



UNIVERSITÀ
DEGLI STUDI
FIRENZE

FLORE

Repository istituzionale dell'Università degli Studi di Firenze

Strong local optimality for a bang-bang-singular extremal: the fixed-free case

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

Original Citation:

Strong local optimality for a bang-bang-singular extremal: the fixed-free case / Laura Poggiolini; Gianna Stefani. - In: SIAM JOURNAL ON CONTROL AND OPTIMIZATION. - ISSN 1095-7138. - STAMPA. - 56:(2018), pp. 2274-2294. [10.1137/17M1140248]

Availability:

The webpage <https://hdl.handle.net/2158/1130218> of the repository was last updated on 2021-03-10T18:40:37Z

Published version:

DOI: 10.1137/17M1140248

Terms of use:

Open Access

La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze (<https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf>)

Publisher copyright claim:

La data sopra indicata si riferisce all'ultimo aggiornamento della scheda del Repository FloRe - The above-mentioned date refers to the last update of the record in the Institutional Repository FloRe

(Article begins on next page)

1 **STRONG LOCAL OPTIMALITY FOR A BANG-BANG-SINGULAR**
2 **EXTREMAL: THE FIXED-FREE CASE***

3 LAURA POGGIOLINI[†] AND GIANNA STEFANI[†]

4 **Abstract.** In this paper we give sufficient conditions for a Pontryagin extremal trajectory,
5 consisting of two bang arcs followed by a partially or totally singular one, to be a strong local
6 minimizer for a Mayer problem. The problem is defined on \mathbb{R}^n and the end-points constraints are of
7 fixed-free type. We use a Hamiltonian approach and its connection with the second order conditions
8 in the form of a LQ accessory problem. An example is proposed. All the results are coordinate free
9 so they also hold on a manifold.

10 **Key words.** Sufficient optimality conditions, singular control, second variation, Hamiltonian
11 methods

12 **AMS subject classifications.** 49K15, 49J15, 93C10

13 **1. Introduction.** In this paper we consider a Mayer problem on a fixed time
14 interval $[0, T]$ and governed by a control affine dynamics. We study the strong local
15 optimality of a trajectory consisting of two bang arcs followed by a singular one.

16 In Optimal Control literature two different kinds of local optimality are usually
17 considered: *weak local optimality*, i.e. with respect to the $C^0 \times L^\infty$ -distance of the
18 couples (trajectory, associated control); *strong local optimality*, i.e. with respect to the
19 C^0 -distance of admissible trajectories, without any localization on the controls and
20 which is defined below. An intermediate kind of local optimality, called *Pontryagin*
21 *local optimality*, is also studied in the literature, see for example [10]. In our case
22 Pontryagin local optimality reduces to local optimality with respect to the $C^0 \times L^1$ -
23 distance of the couples (trajectory, associated control).

24 Here we give sufficient optimality conditions for the reference trajectory to be a
25 strong local minimizer in the case when the end-point constraints are of fixed-free
26 type. We also recall that since a Bolza problem can always be reduced to a Mayer
27 one, sufficient optimality conditions can be also derived for a Bolza problem.

28 Control affine systems can be modeled in different ways; since we want to consider
29 both bang and partially singular arcs (see Definition 1.2), we model the system as
30 follows: let X_1, \dots, X_m be smooth vector fields on \mathbb{R}^n and let \mathcal{X} be their convex hull,
31 i.e.

$$\mathcal{X}(x) = \left\{ \sum_{i=1}^m u_i X_i(x) : u = (u_1, \dots, u_m) \in \Delta \right\},$$

33 where $\Delta := \{u \in \mathbb{R}^m : u_i \geq 0, i = 1, \dots, m, \sum_{i=1}^m u_i = 1\}$ is the simplex of \mathbb{R}^m .

34 Given a smooth function $c: \mathbb{R}^n \rightarrow \mathbb{R}$ and a point $x_0 \in \mathbb{R}^n$, we consider a Mayer
35 problem of the following kind

$$\begin{aligned} 36 \quad (1a) \quad & \text{minimize } c(\xi(T)) \text{ subject to} \\ 37 \quad (1b) \quad & \dot{\xi}(t) \in \mathcal{X}(\xi(t)) \quad \text{a.e. } t \in [0, T], \\ 38 \quad (1c) \quad & \xi(0) = x_0. \end{aligned}$$

*Submitted to the editors 8th March 2018.

Funding: The authors are partially supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

[†]DIMAI, Università degli Studi di Firenze, Italy (laura.poggiolini@unifi.it, gianna.stefani@unifi.it, <http://www.dma.unifi.it/~poggiolini/>).

40 Equivalently, by Filippov's theorem, equation (1b) can also be written as

$$41 \quad \dot{\xi}(t) = \sum_{i=1}^m v_i(t) X_i(\xi(t)), \quad \text{a.e. } t \in [0, T], \quad v \in L^\infty([0, T], \Delta).$$

42 Our aim is to give sufficient conditions for a reference trajectory $\widehat{\xi}: [0, T] \rightarrow \mathbb{R}^n$
 43 of (1b)-(1c) to be indeed a *strong local minimizer* of the problem according to the
 44 following definition

45 DEFINITION 1.1. *An admissible trajectory $\widehat{\xi}: [0, T] \rightarrow \mathbb{R}^n$ for an optimal control
 46 problem is a strong local minimizer if it is a minimizer among the admissible trajec-
 47 tories which are in a neighborhood of $\widehat{\xi}$ with respect to the C^0 topology.*

48 We consider the case when $\widehat{\xi}$ is the concatenation of bang and singular arcs.

49 DEFINITION 1.2. *Given an admissible trajectory ξ and a time interval $(t_1, t_2) \subset$
 50 $[0, T]$, we say that $\xi|_{(t_1, t_2)}$ is*

- 51 • a bang arc of ξ if $\dot{\xi}(t)$ is the same vertex of $\mathcal{X}(\xi(t))$ for any $t \in (t_1, t_2)$,
- 52 • a singular arc of ξ if $\dot{\xi}(t)$ is in the relative interior of a face of $\mathcal{X}(\xi(t))$ for
 53 any $t \in (t_1, t_2)$. A singular arc is called *totally singular* if the dimension of
 54 the face is maximal, otherwise it is called *partially singular*.

55 Notice that in the single input case, $m = 2$, singular means totally singular. This case
 56 was considered in [21] where a proof was only sketched.

57 Here we assume there exist times $\widehat{\tau}_1, \widehat{\tau}_2$ such that $0 < \widehat{\tau}_1 < \widehat{\tau}_2 < T$, vector fields
 58 $h_1, h_2, h_3 \in \{X_1, \dots, X_m\}$, (where h_1 and h_3 might be the same vector field) and a
 59 measurable function $\widehat{v} \in L^\infty([\widehat{\tau}_2, T], (0, 1))$ such that $\widehat{\xi}$ is the absolutely continuous
 60 solution to the following Cauchy problem

$$61 \quad (2) \quad \begin{aligned} \dot{\xi}(t) &= h_1(\xi(t)) & t \in [0, \widehat{\tau}_1), \\ \dot{\xi}(t) &= h_2(\xi(t)) & t \in [\widehat{\tau}_1, \widehat{\tau}_2), \\ \dot{\xi}(t) &= \widehat{v}(t)h_3(\xi(t)) + (1 - \widehat{v}(t))h_2(\xi(t)) & \text{a.e. } t \in [\widehat{\tau}_2, T], \\ \xi(0) &= x_0. \end{aligned}$$

62 Thus, if $m = 2$, then $(\widehat{\tau}_2, T)$ is a totally singular arc, else it is a partially singular one.
 63 Denoting by $f_d := h_3 - h_2$ we can write the dynamics on the singular arc as

$$64 \quad \dot{\xi}(t) = h_2(\xi(t)) + \widehat{v}(t)f_d(\xi(t)), \quad t \in (\widehat{\tau}_2, T).$$

65 We also define the time-dependent reference vector field \widehat{f}_t as

$$66 \quad (3) \quad \widehat{f}_t := \begin{cases} h_1 & t \in [0, \widehat{\tau}_1), \\ h_2 & t \in (\widehat{\tau}_1, \widehat{\tau}_2), \\ h_2 + \widehat{v}(t)f_d & \text{a.e. } t \in (\widehat{\tau}_2, T]. \end{cases}$$

67 REMARK 1.1. *In this paper we consider a case study for our Hamiltonian ap-
 68 proach, i.e. when the final point is not constrained and the initial one is fixed. Indeed
 69 in this case the second order conditions give the possibility of constructing a field of
 70 non intersecting almost extremals and this is sufficient to obtain the result. The ex-
 71 tension to a problem with constrained final point requires adding a penalty term by
 72 taking advantage of a classical result on quadratic forms due to Hestenes, see [8].*

73 In a future paper, [17], we shall extend the result to the case when the end points
 74 are constrained to smooth sub-manifolds of \mathbb{R}^n and the cost depends on both the end
 75 points.

76 Since the main necessary condition for strong local optimality, Pontryagin Maximum
 77 Principle (PMP), is naturally set in the cotangent bundle $(\mathbb{R}^n)^* \times \mathbb{R}^n$, we give our
 78 sufficient conditions in such framework. We then use a Hamiltonian approach and its
 79 connection with the second order conditions.

80 The main idea is to use the symplectic properties of the cotangent bundle to
 81 compare the costs of neighboring admissible trajectories by lifting them to such bun-
 82 dle. To do so we define a suitable Hamiltonian flow \mathcal{H} , emanating from a horizontal
 83 Lagrangian sub-manifold Λ , $\mathcal{H}: (t, \ell) \in [0, T] \times \Lambda \mapsto \mathcal{H}_t(\ell) \in T^*\mathbb{R}^n$. Since the final
 84 point is free, we consider flows backwards in time, with T as a starting time, Section
 85 4.1. The existence of this flow will be ensured by the regularity assumptions on the
 86 extremal $\hat{\lambda}$ given by PMP, see Assumptions 1–5.

87 From a Hamiltonian point of view the sufficient conditions sum up to proving the
 88 existence of a tubular neighborhood of $[0, T] \times \{\hat{\lambda}(T)\}$ in $[0, T] \times \Lambda$ where the map
 89 $\text{id} \times \pi\mathcal{H}$ is locally invertible, see Theorem 4.2. Thanks to the compactness of the
 90 time interval $[0, T]$ it suffices to prove that $\pi\mathcal{H}_t$ is invertible for any t , see Theorem
 91 4.3. The connection with a suitable second order approximation (2^{nd} variation) is
 92 obtained as shown in the following lines.

