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Enumeration of chains and saturated chains in Dyck lattices

Luca Ferrari [a,1], Emanuele Munarini [b,2]

[a] Dipartimento di Matematica e Informatica "U. Dini", viale Morgagni 65, 50134 Firenze, Italy
[b] Politecnico di Milano, Dipartimento di Matematica, Piazza Leonardo da Vinci 32, 20133 Milano, Italy

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ABSTRACT

We find a closed formula for the number of chains in Dyck lattices. Moreover, we determine a general formula to compute the number of saturated chains, and we apply it to find the number of saturated chains of length 2, 3 and 4. We also compute what we call the Hasse index (of order 2, 3 and 4) of Dyck lattices, which is the ratio between the total number of saturated chains (of length 2, 3 and 4) and the cardinality of the underlying poset.

E-mail addresses: luca.ferrari@unifi.it (L. Ferrari), emanuele.munarini@polimi.it (E. Munarini).

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1. Introduction

Given a poset $\mathcal{P}$, a very natural problem is to count how many chains it has. A *chain of length* k (or k-chain) in $\mathcal{P}$ is a $(k+1)$-tuple $(x_0, x_1, \ldots, x_k)$ of elements of $\mathcal{P}$ such that $x_0 < x_1 < \cdots < x_k$. A *saturated chain* in $\mathcal{P}$ is a chain such that, if $x < y$ are consecutive elements in the chain, then y covers x. In the present paper we wish to address the problem of enumerating chains and saturated chains in the case of Dyck lattices. The *Dyck lattice of order* n, to be denoted $\mathcal{D}_n$, is the lattice of Dyck paths of semilength n whose associated partial order relation is given by *containment*: given $\gamma, \gamma' \in \mathcal{D}_n$, it is $\gamma \leq \gamma'$ when, in the usual two-dimensional drawing of Dyck paths, γ lies weakly below γ'. In some sources, Dyck lattices are also called *Stanley lattices* (see, for instance, [3]). Some papers studying properties of Dyck lattices are [6,8]. Counting saturated chains of length 1 is clearly equivalent to enumerating edges in the associated Hasse diagram, which has been considered in [7] not only for Dyck lattices but also for other lattices of paths. Here we start by finding a closed expression for the number of k-chains in a Dyck lattice $\mathcal{D}_n$, which is more precisely a sum of certain Hankel determinants of Catalan numbers. Then we provide a general formula for counting saturated chains of length h, for any fixed h, in a given Dyck lattice. Next we deal with the cases $h = 2, 3, 4$, giving for them detailed enumerative results. We also define the notion of *Hasse index of order* h (thus generalizing the concept of Hasse index proposed in [7]) and compute such an index in the three mentioned special cases.

2. Preliminaries

In this section, we collect some notations and results which will be used later in the paper.

A *multichain of length* k (or k-multichain) in a poset $\mathcal{P}$ is a $(k+1)$-tuple $(x_0, x_1, \ldots, x_k)$ of elements of $\mathcal{P}$ such that $x_0 \leq x_1 \leq \cdots \leq x_k$. Let $c_k(\mathcal{P})$ be the number of the k-chains in $\mathcal{P}$, and $m_k(\mathcal{P})$ be the number of the k-multichains in $\mathcal{P}$. A standard application of the principle of inclusion–exclusion yields the following result, which provides a relation between the numbers introduced above:

$$c_k(P) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} m_i(P). \tag{1}$$

A *Young tableau* is a filling of a Ferrers shape λ using distinct positive integers from 1 to $n = |\lambda|$, with the properties that the values are (strictly) decreasing along each row and each column of the Ferrers shape (here $|\lambda|$ denotes the number of cells of the Ferrers shape λ). This constitutes a slight departure from the classical definition, which requires the word "increasing" instead of the word "decreasing". However, it is clear that all the properties and results on (classical) Young tableaux can be translated into our setting by simply replacing the total order "$\leq$" with the total order "$\geq$" on $\mathbb{N}$. A *skew*

Young tableau is defined exactly as a Young tableau, with the only difference that the underlying shape consists of a Ferrers shape λ with a (possibly empty) Ferrers shape μ removed (starting from the top-left corner), in such a way that the resulting shape is *strongly connected*: this means that every pair of consecutive rows has at least one common column and every pair of consecutive columns has at least one common row (such a shape will be also called a *skew Ferrers shape*).

A *Dyck path* is a path starting from the origin of a fixed Cartesian coordinate system, ending on the x-axis, never going below the x-axis, and using only the two steps $u = (1, 1)$ and $d = (1, -1)$. A *valley* (*peak*) in a Dyck path is a pair of consecutive steps du (ud). The semilength of a Dyck path is just half the number of its steps. The set of all Dyck paths of semilength n will be denoted D_n. The number of Dyck paths of semilength n is the n-th *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The set D_n endowed with the partial order described in the Introduction will be called the *Dyck lattice of order* n and denoted $\mathcal{D}_n$. The generating series of saturated chains of length h in the family of Dyck lattices will be written $SC_h(x)$, whereas the number of saturated chains of length h in $\mathcal{D}_n$ (i.e. the coefficient of x^n in $SC_h(x)$) will be written $sc_h(\mathcal{D}_n)$.

At the end of this section, we propose a generalization of the notion of Hasse index given in [7]. Recall that the *Hasse index* $i(\mathcal{P})$ of a poset $\mathcal{P}$ is given by $i(\mathcal{P}) = \frac{\ell(\mathcal{P})}{|\mathcal{P}|}$, where $\ell(\mathcal{P})$ is the number of covering pairs in $\mathcal{P}$. Given a positive integer h, we now define the *Hasse index of order* h of $\mathcal{P}$ as $i_h(\mathcal{P}) = \frac{sc_h(\mathcal{P})}{|\mathcal{P}|}$, where $sc_h(\mathcal{P})$ denotes the number of saturated chains of length h of the poset $\mathcal{P}$. Of course $i_1(\mathcal{P}) = i(\mathcal{P})$. For instance, for the Boolean algebra $\mathcal{B}_n$ having 2^n elements, $sc_h(\mathcal{B}_n)$ can be computed by taking an arbitrary subset having k elements (for $0 \le k \le n$) and then adding any h of the remaining elements in a specified order. Equivalently, we can choose a subset having h elements, a linear order on it, and a subset of its complement. Therefore we get

$$sc_h(\mathcal{B}_n) = \sum_{k=0}^{n} \binom{n}{k} (n-k)_h = (n)_h 2^{n-h},$$

where $(a)_b = a \cdot (a-1) \cdot \ldots \cdot (a-b+1)$ denotes a *falling factorial*. Thus, the Hasse index of order h of $\mathcal{B}_n$ is given by

$$i_h(\mathcal{B}_n) = \frac{(n)_h \cdot 2^{n-h}}{2^n} = \frac{(n)_h}{2^h}.$$

We will say that the Hasse index of order h of a sequence $\mathscr{P} = \{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n, \ldots\}$ of posets is *Boolean* when $i_h(\mathcal{P}_n) = \frac{(n)_h}{2^h}$ and *asymptotically Boolean* when $i_h(\mathcal{P}_n) \sim \frac{(n)_h}{2^h}$ (or, equivalently, $i_h(\mathcal{P}_n) \sim \frac{n^h}{2^h}$).

