposed to each cell* x_j*, if any, lying between* x_i *and the kernel. In our example, the clauses related to the NW corner are:* $(x_1 \rightarrow x_2)$ *and* $(x_3 \rightarrow x_2)$*. Figure 6,* (c) *shows one of the possible variables' valuations, i.e.,* $x_1 = 0$*,* $x_2 = 1$*, and* $x_3 = 0$*.*

Obviously, it is in Step 2 that one can act and impose the constraints to gain

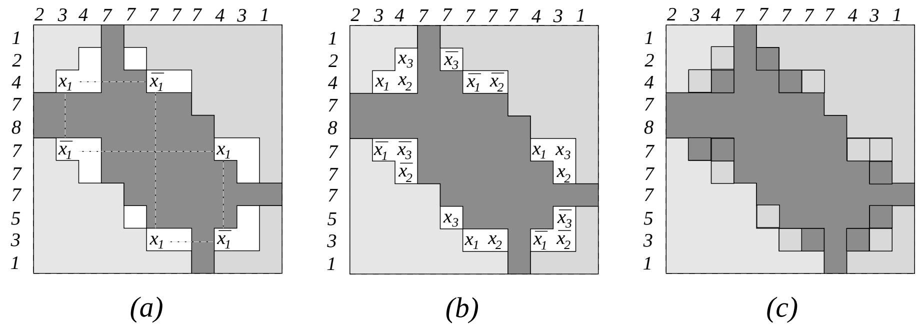


Figure 6: The action of Step 2 on the instance of Fig.5. The variables label each unassigned cell of the NW corner region and the cycles are detected. Then the formula φ is defined and one of its valuations computed to obtain the related solution, if any.

2-convexity. In particular the underlying idea requires to add further clauses to the formula φ according to the 2-convexity characterization of Proposition 3, and, at the same time, to maintain the polynomiality of the process.

From hv-convexity to 2-convexity: definition of the dual Horn clauses

Let $x_{i,j}$ be the label of a cell A in position (i, j) of an hv-convex polyomino; we define the clause:

$$Forb(x_{i,j}) = \ (x_{i,j} \wedge \overline{x}_{i-1,j} \wedge \overline{x}_{i,j-1} \wedge \overline{x}_{i+v_j-1,j'} \wedge \overline{x}_{i',j+h_i-1}) \rightarrow (\overline{\overline{x}}_{i',j'})$$

In the following theorem, we will prove that $Forb(A)$ imposes the 2-convexity on the cell A using the notion of forbidden area, being A a NW corner cell:

Theorem 1. *Let $x_{i,j}$ be the label of a cell A in an hv-convex polyomino P. The clause $Forb(x_{i,j})$ is satisfiable if and only if A is a NW corner cell and its $SE2$ forbidden area has void intersection with P.*

Proof: let us assume that the clause $Forb(x_{i,j})$ has a valuation for the couple of indexes i' and j'. This means that either the premises are false (so nothing has to be proved for the couples of indexes (i, j) and (i', j')), or the premises and the consequence are satisfiable. In this latter case, the premises impose the following constraints:

$i)$ the cell A belongs to the hv-convex polyomino P, since $x_{i,j} = 1$, and it is a NW corner, since $x_{i-1,j} = 0$ and $x_{i,j-1} = 0$;

$ii)$ by $i)$, the cells in position $(i, j + h_i - 1)$ and $(i + v_j - 1, j)$ are the extremal cells on row i and column j of P, being h_i and v_j the horizontal and vertical projections of the row i and the column j of P, respectively;

$iii)$ there exist two cells in positions $(i', j + h_i - 1)$ and $(i + v_j - 1, j')$ that lie outside P, with $i' \geq i + v_j$ and $j' \geq j + h_i$. We point out that the smallest indexes i' and j' for which such a property holds are not known a priori.

The variables' configuration for the indexes i' and j' is depicted in Fig. 7.