- 93 1. The 2^{nd} variation J''_{ext} is in the form of a coordinate-free linear-quadratic (LQ)
 94 problem on the interval $[\hat{\tau}_2, T]$ and it is obtained applying an intrinsic version
 95 of Goh transformation. Indeed we can obtain our sufficient conditions either
 96 when J''_{ext} is coercive or when $L_{f_a}c \equiv 0$ (a fact which prevents the coercivity),
 97 provided a suitable restriction of J''_{ext} is coercive, see Section 3.2.
- 98 2. We show that the derivative of \mathcal{H}_t along $\hat{\lambda}$ is, up to an isomorphism, the
 99 linear Hamiltonian flow associated to the LQ problem, see Section 5.2.
- 100 3. From the coercivity of the 2^{nd} variation we deduce that $\pi\mathcal{H}_t$ is locally invert-
 101 ible (Sections 5.3 and 5.4), so that we can compare the costs of neighboring
 102 admissible trajectories by lifting them to the cotangent bundle. In our case
 103 proving the local invertibility is equivalent to requiring the invertibility of
 104 $\pi\mathcal{H}_t$ for any $t \in [\hat{\tau}_2, T]$ and the sufficient conditions for the optimality of a
 105 bang-bang trajectory of a suitable Mayer problem on $[0, \hat{\tau}_2]$, see Remark 4.3.

106 **REMARK 1.2.** *The Hamiltonian approach allows to prove strong local optimality*
 107 *of the reference trajectory in the case of a partially singular arc, by giving regularity*
 108 *conditions on the reference control, while in the second order conditions only the*
 109 *singular component of the control is considered.*

110 *We also point out that our result applies also to the case $L_{f_a}c \equiv 0$, a case*
 111 *which, up to the authors knowledge, has not been considered so far. In Section 6.1 we*
 112 *provide an example where this condition holds and to which our theory applies.*

113 **REMARK 1.3.** *In [19] we considered the bang-singular-bang case for the minimum*
 114 *time problem in a single input control system with fixed end points. For technical rea-*
 115 *sons the construction of almost-extremals provided in [19] works for Mayer problems*
 116 *only if the singular arc is the first or the last one. The technique can be applied to the*
 117 *concatenation of an arbitrary number of bang arcs and a singular one, provided the*
 118 *singular arc is the initial or the final one. The possibility of modifying the technique*
 119 *in order to consider any concatenation of bang and singular arcs is currently being*
 120 *studied.*

In the case of bang-bang extremals for either a Mayer or a Bolza problem, Hamiltonian methods have been successfully exploited in [3, 12, 14, 15], while bang-bang extremals in the minimum time problem have been studied in [18, 16].

Bang-bang extremals are extensively studied in the literature, also with other methods, see for example [9, 11, 5] and the references therein.

Hamiltonian methods have also been applied to singular extremals, see [22, 6] and to concatenations of bang and singular arcs, see [19] and the references therein.

The literature is rich of results that involve some localization of the control and with different approaches, see e.g. [4] and the references therein.

We should also like to mention that Hamiltonian methods can also be successfully employed to obtain sufficient conditions to structure stability of minimizers, see e.g. [7, 20, 13].

2. Preliminaries.

2.1. Notation. We start by recalling some basic facts and by introducing some specific notations. We identify any bi-linear form Q on a vector space W with a linear form $Q: W \rightarrow W^*$, we write $Q(v, w) = \langle Qv, w \rangle$, and we denote the associate quadratic form as $Q(v, v) = Q[v]^2$.

In this paper we use notation from differential geometry and some basic element of the theory of symplectic manifolds referred to the trivial cotangent bundle $T^*\mathbb{R}^n = (\mathbb{R}^n)^* \times \mathbb{R}^n$, see for example [1]. We take advantage of the intrinsic notation from differential geometry as it is more compact and clear. In particular we distinguish between points in \mathbb{R}^n , usually denoted as x and tangent vectors to \mathbb{R}^n , denoted as δx .

Given a C^1 vector field f on \mathbb{R}^n , we denote as $\text{exp } tf(x)$ the flow at time t emanating from a point x at time 0, i.e. $\text{exp } tf(x)$ is the solution to

$$\dot{\xi}(t) = f(\xi(t)), \quad \xi(0) = x.$$

If g is another C^1 vector field, then the Lie bracket between f and g is denoted as $[f, g]$, i.e. $[f, g](x) := Dg(x)f(x) - Df(x)g(x)$.

If $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^2 function, $d\alpha$ is its differential, while $D^2\alpha$ is the second derivative of α . Moreover $L_f\alpha(x) := \langle d\alpha(x), f(x) \rangle$ is the Lie derivative of α with respect to the vector field f at the point x .

Finally, if G is a C^1 map from a manifold X in a manifold Y , its tangent map at a point $x \in X$ is denoted as T_xG , or simply as G_* if the point x is clear from the context. In particular, if $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^2 function and $\delta x \in \mathbb{R}^n$, we denote by the symbol $d\alpha_*\delta x$ the couple $(D^2\alpha(x)(\delta x, \cdot), \delta x) \in (\mathbb{R}^n)^* \times \mathbb{R}^n$ whenever the point x is clear from the context.

We denote by $\pi: \ell = (p, x) \in T^*\mathbb{R}^n \mapsto x \in \mathbb{R}^n$ the projection on the base space. The symbol \mathbf{s} denotes the canonical Liouville one-form on $T^*\mathbb{R}^n$: $\mathbf{s} := \sum_{i=1}^n p^i dx_i$. The associated canonical symplectic two-form $\sigma = d\mathbf{s} = \sum_{i=1}^n dp^i \wedge dx_i$ allows one to associate to any, possibly time-dependent, smooth Hamiltonian $H_t: T^*\mathbb{R}^n \rightarrow \mathbb{R}$, a Hamiltonian vector field \vec{H}_t , by

$$(4) \quad \sigma \left(V, \vec{H}_t(\ell) \right) = \langle dH_t(\ell), V \rangle, \quad \forall V \in T_\ell T^*\mathbb{R}^n,$$

$$\text{i.e. } \vec{H}_t(\ell) = \left(-\frac{\partial H_t}{\partial x}(\ell), \frac{\partial H_t}{\partial p}(\ell) \right), \quad \forall \ell = (p, x) \in T^*\mathbb{R}^n.$$

We keep this notation throughout the paper, namely the overhead arrow denotes the vector field associated to a Hamiltonian, moreover the script letter denotes its flow from time T , unless otherwise stated.

166 Finally we recall that any vector field f on \mathbb{R}^n defines, by lifting to the cotangent
 167 bundle, a Hamiltonian

$$168 \quad F: \ell = (p, x) \in T^*\mathbb{R}^n \mapsto \langle p, f(x) \rangle \in \mathbb{R}.$$

169 In particular we denote by H_1, H_2, H_3 the Hamiltonians associated with h_1, h_2, h_3 ,
 170 respectively and by $H_{i_1 i_2 \dots i_k}$, $i_1, \dots, i_k \in \{1, 2, 3\}$, the Hamiltonian associated with
 171 the vector field $h_{i_1 i_2 \dots i_k} := [h_{i_1}, [\dots [h_{i_{k-1}}, h_{i_k}] \dots]]$.

172 The flow from time T of the reference vector field \hat{f}_t defined in (3), is a local
 173 diffeomorphism defined in a neighborhood of the point $\hat{x}_T := \hat{\xi}(T)$. For each $t \in [0, T]$
 174 we denote such flow as $\hat{S}_t: \mathbb{R}^n \rightarrow \mathbb{R}^n$, while

$$175 \quad \hat{F}_t = \begin{cases} H_1 & \text{if } t \in [0, \hat{\tau}_1), \\ H_2 & \text{if } t \in (\hat{\tau}_1, \hat{\tau}_2), \\ H_2 + \hat{v}(t)H_3 & \text{if } t \in (\hat{\tau}_2, T], \end{cases}$$

176 denotes the time-dependent reference Hamiltonian and \mathcal{F}_t denotes its flow.

177 **2.2. The necessary conditions.** We start by stating the main necessary condi-
 178 tion of optimality, i.e. Pontryagin Maximum Principle (PMP). Since there is no
 179 constraint on the final point, then PMP must hold in its normal form:

180 ASSUMPTION 1 (Normal PMP). *There exists an absolutely continuous mapping*
 181 $\hat{\mu}: [0, T] \rightarrow (\mathbb{R}^n)^*$ *such that a.e. $t \in [0, T]$*

$$182 \quad \begin{aligned} \hat{\mu}(t) &= -\hat{\mu}(t) D\hat{f}_t(\hat{\xi}(t)), & \hat{\mu}(T) &= -dc(\hat{x}_T), \\ 183 \quad \hat{F}_t(\hat{\mu}(t), \hat{\xi}(t)) &= \max \left\{ \langle \hat{\mu}(t), X \rangle : X \in \mathcal{X}(\hat{\xi}(t)) \right\}. \end{aligned}$$

185 $\hat{\mu}(t)$ is called adjoint covector and the trajectory $\hat{\xi}$ of the system is called a *state*
 186 *extremal* of problem (1) while the couple $\hat{\lambda}(t) := (\hat{\mu}(t), \hat{\xi}(t))$ is called an *extremal* of
 187 problem (1). We use the following notation for the end points and for the switching
 188 points of $\hat{\lambda}(t) \in T^*\mathbb{R}^n$:

$$189 \quad \hat{\ell}_f := \hat{\lambda}(T), \quad \hat{\ell}_2 := \hat{\lambda}(\hat{\tau}_2) = \hat{\mathcal{F}}_{\hat{\tau}_2}(\hat{\ell}_f), \quad \hat{\ell}_1 := \hat{\lambda}(\hat{\tau}_1) = \hat{\mathcal{F}}_{\hat{\tau}_1}(\hat{\ell}_f), \quad \hat{\ell}_0 := \hat{\lambda}(0) = \hat{\mathcal{F}}_0(\hat{\ell}_f).$$

190 We call $\hat{\lambda}|_{[0, \hat{\tau}_1]}$ and $\hat{\lambda}|_{(\hat{\tau}_1, \hat{\tau}_2)}$ bang arcs, while $\hat{\lambda}|_{(\hat{\tau}_2, T]}$ is a singular arc.