In the computation of the Hasse index we will also use the well known *Darboux theorem* (see for instance [2]), which asserts that, given a complex number $\xi \neq 0$ and a complex function $f(x)$ analytic at the origin, if $f(x) = (1 - x/\xi)^{-\alpha}\psi(x)$, where $\psi(x)$ is a series with radius of convergence $R > |\xi|$ and $\alpha \notin \{0, -1, -2, \ldots\}$, then, when $n \to +\infty$,

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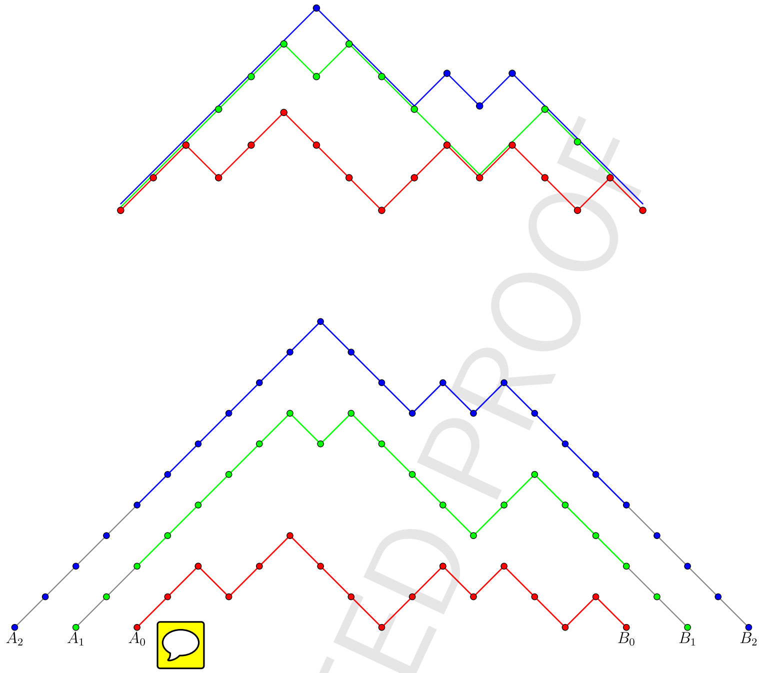

Fig. 1. A 2-multi-chain in $\mathcal{D}_8$ and the corresponding triple of non-intersecting Dyck paths.

$$[x^n]f(x) \sim \frac{\psi(\xi)}{\xi^n}\frac{n^{\alpha-1}}{\Gamma(\alpha)},$$

where Γ is Euler's Gamma function.

3. Enumeration of chains

Our goal is to use formula (1) to count chains in $\mathcal{D}_n$. To this aim, we first need to find an expression for the coefficients $m_k(\mathcal{D}_n)$.

Proposition 3.1. *The number the k-multichains in $\mathcal{D}_n$ is given by*

$$m_k(\mathcal{D}_n) = \det[C_{n+i+j}]_{i,j=0}^{k} = \begin{vmatrix} C_n & C_{n+1} & \cdots & C_{n+k} \\ C_{n+1} & C_{n+2} & \cdots & C_{n+k+1} \\ \vdots & \vdots & & \vdots \\ C_{n+k} & C_{n+k+1} & \cdots & C_{n+2k} \end{vmatrix}. \tag{2}$$

Proof. Given a k-multichain $(\gamma_0, \gamma_1, \ldots, \gamma_k)$ in $\mathcal{D}_n$, we can consider the k-tuple $(\gamma'_0, \gamma'_1, \ldots, \gamma'_k)$ of non-intersecting Dyck paths, defined by setting $\gamma'_0 = \gamma_0$ and $\gamma'_i = u^{2i}\gamma_i d^{2i}$, for all $i \le k$. In this way, the path γ'_i starts from the point $A_i = (-2i, 0)$ and ends at the point $B_i = (2n + 2i, 0)$. See Fig. 1 for an example.

By applying the *Lindström–Gessel–Viennot Theorem* (see, for instance, [1] or the special case reported in [12], Theorem 2, which is precisely the situation which is relevant

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to us), we have that the number of the $k+1$ non-intersecting Dyck paths starting from one point in $\{A_0, A_1, \dots, A_k\}$ and ending at one point in $\{B_0, B_1, \dots, B_k\}$ is

$$\det\left[p(A_i, B_j)\right]_{i,j=0}^{n}$$

where $p(A_i, B_j)$ is the number of all Dyck paths from A_i to B_j. Since $p(A_i, B_j) = C_{n+i+j}$, the proposition is proved. □

Theorem 3.1. *The number of k-chains in $\mathcal{D}_n$ is given by*

$$c_k(\mathcal{D}_n) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \cdot \frac{\prod_{j=1}^{i+1} C_{n+j-1} \cdot \prod_{j=1}^{i} (2j+1)!}{\prod_{j=1}^{i} (n+j+1)^j \cdot (n+2i+2-j)^j} \tag{3}$$

Proof. From formula (1) and Proposition 3.1 we immediately get

$$c_k(\mathcal{D}_n) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \begin{vmatrix} C_n & C_{n+1} & \cdots & C_{n+i} \\ C_{n+1} & C_{n+2} & \cdots & C_{n+i+1} \\ \vdots & \vdots & & \vdots \\ C_{n+i} & C_{n+i+1} & \cdots & C_{n+2i} \end{vmatrix}. \tag{4}$$

To compute the above "Catalan" determinant, we refer to [12]: just apply Theorem 3 with $n = i+1$ and $\alpha_s = n+s$, for $0 \le s \le k$. The expression given in [12] can then be reduced to the one appearing in the statement of this theorem by recognizing that some partial factors give rise to instances of the Catalan numbers. □

Remark. We point out that the case $k = 1$ (chains of length 1 or, equivalently, intervals) appears in [11].

4. Saturated chains: the general enumeration formula

Let $\gamma^{(0)} < \gamma^{(1)} < \cdots < \gamma^{(h)}$ be a saturated chain (of length h) in $\mathcal{D}_n$. It is easy to see that two consecutive paths of the chain only differ by a pair of consecutive steps, namely a valley (a peak) in the smallest (largest) one. More generally, the minimum $\gamma^{(0)}$ and the maximum $\gamma^{(h)}$ differ by a set of steps in such a way that the sum of the areas of the regions delimited by these steps is equal to h. To be more precise, this means that the two paths can be factorized as $\gamma^{(0)} = \alpha_1 \gamma_1^{(0)} \alpha_2 \gamma_2^{(0)} \cdots \alpha_k \gamma_k^{(0)} \alpha_{k+1}$ and $\gamma^{(h)} = \alpha_1 \gamma_1^{(h)} \alpha_2 \gamma_2^{(h)} \cdots \alpha_k \gamma_k^{(h)} \alpha_{k+1}$, where, for every i, the two factors $\gamma_i^{(0)}$ and $\gamma_i^{(h)}$ have the same length, and the sum of the areas of the regions determined by the pairs of factors $(\gamma_i^{(0)}, \gamma_i^{(h)})$ is equal to h (see Fig. 2).

Each of the regions determined by the pairs $(\gamma_i^{(0)}, \gamma_i^{(h)})$ can be regarded as a *skew Ferrers shape*. To fix notations, we will suppose that such a shape is that obtained by rotating the sheet of paper by 45° anticlockwise. Referring again to Fig. 2, the pair of Dyck paths on the left determines the pair of skew Ferrers shapes on the right.

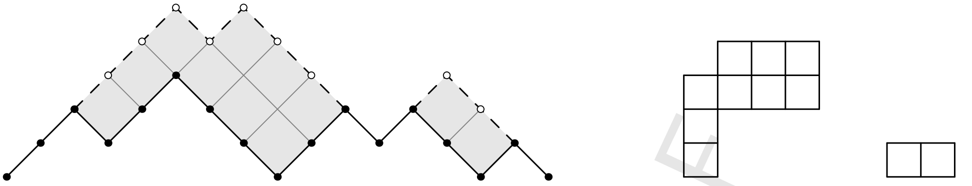

Fig. 2. A pair of Dyck paths γ (thick) and γ' (dashed), with $\gamma < \gamma'$, and the corresponding set of skew Ferrers shapes.