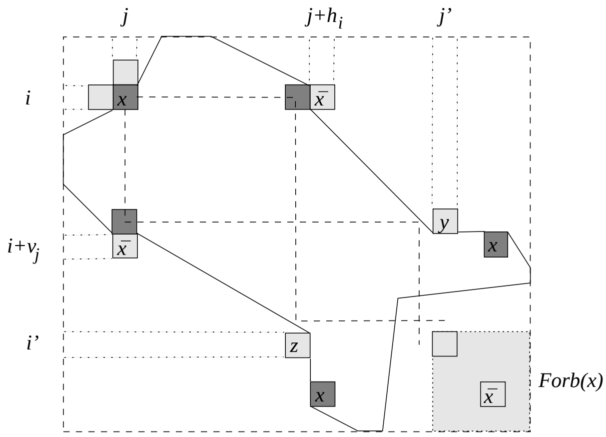


Figure 7: Visualizing the SE forbidden area $Forb(x_{i,j})$ of Theorem 1 up to the variables' renaming $x = x_{i,j}$, $y = x_{i+v_j,j'}$ and $z = x_{i',j+h_i}$.

It is worthwhile that for the smallest indexes i' and j', the cell labeled with $\overline{x}_{i',j'}$ is the upper leftmost corner of the $SE2$ forbidden area of A and its value is 0 (since the consequence of $Forb(A)$, i.e., $\overline{\overline{x}}_{i',j'}$ is true), as desired. In case of bigger values of i' and j', the cell lies inside the forbidden area anyway.

On the other hand, if A is a NW corner cell, then $(x_{i,j} \wedge \overline{x}_{i-1,j} \wedge \overline{x}_{i,j-1})$ is satisfied. The further assumption $(\overline{x}_{i+v_j-1,j'} \wedge \overline{x}_{i',j+h_i-1})$ let $x_{i',j'}$ lie inside the $SE2$ forbidden area of A, so, by hypothesis, $\overline{x}_{i',j'} = 1$, as desired. $\square$

We are now able to specialize the algorithm $RecConv$ into $Rec2Conv$ that reconstructs 2-convex polyominoes from the horizontal and vertical projections H and V. The modification concerns Step 2 of $RecConv$ only, and it is performed in case of oriented placement of the polyomino, according to Property 1: we create a new logic formula ψ that is the conjunction of φ and the clauses of the type $Forb(x_{i,j})$ for each corner cell $x_{i,j}$ in the NW corner area, and for a set of possible couples of indexes i' and j':

$$\psi = \varphi \wedge \bigwedge_{i,j,i' \in R_i, j' \in C_j} Forb(x_{i,j}),$$

where R_i and C_j are the row and column indexes of the unassigned cells in the SW and NE corner areas lying on column $j + h_i - 1$ and on row $i + v_j - 1$, respectively. If one or both the two sets are empty, then i' or j' are the smallest indexes of the cells not belonging to the polyomino.

Proposition 3 and Theorem 1 assure the 2-convexity of the computed solution, if any.

What is left to prove is the polynomiality of $Rec2Conv$: ψ is not a *2-SAT* formula any more, so the related possibility of a polynomial time valuation used in [2] is not available.

Let us recall the notion of *Horn clause*, i.e., a clause containing at most one positive literal and any number of negative literals. A Horn formula is a propositional formula formed by the conjunction of Horn clauses. The problem of finding a valuation of a Horn formula is a *P-complete* problem, [19], sometimes addressed as *HornSat*.

A useful generalization of the class of Horn formulae is that of *Dual-Horn* formulae, i.e., the set of formulae that can be transformed into Horn ones by replacing all the variables with their negation, and whose satisfiability can be still solved in polynomial time.

Theorem 2. *The problem of reconstructing a* 2*-convex polyomino from horizontal and vertical projections is polynomially solvable.*

Proof: the algorithm *Rec2Conv* acts as *RecConv* except for the satisfiability of the formula ψ in Step 2, in case of oriented feet placement. An easy check reveals that all the clauses of ψ contains at most one negative literal, in fact each clause of *varphi* does, as stated in Lemma 3.7 and Proposition 3.8 of [2], while each clause of the form $Forb(x_{i,j})$ has the only negative literal $\overline{x}_{i,j}$. So, ψ is a *Dual-Horn* formula. Finally, we note that *psi* is composed by a polynomial number of clauses, i.e., one for each configuration of the indexes i, j, i', and j', so a valuation can be found in polynomial time. □

Speeding up the reconstruction process

The algorithm *Rec2Conv* has a general behaviour and does not consider strong geometrical regularities exhibited by 2-convex sets. Some of them are a valuable tool to speed up the reconstruction process revealing unexpected shortcuts to the final solution.