191 Thanks to the structure of the reference trajectory, PMP gives the following
 192 necessary conditions:

- 193 1. On the first bang arc, $t \in [0, \hat{\tau}_1]$, we get $H_1(\hat{\lambda}(t)) \geq \langle \hat{\mu}(t), X \rangle$, $\forall X \in \mathcal{X}(\hat{\xi}(t))$.
- 194 2. On the second bang arc, $t \in [\hat{\tau}_1, \hat{\tau}_2]$, we get $H_2(\hat{\lambda}(t)) \geq \langle \hat{\mu}(t), X \rangle$, $\forall X \in \mathcal{X}(\hat{\xi}(t))$,
 195 in particular $H_1(\hat{\ell}_2) = H_2(\hat{\ell}_2)$.
- 196 3. On the singular arc, $t \in [\hat{\tau}_2, T]$, we get

$$197 \quad (H_2 + \hat{v}(t)F_d)(\hat{\lambda}(t)) \geq \langle \hat{\mu}(t), X \rangle, \quad \forall X \in \mathcal{X}(\hat{\xi}(t)),$$

198 which implies $F_d(\hat{\lambda}(t)) \equiv 0$ and, by differentiation,

$$199 \quad (5) \quad \frac{d}{dt} F_d(\hat{\lambda}(t)) = H_{23}(\hat{\lambda}(t)) = 0, \quad \frac{d^2}{dt^2} F_d(\hat{\lambda}(t)) = -H_{232}(\hat{\lambda}(t)) + \hat{v}(t)L(\hat{\lambda}(t)) = 0,$$

200

$$201 \quad \mathbb{L}(\ell) := (H_{323} + H_{232})(\ell) = \langle p, [f_d, [h_2, f_d]](x) \rangle, \quad \ell = (p, x) \in T^*\mathbb{R}^n.$$

202 4. At the first switching time we get $H_{12}(\widehat{\ell}_1) = \left. \frac{d}{dt} (H_2 - H_1) \circ \widehat{\lambda}(t) \right|_{t=\widehat{\tau}_1} \geq 0$, see [3].

203 5. At the second switching time we get $H_{232}(\widehat{\ell}_2) = - \left. \frac{d^2}{dt^2} F_d \circ \widehat{\lambda}(t) \right|_{t=\widehat{\tau}_2^-} \geq 0$, see [19].

204 Moreover, other necessary conditions are known to hold along singular arcs, namely
 205 the Goh condition (which in this case is automatically satisfied) and the *generalized*
 206 *Legendre condition* (GLC), see e.g. [1], applied to the sub-problem where the controlled
 207 vector field is constrained on the edge whose extrema are h_2 and h_3

$$208 \quad (6) \quad \mathbb{L}(\widehat{\lambda}(t)) = \langle \widehat{\mu}(t), (h_{323} + h_{232})(\widehat{\xi}(t)) \rangle \geq 0 \quad \forall t \in [\widehat{\tau}_2, T].$$

209 We recall that the generalized Legendre condition (GLC) takes this form because we
 210 deal with Pontryagin Maximum Principle. If one considers the minimum principle,
 211 as in [4] and in [25], then GLC is given by the reverse inequality.

212 3. Assumptions and main result.

213 **3.1. Regularity conditions.** We now state regularity conditions by requiring
 214 strict inequalities to hold whenever necessary conditions yield mild inequalities.

215 ASSUMPTION 2 (Regularity along the bang arcs).

$$216 \quad H_1(\widehat{\lambda}(t)) > \langle \widehat{\mu}(t), X \rangle, \quad \forall X \in \mathcal{X}(\widehat{\xi}(t)) \setminus \{h_1(\widehat{\xi}(t))\}, \quad \forall t \in [0, \widehat{\tau}_1),$$

$$217 \quad H_2(\widehat{\lambda}(t)) > \langle \widehat{\mu}(t), X \rangle, \quad \forall X \in \mathcal{X}(\widehat{\xi}(t)) \setminus \{h_2(\widehat{\xi}(t))\}, \quad \forall t \in (\widehat{\tau}_1, \widehat{\tau}_2),$$

219 namely we require that the reference control is the only maximizing control along
 220 each bang arc.

221 ASSUMPTION 3 (Regularity along the singular arc). *For any $a \in [0, 1]$ and any*
 222 *$t \in [\widehat{\tau}_2, T]$*

$$223 \quad H_2(\widehat{\lambda}(t)) + \widehat{v}(t)f_d(\widehat{\lambda}(t)) > \langle \widehat{\mu}(t), X \rangle, \quad \forall X \in \mathcal{X}(\widehat{\xi}(t)), \quad X(\widehat{\xi}(t)) \neq (h_2 + af_d)(\widehat{\xi}(t)),$$

224 i.e. we require that the set of maximizers along the singular arc is only the edge defined
 225 by h_2 and h_3 .

226 ASSUMPTION 4 (Regularity at the switching points).

$$227 \quad (7) \quad H_{12}(\widehat{\ell}_1) > 0, \quad H_{232}(\widehat{\ell}_2) > 0.$$

228 ASSUMPTION 5 (Strong generalised Legendre condition).

$$229 \quad (\text{SGLC}) \quad R(t) := \mathbb{L}(\widehat{\lambda}(t)) = \langle \widehat{\mu}(t), [f_d, [h_2, f_d]](\widehat{\xi}(t)) \rangle > 0 \quad t \in [\widehat{\tau}_2, T].$$

230 Thanks to (SGLC) from (5) we can recover the control along the singular arc:

$$231 \quad \widehat{v}(t) = \frac{H_{232}}{\mathbb{L}}(\widehat{\lambda}(t)) \quad \forall t \in (\widehat{\tau}_2, T],$$

232 so that, by recurrence, one can easily prove that $\widehat{v} \in C^\infty([\widehat{\tau}_2, T], (0, 1))$.

233 The condition $\widehat{v}(t) \in (0, 1)$ reads

$$234 \quad (8) \quad H_{232}(\widehat{\lambda}(t)) > 0, \quad H_{323}(\widehat{\lambda}(t)) > 0 \quad \forall t \in [\widehat{\tau}_2, T].$$

235 Thus the reference vector field is discontinuous at $\widehat{\xi}(\widehat{\tau}_2)$ if and only if $\widehat{v}(\widehat{\tau}_2^+) > 0$, i.e. if
 236 and only if the regularity condition at $\widehat{\tau}_2$, equation (7), holds.

237 **3.2. The extended second variation.** The second order conditions will be
 238 derived studying a sub-problem of the given one. Namely we consider the reference
 239 vector field \hat{f}_t and allow only for perturbations of \hat{v} on the singular interval $(\hat{\tau}_2, T)$ and
 240 for perturbations of the switching time $\hat{\tau}_1$. Following the ideas of [19], we represent
 241 the perturbations of the first switching time $\hat{\tau}_1$ by a new positive control v_0 which is
 242 a reparametrization of time. The sub-problem can be written as

243 (9a) Minimize $c(\xi(T))$ subject to

244 (9b)
$$\dot{\xi}(t) = \begin{cases} v_0(t)h_1(\xi(t)) & t \in (0, \hat{\tau}_1), \\ v_0(t)h_2(\xi(t)) & t \in (\hat{\tau}_1, \hat{\tau}_2), \\ h_2(\xi(t)) + v(t)f_d(\xi(t)) & t \in (\hat{\tau}_2, T), \end{cases}$$

245 (9c)
$$\xi(0) = x_0, \quad v_0(t) > 0, \quad \int_0^{\hat{\tau}_2} v_0(t) dt = \hat{\tau}_2, \quad v(t) \in (0, 1).$$

 246

247 Problem (9) gives rise to a linear quadratic problem on the singular arc $[\hat{\tau}_2, T]$ where
 248 the variation of the first switching time τ_1 produces a cost at time $\hat{\tau}_2$. Set

249 (10)
$$g_t := \hat{S}_{t*}^{-1} f_d \circ \hat{S}_t, \quad t \in [\hat{\tau}_2, T], \quad k_i := \hat{S}_{\hat{\tau}_1*}^{-1} h_i \circ \hat{S}_{\hat{\tau}_1}, \quad i = 1, 2, \quad k := k_1 - k_2,$$

250 i.e. g_t is the push-forward of f_d from time $t \in [\hat{\tau}_2, T]$ to time T while k_i is the push-
 251 forwards of h_i , $i = 1, 2$, from the first switching time $\hat{\tau}_1$ to T . With this notation the
 252 second variation of (9) as defined in [2] and written in terms of the push-forwards to
 253 time T instead of pullbacks to time 0, is given by

254 (11)
$$J''[(\delta x, \delta v_0, \delta v)]^2 = \int_{\hat{\tau}_2}^T \delta v(t) L_{\delta \eta(t)} L_{g_t} c(\hat{x}_T) dt + \frac{\varepsilon_0^2}{2} \left(L_k^2 c(\hat{x}_T) + H_{12}(\hat{\ell}_1) \right)$$

255 subject to

256
$$\delta \dot{\eta}(t) = \delta v(t) g_t(\hat{x}_T), \quad \delta \eta(T) = \delta x \in \mathbb{R}^n, \quad \delta \eta(\hat{\tau}_2) = \varepsilon_0 k(\hat{x}_T),$$

 257
$$\varepsilon_0 = \int_0^{\hat{\tau}_1} \delta v_0(t) dt = - \int_{\hat{\tau}_1}^{\hat{\tau}_2} \delta v_0(t) dt.$$

 258

259 We then extend the second variation to a new quadratic form called *extended second*
 260 *variation*. Following the same lines as in the appendix of [19] and setting

261
$$w(t) := \int_t^{\hat{\tau}_2} \delta v(s) ds, \quad \varepsilon_1 := w(T),$$

262 the extended second variation of (9) is given by the following LQ problem on $[\hat{\tau}_2, T]$.

263 (12)
$$J''_{\text{ext}}[(\delta x, \varepsilon_0, \varepsilon_1, w)]^2 = -\varepsilon_1 L_{\delta x} L_{f_d} c(\hat{x}_T) - \frac{\varepsilon_1^2}{2} L_{f_d}^2 c(\hat{x}_T) +$$

$$+ \frac{\varepsilon_0^2}{2} \left(L_k^2 c(\hat{x}_T) + H_{12}(\hat{\ell}_1) \right) + \frac{1}{2} \int_{\hat{\tau}_2}^T (2w(t) L_{\zeta(t)} L_{g_t} c(\hat{x}_T) + w(t)^2 R(t)) dt$$

264 subject to

265 (13)
$$\dot{\zeta}(t) = w(t) \dot{g}_t(\hat{x}_T), \quad \zeta(\hat{\tau}_2) = \varepsilon_0 k(\hat{x}_T), \quad \zeta(T) = \delta x + \varepsilon_1 f_d(\hat{x}_T).$$

266 This means that we consider the quadratic form J''_{ext} defined by (12) on the linear
267 space, called *space of admissible variations*, given by

$$268 \quad \mathcal{W}_{\text{ext}} := \{(\delta x, \varepsilon_0, \varepsilon_1, w) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times L^2([\widehat{\tau}_2, T]): (13) \text{ admits a solution}\}.$$

269 Notice that

$$270 \quad (14) \quad \dot{g}_t = \widehat{S}_{t*}^{-1} h_{23} \circ \widehat{S}_t, \quad t \in [\widehat{\tau}_2, T].$$

271 We consider two cases:

272 • $L_{f_d} c \equiv 0$. J''_{ext} cannot be coercive and we only require its coercivity on the
273 subspace of \mathcal{W}_{ext} given by $\varepsilon_1 = 0$. Notice that $f_d(\widehat{x}_T) \neq 0$. Indeed, if $f_d(\widehat{x}_T) = 0$,
274 then $\mathbb{L}(\widehat{\ell}_f) = \langle \widehat{\mu}(T), [f_d, h_{23}](\widehat{x}_T) \rangle = \langle \text{dc}, \text{D}f_d h_{23}(\widehat{x}_T) \rangle = L_{h_{23}} L_{f_d} c(\widehat{x}_T) = 0$, a
275 contradiction to (SGLC).