Now suppose to select a saturated chain from $\gamma^{(0)}$ to $\gamma^{(h)}$. This corresponds to choosing, one at a time, the h cells belonging to the skew Ferrers shapes described above. More formally, this defines a linear order on the set of all cells of the skew Ferrers shapes determined by the two paths such that, on each row and on each column, cells are in decreasing order. This means that a saturated chain essentially generates a set of *skew Young tableaux*.

Let now $\gamma \in \mathcal{D}_n$. We want to determine the number of saturated chains of length h starting from γ in $\mathcal{D}_n$. According to the above considerations, to describe any such chain we start by giving a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of h. Next we have to choose a set $\gamma_1, \ldots, \gamma_k$ of factors of γ such that, for any $i \leq k$, we can build a skew Ferrers shape φ_i on γ_i having area λ_i. Finally, to determine the saturated chain, we just have to linearly order the cells of the Ferrers shapes thus obtained, or, equivalently, to endow each of the shapes with a skew Young tableaux structure.

Now we will try to describe more formally the above argument. Denote by $SkFS$ the set of all skew Ferrers shapes. Given $\varphi \in SkFS$, we write $A(\varphi)$ for the *area* of φ, i.e. the number of cells of φ. We also define $SkFS(n) = \{\varphi \in SkFS \mid A(\varphi) = n\}$. Given a set of words $\gamma_1, \ldots, \gamma_n$ on the alphabet $\{u, d\}$, we say that they are a set of *pairwise disjoint occurrences (p.d.o.)* in γ when they appear as factors of γ having no pairwise intersection. A skew Ferrers shape φ is delimited by two paths, both starting at its bottom left corner and ending at its top right corner. Each of such paths can be seen as a word on $\{u, d\}$, by simply encoding a horizontal step with the letter d and a vertical step with the letter u. The word having d as its first letter is called the *lower border* of φ and is denoted $b(\varphi)$. Finally, for any given $\varphi \in SkFS$, let $t(\varphi)$ be the number of skew Young tableaux of shape φ.

For any path $\gamma \in \mathcal{D}_n$, let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of the positive integer h (this will also be written as $\lambda \vdash h$). Next we have to choose a set $\gamma_1, \ldots, \gamma_k$ of pairwise disjoint occurrences in γ such that, for any $i \leq k$, there exists a skew Ferrers shape $\varphi_i \in SkFS(\lambda_i)$ for which $b(\varphi_i) = \gamma_i$. Now, to get a saturated chain, we have to select a k-tuple $(\varphi_1, \ldots, \varphi_k) \in SkFS^k$ such that $b(\varphi_i) = \gamma_i$ and $A(\varphi_i) = \lambda_i$, and for each component φ_i we have to choose one among the $t(\varphi_i)$ possible skew Young tableaux. Finally, since the set of integers actually used to fill in the cells of each φ_i can be any possible set of $|\lambda_i|$ integers less than or equal to h, we have proved the following result.

Theorem 4.1. *The number $sc_h(\mathcal{D}_n)$ of saturated chains of length h of the lattice $\mathcal{D}_n$ is given by*

$$\sum_{\gamma\in\mathcal{D}_n}\sum_{\lambda\vdash h}\sum_{\substack{\gamma_1,\dots,\gamma_k\, p.d.o.\\(\forall i)(\exists\varphi_i\in SkFS(\lambda_i))b(\varphi_i)=\gamma_i}}\sum_{\substack{(\varphi_1,\dots,\varphi_k)\in SkFS^k\\(\forall i)(b(\varphi_i)=\gamma_i,A(\varphi_i)=\lambda_i)}}\binom{h}{A(\varphi_1),\dots,A(\varphi_k)}$$
$$\times\, t(\varphi_1)\cdots t(\varphi_k). \tag{5}$$

In the rest of the paper, our main aim is to apply the above formula to the special cases $h=2$, $h=3$ and $h=4$ (the case $h=1$ having already been examined in [7]), thus finding some new results on the poset structure of Dyck lattices.

We end the present section by recalling that this problem could also be tackled from a slightly different point of view. Indeed, given two Dyck paths of the same length γ_1 and γ_2 such that $\gamma_1\le\gamma_2$, the set of all saturated chains between γ_1 and γ_2 can be represented by means of a suitable *Polya festoon*, more precisely a Polya festoon whose components cannot be "−polygons" (see [9]). It seems that this approach could be more elegant, but should lead to more difficult computations.

We also remark that, in the paper [4], pairs of noncrossing free Dyck paths (also called Grand–Dyck paths in different sources) are considered, also in connection with several different combinatorial structures, such as noncrossing partitions and vacillating tableaux. It could be of some interest to extend our results to the case of free Dyck paths and successively interpret them on the above mentioned combinatorial objects via the bijections described in [4].

Finally, a similar problem is addressed by Gessel in [10]. In this paper the author uses the language of Young's lattice (rather than the language of lattices of paths), and deals with a similar problem. However, Gessel's approach is completely different, relying heavily on algebraic notions, whereas we tackle the problem using only direct combinatorial methods.

5. Saturated chains of length 2

In order to apply formula (5) to the case of saturated chains of length 2 we simply have to set $h=2$. Doing this way, one immediately observes that there are only two partitions of 2, namely $(1,1)$ and (2), and that there exists one pair of "admissible" skew Ferrers shapes of area 1, i.e. $(\square,\square)$, and two different skew Ferrers shapes of area 2, i.e. $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. Since each of these shapes can be endowed with only one Young tableau structure, we arrive at the following result.

Proposition 5.1. *The generating series for the number of saturated chains of length 2 of Dyck lattices is given by*

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$$SC_2(x) = \sum_{n\geq 0}\left(\sum_{\gamma\in\mathcal{D}_n}\left(2\cdot\#(du,du)_\gamma + \#(ddu)_\gamma + \#(duu)_\gamma\right)\right)x^n, \tag{6}$$

where with $\#(\gamma_1,\dots,\gamma_k)_\gamma$ *we denote the number of pairwise disjoint occurrences of the* γ_i*'s in* γ*.*

All we have to do now is to evaluate the three unknown quantities appearing in (6). The following proposition translates formula (6) into an expression more suitable for computing.

Proposition 5.2. *Denote with* $F(q,x)$ *the generating series of all Dyck paths where* x *keeps track of the semilength and* q *keeps track of the factor* duu *(resp.* ddu*), that is*

$$F(q;x) = \sum_{n\geq 0}\sum_{\gamma\in\mathcal{D}_n} q^{\#(duu)_\gamma}x^n = \sum_{n\geq 0}\sum_{\gamma\in\mathcal{D}_n} q^{\#(ddu)_\gamma}x^n$$

where $\#(duu)_\gamma$ *(resp.* $\#(ddu)_\gamma$*) is the number of occurrences of the factor* duu *(resp.* ddu*) in the Dyck path* γ*. Similarly, denote with* $V(q,x)$ *the generating series of all Dyck paths where* x *keeps track of the semilength and* q *keeps track of the factor* du *(i.e. valleys). Then*

$$SC_2(x) = 2\cdot\left[\frac{\partial F}{\partial q}\right]_{q=1} + \left[\frac{\partial^2 V}{\partial q^2}\right]_{q=1}. \tag{7}$$

Proof. Since the factors ddu and duu are obviously equidistributed on the set of Dyck paths, the expression

$$\left[\frac{\partial F}{\partial q}\right]_{q=1} = \sum_{n\geq 0}\sum_{\gamma\in\mathcal{D}_n}\#(duu)_\gamma x^n = \sum_{n\geq 0}\sum_{\gamma\in\mathcal{D}_n}\#(ddu)_\gamma x^n$$

gives the generating series of Dyck paths with respect to the number of factors duu, or, equivalently, with respect to the number of factors ddu. Similarly, the expression $[\frac{\partial V}{\partial q}]_{q=1}$ gives the generating series of Dyck paths with respect to the number of valleys, and the expression $[\frac{\partial^2 V}{\partial q^2}]_{q=1}$ gives the generating series of Dyck paths with respect to the number of (non-ordered) pairs of valleys. All this implies formula (7). □

We are now in a position to find a neat expression for the generating series $SC_2(x)$.

