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Calculus of Variations

A natural approach to the asymptotic mean value property for the p-Laplacian

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Abstract Let $1 \leq p \leq \infty$. We show that a function $u \in C(\mathbb{R}^N)$ is a viscosity solution to the normalized p-Laplace equation $\Delta_p^n u(x) = 0$ if and only if the asymptotic formula

$$u(x) = \mu_p(\varepsilon, u)(x) + o(\varepsilon^2)$$

holds as $\varepsilon \to 0$ in the viscosity sense. Here, $\mu_p(\varepsilon, u)(x)$ is the p-mean value of u on $B_\varepsilon(x)$ characterized as a unique minimizer of

$$\|u - \lambda\|_{L^p(B_\varepsilon(x))}$$

with respect to $\lambda \in \mathbb{R}$. This kind of asymptotic mean value property (AMVP) extends to the case $p = 1$ previous (AMVP)'s obtained when $\mu_p(\varepsilon, u)(x)$ is replaced by other kinds of mean values. The natural definition of $\mu_p(\varepsilon, u)(x)$ makes sure that this is a monotonic and continuous (in the appropriate topology) functional of u. These two properties help to establish a fairly general proof of (AMVP), that can also be extended to the (normalized) parabolic p-Laplace equation.

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1 Introduction and main theorems

It is well-known that the classical *mean value property* characterizes harmonic functions and helps to derive most of their salient properties, such as weak and strong maximum principles, analyticity, Liouville's theorem, Harnack's inequality and more. In fact, we know that a continuous function u is harmonic in an open set $\Omega \subseteq \mathbb{R}^N$ if and only if

$$u(x) = \fint_{B_\varepsilon(x)} u(y)\,dy = \fint_{\partial B_\varepsilon(x)} u(y)\,dS_y \tag{1.1}$$

for every ball $B_\varepsilon(x)$ with $\overline{B_\varepsilon(x)} \subset \Omega$; here, $\fint_E u$ denotes the *mean value* of u over a set E with respect to the relevant measure (see Evans [4] for instance). The relation (1.1) can also be regarded as a statistical characterization of solutions of the Laplace equation, without an explicit appearance of derivatives of u. A similar mean value property can also be obtained for linear elliptic equations with constant coefficients, by replacing balls by appropriate ellipsoids (see [2,3]).

Recently, starting with the work of Manfredi et al. [14], a great attention has been paid to the so-called *asymptotic mean value property (AMVP)* and its applications to game theory. In [14], based on the formula

$$\fint_{B_\varepsilon(x)} v(y)\,dy = v(x) + \frac{1}{2}\frac{\Delta v(x)}{N+2}\varepsilon^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0, \tag{1.2}$$

that holds for any smooth function v not necessarily harmonic, it is shown that the characterization (1.1) for the harmonicity of u can be replaced by the weaker (AMVP):

$$u(x) = \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0, \tag{1.3}$$

for all $x \in \Omega$.

Nonetheless, the decisive contribution of [14] is the observation that, provided the mean value in (1.3) is replaced by a suitable (nonlinear) statistical value related to u, an (AMVP) also characterizes *p-harmonic* functions, that is the (viscosity) solutions of the *normalized p-Laplace equation* $\Delta_p^n u = 0$. Here,

$$\Delta_p^n u = \frac{\nabla \cdot (|\nabla u|^{p-2}\nabla u)}{|\nabla u|^{p-2}} \quad \text{for } 1 \le p < \infty, \quad \Delta_\infty^n u = \frac{\langle \nabla^2 u\, \nabla u, \nabla u\rangle}{|\nabla u|^2},$$

denotes the so-called *normalized or homogeneous p-Laplacian*.

In fact, in the same spirit of (1.2), for $1 < p \le \infty$ and for any smooth function with $\nabla v(x) \neq 0$, they proved the formula:

$$\mu_p^*(\varepsilon, v) = v(x) + \frac{1}{2}\frac{\Delta_p^n v(x)}{N+p}\varepsilon^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0,$$

where

$$\begin{aligned}\mu_p^*(\varepsilon, u) = {} & \frac{N+2}{N+p}\fint_{B_\varepsilon(x)} u(y)\,dy \\ & + \frac{1}{2}\frac{p-2}{N+p}\left(\max_{\overline{B_\varepsilon(x)}} u + \min_{\overline{B_\varepsilon(x)}} u\right).\end{aligned} \tag{1.4}$$

(The average of the minimum and the maximum will be referred to as the *min–max mean* of u.)

That formula allowed them to prove that u is p-harmonic in the viscosity sense in Ω if and only if

$$u(x) = \mu_p^*(\varepsilon, u) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0, \tag{1.5}$$

in the viscosity sense for every $x \in \Omega$ (see Sect. 3 for the relevant definitions), thus obtaining an (AMVP) for p-harmonic functions. It is also worth a mention that, for $N = 2$ and small values of the parameter $p > 1$, in [11] it is proved that the (AMVP) holds directly for weak solutions of the p-Laplace equation, without the need to interpret the formula in the viscosity sense.

Thus, the mean $\mu_p^*(\varepsilon, u)$ is an example of the desired (nonlinear) statistical value mentioned above. By similar arguments, one can obtain an (AMVP) with the ball $B_\varepsilon(x)$ replaced by the sphere $\partial B_\varepsilon(x)$ simply by replacing $(N+2)/(N+p)$ and $(p-2)/(N+p)$ by the numbers $N/(N+p-2)$ and $(p-2)/(N+p-2)$.

In the quest of extending this type of result to the case $p = 1$, which is not covered by the choice (1.4), other kinds of means were proposed by several authors. Here, we mention the ones considered by Hartenstine and Rudd [7], based on the *median* of a function,

$$\mu_p'(\varepsilon, u) = \frac{1}{p} \operatorname*{med}_{\partial B_\varepsilon(x)} u + \frac{p-1}{2p} \left(\min_{\overline{B_\varepsilon(x)}} u + \max_{\overline{B_\varepsilon(x)}} u \right), \tag{1.6}$$

$$\mu_p''(\varepsilon, u) = \frac{2-p}{p} \operatorname*{med}_{\partial B_\varepsilon(x)} u + \frac{2(p-1)}{p} \fint_{\partial B_\varepsilon(x)} u(y)\, dS_y, \tag{1.7}$$

and that considered by Kawohl, Manfredi and Parviainen [9],

$$\mu_p^{**}(\varepsilon, u) = \frac{N+1}{N+p} \operatorname{av}_\varepsilon(u)(x) + \frac{1}{2} \frac{p-1}{N+p} \left(\min_{\overline{B_\varepsilon(x)}} u + \max_{\overline{B_\varepsilon(x)}} u \right), \tag{1.8}$$

where

$$\operatorname{av}_\varepsilon(u)(x) = \fint_{L_\varepsilon} u(x+y)\, dS_y,$$

$L_\varepsilon = \{y \in B_\varepsilon(x) : (y-x) \cdot \nu = 0\}$ and

$$\nu = \nu_{x,\varepsilon} \in \partial B_1(0) \ \text{ is such that } \ u(x + \varepsilon\nu) = \min_{\overline{B_\varepsilon(x)}} u.$$

Both $\mu_p'(\varepsilon, u)$ and $\mu_p''(\varepsilon, u)$ yield an (AMVP) for all the cases $1 \le p \le \infty$, but *only when* $N = 2$, and $\mu_p^{**}(\varepsilon, u)$ produces an (AMVP) for any $1 \le p \le \infty$ and $N \ge 2$.

In this paper, for $1 \le p \le \infty$, we propose one more mean that helps us to characterize—in an intrinsic way—p-harmonic functions by an (AMVP). Its definition was inspired by the simple remark that the median, the mean value and the min–max mean of a continuous function u on a compact topological space X equipped with a positive Radon measure ν are respectively the unique real values $\mu_p^X(u)$ that solve the variational problem

$$\|u - \mu_p^X(u)\|_{L^p(X,\nu)} = \min_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L^p(X,\nu)}, \tag{1.9}$$

for $p = 1, 2$, and ∞. Thus, it is natural to ask whether the solution of (1.9) yields a characterization of viscosity solutions of $\Delta_p^n u = 0$ by means of an (AMVP), *for each fixed* $1 \le p \le \infty$.

Therefore, for each $1 \le p \le \infty$, we consider the *p-mean of* u in $B_\varepsilon(x)$, that is the number defined as

$$\mu_p(\varepsilon, u)(x) = \text{ the unique } \mu \in \mathbb{R} \text{ satisfying (1.9) with } X = \overline{B_\varepsilon(x)}. \tag{1.10}$$

The main result of this paper is the following characterization.

Theorem 1.1 *Let $1 \le p \le \infty$ and let Ω be an open subset of $\mathbb{R}^N$. For a function $u \in C(\Omega)$ the following assertions are equivalent:*

(i) *u is a viscosity solution of $\Delta_p^n u = 0$ in Ω;*
(ii) *$u(x) = \mu_p(\varepsilon, u)(x) + o(\varepsilon^2)$ as $\varepsilon \to 0$, in the viscosity sense for every $x \in \Omega$.*

As a by-product, this theorem confirms the (AMVP) for $\mu'_p(\varepsilon, u)$ and $\mu^*_p(\varepsilon, u)$ for the case $p = 1$ in *any dimension* $N \ge 2$.

Theorem 1.1 is based on the asymptotic formula

$$\mu_p(\varepsilon, v)(x) = v(x) + \frac{1}{2}\frac{\Delta_p^n v(x)}{N+p}\varepsilon^2 + o(\varepsilon^2) \text{ as } \varepsilon \to 0,$$

that holds for any smooth function v such that $\nabla v(x) \neq 0$.