276 • $L_{f_d} c$ is not identically zero. Choosing the variation $\delta e = (-f_d(\widehat{x}_T), 0, 1, 0)$ in
277 (12) we get $J''_{\text{ext}}[\delta e]^2 = L_{f_d}^2 c(\widehat{x}_T)$ and we require $L_{f_d}^2 c(\widehat{x}_T) > 0$.

278 In this case the set, locally defined near \widehat{x}_T in \mathbb{R}^n ,

$$279 \quad \widetilde{M} := \{x \in \mathbb{R}^n : L_{f_d} c(x) = 0\},$$

280 is a hyper-surface whose tangent space at \widehat{x}_T is

$$281 \quad T_{\widehat{x}_T} \widetilde{M} = \{\delta z \in \mathbb{R}^n : L_{\delta z} L_{f_d} c(\widehat{x}_T) = 0\}.$$

282 For $x = \exp(r f_d)(z)$, $z \in \widetilde{M}$ set $\widetilde{c}(x) := c(z)$, i.e. we extend $c|_{\widetilde{M}}$ as a constant
283 function along the integral lines of f_d . In a sufficiently small neighborhood \mathcal{O} of \widehat{x}_T ,
284 the function $\widetilde{c}: \mathcal{O} \rightarrow \mathbb{R}$ is smooth and it enjoys the following properties

$$285 \quad (15) \quad \begin{aligned} \widetilde{c}(\widehat{x}_T) &= c(\widehat{x}_T), & \text{d}\widetilde{c}(\widehat{x}_T) &= \text{dc}(\widehat{x}_T), \\ \widetilde{c}(x) &\leq c(x), & L_{f_d} \widetilde{c}(x) &= 0 \quad \forall x \in \mathcal{O}. \end{aligned}$$

286 Following [22] it can be shown that the coercivity of (12) on \mathcal{W}_{ext} is equivalent to
287 $L_{f_d}^2 c(\widehat{x}_T) > 0$ plus the coercivity of

$$288 \quad (16) \quad \begin{aligned} \widetilde{J}[(\delta x, \varepsilon_0, w)]^2 &= \frac{\varepsilon_0^2}{2} \left(L_k^2 \widetilde{c}(\widehat{x}_T) + H_{12}(\widehat{\ell}_1) \right) + \\ &+ \frac{1}{2} \int_{\widehat{\tau}_2}^T (2w(t) L_{\zeta(t)} L_{\dot{g}_t} \widetilde{c}(\widehat{x}_T) + R(t)w(t)^2) dt \end{aligned}$$

289 subject to

$$290 \quad (17) \quad \dot{\zeta}(t) = w(t) \dot{g}_t(\widehat{x}_T), \quad \zeta(\widehat{\tau}_2) = \varepsilon_0 k(\widehat{x}_T), \quad \zeta(T) = \delta x \in \mathbb{R}^n.$$

291 This is exactly the same formula we obtain in the case $L_{f_d} c \equiv 0$ setting $\widetilde{c} := c$.

292 We can now state our final assumption, concerning the second variation \widetilde{J} .

293 ASSUMPTION 6. *We assume the following conditions hold:*

294 a) *The quadratic form \widetilde{J} , (16), is coercive on*

$$295 \quad \widetilde{\mathcal{W}} := \{(\delta x, \varepsilon_0, w) \in \mathbb{R}^n \times \mathbb{R} \times L^2([\widehat{\tau}_2, T], \mathbb{R}): (17) \text{ admits a solution}\}.$$

296 b) *Either $L_{f_d}^2 c(\widehat{x}_T) > 0$ or $L_{f_d} c \equiv 0$ in a neighborhood \mathcal{O} of \widehat{x}_T in \mathbb{R}^n .*

297 **3.3. The main result.** We can now state the main result of this paper

298 **THEOREM 3.1.** *Let $\widehat{\xi}$ be the admissible trajectory defined in (2). Assume that*
 299 *$\widehat{\xi}$ is a state extremal (Assumption 1) satisfying the regularity Assumptions 2–5. If*
 300 *Assumption 6 is satisfied, then $\widehat{\xi}$ is a strict strong local optimal trajectory of (1).*

301 Indeed, in Section 5 we prove that Assumptions 1-5 plus a) of Assumption 6 imply
 302 that $\widehat{\xi}$ is a strict strong locally optimal trajectory for the cost $\widetilde{c}(\xi(T))$.

303 This concludes the proof in the case $L_{f_d}c \equiv 0$. When $L_{f_d}^2c(\widehat{x}_T) > 0$, the claim is
 304 proved by (15).

305 **4. Hamiltonian approach.** The first step in applying the Hamiltonian ap-
 306 proach described in the Introduction, is the construction of an over-maximized Hamil-
 307 tonian flow. Indeed the presence of a singular arc prevents us from using the maxi-
 308 mized Hamiltonian (see [19]) which can be used in the classical case, i.e. when it is
 309 C^2 , see [1]. The over-maximized Hamiltonian was introduced in [22] and then used in
 310 [19, 20, 6]. In [24] the authors give a systematic extension of the classical techniques
 311 to the case of an over-maximized Hamiltonian whose flow is only Lipschitz continuous.

312 **4.1. The over-maximized flow.** In this section we describe how the regularity
 313 conditions allow to define in a neighborhood \mathcal{U} of the graph of $\widehat{\lambda}$ in $[0, T] \times T^*\mathbb{R}^n$, a
 314 time-dependent Hamiltonian function $H: \mathcal{U} \rightarrow \mathbb{R}$ whose flow satisfies the assumptions
 315 stated in [24]. We consider the flow of the over-maximized Hamiltonian emanating
 316 from the following Lagrangian manifold:

$$317 \quad (18) \quad \Lambda := \{(d(-\widetilde{c})(x), x) : x \in \mathcal{O}\}.$$

318 In our case the assumptions of [24] read as follows.

- 319 1. The flow $(t, \ell) \in [0, T] \times \Lambda \mapsto \mathcal{H}_t(\ell) := (\mu_t(\ell), \xi_t(\ell)) \in T^*\mathbb{R}^n$ is Lipschitz contin-
 320 uous.
- 321 2. The function $\Phi: (t, \ell) \in [0, T] \times \Lambda \mapsto \langle \mu_t(\ell), \overrightarrow{\pi H}_t \circ \mathcal{H}_t(\ell) \rangle - H_t \circ \mathcal{H}_t(\ell) \in \mathbb{R}$ is
 322 Lipschitz-Caratheodory i.e.
 - 323 • For almost every $t \in [0, T]$ the map $\ell \in \Lambda \mapsto \Phi(t, \ell) \in \mathbb{R}$ is locally Lipschitz.
 - 324 • For each $\ell \in \Lambda$ the map $t \in [0, T] \mapsto \Phi(t, \ell) \in \mathbb{R}$ is bounded measurable.
 - 325 • For any compact set $K \subset \Lambda$ there is an essentially bounded measurable
 326 function $m: [0, T] \rightarrow \mathbb{R}$ such that

$$327 \quad \|\Phi(t, \ell_1) - \Phi(t, \ell_2)\| \leq m(t) \|\ell_1 - \ell_2\|, \quad \forall \ell_1, \ell_2 \in \Lambda.$$

- 328 3. The following over-maximality conditions hold:

- 329 (a) $H_t \circ \mathcal{H}_t(\ell) \geq H^{\max} \circ \mathcal{H}_t(\ell)$, for any $(t, \ell) \in [0, T] \times \Lambda$,
- 330 (b) $H_t \circ \widehat{\lambda}(t) = \widehat{F}_t \circ \widehat{\lambda}(t) = H^{\max} \circ \widehat{\lambda}(t)$, for a.e. $t \in [0, T]$,
- 331 (c) $\overrightarrow{H}_t \circ \widehat{\lambda}(t) = \overrightarrow{F}_t \circ \widehat{\lambda}(t)$, for a.e. $t \in [0, T]$.

332 The coercivity of the second variation will then guarantee the invertibility of the
 333 projected over-maximized flow of such Hamiltonian.

334 In order to describe H_t , $t \in [\widehat{\tau}_2, T]$ notice that the **SGLC** condition (Assumption 5)
 335 implies that there exists a neighborhood \mathcal{O}_s of the range of the singular arc $\widehat{\lambda}([\widehat{\tau}_2, T])$
 336 in $T^*\mathbb{R}^n$ such that the sets

$$337 \quad \Sigma := \{\ell \in \mathcal{O}_s : F_d(\ell) = 0\} = \{\ell \in \mathcal{O}_s : H_2(\ell) = H_3(\ell)\},$$

$$338 \quad \mathcal{S} := \{\ell \in \Sigma : H_{23}(\ell) = 0\} = \{\ell \in \mathcal{O}_s : H_2(\ell) = H_3(\ell), H_{23}(\ell) = 0\}$$

340 are smooth simply connected manifolds of codimension 1 and 2, respectively. More
 341 precisely $\overrightarrow{H_{23}}$ is transverse to Σ in \mathcal{O}_s , while $\overrightarrow{F_d}$ is tangent to Σ and transverse to \mathcal{S} in
 342 Σ , see [19]. Moreover we can define a smooth function $u_s: \mathcal{O}_s \rightarrow \mathbb{R}$ and the feedback
 343 Hamiltonian as follows:

$$344 \quad u_s(\ell) = \frac{H_{232}}{\mathbb{L}}(\ell), \quad H^S(\ell) = H_2(\ell) + u_s(\ell)F_d(\ell).$$

345

346 **REMARK 4.1.** *On the singular interval $[\widehat{\tau}_2, T]$, $\widehat{v}(t) = u_s(\widehat{\lambda}(t))$ and $\widehat{\lambda}$ is the solu-*
 347 *tion to the Cauchy problem*

$$348 \quad (19) \quad \dot{\lambda}(t) = \overrightarrow{H^S}(\lambda(t)), \quad \lambda(\widehat{\tau}_2) = \widehat{\ell}_2.$$

349 In [19] the authors prove that possibly restricting \mathcal{O}_s , the following implicit func-
 350 tion problem has a solution $\theta: \mathcal{O}_s \rightarrow \mathbb{R}$:

$$351 \quad \begin{cases} H_{23} \circ \exp(\theta(\ell)\overrightarrow{F_d})(\ell) = 0, \\ \theta(\ell) = 0 \quad \text{if } H_{23}(\ell) = 0, \end{cases}$$