Theorem 5.1. *The generating series for the number of saturated chains of length 2 of Dyck lattices is given by*

$$SC_2(x) = \sum_{n\geq 0} sc_2(\mathcal{D}_n)x^n = \frac{1-6x+6x^2-(1-4x)\sqrt{1-4x}}{-(1-4x)\sqrt{1-4x}}, \tag{8}$$

where

$$sc_2(\mathcal{D}_n) = \binom{2n}{n} \frac{(n-1)(n-2)}{2(2n-1)} \quad (n \geq 1). \tag{9}$$

Proof. Let $G(q,x)$, $H(q,x)$ be the generating series of Dyck paths starting with a peak and Dyck paths starting with two consecutive up steps, respectively, where x keeps track of the semilength and q keeps track of the factor duu. Since any non-empty Dyck path γ decomposes uniquely as $\gamma = U\gamma' D\gamma''$, where $\gamma', \gamma'' \in \mathcal{D}$, and γ'' starts either with a peak or with two consecutive up steps (if it is not the empty path), we arrive at the following system (where F is defined as in the previous proposition):

$$\begin{cases} F = 1 + xF(1 + G + qH) \\ G = x(1 + G + qH) \\ H = x^2 F(1 + G + qH)^2. \end{cases} \tag{10}$$

Solving for F, we find the following expression:

$$F(q,x) = \frac{1 - 2(1-q)x - \sqrt{1 - 4x + 4x^2 - 4qx^2}}{2qx}.$$

Moreover, the explicit expression of $V(q,x)$ (see again the previous proposition) can be found in [5,7], and is the following:

$$V(q,x) = \frac{1 - (1-q)x - \sqrt{1 - 2(1+q)x + (1-q)^2 x^2}}{2qx}. \tag{11}$$

We can therefore apply the previous proposition, thus obtaining formula (8). □

The integer sequence associated with $SC_2(x)$ starts $0, 0, 0, 4, 30, 168, 840, 3960, 18\,018, 80\,080, \ldots$. We observe that the terms of the above sequence divided by 2 yield sequence A002740 of [14]. In terms of Dyck paths, this sequence gives the sum of the abscissae of the valleys in all Dyck paths of semilength $n-1$. It would be nice to have a combinatorial explanation of this fact.

The results of the present section allow us to compute the Hasse index of order 2 of Dyck lattices. Recall that in [7] it is shown that the Hasse index of order 1 is asymptotically Boolean.

Proposition 5.3. *The Hasse index of order 2 of the class of Dyck lattices is asymptotically Boolean.*

Proof. Since $|\mathcal{D}_n| = \binom{2n}{n}\frac{1}{n+1}$, from formula (9) we get

$$i_2(\mathcal{D}_n) = \frac{sc_2(\mathcal{D}_n)}{|\mathcal{D}_n|} = \frac{(n-1)(n-2)(n+1)}{2(2n-1)} \sim \frac{n^2}{4},$$

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which means precisely that the Hasse index of order 2 is asymptotically Boolean. □

6. Saturated chains of length 3

Setting $h = 3$ in (5) we obtain a formula for the enumeration of saturated chains of length 3 of Dyck lattices. Similarly to what we did in the previous section, we observe that there are three partitions of the integer 3, namely (1,1,1), (2,1) and (3). Moreover, the unique "admissible" triple of skew Ferrers shapes of area 1 is $(\square,\square,\square)$, whereas there are two pairs of skew Ferrers shapes whose first component have area 1 and whose second component has area 2, namely $(\square,\begin{smallmatrix}\square\\\square\end{smallmatrix})$ and $(\square,\square\square)$, and there are four skew Ferrers shapes having area 3, i.e. $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, $\square\square\square$, $\begin{smallmatrix}&\square\\\square&\square\end{smallmatrix}$ and $\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}$. Unlike the previous case, now we have two of the shapes of area 3 that have two different Young tableaux structures. More precisely, we have to consider the two skew Young tableaux $\begin{smallmatrix}&3\\2&1\end{smallmatrix}$, $\begin{smallmatrix}&2\\3&1\end{smallmatrix}$ and the two (skew) Young tableaux $\begin{smallmatrix}3&1\\2&\end{smallmatrix}$, $\begin{smallmatrix}3&2\\1&\end{smallmatrix}$. Thus, a direct application of formula (5) leads to the following statement.

Proposition 6.1. *The generating series for the number of saturated chains of length 3 of Dyck lattices is given by*

$$\begin{aligned} SC_3(x) = \sum_{n\geq 0}\sum_{\gamma\in\mathcal{D}_n}\big(&6\cdot\#(du,du,du)_\gamma + 3\cdot\#(du,ddu)_\gamma \\ &+3\cdot\#(du,duu)_\gamma + \#(dddu)_\gamma + \#(duuu)_\gamma \\ &+2\cdot\#(dduu)_\gamma + 2\cdot\#(dudu)_\gamma\big)x^n. \end{aligned} \tag{12}$$

Our next step will be the evaluation of the unknown quantities appearing in (12).

Analogously to the case of saturated chains of length 2, we start by finding an expression of (12) better suited for computation.

Proposition 6.2. *Denote with $A(q,x)$, $B(q,x)$ and $D(q,x)$ the generating series of Dyck paths where x keeps track of the semilength and q keeps track of the factors $dduu, dudu$ and $duuu$, respectively. Moreover, let $V(q,x)$ be defined as in the previous section. Finally, let $F(p,q,x)$ be the generating series of Dyck paths obtained from the series $F(q,x)$ defined in the previous section by adding the indeterminate p keeping track of valleys (i.e. of the factor du). Then*

$$\begin{aligned} SC_3(x) = 2\cdot\left[\frac{\partial A}{\partial q}\right]_{q=1} + 2\cdot\left[\frac{\partial B}{\partial q}\right]_{q=1} + 2\cdot\left[\frac{\partial D}{\partial q}\right]_{q=1} \\ + \left[\frac{\partial^3 V}{\partial q^3}\right]_{q=1} + 6\cdot\left[\frac{\partial^2 F}{\partial p\partial q} - \frac{\partial F}{\partial q}\right]_{p=q=1}. \end{aligned} \tag{13}$$

Proof. We start by observing that the knowledge of the generating series $F(p, q, x)$ allows us to compute the term of (12) associated with the pair (du, duu). Indeed, it is clear that, if we differentiate F with respect to p and q and then evaluate at $p = q = 1$, we obtain the generating series of Dyck paths with respect to semilength and number of pairs (du, duu). However, in this way we are going to consider also those pairs in which the valley du is part of the factor duu. Thus, to obtain what we need, we have to subtract the derivative of F with respect to q, then evaluate at $p = q = 1$, which yields the expression

$$\left[\frac{\partial^2 F}{\partial p \partial q} - \frac{\partial F}{\partial q}\right]_{p=q=1}.$$

Moreover, it is clear that the generating series describing the distribution of the pair (du, ddu) is the same, and this explains the coefficient 6 in front of the above displayed expression in formula (13).