We mention in passing that the mean $\mu_p^X(u)$ has also been considered in [5], when $p \ge 2$ and $N = 2$, when X is a finite set and ν is the counting measure and has proved to be effective in the numerical approximation of the operator Δ_p^n. Another type of (AMVP) has been proved in [6] for $N = 2$ and $1 < p < \infty$; however, the mean considered there, besides the values of the function u on $B_\varepsilon(x)$, also depends on the value of ∇u at x.

Compared to the means defined in (1.4), (1.6), (1.7), and (1.8) (and that in [6]), $\mu_p(\varepsilon, u)$ has a drawback, since it cannot be defined explicitly, unless $p = 1, 2, \infty$. Nevertheless, we think it is somewhat *natural,* for a number of reasons, that we list below.

First of all, it provides a unified proof of the (AMVP) for all the range $1 \le p \le \infty$.

Secondly, it has desirable and advantageous properties that the means defined in (1.4), (1.6), (1.7), and (1.8) do not always have. Those properties are the consequences of the fact that $\mu_p(\varepsilon, u)$ is the *projection* of u on the linear sub-space $\Lambda \subset L^p(B_\varepsilon(x))$ of constant functions on $\overline{B_\varepsilon(x)}$, thus making it in a sense a best possible approximation of u. As a matter of fact, we shall show that the functional $L^p(B_\varepsilon(x)) \ni u \mapsto \mu_p(\varepsilon, u)(x) \in \mathbb{R}$ is *continuous* in the corresponding L^p-topology and *monotonic*, in the sense that

$$u \le v \text{ pointwise implies that } \mu_p(\varepsilon, u)(x) \le \mu_p(\varepsilon, v)(x).$$

In this respect, it is worthwhile noticing that the functionals defined by $\mu'_p(\varepsilon, u)$ and $\mu^{**}_p(\varepsilon, u)$ are always monotonic, but never continuous for $p \in (1, \infty)\backslash\{2\}$, while those defined by $\mu^*_p(\varepsilon, u)$ and $\mu''_p(\varepsilon, u)$ are not always monotonic (the former for $p > 2$, the latter for $1 < p < 2$) and never continuous for $p \in (1, \infty)\backslash\{2\}$, due to the presence of the min–max mean in their definition.

We shall see that the properties of continuity and monotonicity play an essential role in the proof of Theorem 1.1, since they allow to reduce the argument to the simpler case of a *quadratic polynomial* (see Lemma 3.1 and Theorem 3.2). We hope that these important properties will also help to find better proofs in the development of the theory as in [12,13,15].

With a few technical adjustments, it is not difficult to treat the case of the parabolic p-Laplace operator. It is just the matter of replacing the euclidean ball and the Lebesgue measure by a suitable measure space. The appropriate choice is the so-called *heat ball*,

$$E_\varepsilon(x, t) = \left\{(y, s) \in \mathbb{R}^{N+1} : s < t,\ \Phi(x - y, t - s) > \varepsilon^{-N}\right\},$$

where

$$\Phi(y,s) = (4\pi s)^{-N/2} e^{-\frac{|y|^2}{4s}} \mathcal{X}_{(0,\infty)}(s) \text{ for } (y,s) \in \mathbb{R}^N \times (-\infty,\infty)$$

is the fundamental solution for the heat equation, equipped with the space-time measure

$$d\nu(y,s) = \frac{|x-y|^2}{(t-s)^2}\, dy\, ds.$$

Thus, by arguing in a similar spirit, we shall consider the value $\pi_p(\varepsilon,u)(x,t)$ as the unique solution of the variational problem

$$\|u - \pi_p(\varepsilon,u)(x,t)\|_{L^p(E_\varepsilon(x,t),\nu)} = \min_{\lambda\in\mathbb{R}} \|u-\lambda\|_{L^p(E_\varepsilon(x,t),\nu)}. \tag{1.11}$$

Notice that the value $\pi_p(\varepsilon,u)(x,t)$ in (1.11) can be easily computed for $p=2$ as the *caloric mean value* of u, for which a classical mean value property holds true for solutions of the heat equation ([4] [pp. 52–54]). If we define the space-time cylinder $\Omega_T = \Omega \times (0,T)$, we can prove the following companion of Theorem 1.1 .

Theorem 1.2 *Let $1 \le p \le \infty$. For a function $u \in C(\Omega_T)$, the following assertions are equivalent:*

(i) $u_t = \frac{N}{N+p-2}\Delta_p^n u$ *in Ω_T in the viscosity sense;*
(ii) $u(x,t) = \pi_p(\varepsilon,u)(x,t) + o(\varepsilon^2)$ *as $\varepsilon \to 0$ in the viscosity sense for every $(x,t) \in \Omega_T$.*

A similar result can be obtained if we replace $\pi_p(\varepsilon,u)(x,t)$ by the *cylindrical mean* defined by

$$\pi_p'(\varepsilon,u)(x,t) = \frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} \mu_p(\varepsilon,u(\cdot,s))(x)\, ds,$$

in analogy with some results contained in [15]. We chose to use $\pi_p(\varepsilon,u)$ to stress its (natural) connection to the fundamental solution of the heat equation (as it also happens for $\mu_p(\varepsilon,u)$ in the elliptic case).

For further developments and applications of (AMVP)'s, we refer the reader to [1,8,11,12,15,17], and references therein. For numerical applications to game theory, we refer to [5,6].

This paper is organized as follows. In Sect. 2, we derive the pertinent properties of the p-mean value of a continuous function u: continuity and monotonicity will be the most important. Then, we shall prove Theorems 1.1 and 1.2 in Sects. 3 and 4, respectively. Finally, Sect. 5 is devoted to the calculation of some relevant integrals.

2 Properties of p-mean values

Let X be a compact topological space which is also a measure space with respect to a positive Radon measure ν such that $\nu(X) < \infty$. We recall that, if $u \in C(X)$, the *median* $\operatorname{med}_X u$ of u in X is defined as the unique solution λ of the equation

$$\nu(\{y \in X : u(y) \ge \lambda\}) = \nu(\{y \in X : u(y) \le \lambda\}). \tag{2.1}$$

We start by showing that the definitions (1.10) and (1.11) of $\mu_p(\varepsilon,u)$ and $\pi_p(\varepsilon,u)$ are well posed.

Theorem 2.1 *Let $1 \le p \le \infty$ and $u \in C(X)$. There exists a unique real value $\mu_p^X(u)$ such that*

$$\|u - \mu_p^X(u)\|_{L^p(X,\nu)} = \min_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L^p(X,\nu)}.$$

In particular,

$$\mu_1^X(u) = \operatorname*{med}_X u, \qquad \mu_2^X(u) = \fint_X u(y)\, d\nu$$
$$\text{and } \mu_\infty^X(u) = \frac{1}{2}\left(\min_X u + \max_X u\right).$$

Furthermore, for $1 \le p < \infty$, $\mu_p^X(u)$ is characterized by the equation

$$\int_X \left|u(y) - \mu_p^X(u)\right|^{p-2} \left[u(y) - \mu_p^X(u)\right] d\nu = 0, \tag{2.2}$$

where, for $1 \le p < 2$, we mean that the integrand is zero if $u(y) - \mu_p^X(u) = 0$.

Proof The case $p = 1$ is a straightforward extension of the proofs in [16, Theorems 2.3–2.4], [18, Exercise 2.6.92], [19, Exercise 1.4.27].

If $p = \infty$, the assertion follows at once by observing that

$$\max_X |u - \lambda| = \max\left(\max_X u - \lambda,\ \lambda - \min_X u\right).$$

Next, in the case $1 < p < \infty$, we observe that

$$\min_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L^p(X,\nu)} = \min_{v \in \Lambda} \|u - v\|_{L^p(X,\nu)},$$

where Λ is the subspace of constant functions on X; in other words $\mu_p^X(u)$ is a projection of u on Λ. Thus, the existence, uniqueness and characterization of $\mu_p^X(u)$ are guaranteed by the theorem of the projection, since $L^p(X, \nu)$ is uniformly convex and Λ is a closed subspace, and the differentiability of the function $\lambda \mapsto \|u - \lambda\|_{L^p(X,\nu)}$ (see [10, Lemma 2.8]).

The expression of $\mu_2^X(u)$ is readily computed as the minimum point of a quadratic polynomial. □

Remark 2.2 In the case $p = 1$, the fact that λ satisfies (2.1) is equivalent to $\lambda = \mu_1^X(u)$, see [18, Exercise 2.6.92] and [19, Exercise 1.4.27]. As a result, we see that $\mu_1^X(u)$ in Theorem 2.1 is the unique value satisfying (2.1).

Note that, for $1 < p \le \infty$, Theorem 2.1 extends to the case in which $u \in L^p(X, \nu)$, provided the minimum and the maximum are replaced by

$$\operatorname*{ess\,inf}_X u \quad \text{and} \quad \operatorname*{ess\,sup}_X u,$$

when $p = \infty$.

If $u \in L^1(X, \nu) \backslash C(X)$, it is known that the median of u in X may not be unique (see [16]).

The following corollary will be very useful for further computations. From now on, we set $B = B_1(0)$.