352 and

$$353 \quad \langle d\theta(\ell_S), \delta\ell \rangle = \frac{-\sigma(\delta\ell, \overrightarrow{H_{23}}(\ell_S))}{\mathbb{L}(\ell_S)}, \quad \forall \ell_S: H_{23}(\ell_S) = 0.$$

354 The over-maximized Hamiltonian is defined starting from

$$355 \quad \widetilde{H}_2(\ell) := H_2 \circ \exp(\theta(\ell)\overrightarrow{F_d})(\ell).$$

356

357 **LEMMA 4.1.** *Possibly restricting \mathcal{O}_s the following properties hold*

- 358 1. $\widetilde{H}_2(\ell) \geq H_2(\ell)$ for any $\ell \in \Sigma$ and equality holds if and only if $\ell \in \mathcal{S}$.
 359 2. For any $\ell_S \in \mathcal{S}$ and $\delta\ell \in T_{\ell_S}\Sigma$

$$360 \quad (20) \quad d(\widetilde{H}_2 - H_2)(\ell_S) = 0, \quad D^2(\widetilde{H}_2 - H_2)(\ell_S)[\delta\ell]^2 = \frac{(\sigma(\delta\ell, \overrightarrow{H_{23}}(\ell_S)))^2}{\mathbb{L}(\ell_S)}.$$

- 361 3. $\overrightarrow{\widetilde{H}_2}$ is tangent to Σ and, setting,

$$362 \quad (21) \quad H_t(\ell) := \widetilde{H}_2(\ell) + \widehat{v}(t)F_d(\ell), \quad \forall (t, \ell) \in [\widehat{\tau}_2, T] \times \mathcal{O}_s$$

363 we easily get that $\overrightarrow{H_t}$ is tangent to Σ .

- 364 4. $\widehat{\lambda}|_{[\widehat{\tau}_2, T]}$ solves the Cauchy problem $\dot{\lambda}(t) = \overrightarrow{H_t}(\lambda(t))$, $\lambda(T) = \widehat{\ell}_f$.

- 365 5. $[\overrightarrow{F_d}, \overrightarrow{H_t}] \equiv 0$ on Σ hence $\overrightarrow{F_d}$ is invariant on Σ with respect to the flow of $\overrightarrow{H_t}$:

$$366 \quad (22) \quad \overrightarrow{F_d} \circ \mathcal{H}_t(\ell) = \mathcal{H}_{t*}\overrightarrow{F_d}(\ell), \quad \forall (t, \ell) \in [\widehat{\tau}_2, T] \times \Sigma.$$

367 The proof of Lemma 4.1 can be done adapting the results in [19] and completes the
 368 analysis of the singular arc.

369 The bang arcs present problems of a different kind. Namely we need to define the
 370 switching times near the reference switching points $\widehat{\ell}_1$ and $\widehat{\ell}_2$.

371 For (t, ℓ) near the graph of $\widehat{\lambda}$ we need $H_t(\ell) \geq H^{\max}(\ell)$, but in [19] it is shown
 372 that the backwards flow of $\overrightarrow{H_2}$ from time $\widehat{\tau}_2$ is the maximized one *if and only if*
 373 $H_{23}(\ell) \geq 0$. In order to overcome this problem we introduce a *correction* of such a
 374 flow from time $\widehat{\tau}_2$ by keeping it on Σ whenever $H_{23}(\ell) < 0$.

375 In [19] it is shown that, thanks to the second inequality in Assumption 4, the
 376 implicit function theorem applied to the problem:

$$377 \quad \begin{cases} H_{23} \circ \exp(t_2 - \widehat{\tau}_2) \overrightarrow{H_2}(\ell) = 0, \\ t_2(\ell) = \widehat{\tau}_2 \quad \text{if } H_{23}(\ell) = 0. \end{cases}$$

378 defines, in a neighborhood \mathcal{O}_2 of $\widehat{\ell}_2$, a function $t_2: \mathcal{O}_2 \rightarrow \mathbb{R}$ such that if $\ell \in \Sigma$, then
 379 $t_2(\ell) = \widehat{\tau}_2$ if and only if $\ell \in \mathcal{S}$; moreover

$$380 \quad \langle dt_2(\widehat{\ell}_2), \delta\ell \rangle = \frac{-\sigma\left(\delta\ell, \overrightarrow{H_{23}}(\widehat{\ell}_2)\right)}{H_{223}(\widehat{\ell}_2)} \quad \forall \delta\ell \in T^*\mathbb{R}^n.$$

381 We set

$$382 \quad \tau_2(\ell) := \min\{t_2(\ell), \widehat{\tau}_2\} = \begin{cases} t_2(\ell) & \text{if } H_{23}(\ell) < 0, \\ \widehat{\tau}_2 & \text{if } H_{23}(\ell) \geq 0. \end{cases}$$

383 The next step is the definition of the switching time $\tau_1: \mathcal{O}_2 \rightarrow \mathbb{R}$, possibly shrinking
 384 \mathcal{O}_2 . Actually, thanks to the first inequality in Assumption 4, the implicit function
 385 theorem applies also to

$$386 \quad \begin{cases} (H_2 - H_1) \circ \exp(\tau_1 - \tau_2(\ell)) \overrightarrow{H_2} \circ \exp(\tau_2(\ell) - \widehat{\tau}_2) \overrightarrow{H_2}(\ell) = 0, \\ \tau_1(\widehat{\ell}_2) = \widehat{\tau}_1, \end{cases}$$

387 see e.g. [3]. Setting

$$388 \quad (23) \quad \widetilde{k}(x) := \widehat{S}_{\widehat{\tau}_2}^{-1} \circ k \circ \widehat{S}_{\widehat{\tau}_2}^{-1}(x), \quad \widetilde{K}(p, x) := \langle p, \widetilde{k}(x) \rangle,$$

389 the linearization of τ_1 at $\widehat{\ell}_2$ is given by

$$390 \quad (24) \quad \langle d\tau_1(\widehat{\ell}_2), \delta\ell \rangle = \frac{\sigma\left(\exp(\widehat{\tau}_1 - \widehat{\tau}_2) \overrightarrow{H_2} \circ \delta\ell, \left(\overrightarrow{H_1} - \overrightarrow{H_2}\right)(\widehat{\ell}_1)\right)}{H_{12}(\widehat{\ell}_1)} = \frac{\sigma\left(\delta\ell, \overrightarrow{K}(\widehat{\ell}_2)\right)}{H_{12}(\widehat{\ell}_1)}.$$

391 We can now define the flow $(t, \ell) \mapsto \mathcal{H}_t(\ell)$ backwards in time emanating from a
 392 neighborhood \mathcal{O}_f of $\widehat{\ell}_f$ in $T^*\mathbb{R}^n$ at time T .

393 For any $t \in [\widehat{\tau}_2, T]$ we choose as $\mathcal{H}_t(\ell)$ the flow of $\overrightarrow{H_t}$ defined in (21).

394 For $t < \widehat{\tau}_2$, setting $\widetilde{\ell} := \mathcal{H}_{\widehat{\tau}_2}(\ell)$, we define

$$395 \quad (25) \quad \mathcal{H}_t(\ell) := \begin{cases} \exp(t - \widehat{\tau}_2) \overrightarrow{H_2}(\widetilde{\ell}) & t \in [\tau_2(\widetilde{\ell}), \widehat{\tau}_2], \\ \exp(t - \tau_2(\widetilde{\ell})) \overrightarrow{H_2} \circ \mathcal{H}_{\tau_2(\widetilde{\ell})}(\widetilde{\ell}) & t \in [\tau_1(\widetilde{\ell}), \tau_2(\widetilde{\ell})], \\ \exp(t - \tau_1(\widetilde{\ell})) \overrightarrow{H_1} \circ \mathcal{H}_{\tau_1(\widetilde{\ell})}(\widetilde{\ell}) & t \in [0, \tau_1(\widetilde{\ell})], \end{cases}$$

396 see Figure 1.

397 **REMARK 4.2.** Notice that \mathcal{H} is C^∞ on $[\widehat{\tau}_2^+, T] \times \mathcal{O}_f$ and it is Lipschitz continuous
 398 on $[0, \widehat{\tau}_2^-] \times \mathcal{O}_f$. Actually it is C^1 but on $\{(t, \mathcal{H}_t(\ell)): t = \tau_1(\widetilde{\ell})\}$. Indeed on the
 399 set $\{(t, \mathcal{H}_t(\ell)): t = \tau_2(\widetilde{\ell})\}$ it is C^1 since $\mathcal{H}_{\tau_2(\widetilde{\ell})}(\widetilde{\ell}) \in \mathcal{S}$, so that $\overrightarrow{H_2}(\mathcal{H}_{\tau_2(\widetilde{\ell})}(\widetilde{\ell})) =$
 400 $\overrightarrow{H_2}(\mathcal{H}_{\tau_2(\widetilde{\ell})}(\widetilde{\ell}))$, by (20).

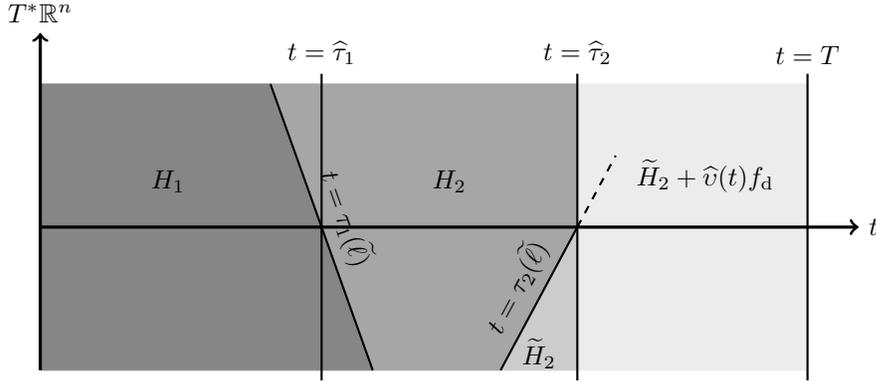


FIG. 1. The over-maximized Hamiltonian along its flow emanating from Λ at time T . The picture shows when the transition from one smooth piece to another is defined and where the over-maximized Hamiltonian is actually greater than the maximized Hamiltonian of the control system.

401 **4.2. Hamiltonian sufficient conditions.** In this section we state and prove
 402 the sufficient conditions for strong local optimality of $\hat{\xi}$ in terms of Hamiltonian flow.

403 First we prove that if the projected over-maximized flow emanating from Λ is
 404 locally Lipschitz invertible then $\hat{\xi}$ is a strong local minimizer, Theorem 3.1. Afterwards
 405 we give the second order conditions that ensure this invertibility property, Theorem
 406 4.2.