Finally, the meaning of the partial derivatives of the generating series A, B and D are obvious (notice, in particular, that the factors $dddu$ and $duuu$ are clearly equidistributed, so they are both described by series D), as well as the triple partial derivative of V evaluated in $q = 1$, which gives 6 times the distribution of triples of valleys in Dyck paths. □

Theorem 6.1. *The generating series for the number of saturated chains of length 3 of $\mathcal{D}_n$ is given by*

$$SC_3(x) = \sum_{n \geq 0} sc_3(\mathcal{D}_n)x^n = \frac{P(x) - Q(x)\sqrt{1-4x}}{x(1-4x)^3}, \tag{14}$$

where

$$P(x) = 1 - 13x + 59x^2 - 100x^3 + 16x^4 + 64x^5 = (1-4x)^3(1 - x - x^2),$$
$$Q(x) = 1 - 11x + 39x^2 - 40x^3 - 22x^4.$$

The coefficients $sc_3(\mathcal{D}_n)$ can be expressed as

$$sc_3(\mathcal{D}_n) = \binom{2n}{n} \frac{(n^3 - 7n + 2)(n-2)}{4(n+1)(2n-1)} \quad (n \geq 2).$$

Proof. We start by considering the generating series F, G, H defined in the previous section. Similarly to what we did in the above proposition, we need to add an indeterminate p which will keep track of valleys. Thus, in the following, we will have $F = F(p, q, x)$, and the same for G and H.

Using the same decomposition of Dyck paths described in Theorem 5.1, we can now rewrite system (10) taking into account the presence of the indeterminate p, thus obtaining

$$\begin{cases} F = 1 + xF(1 + pG + pqH) \\ G = x(1 + pG + pqH) \\ H = x^2F(1 + pG + pqH)^2. \end{cases} \tag{15}$$

The solution of such a system is the following:

$$\begin{cases} F = \dfrac{1 - (1 + p - 2pq)x - \sqrt{(1 + 2p + p^2 - 4pq)x^2 - 2(1 + p)x + 1}}{2pqx} \\ G = \dfrac{(1 - (1 + p)x - \sqrt{\Delta}\,)(1 - (1 + p - 2pq)x + \sqrt{\Delta}\,)}{4pqx - 4p^2q(1 - q)x^2} \\ H = \dfrac{(-1 + (1 + p)x + \sqrt{\Delta}\,)(1 - (1 + p - 2pq)x + \sqrt{\Delta}\,)}{2pqx(4pqx - 4p^2q(1 - q)x^2)}, \end{cases}$$

where $\Delta = 1 - 2(1 + p)x + (1 + 2p + p^2 - 4pq)x^2$.

The expression of F allows us to compute the term of (12) associated with the pair (du, duu):

$$\left[\frac{\partial^2 F}{\partial p \partial q} - \frac{\partial F}{\partial q}\right]_{p=q=1} = \frac{-2 + 15x - 30x^2 + 10x^3 + (2 - 11x + 12x^2)\sqrt{1 - 4x}}{2x(1 - 4x)\sqrt{1 - 4x}}.$$

Recalling the expression of the generating series V reported in (11), we obtain:

$$\left[\frac{\partial^3 V}{\partial q^3}\right]_{q=1} = \frac{3(1 - 11x + 40x^2 - 50x^3 + 10x^4 - (1 - 9x + 24x^2 - 16x^3)\sqrt{1 - 4x})}{x(1 - 4x)^2\sqrt{1 - 4x}}.$$

The generating series A and B can be easily computed starting from the functional equations they satisfy, which can be found in [13] and are reported below for the reader's convenience:

$$\begin{cases} x\big(q + (1 - q)x\big)A^2 - \big(1 + (1 - q)(x - 2)x\big)A + 1 - (1 - q)x = 0 \\ xB^2 + \big((1 - q)(x - 1)x - 1\big)B + (1 - q)x + 1 = 0. \end{cases}$$

More precisely, we obtain the following expressions:

$$\begin{cases} A(q, x) = \frac{-1+2(1-q)x-(1-q)x^2+\sqrt{1-4x+2(1-q)x^2+(1-q)^2x^4}}{-2x(q+(1-q)x)} \\ B(q, x) = \frac{1+(1-q)x-(1-q)x^2-\sqrt{1-2(1+q)x-(5-4q-q^2)x^2-2(1-q)^2x^3+(1-q)^2x^4}}{2x}. \end{cases} \tag{16}$$

Differentiating with respect to q and evaluating at $q = 1$ we then obtain:

$$\begin{cases} \left[\dfrac{\partial A}{\partial q}\right]_{q=1} = \dfrac{1 - 5x + 5x^2 - (1 - 3x + x^2)\sqrt{1 - 4x}}{2x\sqrt{1 - 4x}} \\ \left[\dfrac{\partial B}{\partial q}\right]_{q=1} = \dfrac{1 - 3x - (1 - x)\sqrt{1 - 4x}}{2\sqrt{1 - 4x}}. \end{cases}$$

Instead the computations related to the generating series D are a little bit more complicated. Again in [13] we find the following functional equation satisfied by D:

$$qxD^3 + \big(3(1-q)x - 1\big)D^2 - \big(3(1-q)x - 1\big)D + (1-q)x = 0.$$

Differentiating both sides with respect to q and then solving for $\frac{\partial D}{\partial q}$ yields:

$$\frac{\partial D}{\partial q} = -\frac{xD^3 - 3xD^2 + 3xD - x}{3qxD^2 + 2(3(1-q)x-1)D - 3(1-q)x + 1}.$$

Now, evaluating at $q = 1$ and recalling that $D(1,x) = C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating series of Catalan numbers, we get the following series:

$$\left[\frac{\partial D}{\partial q}\right]_{q=1} = \frac{-1 + 6x - 9x^2 + 2x^3 + (1 - 4x + 3x^2)\sqrt{1-4x}}{x(1 - 4x - \sqrt{1-4x})}.$$

We finally have all the information needed to compute $SC_3(x)$ using (13), and we obtain formula (14). A careful algebraic manipulation of this series yields the stated expression for the coefficients $sc_3(\mathcal{D}_n)$. □

The integer sequence $sc_3(\mathcal{D}_n)$ starts $0, 0, 0, 2, 38, 322, 2112, 12\,210, 65\,494, 334\,334, \ldots$. Neither this sequence nor such a sequence divided by 2 appear in [14].

Proposition 6.3. *The Hasse index of order 3 of the class of Dyck lattices is asymptotically Boolean.*

Proof. Since we have not fully explained the computations needed to derive the coefficients $sc_3(\mathcal{D}_n)$, we will provide a proof independent from the explicit knowledge of such coefficients.

Since series (14) can be rewritten as:

$$SC_3(x) = \frac{1}{x}\left(1 - x - x^2 - \frac{Q(x)}{(1-4x)^{5/2}}\right),$$

when n is sufficiently large we have

$$sc_3(\mathcal{D}_n) = \left[x^n\right]SC_3(x) = -\left[x^{n+1}\right]Q(x)(1-4x)^{-5/2}.$$

Using Darboux's theorem, we get

$$sc_3(\mathcal{D}_n) \sim -\frac{Q(\xi)}{\xi^{n+1}}\frac{(n+1)^{5/2-1}}{\Gamma(5/2)},$$

where $\xi = \frac{1}{4}$. Since $Q(\xi) = \frac{3}{128}$ and $\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}$, we obtain