Corollary 2.3 *Let $u \in L^p(B_\varepsilon(x))$, for $1 < p \leq \infty$, and $u \in C(B_\varepsilon(x))$, for $p = 1$.*

If we let $u_\varepsilon(z) = u(x + \varepsilon z)$ for $z \in \overline{B}$ and set

$$\mu_p(\varepsilon, u)(x) = \mu_p^{B_\varepsilon(x)}(u), \tag{2.3}$$

then it holds that

$$\mu_p(\varepsilon, u)(x) = \mu_p(1, u_\varepsilon)(0). \tag{2.4}$$

Proof It suffices to observe that, for every $\lambda \in \mathbb{R}$, it holds that

$$\|u - \lambda\|_{L^p(B_\varepsilon(x))} = \varepsilon^{N/p} \|u_\varepsilon - \lambda\|_{L^p(B)},$$

for $1 \leq p < \infty$, and

$$\|u - \lambda\|_{L^\infty(B_\varepsilon(x))} = \|u_\varepsilon - \lambda\|_{L^\infty(B)},$$

and hence invoke the uniqueness part of Theorem 2.1. □

In the next two theorems we regard $\mu_p^X(u)$ as the value at u of a functional μ_p^X on $L^p(X)$. If $p = 1$ and $u \in L^1(X) \backslash C(X)$, we allow $\mu_1^X(u)$ to be any minimizing value of $\lambda \mapsto \|u - \lambda\|_{L^1(X)}$ on $\mathbb{R}$, whenever it is convenient.

Theorem 2.4 (Continuity) *Let $1 \leq p \leq \infty$. It holds that*

$$\left| \left\| u - \mu_p^X(u) \right\|_{L^p(X)} - \left\| v - \mu_p^X(v) \right\|_{L^p(X)} \right| \leq \|u - v\|_{L^p(X)}, \tag{2.5}$$

for any $u, v \in L^p(X)$.

Moreover, if $u_n \to u$ in $L^p(X)$ for $1 \leq p \leq \infty$ and $u_n, u \in C(X)$ for $p = 1$, then $\mu_p^X(u_n) \to \mu_p^X(u)$ as $n \to \infty$.

In particular, the same conclusion holds for any $p \in [1, \infty]$, if $\{u_n\}_{n\in\mathbb{N}} \subset C(X)$ converges to u uniformly on X as $n \to \infty$.

Proof The inequality (2.5) simply follows by observing that $\left\| u - \mu_p^X(u) \right\|_{L^p(X)}$ is nothing else than the distance of u from the subspace $\Lambda \subset L^p(X)$ of constant functions on X.

Next, if $u_n \to u$ in $L^p(X)$ as $n \to \infty$, (2.5) implies that

$$\|u_n - \mu_p^X(u_n)\|_{L^p(X)} \to \|u - \mu_p^X(u)\|_{L^p(X)} \text{ as } n \to \infty, \tag{2.6}$$

and hence, in particular, that the sequence $\{\mu_p^X(u_n)\}_{n\in\mathbb{N}}$ is bounded. Now, set

$$\mu = \limsup_{n\to\infty} \mu_p^X(u_n);$$

by taking a subsequence, if necessary, we may assume that $\mu_p^X(u_n) \to \mu$ as $n \to \infty$. Thus, (2.6) gives that $\|u - \mu\|_{L^p(X)} = \|u - \mu_p^X(u)\|_{L^p(X)}$ and hence we obtain that $\mu_p^X(u) = \mu$, since $\mu_p(u)$ is unique, by virtue of Theorem 2.1. At the same conclusion we arrive, if we replace lim sup by lim inf, and this means that $\mu_p(u_n) \to \mu_p^X(u)$ as $n \to \infty$. □

Theorem 2.5 [Monotonicity] *Let u and v be two functions in $L^p(X)$, for $1 < p \leq \infty$, or in $C(X)$ for $p = 1$.*

If $u \leq v$ a.e. in X, then $\mu_p^X(u) \leq \mu_p^X(v)$.

Proof The case $p = \infty$ follows by an inspection, since we know that

$$\mu_\infty^X(u) = \frac{1}{2}\left(\operatorname*{ess\,inf}_X u + \operatorname*{ess\,sup}_X u\right).$$

For $1 < p < \infty$, the function $\mathbb{R} \times \mathbb{R} \ni (u, \lambda) \mapsto |u - \lambda|^{p-2}(u - \lambda) \in \mathbb{R}$ is strictly increasing in u for fixed λ and strictly decreasing in λ for fixed u. Thus, if $u \le v$ a.e. in X, we have that

$$\int_X |u(y) - \lambda|^{p-2}[u(y) - \lambda]\, d\nu_y \le \int_X |v(y) - \lambda|^{p-2}[v(y) - \lambda]\, d\nu_y$$

for every $\lambda \in \mathbb{R}$. Since the two functions of λ in this inequality are both decreasing, we easily obtain that $\mu_p^X(u) \le \mu_p^X(v)$, by the characterization of p-means as unique zeroes of (2.2).

The proof in the case in which $p = 1$ runs similarly. We consider the function defined by

$$G_u(\lambda) = \nu(\{y \in X : u(y) \ge \lambda\}) - \nu(\{y \in X : u(y) \le \lambda\}) \quad \text{for } \lambda \in \mathbb{R};$$

it is clear that G_u is decreasing in λ. It is also clear that $G_u(\lambda) \le G_v(\lambda)$ for any fixed λ, if $u \le v$ a.e. in X. Thus, we conclude as in the case $1 < p < \infty$, by the characterization of $\mu_1^X(u)$ and $\mu_1^X(v)$ as the unique zeroes of G_u and G_v, as shown in Theorem 2.1 and Remark 2.2. □

Remark 2.6 Notice that the mean $\mu_p^*(\varepsilon, u)$ in (1.4) is not monotonic when $1 \le p < 2$ and is continuous in $L^p(B_\varepsilon(x))$ only for $p = 2, \infty$.

The mean $\mu_p''(\varepsilon, u)$ in (1.7) is not monotonic for $2 < p < \infty$ and the mean $\mu_p'(\varepsilon, u)$ in (1.6) is not continuous unless $p = \infty$.

Finally, the mean $\mu_p^{**}(\varepsilon, u)$ in (1.8) is not continuous unless $p = \infty$.

To disprove continuity, it is sufficient to take the sequence of functions $u_n(y) = (|y - x|/\varepsilon)^n$ for $y \in B_\varepsilon(x)$: this converges to zero in $L^p(B_\varepsilon(x))$ for $1 \le p < \infty$, but the average of its maximum and minimum is always $1/2$.

The proof of the following proposition is straightforward.

Proposition 2.7 *We have that*

(i) $\mu_p^X(u + c) = c + \mu_p^X(u)$ *for every* $c \in \mathbb{R}$;
(ii) $\mu_p^X(\alpha u) = \alpha\, \mu_p^X(u)$ *for every* $\alpha \in \mathbb{R}$.

3 The (AMVP) for the elliptic case

This section is devoted to prove Theorem 1.1. We first give a proof of the (AMVP) for smooth functions. The following lemma is the crucial step of that proof.

Lemma 3.1 *Let* $1 \le p \le \infty$, *pick* $\xi \in \mathbb{R}^N \setminus \{0\}$, *and let* A *be a symmetric* $N \times N$ *matrix. Consider the quadratic function* $q : B_\varepsilon(x) \to \mathbb{R}$ *defined by*

$$q(y) = q(x) + \xi \cdot (y - x) + \frac{1}{2}\langle A(y - x), y - x\rangle, \quad y \in B_\varepsilon(x).$$

Then it holds that

$$\mu_p(\varepsilon, q)(x) = q(x) + \frac{1}{2(N + p)}\left\{\operatorname{tr}(A) + (p - 2)\frac{\langle A\xi, \xi\rangle}{|\xi|^2}\right\}\varepsilon^2 + o(\varepsilon^2)$$

as $\varepsilon \to 0$.

Proof Set

$$q_\varepsilon(z) = q(x + \varepsilon z), \quad v_\varepsilon(z) = \frac{q_\varepsilon(z) - q(x)}{\varepsilon}, \quad \text{and} \quad v(z) = \xi \cdot z \ \text{ for } \ z \in B.$$

We know that $\mu_p(\varepsilon, q)(x) = \mu_p(1, q_\varepsilon)(0)$, from Corollary 2.3. Then Proposition 2.7 implies that

$$\frac{\mu_p(\varepsilon, q)(x) - q(x)}{\varepsilon} = \mu_p(1, v_\varepsilon)(0).$$

Since v_ε converges to v uniformly on B as $\varepsilon \to 0$, Theorem 2.4 yields that

$$\lim_{\varepsilon \to 0} \frac{\mu_p(\varepsilon, q)(x) - q(x)}{\varepsilon} = \mu_p(1, v)(0).$$

We now claim $\mu_p(1, v)(0) = 0$. Indeed, if $1 \le p < \infty$, by Theorem 2.1, $\lambda = \mu_p(1, v)(0)$ is the unique root of equation

$$\int_B |v(z) - \lambda|^{p-2}[v(z) - \lambda]\, dz = 0.$$

This implies that $\mu_p(1, v)(0) = 0$, since

$$\int_B |v(z)|^{p-2} v(z) dz = \int_B |\xi \cdot z|^{p-2}(\xi \cdot z)\, dz = 0,$$

being $\xi \cdot z$ symmetric in z with respect to the center of B.