407 **THEOREM 4.2.** Let Λ be defined in (18). Assume that

$$408 \quad (26) \quad \text{id} \times \pi\mathcal{H}: (t, \ell) \in [0, T] \times \Lambda \mapsto (t, \pi\mathcal{H}_t(\ell)) \in \mathcal{U}.$$

409 is locally Lipschitz invertible onto a neighborhood \mathcal{U} of the graph of $\hat{\xi}$ in $[0, T] \times \mathbb{R}^n$.
 410 Then $\hat{\xi}$ is a strict strong locally optimal trajectory for the cost $\tilde{c}(\xi(T))$ subject to
 411 (1b)-(1c).

412 *Proof.* Clearly $(\text{id} \times \pi\mathcal{H})^{-1}(t, \hat{\xi}(t)) = (t, \hat{\ell}_f)$ for any $t \in [0, T]$. Let $\xi: [0, T] \rightarrow \mathbb{R}^n$
 413 be an admissible trajectory for (1) whose graph is in \mathcal{U} and let

$$414 \quad (t, \ell(t)) := (\text{id} \times \pi\mathcal{H})^{-1}(t, \xi(t)), \quad \lambda(t) := \mathcal{H}_t(\ell(t)) = (\mu(t), \xi(t)), \quad t \in [0, T].$$

415 If $\varphi: [0, 1] \rightarrow \Lambda$ is a smooth curve such that $\varphi(0) = \ell(T)$, $\varphi(1) = \hat{\ell}_f$ then we can
 416 consider the closed path in $[0, T] \times \Lambda$ obtained by the concatenation of the curves

$$417 \quad t \mapsto (t, \ell(t)), \quad s \mapsto (T, \varphi(s)), \quad t \mapsto (T - t, \hat{\ell}_f).$$

418 Integrating the one-form $\omega := \mathcal{H}^*(s - H_t dt)$ (which is exact on $[0, T] \times \Lambda$, see
 419 [24]), we obtain

$$420 \quad (27) \quad 0 = \oint \omega = \int_{\text{id} \times \ell} \left(\langle \mu(t), \dot{\xi}(t) \rangle - H_t(\lambda(t)) \right) dt + \int_{\varphi} \mathcal{H}_t^* s +$$

$$421 \quad - \int_{\text{id} \times \hat{\ell}_f} \left(\langle \hat{\mu}(t), \dot{\hat{\xi}}(t) \rangle - H_t(\hat{\lambda}(t)) \right) dt.$$

422 By construction of the over-maximized Hamiltonian H_t , the integrand is non positive
 423 along the curve $\text{id} \times \ell$ and is identically zero along the curve $\text{id} \times \hat{\ell}_f$. Therefore

$$\begin{aligned}
 427 \quad (28) \quad 0 &\leq \int_{\varphi} \mathcal{H}_t^* \mathbf{s} = \int_0^1 \langle \varphi(s), \frac{d}{ds}(\pi\varphi)(s) \rangle ds \\
 428 \quad &= \int_0^1 \langle d(-\tilde{c})(\pi\varphi(s)), \frac{d}{ds}(\pi\varphi)(s) \rangle ds = \tilde{c}(\xi(T)) - \tilde{c}(\widehat{x}_T). \\
 429
 \end{aligned}$$

430 Thus $\tilde{c}(\xi(T)) \geq \tilde{c}(\widehat{x}_T)$, i.e. $\widehat{\xi}$ is a strong local minimizer for the cost \tilde{c} . Let us show
431 that in fact it is a strict one.

432 If $\tilde{c}(\xi(T)) = \tilde{c}(\widehat{x}_T)$, then (27)-(28) imply that

$$433 \quad (29) \quad \langle \mu(t), \dot{\xi}(t) \rangle - H_t(\lambda(t)) = 0 \quad \text{a.e. } t \in [0, T]$$

434 so that

$$435 \quad (30) \quad \langle \mu(t), \dot{\xi}(t) \rangle = H^{\max}(\lambda(t)) = H_t(\lambda(t)) \quad \text{a.e. } t \in [0, T].$$

436 Since $\xi(0) = \widehat{\xi}(0)$, we also have $\lambda(0) = \widehat{\ell}_0$. On the interval $[0, \widehat{\tau}_2]$, equation (30)
437 and the maximality condition imply that $d(H_t - H^{\max})|_{\lambda(t)} = 0$ a.e. $t \in [0, \widehat{\tau}_2]$, so
438 that $\overrightarrow{H}^{\max}(\lambda(t)) = \overrightarrow{H}_t(\lambda(t))$ a.e. $t \in [0, \widehat{\tau}_2]$. Thus $\lambda(t) = \widehat{\lambda}(t)$ for any $t \in [0, \widehat{\tau}_2]$, in
439 particular $\lambda(\widehat{\tau}_2) = \widehat{\ell}_2$.

440 For $t \in [\widehat{\tau}_2, T]$, equation (29) yields $\widetilde{H}_2(\lambda(t)) = H_2(\lambda(t))$, i.e. $\lambda(t) \in \mathcal{S}$. Define

$$441 \quad \Sigma_{\xi(t)} := \{p \in (\mathbb{R}^n)^* : \langle p, f_d(\xi(t)) \rangle = 0\}$$

442 and consider the function

$$443 \quad \Omega : p \in \Sigma_{\xi(t)} \mapsto \langle p, \dot{\xi}(t) \rangle - H_t(p, \xi(t)) \in \mathbb{R}.$$

444 By PMP the function Ω is non positive and by (29) it is null in $\mu(t)$, therefore

$$445 \quad (31) \quad \langle \delta p, \dot{\xi}(t) - \pi \overrightarrow{H}_t(\lambda(t)) \rangle = 0, \quad \forall \delta p \in (\mathbb{R}^n)^*, \text{ such that } \langle \delta p, f_d(\xi(t)) \rangle = 0.$$

446 Hence there exists $b(t) \in \mathbb{R}$ such that

$$447 \quad \dot{\xi}(t) = \pi \overrightarrow{H}_t(\lambda(t)) + b(t) f_d(\xi(t)) \quad \forall t \in [\widehat{\tau}_2, T].$$

448 By (22), $(\pi \mathcal{H}_t)_*^{-1} f_d \circ (\pi \mathcal{H}_t)(\ell(t)) = \overrightarrow{F}_d(\ell(t))$ so that

$$449 \quad \dot{\ell}(t) = (\pi \mathcal{H}_t)_*^{-1} \left(\dot{\xi}(t) - \pi \overrightarrow{H}_t(\lambda(t)) \right) = b(t) (\pi \mathcal{H}_t)_*^{-1} f_d(\xi(t)) = b(t) \overrightarrow{F}_d(\ell(t)),$$

$$450 \quad \dot{\lambda}(t) = \overrightarrow{H}_t(\lambda(t)) + \mathcal{H}_{t*} \dot{\ell}(t) = \overrightarrow{H}_2(\lambda(t)) + (\widehat{v}(t) + b(t)) \overrightarrow{F}_d(\lambda(t)).$$

452 Finally, since $\lambda(t) \in \mathcal{S}$, we get

$$453 \quad (32) \quad 0 = \boldsymbol{\sigma} \left(\dot{\lambda}(t), \overrightarrow{H}_{23}(\lambda(t)) \right) = -H_{232}(\lambda(t)) + (\widehat{v}(t) + b(t)) \mathbb{L}(\lambda(t)),$$

454 so that $\widehat{v}(t) + b(t) = u_s(\lambda(t))$. Therefore $\lambda(t)$ and $\widehat{\lambda}(t)$ solve the same Cauchy problem
455 (19). This proves that $\lambda \equiv \widehat{\lambda}$ and hence the strict strong local optimality of $\widehat{\xi}$. \square

456 As we want to obtain second order sufficient conditions, we take a Hamiltonian
457 approach based on the linearization of the flow from

$$458 \quad L_T := T_{\widehat{\ell}_T} \Lambda = \{d(-\tilde{c})_* \delta x : \delta x \in \mathbb{R}^n\}.$$

459 Our construction is naturally split in two parts by the time $t = \widehat{\tau}_2$. In particular we
 460 point out that if $(\pi\mathcal{H}_{\widehat{\tau}_2})_* : L_T \rightarrow \mathbb{R}^n$ is an isomorphism, then there exists at least a
 461 smooth function $\alpha_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$462 \quad (33) \quad d\alpha_2(\widehat{x}_2) = \widehat{\mu}(\widehat{\tau}_2), \quad \mathcal{H}_{\widehat{\tau}_2} L_T = \{d\alpha_2 \delta x : \delta x \in \mathbb{R}^n\}.$$

463 We explicitly point out that α_2 is not uniquely determined but only its first and
 464 second order derivatives at \widehat{x}_2 are determined by (33).

465 We can now state a second order sufficient condition for the strong local optimality
 466 of the reference trajectory $\widehat{\xi}$.

467 **THEOREM 4.3.** *Let $\widehat{\xi}$ be the admissible trajectory defined in (2). Assume that $\widehat{\xi}$
 468 is a state extremal (Assumption 1) satisfying the regularity Assumptions 2–5.*

469 *Assume moreover*

- 470 1. $(\pi\mathcal{H}_t)_* : L_T \rightarrow \mathbb{R}^n$ is an isomorphism for any $t \in [\widehat{\tau}_2, T]$, i.e. the kernel of
 471 the map is trivial;
- 472 2. $H_{12}(\widehat{\ell}_1) - L_k^2 \alpha_2(\widehat{x}_2) > 0$ where α_2 is any smooth function on \mathbb{R}^n satisfying
 473 (33) and \widetilde{k} is defined in (23).

474 Then $\widehat{\xi}$ is a strict strong local minimizer for problem (1).

475 *Proof.* According to Theorem 4.2 we only need to prove that the map $\text{id} \times$
 476 $\pi\mathcal{H} : [0, T] \times \Lambda \rightarrow [0, T] \times \mathbb{R}^n$ is one-to-one onto a neighborhood \mathcal{U} of the graph of
 477 $\widehat{\xi}$. Since $[0, T]$ is a compact interval, it suffices to prove that $\text{id} \times \pi\mathcal{H}_t$ is locally
 478 bi-Lipschitz in a neighborhood of $(t, \widehat{\ell}_f)$ for any $t \in [0, T]$.

479 For $t \neq \widehat{\tau}_1$ Remark 4.2 implies that it suffices to prove that $(\pi\mathcal{H}_t)_* : L_T \rightarrow \mathbb{R}^n$
 480 is an isomorphism while, for $t = \widehat{\tau}_1$ we take advantage of Clarke inverse function
 481 theorem.