$$sc_3(\mathcal{D}_n) \sim \frac{2^{2n-3}n^{3/2}}{\sqrt{\pi}}.$$

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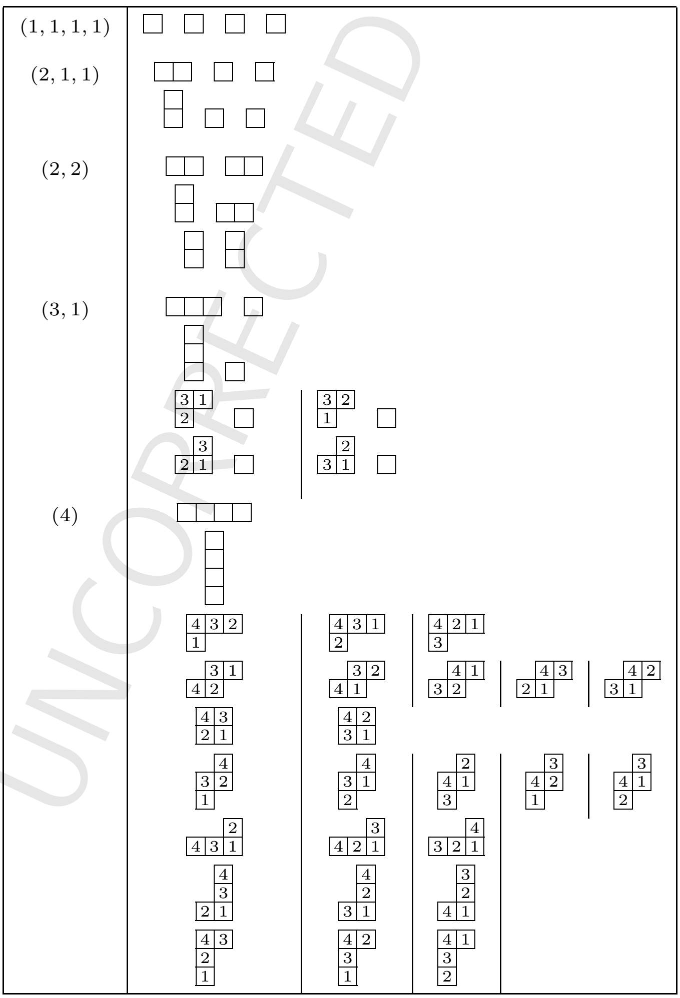
Recalling that $|\mathcal{D}_n| \sim \frac{4^n}{n\sqrt{n\pi}}$, we finally have

$$i_3(\mathcal{D}_n) = \frac{sc_3(\mathcal{D}_n)}{|\mathcal{D}_n|} \sim \frac{n^3}{8}. \qquad \square$$

7. Saturated chains of length 4

Our last application of the general formula (5) concerns the enumeration of saturated chains of length 4. This corresponds to setting $h = 4$ in (5).

The integer partitions of 4 are $(1,1,1,1)$, $(2,1,1)$, $(2,2)$, $(3,1)$ and (4). In the table below, for each of such partitions, we depict the possible tuples of skew Ferrers shapes, also specifying all the different Young tableau structures they can be endowed with (in case there is more than one).



Also in this case, applying Theorem 4.1, we obtain the following result, in which the generating series of saturated chains of length 4 is expressed in terms of the number of occurrences of certain tuples of factors.

Proposition 7.1. *The generating series for the number of saturated chains of length 4 of Dyck lattices is given by*

$$\begin{aligned} SC_4(x) = \sum_{n\geq 0}\sum_{\gamma\in\mathcal{D}_n}\big(&24\cdot\#(du,du,du,du)_\gamma + 12\cdot\#(du,du,ddu)_\gamma \\ &+ 12\cdot\#(du,du,duu)_\gamma + 6\cdot\#(ddu,ddu)_\gamma \\ &+ 6\cdot\#(ddu,duu)_\gamma + 6\cdot\#(duu,duu)_\gamma \\ &+ 4\cdot\#(du,dddu)_\gamma + 4\cdot\#(du,duuu)_\gamma \\ &+ 8\cdot\#(du,dudu)_\gamma + 8\cdot\#(du,dduu)_\gamma \\ &+ \#(duuuu)_\gamma + \#(ddddu)_\gamma) + 3\cdot\#(duddu)_\gamma \\ &+ 5\cdot\#(ddudu)_\gamma + 2\cdot\#(dduu)_\gamma) + 5\cdot\#(duduu)_\gamma \\ &+ 3\cdot\#(ddduu)_\gamma + 3\cdot\#(dduuu)_\gamma) + 3\cdot\#(duudu)_\gamma\big)x^n. \end{aligned} \tag{17}$$

In the rest of this section we will determine all the bivariate generating series which describe the distribution of each of the set of pairwise disjoint occurrences appearing in the above proposition among all Dyck paths of fixed semilength. Actually, to be more precise, we reduce the number of generating series to compute by observing that some of them trivially coincide (by a simple application of the involution which maps every Dyck path γ to the Dyck path obtained by reading γ from right to left and exchanging u and d steps, as we already did in previous sections). For instance, the distribution of the set $(du, dddu)$ is equal to the distribution of the set $(du, duuu)$; similarly, the distribution of the factor $ddudu$ is equal to the distribution of the factor $duduu$.

In order to keep the length of the paper to a minimum, we will give details only for some of the above cases, leaving the analysis of remaining ones to the reader. In particular, in the first case we examine (which is that of the factor $duuuu$), we also carefully describe how to obtain a system of equations satisfied by certain generating series (see below) from purely combinatorial considerations. In all the remaining cases, the reader is invited to employ similar arguments to verify that the displayed systems are indeed correct.

Before starting, here is a list of general notations we will use in what follows. We warn the reader that some of the letters we use below are the same we have used in previous sections, but the meaning may in general be very different.

- $S_{\gamma_1,\ldots,\gamma_r}(x)$: generating series of Dyck paths with respect to the number of pairwise disjoint occurrences of $\gamma_1,\ldots,\gamma_r$ ($\gamma_1,\ldots,\gamma_r$ are words on the alphabet $\{u,d\}$).

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- p, q: names for indeterminates keeping track of the number of occurrences of several kinds of words on $\{u, d\}$ (in each case, we will specify which words is associated with p and q, except for the cases in which we deal with a single factor).
- A: generating series for unrestricted Dyck paths.
- B_i: generating series for Dyck paths starting with exactly i u steps.
- D: generating series for Dyck paths starting with at least a certain number of u steps (such number will be clear in each case).
- C: generating series of Catalan numbers, i.e. $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.
- $\mathcal{D}$: set of all Dyck paths.

(**duuuu**) It is clear that any nonempty Dyck path is counted by either B_1, B_2, B_3 or D, and an analogous fact holds for those paths starting with ud (if we remove the starting peak). Moreover, every Dyck path γ starting with uud can be decomposed as $\gamma = u\gamma' d\gamma''$, with $\gamma', \gamma'' \in \mathcal{D}$ and γ' starting with a peak ud. In this decomposition we observe that, if we remove γ', we are left with another path starting with ud. An analogous argument can be employed for Dyck paths starting with $uuud$. Finally, if a Dyck path starts with at least 4 u steps, then we have two cases: either it start with exactly 4 u steps, in which case it starts with an elevated path starting with $uuud$ (and if we remove it we are left with a path starting with ud), or it starts with at least 5 u steps, in which case it starts with an elevated path with at least 4 u steps (and if we remove it we are left with a path starting with ud). These considerations lead us to the following system:

$$\begin{cases} A = 1 + B_1 + B_2 + B_3 + D \\ B_1 = x(1 + B_1 + B_2 + B_3 + qD) \\ B_2 = B_1^2 \\ B_3 = B_2 \cdot B_1 \\ D = (B_3 + D) \cdot B_1. \end{cases} \tag{18}$$

Replacing $D = \frac{B_1^4}{1-B_1}$ in the first equation gives $A = \frac{1}{1-B_1}$; the same substitution in the second equation leads to the following equality:

$$(1-q)xB_1^4 - B_1^2 + B_1 - x = 0. \tag{19}$$

Setting $q = 1$ in (19) gives

$$[B_1]_{q=1} = \frac{1-\sqrt{1-4x}}{2} = xC(x).$$

We now need to compute $[\frac{\partial A}{\partial q}]_{q=1}$. We have:

$$\left[\frac{\partial A}{\partial q}\right]_{q=1} = \left[\frac{1}{(1-B_1)^2}\frac{\partial B_1}{\partial q}\right]_{q=1} = \frac{1}{(1-xC(x))^2}\left[\frac{\partial B_1}{\partial q}\right]_{q=1}.$$