If $p = \infty$ instead, by using Theorem 2.1, we can directly compute:

$$2\,\mu_\infty(1, v)(0) = \min_{z \in \overline{B}}(\xi \cdot z) + \max_{z \in \overline{B}}(\xi \cdot z) = -|\xi| + |\xi| = 0.$$

Next, for $1 \le p \le \infty$, set

$$\delta_\varepsilon = \frac{\mu_p(\varepsilon, q)(x) - q(x)}{\varepsilon^2}. \tag{3.1}$$

Case 1 < p < ∞. To simplify notations, for $s \in \mathbb{R}$, we set $h(s) = |s|^{p-2}s$. From the characterization of $\mu = \mu_p(1, v_\varepsilon)(0)$, we know that

$$\int_B h(v_\varepsilon(z) - \mu)\, dz = 0;$$

then, noting $v_\varepsilon(z) = \xi \cdot z + \frac{\varepsilon}{2}\langle Az, z\rangle$ and $\mu_p(1, v_\varepsilon)(0) = \varepsilon\,\delta_\varepsilon$, we obtain that

$$\int_B h(\xi \cdot z + \varepsilon\,[\langle Az, z\rangle/2 - \delta_\varepsilon])\, dz = 0.$$

Without loss of generality, we assume that $|\xi| = 1$. By applying the change of variables $z = R\,y$, where R is a rotation matrix such that ${}^tR\,\xi = e_1$, and setting $C = {}^tR\,A\,R$, we obtain the equation

$$\int_B \frac{h(y_1 + \varepsilon[\langle Cy, y\rangle/2 - \delta_\varepsilon]) - h(y_1)}{\varepsilon}\, dy = 0,$$

since $\int_B h(y_1)\, dy = 0$.

Then, we apply the fundamental theorem of calculus to the function $[0,1] \ni \tau \mapsto h(y_1 + \tau\,\varepsilon\,[\langle Cy, y\rangle/2 - \delta_\varepsilon])$ to infer that

$$\int_B \left\{ \int_0^1 h'(y_1 + \tau\,\varepsilon\,[\langle Cy, y\rangle/2 - \delta_\varepsilon])\,d\tau \right\} [\langle Cy, y\rangle/2 - \delta_\varepsilon]\,dy = 0. \tag{3.2}$$

This equation implies that δ_ε is a weighted average of the function $\langle Cy, y\rangle/2$, and hence, by the mean value theorem, is bounded by its maximum on $\overline{B}$, that we shall denote by c (c is equal to half of the norm of the matrix C).

If $2 \le p < \infty$, it is easy to prove that, by the dominated convergence theorem, (any converging subsequence of) δ_ε converges to the number δ_0 defined by

$$\int_B h'(y_1)\,[\langle Cy, y\rangle/2 - \delta_0]\,dy = 0. \tag{3.3}$$

If $1 < p < 2$, we observe that

$$\left| \int_0^1 h'(y_1 + \tau\,\varepsilon\,[\langle Cy, y\rangle/2 - \delta_\varepsilon])\,[\langle Cy, y\rangle/2 - \delta_\varepsilon]\,d\tau \right| \le 2\,c\,\Big| |y_1| - 2c\,\varepsilon \Big|^{p-2}$$

and

$$\lim_{\varepsilon\to 0} \int_B ||y_1| - 2c\,\varepsilon|^{p-2}\,dy = \int_B |y_1|^{p-2}\,dy.$$

If (any converging subsequence of) δ_ε converges to a number δ_0, then the integrand in (3.2) converges pointwise to $h'(y_1)\,[\langle Cy, y\rangle/2 - \delta_0]$, and hence we can conclude that (3.3) also holds in this case, by the generalized dominated convergence theorem (see Theorem 5.4).

Therefore, by Lemma 5.1 we have that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \delta_\varepsilon &= \frac{1}{2}\,\frac{\int_B |y_1|^{p-2}\,\langle Cy, y\rangle\,dy}{\int_B |y_1|^{p-2}\,dy} \\
&= \frac{1}{2\,(N+p)}\,\{\mathrm{tr}(C) + (p-2)\,\langle C\,e_1, e_1\rangle\} \\
&= \frac{1}{2\,(N+p)}\,\left\{\mathrm{tr}(A) + (p-2)\,\frac{\langle A\xi, \xi\rangle}{|\xi|^2}\right\},
\end{aligned}$$

since $\langle C\,e_1, e_1\rangle = \langle AR\,e_1, R\,e_1\rangle$, with $R\,e_1 = \xi/|\xi|$.

Case $\mathbf{p = 1}$. We know that $\mu_1(\varepsilon, q)$ is the unique root of the equation

$$|\{y \in B_\varepsilon(x) : q(y) > \mu_1(\varepsilon, q)\}| = |\{y \in B_\varepsilon(x) : q(y) < \mu_1(\varepsilon, q)\}|. \tag{3.4}$$

Then, by the change of variables $y = x + \varepsilon\,z$ in (3.4), we can write that

$$\left|\left\{z \in B : \xi\cdot z + \frac{\varepsilon}{2}\,\langle Az, z\rangle > \varepsilon\delta_\varepsilon\right\}\right| = \left|\left\{z \in B : \xi\cdot z + \frac{\varepsilon}{2}\,\langle Az, z\rangle < \varepsilon\delta_\varepsilon\right\}\right|$$

and, by applying the substitution $z = R\,y$, where R is a rotation matrix such that ${}^t R\,\xi = |\xi|\,e_1$, we can infer:

$$\begin{aligned}
&\left|\left\{y \in B : |\xi|\,y_1 + \frac{\varepsilon}{2}\,\langle C\,y, y\rangle > \varepsilon\delta_\varepsilon\right\}\right| \\
&\quad = \left|\left\{y \in B : |\xi|\,y_1 + \frac{\varepsilon}{2}\,\langle C\,y, y\rangle < \varepsilon\delta_\varepsilon\right\}\right|;
\end{aligned} \tag{3.5}$$

here $C = {}^t RAR$, as before.

Now, consider the right-hand side of the last formula, set

$$f_\varepsilon(y) = |\xi|\, y_1 + \frac{\varepsilon}{2}\,\langle C\, y, y\rangle,$$

and

$$c_\varepsilon = |\{y \in B : f_\varepsilon(y) < \varepsilon\delta_\varepsilon\}| - |B^-|,$$

where $B^- = \{y \in B : y_1 \le 0\}$. The use of the change of variables

$$y = \frac{\varepsilon\, z_1}{|\xi|}\, e_1 + z' \ \text{ where } \ z' = (0, z_2, \dots, z_N),$$

yields that

$$\frac{|\xi|}{\varepsilon}\, c_\varepsilon = \left|\left\{(z_1, z') \in B^\varepsilon : \varepsilon^{-1} f_\varepsilon\left(\frac{\varepsilon\, z_1}{|\xi|}\, e_1 + z'\right) < \delta_\varepsilon\right\}\right| - |B^{\varepsilon,-}|,$$

where

$$B^\varepsilon = \{(z_1, z') \in \mathbb{R}^N : (\varepsilon\, z_1/|\xi|)^2 + |z'|^2 < 1\},$$
$$\text{and } \ B^{\varepsilon,-} = \{z \in B^\varepsilon : z_1 \le 0\}.$$

Now, set $B' = \{z' \in \mathbb{R}^{N-1} : |z'| < 1\}$ and notice that, if ε is small enough, by the implicit function theorem, there is a unique function $g_\varepsilon : B' \to \mathbb{R}$ such that

$$\varepsilon^{-1} f_\varepsilon\left(\frac{\varepsilon\, g_\varepsilon(z')}{|\xi|}\, e_1 + z'\right) = \delta_\varepsilon \ \text{ for } \ z' \in B'.$$

We can then infer that

$$\begin{aligned} c_\varepsilon = \frac{\varepsilon}{|\xi|} \int_{B'} \Big\{ &\min\left[g_\varepsilon^+(z'), (|\xi|/\varepsilon)\sqrt{1 - |z'|^2}\right] \\ &- \min\left[g_\varepsilon^-(z'), (|\xi|/\varepsilon)\sqrt{1 - |z'|^2}\right]\Big\}\, dz', \end{aligned}$$

where g_ε^+ and g_ε^- denote the positive and negative parts of g_ε. Thus, since $g_\varepsilon(z') \to \delta_0 - \langle Cz', z'\rangle/2$ pointwise (possibly passing to a subsequence), by the dominated convergence theorem, we obtain that

$$\lim_{\varepsilon\to 0} \frac{|\xi|}{\varepsilon}\, c_\varepsilon = \int_{B'} \left[\delta_0 - \frac{1}{2}\,\langle Cz', z'\rangle\right] dz'.$$

We can repeat the same arguments for the left-hand side of (3.5) and obtain that

$$\lim_{\varepsilon\to 0} \frac{|\xi|}{\varepsilon}\, \left\{|\{y \in B : f_\varepsilon(y) > \varepsilon\delta_\varepsilon\}| - |B^+|\right\} = \int_{B'} \left[\frac{1}{2}\,\langle Cz', z'\rangle - \delta_0\right] dz'.$$

Therefore, (3.5) implies that

$$\int_{B'} \left[\delta_0 - \frac{1}{2}\,\langle C\, z', z'\rangle\right] dz' = 0,$$

and hence

$$\delta_0\, \frac{\omega_{N-1}}{N-1} = \frac{1}{2} \int_{B'} \langle C\, z', z'\rangle\, dz' = \frac{\omega_{N-1}}{2\,(N^2-1)} \sum_{j=2}^{N} C_{jj}.$$

Finally, the desired conclusion follows from

$$\lim_{\varepsilon\to 0}\frac{\mu_1(\varepsilon,q)(x)-q(x)}{\varepsilon^2}=\lim_{\varepsilon\to 0}\delta_\varepsilon=\delta_0,$$

where

$$2\,(N+1)\,\delta_0^2=\mathrm{tr}(C)-\langle C\,e_1,e_1\rangle=\mathrm{tr}(A)-\frac{\langle A\,\xi,\xi\rangle}{|\xi|^2},$$

since $\langle C\,e_1,e_1\rangle=\langle AR\,e_1,R\,e_1\rangle$, with $R\,e_1=\xi/|\xi|$.