Let us focus on the cycles detected in Step 2 of *RecConv*: we call k-cycle each cycle whose length, i.e., number of changes of direction in the path leading from a variable in the NW corner area to the corresponding one in the SE corner area, equals k; as an example all the cycles in Fig.6, (b) are 2-cycles.

It is easy to figure out that the presence of k-cycles relates to the degree of convexity of the polyomino to be reconstructed. In particular it holds

Proposition 4. *If an hv-convex polyomino admits a* 1*-cycle, then all its cycles are* 1*-cycles.*

Proof: let us assume a 1-cycle is detected in an *hv*-convex polyomino during Step 2 of its reconstruction process; by definition the cells of the 1-cycle do not belong to the feet and three of them lie, one in the NW area, say cell $x_{i,j}$, one in the SW area, say $\overline{x}_{i+s,j}$, and one in the NE area, say $\overline{x}_{i,j+e}$. Let us proceed by contradiction assuming the presence of a 2-cycle composed by six cells, i.e., the minimum admissible number: since the polyomino is supposed $NW \rightarrow SE$ oriented, then one cell lies in the NW area, say y, one in the SE area, two cells in the NE area, and two in the SW area.

By monotonicity of the NW path the cell y either is in position $(i-k, j+k')$ or $(i+k, j-k')$ for some non negative integers k and k' with $k, k' \geq 0$. Without loss of generality, we consider y to be in position $(i-k, j+k')$. Let the first of these points be $\overline{y}_{i-k,j+k'+l'}$, with $j+k'+l' < j+e$. Again by monotonicity of the SW path, let the first point of the 2-cycle in the SW area be $\overline{y}_{i-k+l,j+k'}$, with $i-k+l > i+s$.

By construction the point in position $(i+e, j+s)$ lies in the SE area and consequently there are no further points of the 2-cycle in the SW area, so we reach a contradiction. To end the proof we remark that the reasoning is maid for 2-cycles and could be extend directly for k-cycles with $k > 2$. □

Proposition 5. *Let RecConv(H, V) computes a kernel that admits* 1*-cycles only. All the valuations of* φ*, if any, are* 2*-convex.*

Proof: the result directly follows by definition of 1-cycle, after observing that from each NW-corner cell $x_{i,j}$ belonging to a 1-cycle, it starts a $S1$-path reaching the SE area, so its $SE2$ forbidden area has no intersection with the polyomino. □

Analogously, one can easily check that the configuration of the variable x of a 3-cycle prevents a non intersecting $SE2$ forbidden area for the variable. So it holds:

Proposition 6. *Let RecConv(H, V) computes a kernel that admits a* 3*-cycle. All the valuations of* φ*, if any, are not* 2*-convex.*

4. Final comments

The problem we have investigated here integrates the researches of Maurice Nivat about the notion of convexity in polyominoes and the ambiguity of their reconstruction from a small number of projections. What in our intent was to specialize the well known algorithm by Maurice Nivat and co-authors in [2] to reconstruct hv-convex polyominoes by orthogonal projections. In particular, we maintained the first preprocessing step where the cells common to all the solutions of a given instance are computed, and we acted on the second part of the algorithm, where the values of the undetermined cells are linked in a boolean formula to impose the desired convexity.

The *Dual-Horn* formula ψ we defined, includes a new characterization of 2-convexity that we obtained throughout the notion of forbidden area. We underline that the formula ψ and the related Theorem 1 are open to a generalization to k-convexity: in particular the idea of bouncing inside a polyomino using maximal internal paths, starting from a corner area till reaching the opposite one seems exactly what needed to this purpose.

Nevertheless, despite Theorem 1 states a general result on k-convexity, it is not trivial to express k-convexity, with $k > 3$, by a formula still belonging to *Dual-Horn*, since positive and negative variables mix in the NE and SW areas.

On the other hand, for those same values of k, a deeper study deserves the possible to define some gadgets to describe each 3-sat formula by means of a k-convex polyomino compatible with a couple of projections: the very many possibilities to code logic gates starting from the NW or the SE border independently and letting them interfere in the remaining corner areas may lead to a result of non polynomiality of the related reconstruction problem.

The present study also reveals a strong connection between k-convexity and cycles in polyominoes that realizes again by the notion of forbidden area. It would be desirable to characterize the types of cycles allowed by a k-convex polyominoes in order to bound the degrees of convexity of the solutions of a reconstruction instance, and study the related enumerative problems.

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