- 482 • Condition 1 ensures the invertibility on $[\widehat{\tau}_2, T]$.
- 483 • For $t \in (\widehat{\tau}_1, \widehat{\tau}_2)$, $(\pi\mathcal{H}_t)_* = \exp(t - \widehat{\tau}_2) h_{2*} (\pi\mathcal{H}_{\widehat{\tau}_2})_*$ which is invertible.
- 484 • If $t = \widehat{\tau}_1$, for any $\delta\ell \in L_T$, set $\widetilde{\delta\ell} = \mathcal{H}_{\widehat{\tau}_2} \delta\ell$. The linearization of $\pi\mathcal{H}_{\widehat{\tau}_1}$ at $\widehat{\ell}_f$ is
 485 given by

$$486 \quad (\pi\mathcal{H}_{\widehat{\tau}_1})_* \delta\ell = \begin{cases} \exp(\widehat{\tau}_1 - \widehat{\tau}_2) h_{2*} \pi \widetilde{\delta\ell}, & \langle d\tau_1(\widehat{\ell}_2), \widetilde{\delta\ell} \rangle < 0, \\ \exp(\widehat{\tau}_1 - \widehat{\tau}_2) h_{2*} \left(\pi \widetilde{\delta\ell} - \langle d\tau_1(\widehat{\ell}_2), \widetilde{\delta\ell} \rangle \widetilde{k}(\widehat{x}_2) \right), & \langle d\tau_1(\widehat{\ell}_2), \widetilde{\delta\ell} \rangle > 0. \end{cases}$$

487 If $\langle d\tau_1(\widehat{\ell}_2), \widetilde{\delta\ell} \rangle \widetilde{k}(\widehat{x}_2) = 0$ for any $\widetilde{\delta\ell} \in L_T$, we are done. Otherwise, it suffices to prove
 488 that for any $a \in [0, 1]$ and any $\delta\ell \in L_T$, $\delta\ell \neq 0$

$$489 \quad \pi(\widetilde{\delta\ell}) - a \langle d\tau_1(\widehat{\ell}_2), \widetilde{\delta\ell} \rangle \widetilde{k}(\widehat{x}_2) \neq 0.$$

490 For $a = 0$ the claim is obvious thanks to assumption 1. Assume by contradiction
 491 there exists $a \in (0, 1]$ and $\delta\ell \in L_T$, $\delta\ell \neq 0$ such that

$$492 \quad (34) \quad \pi \widetilde{\delta\ell} - a \langle d\tau_1(\widehat{\ell}_2), \widetilde{\delta\ell} \rangle \widetilde{k}(\widehat{x}_2) = 0.$$

493 By (34) there exists $\rho \neq 0$ such that $\pi \widetilde{\delta\ell} = \rho \widetilde{k}(\widehat{x}_2)$ so that $\widetilde{\delta\ell} = \rho d\alpha_2 \widetilde{k}(\widehat{x}_2)$ and

$$494 \quad 0 = \rho \widetilde{k}(\widehat{x}_2) - a \langle d\tau_1(\widehat{\ell}_2), \rho d\alpha_2 \widetilde{k}(\widehat{x}_2) \rangle \widetilde{k}(\widehat{x}_2).$$

495 Thus $1 - a \langle d\tau_1(\widehat{\ell}_2), d\alpha_2 \widetilde{k}(\widehat{x}_2) \rangle = 0$, so that

$$496 \quad 0 = 1 - \frac{a}{H_{12}(\widehat{\ell}_1)} \sigma \left(d\alpha_2 \widetilde{k}(\widehat{x}_2), \vec{K}(\widehat{\ell}_2) \right) = \frac{1}{H_{12}(\widehat{\ell}_1)} \left\{ H_{12}(\widehat{\ell}_1) - a L_k^2 \alpha_2(\widehat{x}_2) \right\}.$$

497 Since this quantity is positive both for $a = 0$ and for $a = 1$, it is positive for any
 498 $a \in [0, 1]$. \square

499 **REMARK 4.3.** *As already said, the switching time $\widehat{\tau}_2$ naturally splits our construc-*
 500 *tion in two parts. In particular Assumption 1 of Theorem 4.3 takes into account only*
 501 *the problem restricted to the singular interval $[\widehat{\tau}_2, T]$. Assumption 2 coincides with the*
 502 *sufficient condition in [3] for a fixed-free Mayer problem on $[0, \widehat{\tau}_2]$ with cost $\alpha_2(\xi(\widehat{\tau}_2))$.*

503 **5. Proof of the main result.** In this section we prove that the coercivity of \widetilde{J}
 504 (Assumption 6 a)) guarantees that Assumptions 1 and 2 of Theorem 4.3 hold true.
 505 In particular Assumption 1 will be proven to hold by exploiting the coercivity of \widetilde{J}
 506 on the subspace $\widetilde{\mathcal{V}}$ of the admissible variations such that $\varepsilon_0 = 0$, while Assumption 2
 507 is proven to hold by exploiting the coercivity of \widetilde{J} on the subspace of the admissible
 508 variations which are \widetilde{J} -orthogonal to $\widetilde{\mathcal{V}}$.

509 **5.1. Coercivity of \widetilde{J} in Hamiltonian formalism.** We start by exploiting the
 510 coercivity of \widetilde{J} on $\widetilde{\mathcal{V}} := \left\{ \delta e = (\delta x, 0, w) \in \widetilde{\mathcal{W}} \right\}$, i.e.

$$511 \quad (35) \quad \widetilde{J}[\delta e]^2 = \frac{1}{2} \int_{\widehat{\tau}_2}^T (2w(t)L_{\zeta(t)}L_{\dot{g}_t}\widetilde{c}(\widehat{x}_T) + R(t)w(t)^2) dt$$

512 subject to

$$513 \quad (36) \quad \dot{\zeta}(t) = w(t)\dot{g}_t(\widehat{x}_T), \quad \zeta(\widehat{\tau}_2) = 0, \quad \zeta(T) = \delta x \in \mathbb{R}^n.$$

514 The associated Hamiltonian is given by the quadratic form

$$515 \quad (37) \quad H_t''(\delta p, \delta x) = -\frac{1}{2R(t)} (\langle \delta p, \dot{g}_t(\widehat{x}_T) \rangle + L_{\delta x}L_{\dot{g}_t}\widetilde{c}(\widehat{x}_T))^2$$

516 and the corresponding Hamiltonian linear system with initial conditions in the La-
 517 grangian subspace of transversality conditions

$$518 \quad L_T'' := \{(0, \delta x) : \delta x \in \mathbb{R}^n\}$$

519 is given by

$$520 \quad (38) \quad \begin{cases} \dot{\mu}''(t) = \frac{1}{R(t)} \left(\langle \mu''(t), \dot{g}_t(\widehat{x}_T) \rangle + L_{\zeta''(t)}L_{\dot{g}_t}\widetilde{c}(\widehat{x}_T) \right) L_{(\cdot)}L_{\dot{g}_t}\widetilde{c}(\widehat{x}_T), \\ \dot{\zeta}''(t) = \frac{-1}{R(t)} \left(\langle \mu''(t), \dot{g}_t(\widehat{x}_T) \rangle + L_{\zeta''(t)}L_{\dot{g}_t}\widetilde{c}(\widehat{x}_T) \right) \dot{g}_t(\widehat{x}_T), \\ \mu''(T) = 0, \quad \zeta''(T) = \delta x. \end{cases}$$

521 We denote the solution of (38) as $\mathcal{H}_t''(0, \delta x)$.

522 \widetilde{J} is coercive on $\widetilde{\mathcal{V}}$ if and only if for any $t \in [\widehat{\tau}_2, T]$,

$$523 \quad (39) \quad \delta x \neq 0 \implies \zeta''(t) \neq 0$$

524 where ζ'' is defined in (38), see for example [23].

525 If $k(\widehat{x}_T) = 0$, then we get no more information since $\widetilde{\mathcal{V}} = \widetilde{\mathcal{W}}$.

526 Assume $k(\widehat{x}_T) \neq 0$. Since \widetilde{J} is coercive on $\widetilde{\mathcal{V}}$, we just need to express its coercivity on

$$527 \quad \widetilde{\mathcal{V}}^\perp := \left\{ \delta e \in \widetilde{\mathcal{W}} : \widetilde{J}(\delta e, \overline{\delta e}) = 0 \quad \forall \overline{\delta e} \in \widetilde{\mathcal{V}} \right\}.$$

528 For any $\delta e = (\delta x, \varepsilon_0, w)$, $\overline{\delta e} = (\overline{\delta x}, \overline{\varepsilon_0}, \overline{w})$ in $\widetilde{\mathcal{W}}$, let ζ and $\overline{\zeta}$ be the corresponding
529 solutions of system (17). The bilinear form associated with \widetilde{J} is given by

$$530 \quad (40) \quad \begin{aligned} \widetilde{J}(\delta e, \overline{\delta e}) &= \frac{\varepsilon_0 \overline{\varepsilon_0}}{2} \left(H_{12}(\widehat{\ell}_1) + L_k^2 \widetilde{c}(\widehat{x}_T) \right) + \\ &+ \frac{1}{2} \int_{\widehat{\tau}_2}^T \left(\overline{w}(t) L_{\zeta(t)} L_{\dot{g}_t} \widetilde{c}(\widehat{x}_T) + w(t) L_{\overline{\zeta}(t)} L_{\dot{g}_t} \widetilde{c}(\widehat{x}_T) + w(t) \overline{w}(t) R(t) \right) dt. \end{aligned}$$

531 If $p(t)$ is the solution of the Cauchy problem

$$532 \quad \dot{p}(t) = -w(t) L_{(\cdot)} L_{\dot{g}_t} \widetilde{c}(\widehat{x}_T), \quad p(T) = 0,$$

533 then an integration by parts in (40) gives

$$534 \quad (41) \quad \begin{aligned} \widetilde{J}(\delta e, \overline{\delta e}) &= \frac{1}{2} \left(\varepsilon_0 \overline{\varepsilon_0} \left(H_{12}(\widehat{\ell}_1) + L_k^2 \widetilde{c}(\widehat{x}_T) \right) + \langle p(\widehat{\tau}_2), \overline{\zeta}(\widehat{\tau}_2) \rangle \right) + \\ &+ \frac{1}{2} \int_{\widehat{\tau}_2}^T \overline{w}(t) \left(\langle p(t), \dot{g}_t(\widehat{x}_T) \rangle + L_{\zeta(t)} L_{\dot{g}_t} \widetilde{c}(\widehat{x}_T) + w(t) R(t) \right) dt. \end{aligned}$$

535 Since $\overline{\zeta}(T)$ is free, we obtain that \overline{w} may be any function in $L^2([\widehat{\tau}_2, T], \mathbb{R})$. Thus,
536 if $\delta e \in \widetilde{\mathcal{V}}^\perp$ then

$$537 \quad (42) \quad \langle p(t), \dot{g}_t(\widehat{x}_T) \rangle + L_{\zeta(t)} L_{\dot{g}_t} \widetilde{c}(\widehat{x}_T) + w(t) R(t) = 0 \quad \text{a.e. } t \in [\widehat{\tau}_2, T].$$