Case p $=\infty$. For what we already showed at the beginning of this proof, we know that

$$\frac{\mu_\infty(\varepsilon,q)(x)-q(x)}{\varepsilon}=\frac{1}{2}\left\{\min_{z\in B}[\xi\cdot z+\varepsilon\,\langle A\,z,z\rangle/2]+\max_{z\in B}[\xi\cdot z+\varepsilon\,\langle A\,z,z\rangle/2]\right\}.$$

Now, notice that, if ε is sufficiently small, the minimum and the maximum are respectively attained at some points z_ε^+ and z_ε^- on ∂B and

$$z_\varepsilon^\pm=\pm\frac{\xi+\varepsilon\,A\,z_\varepsilon'}{|\xi+\varepsilon\,A\,z_\varepsilon'|}=\pm\frac{\xi}{|\xi|}+o(\varepsilon),$$

as $\varepsilon\to 0$. Thus, we can infer that

$$\frac{\mu_\infty(\varepsilon,q)-q(x)}{\varepsilon^2}=\frac{\langle A\,z_\varepsilon^+,z_\varepsilon^+\rangle+\langle A\,z_\varepsilon^-,z_\varepsilon^-\rangle}{4}+o(1)$$

and conclude that

$$\frac{\mu_\infty(\varepsilon,q)-q(x)}{\varepsilon^2}\to\frac{1}{2}\,\frac{\langle A\,\xi,\xi\rangle}{|\xi|^2}$$

as $\varepsilon\to 0$. □

Theorem 3.2 (Asymptotics for $\mu_p(\varepsilon,u)$ as $\varepsilon\to 0$) *Let $1\le p\le\infty$. Let $\Omega\subseteq\mathbb{R}^N$ be an open set and $x\in\Omega$.*

If $u\in C^2(\Omega)$ with $\nabla u(x)\neq 0$, then

$$\mu_p(\varepsilon,u)(x)=u(x)+\frac{1}{2}\,\frac{\Delta_p^n u(x)}{N+p}\,\varepsilon^2+o(\varepsilon^2)\ \ as\ \ \varepsilon\to 0. \tag{3.6}$$

Proof Let $\varepsilon>0$ be such that $\overline{B_\varepsilon(x)}\subset\Omega$ and consider the function $q(y)$ in Lemma 3.1 with $q(x)=u(x)$, $\xi=\nabla u(x)$ and $A=\nabla^2u(x)$; also, notice that

$$\mathrm{tr}(A)+(p-2)\,\frac{\langle A\,\xi,\xi\rangle}{|\xi|^2}=\Delta_p^n u(x).$$

Set $u_\varepsilon(z)=u(x+\varepsilon z)$ and $q_\varepsilon(z)=q(x+\varepsilon z)$; since $u\in C^2(\Omega)$, then for every $\eta>0$ there exists $\varepsilon_\eta>0$ such that

$$|u_\varepsilon(z)-q_\varepsilon(z)|<\eta\,\varepsilon^2\ \text{ for every }\ z\in\overline{B}\ \text{ and }\ 0<\varepsilon<\varepsilon_\eta.$$

Thus, since by Proposition 2.7

$$\mu_p(\varepsilon,q\pm\eta\varepsilon^2)(x)=\mu_p(\varepsilon,q)(x)\pm\eta\varepsilon^2,$$

Theorem 2.5 and Corollary 2.3 yield that

$$\frac{\mu_p(\varepsilon,q)(x)-u(x)}{\varepsilon^2}-\eta\le\frac{\mu_p(\varepsilon,u)(x)-u(x)}{\varepsilon^2}\le\frac{\mu_p(\varepsilon,q)(x)-u(x)}{\varepsilon^2}+\eta.$$

Therefore, Lemma 3.1 implies that

$$\frac{1}{2}\frac{\Delta_p^n u(x)}{N+p} - \eta \le \liminf_{\varepsilon\to 0} \frac{\mu_p(\varepsilon,u)(x)-u(x)}{\varepsilon^2}$$
$$\le \limsup_{\varepsilon\to 0} \frac{\mu_p(\varepsilon,u)(x)-u(x)}{\varepsilon^2} \le \frac{1}{2}\frac{\Delta_p^n u(x)}{N+p} + \eta.$$

The desired conclusion follows, since η is arbitrary. □

Corollary 3.3 ((AMVP) for smooth functions) *Let* $1 \le p \le \infty$ *and* $u \in C^2(\Omega)$. *The following assertions are equivalent:*

(i) $\Delta_p^n u(x) = 0$ *at any* $x \in \Omega$ *such that* $\nabla u(x) \neq 0$*;*
(ii) $u(x) = \mu_p(\varepsilon,u)(x) + o(\varepsilon^2)$ *as* $\varepsilon \to 0$ *at any* $x \in \Omega$ *such that* $\nabla u(x) \neq 0$.

Remark 3.4 Without any essential modification, we can show a similar result with $\mu_p(\varepsilon,u)(x)$ in Corollary 3.3 replaced by an analogous spherical p-mean value of u on $\partial B_\varepsilon(x)$, that is the minimum value in the variational problem (1.9), where the L^p norm is taken on $\partial B_\varepsilon(x)$. The asymptotic formula (3.6) reads in this case as:

$$\mu_p(\varepsilon,u)(x) = u(x) + \frac{1}{2}\frac{\Delta_p^n u(x)}{N+p-2}\varepsilon^2 + o(\varepsilon^2) \ \text{ as } \ \varepsilon \to 0.$$

We are now going to prove that continuous viscosity solutions of the normalized p-Laplace equation are characterized by an (AMVP) in the viscosity sense. We recall the relevant definitions from [14].

A function $u \in C(\Omega)$ is a *viscosity solution of* $\Delta_p^n u = 0$ *in* Ω, if both of the following requisites hold at every $x \in \Omega$:

(i) for any function ϕ of class C^2 near x such that $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$ and $\nabla\phi(x) \neq 0$, there holds that $\Delta_p^n \phi(x) \le 0$;
(ii) for any function ϕ of class C^2 near x such that $u - \phi$ has a strict maximum at x with $u(x) = \phi(x)$ and $\nabla\phi(x) \neq 0$, there holds that $\Delta_p^n \phi(x) \ge 0$.

We say that a function $u \in C(\Omega)$ satisfies at $x \in \Omega$ the *asymptotic mean value property (AMVP)*

$$u(x) = \mu_p(\varepsilon,u)(x) + o(\varepsilon^2) \ \text{ as } \ \varepsilon \to 0$$

in the viscosity sense if both of the following requisites hold:

(a) for any function ϕ of class C^2 near x such that $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$ and $\nabla\phi(x) \neq 0$, there holds that

$$\phi(x) \ge \mu_p(\varepsilon,\phi)(x) + o(\varepsilon^2) \ \text{ as } \ \varepsilon \to 0;$$

(b) for any function ϕ of class C^2 near x such that $u - \phi$ has a strict maximum at x with $u(x) = \phi(x)$ and $\nabla\phi(x) \neq 0$, there holds that

$$\phi(x) \le \mu_p(\varepsilon,\phi)(x) + o(\varepsilon^2) \ \text{ as } \ \varepsilon \to 0.$$

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1 Let ϕ be of class C^2 near x with $\nabla\phi(x) \neq 0$; by Theorem 3.2, we know that

$$\phi(x) = \mu_p(\varepsilon, \phi)(x) - \frac{1}{2}\frac{\varepsilon^2}{N+p}\Delta_p^n\phi(x) + o(\varepsilon^2) \tag{3.7}$$

as $\varepsilon \to 0$.

Thus, if $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$ and $\Delta_p^n\phi(x) \leq 0$, then (3.7) implies that

$$\phi(x) \geq \mu_p(\varepsilon, \phi)(x) + o(\varepsilon^2) \ \text{ as } \ \varepsilon \to 0.$$

Conversely, if $\phi(x) \geq \mu_p(\varepsilon, \phi)(x) + o(\varepsilon^2)$ as $\varepsilon \to 0$, by (3.7) we infer that

$$-\Delta_p^n\phi(x) \geq o(1) \ \text{ as } \ \varepsilon \to 0,$$

and hence $\Delta_p^n\phi(x) \leq 0$.

We proceed similarly, if $u - \phi$ has a strict maximum at x. □

4 The (AMVP) for the parabolic case

The situation in the parabolic case is similar to that presented in the previous paragraph: we just have to use the proper cost function. As already observed, the choice disclosed in (1.11) is a good candidate since it yields for $p = 2$ the classical mean value property for solutions of the heat equation. Thus, we shall denote:

$$\pi_p(\varepsilon, u)(x, t) = \text{ the unique } \pi \in \mathbb{R} \text{ satisfying (1.11).} \tag{4.1}$$

It is clear that the characterization, continuity and monotonicity of Theorems 2.1, 2.4 and 2.5 apply to $\pi_p(\varepsilon, u)(x, t)$, if we set $X = \overline{E_\varepsilon(x, t)}$ and $d\nu(y, s) = |x-y|^2/(t-s)^2 dy\, ds$. In particular, the *heat mean value of* u is

$$\fint_{E_\varepsilon(x,t)} u(y, s)\, d\nu(y, s) = \frac{1}{4\varepsilon^N}\int_{E_\varepsilon(x,t)} u(y, s)\, d\nu(y, s)$$

and the *heat median of* u, $\underset{E_\varepsilon(x,t)}{\text{h-med}}\, u$, is the unique root of the equation:

$$\int_{E_\varepsilon^{\lambda,+}(x,t)} \frac{|x-y|^2}{(t-s)^2}\, dy\, ds = \int_{E_\varepsilon^{\lambda,-}(x,t)} \frac{|x-y|^2}{(t-s)^2}\, dy\, ds, \tag{4.2}$$

where

$$E_\varepsilon^{\lambda,\pm}(x, t) = \{(y, s) \in E_\varepsilon(x, t) : \lambda \lessgtr u(y, s)\}.$$

The companion of Corollary 2.3 is the following result, that does not need an ad hoc proof.