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Deriving equality (19) with respect to q we obtain:

$$-xB_1^4 + 4(1-q)xB_1^3\frac{\partial B_1}{\partial q} - 2B_1\frac{\partial B_1}{\partial q} + \frac{\partial B_1}{\partial q} = 0.$$

Setting $q = 1$ we get

$$-x^5C(x)^4 - 2xC(x)\left[\frac{\partial B_1}{\partial q}\right]_{q=1} + \left[\frac{\partial B_1}{\partial q}\right]_{q=1} = 0$$

and hence

$$\left[\frac{\partial B_1}{\partial q}\right]_{q=1} = \frac{x^5C(x)^4}{1-2xC(x)}.$$

Standard computations then lead to the following expression for $S_{duuuu}(x) = [\frac{\partial A}{\partial q}]_{q=1}$:

$$S_{duuuu}(x) = -\frac{1-8x-19x^2-13x^3-(1-6x+9x^2-2x^3)\sqrt{1-4x}}{2x(1-4x)}.$$

(**duudu**) The generating series $A(q,x), B_1(q,x), B_2(q,x), D(q,x)$ satisfy the following system:

$$\begin{cases} A = 1 + B_1 + B_2 + D \\ B = x\big(1 + B_1 + xB_1 + qB_1(B_1 - x) + D\big) \\ C = B_1B_1 \\ D = B_1(B_2 + D). \end{cases}$$

Using an argument completely analogous to the previous case, we are able to determine a functional equation satisfied by B_1, hence (differentiating with respect to q and then evaluating in $q = 1$) the following expression for $[\frac{\partial B_1}{\partial q}]_{q=1}$:

$$\left[\frac{\partial B_1}{\partial q}\right]_{q=1} = \frac{-x^4C(x) + x^3(1+x)C(x)^2 - x^3C(x)}{1-2xC(x)}.$$

This result can be used to determine the generating series $S_{duudu}(x)$ as follows:

$$\begin{aligned} S_{duudu}(x) &= \left[\frac{\partial A}{\partial q}\right]_{q=1} = \left[\frac{1}{(1-B_1^2)}\frac{\partial B_1}{\partial q}\right]_{q=1} = \frac{1}{(1-xC(x))^2}\left[\frac{\partial B_1}{\partial q}\right]_{q=1} \\ &= \frac{x^4C(x)^4}{\sqrt{1-4x}} = -\frac{1-6x+8x^2-(1-4x+2x^2)\sqrt{1-4x}}{2(1-4x)}. \end{aligned}$$

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(duduu) The generating series $A(q,x), B_1(q,x), D(q,x)$ satisfy the following system:

$$\begin{cases} A = 1 + B_1 + D \\ B_1 = x + x^2(1 + B_1 + qD) + xD \\ D = B_1(B_1 + D). \end{cases}$$

Solving such a system then leads to the following expression for $S_{duduu}(x) = [\frac{\partial A}{\partial q}]_{q=1}$:

$$S_{duduu}(x) = \frac{1 - 4x + 2x^2 - (1-2x)\sqrt{1-4x}}{2\sqrt{1-4x}}.$$

(dduu) This generating series has already been computed in the previous section (it is $[\frac{\partial A}{\partial q}]_{q=1}$ in the proof of Theorem 6.1).

(dduuu) The generating series $A(q,x), B_1(q,x), B_2(q,x), D(q,x)$ satisfy the following system:

$$\begin{cases} A = 1 + B_1 + B_2 + D \\ B_1 = xA \\ B_2 = xB_1(1 + B_1 + B_2 + qD) \\ D = x(B_2 + D)(1 + B_1 + B_2 + qD). \end{cases}$$

Solving such a system then leads to the following expression for $S_{dduuu}(x) = [\frac{\partial A}{\partial q}]_{q=1}$:

$$S_{dduuu}(x) = \frac{1 - 6x + 9x^2 - 2x^3 - (1 - 4x + 3x^2)\sqrt{1-4x}}{2x\sqrt{1-4x}}.$$

(du, duuu) Here p and q keep track of the factors du and $duuu$, respectively. The generating series $A(p,q,x), B_1(p,q,x), B_2(p,q,x), D(p,q,x)$ satisfy the following system:

$$\begin{cases} A = 1 + B_1 + B_2 + D \\ B_1 = x(1 + pB_1 + pB_2 + pqD) \\ B_2 = B_1 B_1 \\ D = (B_2 + D) \cdot B_1. \end{cases}$$

From the above system we also get the following identities:

$$A = \frac{1}{1 - B_1}, \qquad D = \frac{B_1^3}{1 - B_1}$$

which in turn yield the following identity satisfied by B:

$$p(1-q)xB_1^3 - B_1^2 + \big(1 - (1-p)x\big)B_1 - x = 0. \tag{20}$$

Observe that the generating series we are interested in can be expressed in terms of A (hence of B_1) as follows:

$$S_{duu,duuu}(x) = \left[\frac{\partial^2 A}{\partial p \partial q} - \frac{\partial A}{\partial q}\right]_{p=q=1}$$
$$= \left[\frac{2}{(1-B_1)^3}\frac{\partial B_1}{\partial p}\frac{\partial B_1}{\partial q} + \frac{1}{(1-B_1)^2}\frac{\partial^2 B_1}{\partial p \partial q} - \frac{1}{(1-B_1)^2}\frac{\partial B_1}{\partial q}\right]_{p=q=1}. \tag{21}$$

By differentiating Eq. (20) with respect to p and q (separately), we obtain respectively:

$$(1-q)xB_1^3 + 3p(1-q)xB_1^2\frac{\partial B_1}{\partial p} - 2B\frac{\partial B_1}{\partial p} - \big(1+(1-p)x\big)\frac{\partial B_1}{\partial p} = 0$$
$$pxB_1^3 - 3p(1-q)xB_1^2\frac{\partial B_1}{\partial q} + 2B_1\frac{\partial B_1}{\partial q} + \big(1+(1-p)x\big)\frac{\partial B_1}{\partial q} = 0,$$

hence, by setting $p = q = 1$:

$$\left[\frac{\partial B_1}{\partial p}\right]_{p=q=1} = \frac{x^2C(x)}{1-2xC(x)} \quad \text{and} \quad \left[\frac{\partial B_1}{\partial q}\right]_{p=q=1} = \frac{x^4C(x)^4}{1-2xC(x)}.$$

In a similar fashion, differentiating Eq. (20) with respect to both p and q, and then setting $p = q = 1$, we get

$$\left[\frac{\partial^2 B_1}{\partial p \partial q}\right]_{p=q=1} = \frac{1}{\sqrt{1-4x}}\left(x^4C(x)^3 + \frac{3x^4(1+x)C(x)^3}{\sqrt{1-4x}} + \frac{2x^6C(x)^4}{\sqrt{1-4x}}\right).$$

Plugging into (21) the expressions obtained for the partial derivatives of B_1, after suitable simplifications we finally get the series:

$$S_{du,duuu}(x) = \frac{1-10x+32x^2-32x^3-(1-8x+18x^2-8x^3-2x^4)\sqrt{1-4x}}{x(1-4x)^2}.$$

$(\mathbf{ddu}, \mathbf{duu})$ Here p and q keep track of the factors ddu and duu, respectively. The generating series $A(p,q,x), B_1(p,q,x), D(p,q,x)$ satisfy the following system:

$$\begin{cases} A = 1 + B_1 + D \\ B_1 = x(1 + B_1 + qD) \\ D = x(A-1) + x(A-1)(pB_1 + pqD). \end{cases}$$

Solving such a system then leads to the following expression for $S_{ddu,duu}(x) = [\frac{\partial^2 A}{\partial p \partial q}]_{p=q=1} - S_{dduu}(x)$:

$$S_{ddu,duu}(x) = -\frac{2-17x+44x^2-34x^3+4x^4-(2-13x+22x^2-8x^3)\sqrt{1-4x}}{2x(1-4x)^{3/2}}.$$

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(duu, duu) The generating series $A(q,x), B_1(q,x), D(q,x)$ satisfy the following system:

$$\begin{cases} A = 1 + B_1 + D \\ B_1 = x(1 + B_1 + qD) \\ D = (B_1 + D) \cdot B_1. \end{cases}$$

Solving such a system then leads to the following expression for $S_{duu,duu}(x) = \frac{1}{2}[\frac{\partial^2 A}{\partial q^2}]_{q=1}$:

$$S_{duu,duu}(x) = \frac{1 - 10x + 32x^2 - 32x^3 - (1 - 8x + 18x^2 - 8x^3 - 2x^4)\sqrt{1-4x}}{2x(1-4x)^2}.$$

(du, dudu) Here p and q keep track of the factors du and $dudu$, respectively.