Corollary 4.1 *Let* $1 \leq p < \infty$, $u \in C(\overline{E_\varepsilon(x, t)})$ *and define*

$$u_\varepsilon(z, \sigma) = u(x + \varepsilon z, t - \varepsilon^2\sigma), \ (z, \sigma) \in E,$$

where

$$E = \left\{(z, \sigma) \in \mathbb{R}^{N+1} : 0 < \sigma < \frac{1}{4\pi}, \Phi(z, \sigma) > 1\right\}. \tag{4.3}$$

Then

$$\pi_p(\varepsilon, u)(x, t) = \pi_p(1, u_\varepsilon)(0, 0), \tag{4.4}$$

where $\lambda = \pi_p(1, u_\varepsilon)(0,0)$ *is the unique root of the equation*

$$\int_E |u_\varepsilon(z,\sigma) - \lambda|^{p-2}[u_\varepsilon(z,\sigma) - \lambda]\, d\nu(z,\sigma) = 0. \tag{4.5}$$

Lemma 4.2 *Let* $1 \le p \le \infty$*, pick* $a \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^N \setminus \{0\}$*, and let* A *be a symmetric* $N \times N$ *matrix.*

Consider the quadratic function $q : E_\varepsilon(x,t) \to \mathbb{R}$ *defined by*

$$q(y,s) = q(x,t) + \xi \cdot (y-x) + a\,(s-t) + \frac{1}{2}\langle A(y-x), y-x\rangle$$

for $(y,s) \in E_\varepsilon(x,t)$*. Let* $\pi_p(\varepsilon, q)$ *be the heat* p*-mean of* q *on* $E_\varepsilon(x,t)$*.*

Then it holds that

$$\pi_p(\varepsilon, q) = q(x,t) + \frac{1}{4\pi}\left(1 - \frac{2}{N+p}\right)^{1+\frac{N+p}{2}} \left\{ -a + \frac{N}{N+p-2}\left[\operatorname{tr}(A) + (p-2)\frac{\langle A\xi, \xi\rangle}{|\xi|^2}\right]\right\} \varepsilon^2 + o(\varepsilon^2)$$

as $\varepsilon \to 0$*.*

Proof We proceed similarly to the proof of Lemma 3.1. Set

$$q_\varepsilon(z,\sigma) = q(x + \varepsilon z, t - \varepsilon^2\sigma),$$
$$v_\varepsilon(z,\sigma) = \frac{q(x+\varepsilon z, t - \varepsilon^2\sigma) - q(x,t)}{\varepsilon} \quad \text{and} \quad v(z,\sigma) = \xi \cdot z.$$

We know that

$$\pi_p(\varepsilon, q)(x,t) = \pi_p(1, q_\varepsilon)(0,0);$$

thus, Proposition 2.7 implies that

$$\frac{\pi_p(\varepsilon, q)(x,t) - q(x,t)}{\varepsilon} = \pi_p(1, v_\varepsilon)(0,0),$$

and v_ε converges to v uniformly on E as $\varepsilon \to 0$. Theorem 2.4 then yields that

$$\lim_{\varepsilon \to 0} \frac{\pi_p(\varepsilon, q)(x,t) - q(x,t)}{\varepsilon} = \pi_p(1, v)(0,0),$$

and $\pi_p(1, v) = 0$, since it is the unique solution λ of

$$\int_E |v(z,\sigma) - \lambda|^{p-2}[v(z,\sigma) - \lambda]\, d\nu(z,\sigma) = 0,$$

for $1 \le p < \infty$, and for $p = \infty$ maximizes the quantity

$$\max(|\xi| - \lambda, \lambda + |\xi|)$$

for $\lambda \in \mathbb{R}$.

As before, set

$$\delta_\varepsilon = \frac{\pi_p(\varepsilon, q)(x,t) - q(x,t)}{\varepsilon^2}.$$

Case $1 < \mathbf{p} < \infty$. By some manipulations, similar to those used in the proof of Lemma 3.1, we get that

$$\int_E h\left(\xi \cdot z + \varepsilon\left[-a\,\sigma + \langle Az, z\rangle/2 - \delta_\varepsilon\right]\right) d\nu(z,\sigma) = 0,$$

where h is the function already defined. Without loss of generality, we assume that $|\xi| = 1$, apply the change of variables $z = R\,y$, where R is a rotation matrix such that ${}^tR\,\xi = e_1$, and set $C = {}^tRAR$ to obtain that

$$\int_E \frac{h(y_1 + \varepsilon[-a\,\sigma + \langle Cy, y\rangle/2 - \delta_\varepsilon]) - h(y_1)}{\varepsilon}\, d\nu(y,\sigma),$$

since $\int_E h(y_1)\,|y|^2/\sigma^2\, dy d\sigma = 0$. Thus, by proceeding as before, we have that

$$\begin{aligned}
&\delta_\varepsilon \int_E \left\{\int_0^1 h'(y_1 + \tau\,\varepsilon\,[-a\,\sigma + \langle Cy, y\rangle/2 - \delta_\varepsilon])\, d\tau\right\} d\nu(y,\sigma) \\
&\quad = \int_E \left\{\int_0^1 h'(y_1 + \tau\,\varepsilon\,[-a\,\sigma + \langle Cy, y\rangle/2 - \delta_\varepsilon])\, d\tau\right\} [-a\,\sigma + \langle Cy, y\rangle/2]\, d\nu(y,\sigma),
\end{aligned} \tag{4.6}$$

and this implies that δ_ε is bounded by some constant (this is equal to $c + |a|/4\pi$).

If $2 \le p < \infty$, it is easy to prove that, by the dominated convergence theorem, (any converging subsequence of) δ_ε converges to the number δ_0 defined by

$$\int_E h'(y_1)\,[-a\,\sigma + \langle Cy, y\rangle/2 - \delta_0]\, d\nu(y,\sigma) = 0. \tag{4.7}$$

If $1 < p < 2$, we observe that

$$\begin{aligned}
&\left|\int_0^1 h'\left(y_1 + \tau\,\varepsilon\,[-a\,\sigma + \langle Cy, y\rangle/2 - \delta_\varepsilon]\right)\,[-a\,\sigma + \langle Cy, y\rangle/2 - \delta_\varepsilon]\, d\tau\right| \\
&\quad \le 2\,(c + |a|/4\pi)\left||y_1| - 2\,(c + |a|/4\pi)\,\varepsilon\right|^{p-2}
\end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\int_E \left||y_1| - 2\,(c + |a|/4\pi)\,\varepsilon\right|^{p-2} d\nu(y,\sigma) = \int_B |y_1|^{p-2}\, d\nu(y,\sigma).$$

If (any converging subsequence of) δ_ε converges to a number δ_0, then the integrand in (3.2) converges pointwise to $h'(y_1)\,[-a\,\sigma + \langle Cy, y\rangle/2 - \delta_0]$, and hence we can conclude that (4.7) holds, by the generalized dominated convergence theorem (Theorem 5.4).

Therefore, by Lemma 5.3 we have that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\delta_\varepsilon &= \frac{\int_E |y_1|^{p-2}\,[-a\,\sigma + \langle Cy, y\rangle/2]\, d\nu(y,\sigma)}{\int_E |y_1|^{p-2}\, d\nu(y,\sigma)} \\
&= \frac{1}{4\pi}\left(1 - \frac{2}{N+p}\right)^{\frac{N+p}{2}+1}\left\{-a + \frac{N}{N+p-2}\,[\mathrm{tr}(C) + (p-2)\,\langle Ce_1, e_1\rangle]\right\} \\
&= \frac{1}{4\pi}\left(1 - \frac{2}{N+p}\right)^{\frac{N+p}{2}+1}\left\{-a + \frac{N}{N+p-2}\left[\mathrm{tr}(A) + (p-2)\,\frac{\langle A\xi, \xi\rangle}{|\xi|^2}\right]\right\}
\end{aligned}$$

since $\langle C\, e_1, e_1\rangle = \langle A R\, e_1, R\, e_1\rangle$, with $R\, e_1 = \xi/|\xi|$.

Case p = 1. By proceeding as in the proof of Lemma 3.1, it is easy to show that

$$\int_{E_\varepsilon^+} \frac{|y|^2}{\sigma^2}\, dy\, d\sigma = \int_{E_\varepsilon^-} \frac{|y|^2}{\sigma^2}\, dy\, d\sigma$$

where

$$E_\varepsilon^{\mp} = \{(y,\sigma) \in E : -y_1 \lesseqgtr 0,\, |\xi|\, y_1 + \varepsilon\, [-a\,\sigma + \langle C y, y\rangle/2] \lesseqgtr \varepsilon\delta_\varepsilon\}\,,$$

R is the usual rotation matrix, and $C = {}^t R A R$.

Now, we assume that $|\xi| = 1$ without loss of generality, use the change of variables

$$y = \varepsilon\, z_1\, e_1 + z' \;\text{ where }\; z' = (0, z_2, \ldots, z_N),$$

and take the limit as $\varepsilon \to 0$; similarly to the proof of Lemma 3.1 we obtain that

$$\int_{E_0^+} \frac{|z|^2}{\sigma^2}\, dz\, d\sigma = \int_{E_0^-} \frac{|z|^2}{\sigma^2}\, dz\, d\sigma$$

where δ_0 is, as usual, the limit of δ_ε as $\delta \to 0$ and

$$E_0^{\pm} = \left\{(z,\sigma) \in \mathbb{R}^{N+1} : 0 \le |z'| < \sqrt{-2N\,\sigma\, \log(4\pi\sigma)},\; 0 < \sigma < \frac{1}{4\pi},\right.$$
$$\left. 0 \lesseqgtr z_1,\, \delta_0 \lesseqgtr z_1 - a\,\sigma + \frac{\langle C\, z', z'\rangle}{2}\right\}.$$

Thus, δ_0 results to be the solution of

$$\int_{E'} \left[\delta_0 + a\,\sigma - \frac{1}{2}\,\langle C\, z', z'\rangle\right] \frac{|z'|^2}{\sigma^2}\, dz'\, d\sigma = 0,$$

where

$$E' = \left\{(z',\sigma) \in \mathbb{R}^N : 0 \le |z'| < \sqrt{-2N\,\sigma\, \log(4\pi\sigma)},\; 0 < \sigma < \frac{1}{4\pi}\right\},$$

and hence

$$\delta_0 \int_{E^*} r^N \sigma^{-2}\, dr\, d\sigma = -a \int_{E^*} r^N \sigma^{-1}\, dr\, d\sigma$$
$$+ \frac{1}{2\,(N-1)} \left[\mathrm{tr}(A) - \frac{\langle A\,\xi, \xi\rangle}{|\xi|^2}\right] \int_{E^*} r^{N+2} \sigma^{-2}\, dr\, d\sigma,$$

where E^* is defined in Lemma 5.2.

Finally, Lemma 5.2 gives that

$$\delta_0 = \frac{1}{4\pi} \left(\frac{N-1}{N+1}\right)^{\frac{N+1}{2}+1} \left\{-a + \frac{N}{N-1}\left[\mathrm{tr}(A) - \frac{\langle A\,\xi, \xi\rangle}{|\xi|^2}\right]\right\}.$$

Case p = ∞. For what we already showed at the beginning of this proof, we know that

$$\frac{\pi_\infty(\varepsilon, q) - q(x,t)}{\varepsilon} = \frac{1}{2} \min_{(z,\sigma)\in E} [\xi \cdot z + \varepsilon\,(-a\,\sigma + \langle A\, z, z\rangle/2)]$$
$$+ \frac{1}{2} \max_{(z,\sigma)\in E} [\xi \cdot z + \varepsilon\,(-a\,\sigma + \langle A\, z, z\rangle/2)]\,.$$

Now, notice that if ε is sufficiently small, since $\xi \neq 0$, the minimum and the maximum are attained at some points $(z_\varepsilon^+, \sigma_\varepsilon^+)$ and $(z_\varepsilon^-, \sigma_\varepsilon^-)$ on ∂E. Thus, there exist two Lagrange multipliers λ_ε^+ and λ_ε^- such that the following three equations hold:

$$\begin{gathered} \xi + \varepsilon\, A z_\varepsilon^\pm = \lambda_\varepsilon^\pm z_\varepsilon^\pm, \quad -\varepsilon\, a = \lambda_\varepsilon^\pm N\{\log(4\pi\sigma_\varepsilon^\pm) + 1\}, \\ |z_\varepsilon^\pm|^2 + 2N\sigma_\varepsilon^\pm \log(4\pi\sigma_\varepsilon^\pm) = 0. \end{gathered} \tag{4.8}$$

Since

$$\max_{(z,\sigma)\in E}(\xi \cdot z) = |\xi|\sqrt{\frac{N}{2\pi e}} \quad \text{and} \quad \min_{(z,\sigma)\in E}(\xi \cdot z) = -|\xi|\sqrt{\frac{N}{2\pi e}},$$

a straightforward asymptotic analysis on the system (4.8) informs us that

$$z_\varepsilon^\pm = \pm\sqrt{\frac{N}{2\pi e}}\frac{\xi}{|\xi|} + o(\varepsilon) \ \text{ and } \ \sigma_\varepsilon^\pm = \frac{1}{4\pi e} + o(\varepsilon) \ \text{ as } \ \varepsilon \to 0.$$

Therefore, we obtain:

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{\pi_\infty(\varepsilon, q) - q(x,t)}{\varepsilon^2} &= \lim_{\varepsilon\to 0}\left\{\xi \cdot \frac{z_\varepsilon^- + z_\varepsilon^+}{2\varepsilon} - a\,\frac{\sigma_\varepsilon^- + \sigma_\varepsilon^+}{2} + \frac{\langle Az_\varepsilon^-, z_\varepsilon^-\rangle + \langle Az_\varepsilon^+, z_\varepsilon^+\rangle}{4}\right\} \\ &= \frac{1}{4\pi e}\left(-a + N\,\frac{\langle A\xi, \xi\rangle}{|\xi|^2}\right), \end{aligned}$$

as desired. □

Theorem 4.3 (Asymptotics for $\pi_p(\varepsilon, u)$ as $\varepsilon \to 0$) *Let* $1 \le p \le \infty$. *Assume* $(x,t) \in \Omega_T$, $u \in C^2(\Omega_T)$ *and* $\nabla u(x,t) \neq 0$.

Then

$$\begin{aligned} \pi_p(\varepsilon, u)(x,t) = u(x,t) &+ \frac{1}{4\pi}\left(1 - \frac{2}{N+p}\right)^{\frac{N+p}{2}+1} \\ &\times \left\{-u_t(x,t) + \frac{N}{N+p-2}\,\Delta_p^n u(x,t)\right\}\varepsilon^2 + o(\varepsilon^2), \end{aligned} \tag{4.9}$$

as $\varepsilon \to 0$.

Proof Let $\varepsilon > 0$ be such that $\overline{E_\varepsilon(x,t)} \subset \Omega_T$ and consider the function $q(y,s)$ in Lemma 4.2 with $q(x,t) = u(x,t)$, $a = u_t(x,t)$, $\xi = \nabla u(x,t)$, and $A = \nabla^2 u(x,t)$; then, set $u_\varepsilon(z,\sigma) = u(x + \varepsilon z, t - \varepsilon^2\sigma)$ and $q_\varepsilon(z,\sigma) = u(x + \varepsilon z, t - \varepsilon^2\sigma)$.

Since $u \in C^2(\Omega_T)$, for every $\eta > 0$ there exists $\varepsilon_\eta > 0$ such that

$$|u_\varepsilon(z,\sigma) - q_\varepsilon(z,\sigma)| < \eta\,\varepsilon^2 \ \text{ for every } \ z \in \overline{E} \ \text{ and } \ 0 < \varepsilon < \varepsilon_\eta.$$

Thus, by Proposition 2.7, (4.4) and Theorem 2.5,

$$\begin{aligned} \frac{\pi_p(\varepsilon, q)(x,t) - q(x,t)}{\varepsilon^2} - \eta &\le \frac{\pi_p(\varepsilon, u)(x,t) - q(x,t)}{\varepsilon^2} \\ &\le \frac{\pi_p(\varepsilon, q)(x,t) - q(x,t)}{\varepsilon^2} + \eta. \end{aligned}$$

Therefore, Lemma 4.2 implies that

$$\begin{aligned}
&\frac{1}{4\pi}\left(1-\frac{2}{N+p}\right)^{\frac{N+p}{2}+1}\left\{-u_t(x,t)+\frac{N}{N+p-2}\,\Delta_p^n u(x,t)\right\}-\eta\\
&\le \liminf_{\varepsilon\to 0}\frac{\pi_p(\varepsilon,u)(x,t)-q(x,t)}{\varepsilon^2}\le \limsup_{\varepsilon\to 0}\frac{\pi_p(\varepsilon,u)(x,t)-q(x,t)}{\varepsilon^2}\\
&\le \frac{1}{4\pi}\left(1-\frac{2}{N+p}\right)^{\frac{N+p}{2}+1}\left\{-u_t(x,t)+\frac{N}{N+p-2}\,\Delta_p^n u(x,t)\right\}+\eta.
\end{aligned}$$

The desired conclusion follows at once, since η is arbitrary. □

Corollary 4.4 *Let $u\in C^2(\Omega_T)$. The following assertions are equivalent:*

(i) $-u_t(x,t)+\frac{N}{N+p-2}\,\Delta_p^n u(x,t)=0$,

(ii) $u(x)=\pi_p(\varepsilon,u)(x,t)+o(\varepsilon^2)$ *as* $\varepsilon\to 0$,

at any point $(x,t)\in\Omega_T$ such that $\nabla u(x,t)\neq 0$.

We do not provide the proof of Theorem 1.2, since is a straightforward re-adaptation of that of Theorem 1.1, once the following definitions are established.

A function $u\in C(\Omega_T)$ is a *viscosity solution of* $u_t=\frac{N}{N+p-2}\Delta_p^n u$ *in* Ω_T, if both of the following requisites hold at every $(x,t)\in\Omega_T$:

(i) for any function ϕ of class C^2 near (x,t) such that $u-\phi$ has a strict minimum at (x,t) with $u(x,t)=\phi(x,t)$ and $\nabla\phi(x,t)\neq 0$, there holds that $\frac{N}{N+p-2}\Delta_p^n u(x,t)\le u_t(x,t)$;

(ii) for any function ϕ of class C^2 near (x,t) such that $u-\phi$ has a strict maximum at (x,t) with $u(x,t)=\phi(x,t)$ and $\nabla\phi(x,t)\neq 0$, there holds that $\frac{N}{N+p-2}\Delta_p^n u(x,t)\ge u_t(x,t)$.