The generating series $A(p,q,x), B_1(p,q,x), D(p,q,x)$ satisfy the following system:

$$\begin{cases} A = 1 + B_1 + D \\ B_1 = x\bigl(1 + px + pq(B_1 - x) + pD\bigr) \\ D = B_1(B_1 + D). \end{cases}$$

Solving such a system then leads to the following expression for $S_{du,dudu}(x) = [\frac{\partial^2 A}{\partial p \partial q} - 2\frac{\partial A}{\partial q}]_{p=q=1}$:

$$S_{du,dudu}(x) = -\frac{1 - 7x + 12x^2 - 2x^3 - (1 - 5x + 4x^2)\sqrt{1-4x}}{(1-4x)^{3/2}}.$$

(du, dduu) Here p and q keep track of the factors du and $dduu$, respectively.

The generating series $A(p,q,x), B_1(p,q,x), D(p,q,x)$ satisfy the following system:

$$\begin{cases} A = 1 + B_1 + D \\ B_1 = x(1 + pB_1 + pD) \\ D = x(A - 1)(1 + pB_1 + pqD). \end{cases}$$

Solving such a system then leads to the following expression for $S_{du,dduu}(x) = [\frac{\partial^2 A}{\partial p \partial q} - \frac{\partial A}{\partial q}]_{p=q=1}$:

$$S_{du,dduu}(x) = -\frac{1 - 8x + 18x^2 - 8x^3 - 2x^4 - (1 - 6x + 8x^2)\sqrt{1-4x}}{x(1-4x)^{3/2}}.$$

(du, du, duu) Here p and q keep track of the factors du and duu, respectively.

The generating series $A(p,q,x), B_1(p,q,x), D(p,q,x)$ satisfy the following system:

$$\begin{cases} A = 1 + B_1 + D \\ B_1 = x(1 + pB_1 + pqD) \\ D = (B_1 + D) \cdot B_1. \end{cases}$$

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Solving such a system then leads to the following expression for $S_{du,du,duu}(x) = [\frac{1}{2}\frac{\partial^3 A}{\partial^2 p \partial q} - \frac{\partial^2 A}{\partial p \partial q} + \frac{\partial A}{\partial q}]_{p=q=1}$:

$$S_{du,du,duu}(x) = \frac{3 - 34x + 130x^2 - 180x^3 + 50x^4 + 4x^5 - (3 - 28x + 80x^2 - 64x^3)\sqrt{1-4x}}{2x(1-4x)^{5/2}}.$$

(**du**, **du**, **du**, **du**) In this case we simply have to compute $[\frac{\partial^4 V}{\partial q^4}]_{q=1}$, where $V(q,x)$ is the generating series given in (11), thus obtaining:

$$S_{du,du,du,du}(x) = \frac{12(-1+15x-84x^2+210x^3-210x^4+42x^5+2x^6+(1-13x+60x^2-112x^3+64x^4)\sqrt{1-4x}\,)}{x(1-4x)^{7/2}}.$$

We are now ready to state and prove the formula for the number of saturated chains of length 4.

Theorem 7.1. *The generating series for the number of saturated chains of length 4 of $\mathcal{D}_n$ is*

$$SC_4(x) = \sum_{n\geq 0} sc_4(\mathcal{D}_n)x^n = \frac{p(x) - q(x)\sqrt{1-4x}}{x(1-4x)^4}, \tag{22}$$

where

$$p(x) = 1 - 12x + 31x^2 + 144x^3 - 864x^4 + 1280x^5 - 256x^6,$$
$$q(x) = 1 - 10x + 13x^2 + 154x^3 - 560x^4 + 488x^5 + 88x^6.$$

Moreover, the coefficients $sc_4(\mathcal{D}_n)$ can be expressed as

$$sc_4(\mathcal{D}_n) = \binom{2n}{n}\frac{(n^4 + 2n^3 - 13n^2 + 4n + 8)(n-3)(n-2)}{4(2n-1)(2n-3)(n+1)} \quad (n \geq 2), \tag{23}$$

which also implies that the Hasse index of order 4 of the class of Dyck lattices is asymptotically Boolean.

Proof. Just collect all the results obtained above and plug them into the series $SC_3(x)$ given in (17) to get (22). Observe that this series can be decomposed as

$$SC_3(x) = -4 + x + \frac{9}{4}\sqrt{1-4x} + \frac{11}{32}x\sqrt{1-4x} - \frac{1-\sqrt{1-4x}}{x} + \frac{15}{128}\frac{1}{(1-4x)^{3+1/2}}$$
$$- \frac{39}{128}\frac{1}{(1-4x)^{2+1/2}} - \frac{21}{64}\frac{1}{(1-4x)^{1+1/2}} + \frac{273}{64}\frac{1}{\sqrt{1-4x}}.$$

Now, since

$$[x^n]\frac{1}{(1-4x)^{k+1/2}} = \binom{2n}{n}\frac{\binom{2n+2k}{2k}}{\binom{n+k}{k}},$$

we have (for $n \geq 2$)

$$sc_3(\mathcal{D}_n) = \binom{2n}{n}\Bigg[-\frac{9}{4}\frac{1}{2n-1} - \frac{11}{32}\frac{n}{2(2n-1)(2n-3)} - \frac{2}{n+1} + \frac{15}{128}\frac{\binom{2n+6}{6}}{\binom{n+3}{3}} - \frac{39}{128}\frac{\binom{2n+4}{4}}{\binom{n+2}{2}} - \frac{21}{64}\frac{\binom{2n+2}{2}}{\binom{n+1}{1}} + \frac{273}{64}\Bigg]$$

which can be reduced to (23). □

8. Conclusions and further work

We have derived a general formula for the enumeration of saturated chains of any fixed length h in Dyck lattices. However, we have applied such a formula only when h is small (namely $h = 2, 3, 4$). When h becomes bigger, computations become much more complicated. Is it possible to conceive a different approach more suitable for effective computation? Notice that, from the results we have found, as well as from some numerical experiments, it seems plausible to conjecture that, for $n \geq h$:

$$sc_h(\mathcal{D}_n) = \frac{\binom{2n-h}{n}}{\binom{2n}{n}}\frac{1}{n-h+1}p_h(n)C_n = \binom{2n-h}{n}\frac{p_h(n)}{(n+1)(n-h+1)},$$

where $p_h(n)$ is a monic polynomial of degree $h+1$.

We have proved that the Hasse indexes of order 1, 2, 3 and 4 of Dyck lattices are asymptotically Boolean. The natural conjecture is that the Hasse index of any order $h \geq 1$ is asymptotically Boolean, i.e. that $i_h(\mathcal{D}_n) = \frac{sc_h(\mathcal{D}_n)}{|\mathcal{D}_n|} \sim \frac{n^h}{2^h}$ (for $n \to +\infty$) for every $h \geq 1$.

The problem of enumerating (saturated) chains can also be posed for other classes of posets. In this context, it would be interesting to find analogous results in the case of Motzkin and Schröder lattices.

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