We say that a function $u\in C(\Omega_T)$ satisfies at $(x,t)\in\Omega_T$ the *asymptotic mean value property (AMVP)*

$$u(x,t)=\pi_p(\varepsilon,u)(x,t)+o(\varepsilon^2)\ \text{ as }\ \varepsilon\to 0$$

in the viscosity sense if both of the following requisites hold:

(a) for any function ϕ of class C^2 near (x,t) such that $u-\phi$ has a strict minimum at (x,t) with $u(x,t)=\phi(x,t)$ and $\nabla\phi(x,t)\neq 0$, there holds that

$$\phi(x,t)\ge\pi_p(\varepsilon,\phi)(x,t)+o(\varepsilon^2)\ \text{ as }\ \varepsilon\to 0;$$

(b) for any function ϕ of class C^2 near (x,t) such that $u-\phi$ has a strict maximum at (x,t) with $u(x,t)=\phi(x,t)$ and $\nabla\phi(x,t)\neq 0$, there holds that

$$\phi(x,t)\le\pi_p(\varepsilon,\phi)(x,t)+o(\varepsilon^2)\ \text{ as }\ \varepsilon\to 0.$$

5 Useful integrals

We begin with the computation of some useful integrals.

Lemma 5.1 *Let $\mathbb{S}^{N-1}$ be the unit sphere in $\mathbb{R}^N$. Let $\xi\in\mathbb{R}^N\backslash\{0\}$ and A be an $N\times N$ symmetric matrix. Then for $1<p<\infty$ we have that*

$$\frac{\int_{\mathbb{S}^{N-1}}|\xi\cdot y|^{p-2}\langle Ay,y\rangle\,dS_y}{\int_{\mathbb{S}^{N-1}}|\xi\cdot y|^{p-2}\,dS_y}=\frac{1}{N+p-2}\left\{\mathrm{tr}(A)+(p-2)\,\frac{\langle A\xi,\xi\rangle}{|\xi|^2}\right\}\tag{5.1}$$

and

$$\frac{\int_B |\xi\cdot y|^{p-2}\langle Ay,y\rangle\,dy}{\int_B |\xi\cdot y|^{p-2}\,dy}=\frac{1}{N+p}\left\{\mathrm{tr}(A)+(p-2)\,\frac{\langle A\xi,\xi\rangle}{|\xi|^2}\right\}. \tag{5.2}$$

Proof Let R be a rotation matrix such that ${}^tR\xi=|\xi|\,e_1$; by the change of variables $y=R\theta$, we have that

$$\frac{\int_{\mathbb{S}^{N-1}} |\xi\cdot y|^{p-2}\langle Ay,y\rangle\,dS_y}{\int_{\mathbb{S}^{N-1}} |\xi\cdot y|^{p-2}\,dS_y}=\frac{\int_{\mathbb{S}^{N-1}} |\theta_1|^{p-2}\langle({}^tRAR)\,\theta,\theta\rangle\,dS_\theta}{\int_{\mathbb{S}^{N-1}} |\theta_1|^{p-2}\,dS_\theta}.$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} |\theta_1|^{p-2}\theta_i\,\theta_j\,dS_\theta &= \int_B \frac{\partial}{\partial y_j}(|y_1|^{p-2}y_i)\,dy=[\delta_{ij}+(p-2)\delta_{i1}\delta_{1j}]\int_B |y_1|^{p-2}dy\\ &=\frac{\delta_{ij}+(p-2)\delta_{i1}\delta_{1j}}{N+p-2}\int_{\mathbb{S}^{N-1}} |\theta_1|^{p-2}\,dS_\theta,\end{aligned}$$

where we have used the divergence theorem in the first equality. Therefore, we obtain that

$$\begin{aligned}\frac{\int_{\mathbb{S}^{N-1}} |\xi\cdot y|^{p-2}\langle Ay,y\rangle\,dS_y}{\int_{\mathbb{S}^{N-1}} |\xi\cdot y|^{p-2}\,dS_y} &= \frac{\mathrm{tr}({}^tRAR)+(p-2)\langle({}^tRAR)\,e_1,e_1\rangle}{N+p-2}\\ &=\frac{1}{N+p-2}\left\{\mathrm{tr}(A)+(p-2)\,\frac{\langle A\,\xi,\xi\rangle}{|\xi|^2}\right\}.\end{aligned}$$

Formula (5.2) easily follows from (5.1). □

Lemma 5.2 *Let $\alpha>0$ and $\beta<\alpha+1$ be real numbers and let*

$$E_*=\left\{(r,\sigma)\in\mathbb{R}^2: 0<r<\sqrt{-2N\sigma\,\log(4\pi\sigma)},\ 0<\sigma<\frac{1}{4\pi}\right\}.$$

Then

$$\int_{E_*} r^{2\alpha-1}\sigma^{-\beta}\,drd\sigma=\frac{2^{2\beta-\alpha-3}\,\pi^{\beta-\alpha-1}\,N^\alpha}{\alpha\,(\alpha-\beta+1)^{\alpha+1}}\,\Gamma(\alpha+1). \tag{5.3}$$

Proof The result follows from the calculations:

$$\begin{aligned}\int_{E_*} r^{2\alpha-1}\sigma^{-\beta}\,drd\sigma &= \frac{1}{2\alpha}\int_0^{\frac{1}{4\pi}}\sigma^{-\beta}\{-2N\sigma\,\log(4\pi\sigma)\}^\alpha\,d\sigma\\ &=\frac{2^{2\beta-\alpha-3}\,\pi^{\beta-\alpha-1}\,N^\alpha}{\alpha}\int_0^\infty \tau^\alpha\,e^{-(\alpha-\beta+1)\tau}\,d\tau\\ &=\frac{2^{2\beta-\alpha-3}\,\pi^{\beta-\alpha-1}\,N^\alpha}{\alpha\,(\alpha-\beta+1)^{\alpha+1}}\int_0^\infty \tau^\alpha\,e^{-\tau}\,d\tau;\end{aligned}$$

in the second equality we used the substitution $4\pi\sigma=e^{-\tau}$. □

Lemma 5.3 *Let ξ and A be as in Lemma 5.1. Then for $1<p<\infty$ we have that*

$$\frac{\int_E |\xi\cdot z|^{p-2}\,\sigma\,d\nu(z,\sigma)}{\int_E |\xi\cdot z|^{p-2}\,d\nu(z,\sigma)}=\frac{1}{4\pi}\left(\frac{N+p-2}{N+p}\right)^{1+\frac{N+p}{2}} \tag{5.4}$$

and

$$\frac{\int_E |\xi \cdot z|^{p-2} \langle Az, z\rangle \, d\nu(z,\sigma)}{\int_E |\xi \cdot z|^{p-2} \, d\nu(z,\sigma)} = \frac{1}{2\pi} \frac{N}{N+p-2} \left(\frac{N+p-2}{N+p}\right)^{1+\frac{N+p}{2}} \left\{ \mathrm{tr}(A) + (p-2) \frac{\langle A\xi, \xi\rangle}{|\xi|^2} \right\}. \tag{5.5}$$

Proof By using spherical coordinates, we calculate that

$$\frac{\int_E |\xi \cdot z|^{p-2} \sigma \, d\nu(z,\sigma)}{\int_E |\xi \cdot z|^{p-2} \, d\nu(z,\sigma)} = \frac{\int_{E_*} r^{p+N-1} \sigma^{-1} dr \, d\sigma}{\int_{E_*} r^{p+N-1} \sigma^{-2} dr \, d\sigma}$$

and

$$\frac{\int_E |\xi \cdot z|^{p-2} \langle Az, z\rangle \, d\nu(z,\sigma)}{\int_E |\xi \cdot z|^{p-2} \, d\nu(z,\sigma)} = \frac{\int_{E_*} r^{p+N+1} \sigma^{-2} dr \, d\sigma}{\int_{E_*} r^{p+N-1} \sigma^{-2} dr \, d\sigma} \frac{\int_{\mathbb{S}^{N-1}} |\xi \cdot y|^{p-2} \langle Ay, y\rangle \, dS_y}{\int_{\mathbb{S}^{N-1}} |\xi \cdot y|^{p-2} \, dS_y}.$$

Thus, (5.4) and (5.5) follow from the calculations

$$\frac{\int_{E_*} r^{p+N-1} \sigma^{-1} dr \, d\sigma}{\int_{E_*} r^{p+N-1} \sigma^{-2} dr \, d\sigma}$$

and (5.1). □

For the reader's convenience, we recall the generalized dominated convergence theorem (see [10] for instance), that is needed for the proofs of Lemmas 3.1 and 4.2.

Theorem 5.4 (Generalized dominated convergence theorem) *Let (X, ν) be a measure space and let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be sequences of measurable functions on X such that*

(i) *f_n converges to a measurable function f a.e. on X as $n \to \infty$;*
(ii) *each $g_n \in L^1(X, \nu)$ and g_n converges to a function g in $L^1(X, \nu)$ a.e. on X as $n \to \infty$;*
(iii) *$|f_n| \le g_n$ a.e. on X for all $n \in \mathbb{N}$;*
(iv) *$\lim_{n\to\infty} \int_X g_n \, d\nu = \int_X g \, d\nu$.*

Then, we have that

$$\lim_{n\to\infty} \int_X f_n \, d\nu = \int_X f \, d\nu.$$

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