538 Comparing (42) and (38) we get $(p(t), \zeta(t)) = \mathcal{H}_t''(0, \delta x) = (\mu''(t), \zeta''(t))$ so that for
539 any $\delta e \in \widetilde{\mathcal{V}}^\perp$ we get $\zeta''(\widehat{\tau}_2) = \varepsilon_0 k(\widehat{x}_T)$ and

$$540 \quad (43) \quad \widetilde{J}[\delta e]^2 = \frac{\varepsilon_0^2}{2} \left(H_{12}(\widehat{\ell}_1) + L_k^2 \widetilde{c}(\widehat{x}_T) \right) + \frac{1}{2} \langle \mu''(\widehat{\tau}_2), \zeta''(\widehat{\tau}_2) \rangle.$$

541 Without loss of generality we can choose $\varepsilon_0 = 1$, so that the coercivity of \widetilde{J} can be
542 expressed as

$$543 \quad (44) \quad H_{12}(\widehat{\ell}_1) - L_k^2(-\widetilde{c})(\widehat{x}_T) + \sigma \left(\mathcal{H}_{\widehat{\tau}_2}'' (\pi \mathcal{H}_{\widehat{\tau}_2}'')^{-1} k(\widehat{x}_T), (0, k(\widehat{x}_T)) \right) > 0.$$

544 **5.2. The anti-symplectic isomorphism.** In order to relate the coercivity of
545 \widetilde{J} with the properties of the flow \mathcal{H}_t , we define

$$546 \quad \iota: (\delta p, \delta x) \in (\mathbb{R}^n)^* \times \mathbb{R}^n \mapsto \delta \ell := (-\delta p + D^2(-\widetilde{c})(\widehat{x}_T)(\delta x, \cdot), \delta x) \in (\mathbb{R}^n)^* \times \mathbb{R}^n$$

547 so that $\iota^{-1} = \iota$. The mapping ι is an anti-symplectic linear isomorphism, i.e.

$$548 \quad \sigma(\iota(\delta p, \delta x), \iota(\overline{\delta p}, \overline{\delta x})) = \sigma((\overline{\delta p}, \overline{\delta x}), (\delta p, \delta x)), \quad \forall (\delta p, \delta x), (\overline{\delta p}, \overline{\delta x}) \in (\mathbb{R}^n)^* \times \mathbb{R}^n.$$

549 The choice of the anti-symplectic isomorphism ι , instead of a symplectic one, depends
550 on the fact that we are using PMP while for the accessory problem we are studying
551 a minimization problem.

552 With this notation we get

$$553 \quad \iota L_T'' = \{d(-\widetilde{c})_* \delta x : \delta x \in T_{\widehat{x}_T} \mathbb{R}^n\} = L_T.$$

554 Following the lines of Lemma 9 in [19] one can prove the following Lemma:

555 **LEMMA 5.1.** *Let \mathcal{H}_t'' and \mathcal{H}_t be the Hamiltonian flows associated to the quadratic*
556 *Hamiltonian H_t'' defined in (37) and to the over-maximized Hamiltonian H_t defined*
557 *in (21), respectively. Then*

$$558 \quad (45) \quad \iota \mathcal{H}_t'' \iota^{-1} = \widehat{\mathcal{F}}_{t*}^{-1} \mathcal{H}_{t*} \quad \forall t \in [\widehat{\tau}_2, T].$$

559 **5.3. Proof of Assumption 1 of Theorem 4.3.** Applying (45) to $\mathcal{H}_t''|_{L_T''}$ we
 560 get for $\delta\ell \in L_T$

$$561 \quad \pi\mathcal{H}_t''\iota^{-1}\delta\ell = \pi\widehat{\mathcal{F}}_{t*}^{-1}\mathcal{H}_{t*}\delta\ell.$$

562 Since the Hamiltonian \widehat{F}_t is the lift of the vector field \widehat{f}_t , we get that $\pi\widehat{\mathcal{F}}_{t*}^{-1} = \widehat{S}_{t*}^{-1}\pi$.
 563 Thus from (39) we obtain that Assumption 1 of Theorem 4.3 holds true. This implies
 564 the existence of a function α_2 as defined in (33).

565 **5.4. Proof of Assumption 2 of Theorem 4.3.** Notice that

$$566 \quad (46) \quad \begin{aligned} L_k^2(-\tilde{c})(\widehat{x}_T) &= \sigma\left(d(-\tilde{c})_*k(\widehat{x}_T), \vec{K}(\widehat{\ell}_f)\right), \\ (\pi\mathcal{H}_{\widehat{\tau}_2}'')^{-1}k(\widehat{x}_T) &= \iota^{-1}\left(\pi\widehat{\mathcal{F}}_{\widehat{\tau}_2}^{-1}\mathcal{H}_{\widehat{\tau}_2}\right)_*^{-1}k(\widehat{x}_T) = \iota^{-1}\left(\pi\mathcal{H}_{\widehat{\tau}_2}\right)_*^{-1}\tilde{k}(\widehat{x}_2) \end{aligned}$$

567 where the vector field \tilde{k} is defined in (23), as well as the associated Hamiltonian \tilde{K} .
 568 We can compute

$$569 \quad \begin{aligned} \sigma\left(\mathcal{H}_{\widehat{\tau}_2}''\left(\pi\mathcal{H}_{\widehat{\tau}_2}''\right)^{-1}k(\widehat{x}_T), (0, k(\widehat{x}_T))\right) &= \sigma\left(\iota^{-1}\widehat{\mathcal{F}}_{\widehat{\tau}_2*}^{-1}\mathcal{H}_{\widehat{\tau}_2*}\iota\left(\pi\mathcal{H}_{\widehat{\tau}_2}''\right)^{-1}k(\widehat{x}_T), (0, k(\widehat{x}_T))\right) \\ 570 \quad &= \sigma\left(d(-\tilde{c})_*k(\widehat{x}_T), \widehat{\mathcal{F}}_{\widehat{\tau}_2*}^{-1}\mathcal{H}_{\widehat{\tau}_2*}\left(\pi\mathcal{H}_{\widehat{\tau}_2*}\right)^{-1}\tilde{k}(\widehat{x}_2)\right) = \sigma\left(d(-\tilde{c})_*k(\widehat{x}_T), \widehat{\mathcal{F}}_{\widehat{\tau}_2*}^{-1}d\alpha_{2*}\tilde{k}(\widehat{x}_2)\right). \end{aligned}$$

572 Thus

$$573 \quad (47) \quad \begin{aligned} \sigma\left(\mathcal{H}_{\widehat{\tau}_2}''\left(\pi\mathcal{H}_{\widehat{\tau}_2}''\right)^{-1}k(\widehat{x}_T), (0, k(\widehat{x}_T))\right) - L_k^2(-\tilde{c})(\widehat{x}_T) &= \\ 574 \quad &= \sigma\left(d(-\tilde{c})_*k(\widehat{x}_T), \widehat{\mathcal{F}}_{\widehat{\tau}_2*}^{-1}d\alpha_{2*}\tilde{k}(\widehat{x}_2) - \vec{K}(\widehat{\ell}_f)\right) = \\ 575 \quad &= \sigma\left(d(-\tilde{c})_*\widehat{S}_{\widehat{\tau}_2}^{-1}\tilde{k}(\widehat{x}_2), d\alpha_{2*}\tilde{k}(\widehat{x}_2) - \vec{K}(\widehat{\ell}_2)\right) = \\ 576 \quad &= -\left(D^2\alpha_2(\widehat{x}_2)[\tilde{k}(\widehat{x}_2)]^2 + \langle\widehat{\mu}(\widehat{\tau}_2), D\tilde{k}(\widehat{x}_2)\tilde{k}(\widehat{x}_2)\rangle\right) = -L_k^2\alpha_2(\widehat{x}_2). \end{aligned}$$

577 Equations (47) and (44) complete the proof of Assumption 2 of Theorem 4.3.
 578

580 **6. The state-feedback single input case.** The standard form for single input
 581 control affine systems is

$$582 \quad (48) \quad \dot{\xi}(t) = f_0(\xi(t)) + u(t)f_1(\xi(t)), \quad |u(t)| \leq 1.$$

583 This case was dealt with by the authors in [21]. We now consider the case when
 584 there is a state-feedback control for singular extremals, namely there exists a function
 585 $v_s: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $v_s(\widehat{\xi}(t)) = \widehat{v}(t)$ for any $t \in [\widehat{\tau}_2, T]$. Indeed this is the case when
 586 the ratio $\frac{-F_{001}}{F_{101}}(p, x)$ does not depend on p whenever $(p, x) \in \mathcal{S}$.

587 Under this assumption, sufficient second order conditions have been given for
 588 optimality of trajectories containing both bang and singular arcs with respect to
 589 trajectories with the same switching structure, see [25] and the references therein.

590 Without any loss of generality we can assume that the dynamics driving a bang-
 591 bang-singular trajectory is given by

$$592 \quad (49) \quad \widehat{f}_t = \begin{cases} h_1 = f_0 - f_1 & t \in [0, \widehat{\tau}_1), \\ h_2 = f_0 + f_1 & t \in [\widehat{\tau}_1, \widehat{\tau}_2), \\ f_s := f_0 + v_s f_1 & t \in [\widehat{\tau}_2, T], \end{cases}$$

593 hence for any $t \in [\widehat{\tau}_2, T]$, \widehat{S}_t coincides with $\exp(t - T)f_s$.

594 In this case we denote by F_0 , F_s and F_1 the Hamiltonians associated to f_0 , f_s and
595 f_1 and by $F_{i_1 i_2 \dots i_k}$, $i_1, \dots, i_k \in \{0, s, 1\}$ according to the rules stated in Section 2.1.

596 In sub-problem (9) the dynamics in the interval $[\widehat{\tau}_2, T]$ can be written as

$$597 \quad \dot{\xi}(t) = f_s(\xi(t)) + v(t)f_1(\xi(t))$$

598 with $v(t)$ taking values in a neighborhood of zero so that the second variation reads
599

$$600 \quad \widetilde{J}[(\delta x, \varepsilon_0, w)]^2 = \frac{\varepsilon_0^2}{2} \left(L_k^2 \widetilde{c}(\widehat{x}_T) + H_{12}(\widehat{\ell}_1) \right) + \\ 601 \quad \quad \quad + \int_{\widehat{\tau}_2}^T \left(w(t)^2 F_{1s1}(\widehat{\lambda}(t)) + 2w(t)L_{\zeta(t)}L_{\dot{g}_t} \widetilde{c}(\widehat{x}_T) \right) dt \\ 602$$

603 subject to $\dot{\zeta}(t) = w(t)\dot{g}_t(\widehat{x}_T)$, $\zeta(\widehat{\tau}_2) = \varepsilon_0 k(\widehat{x}_T)$, $\zeta(T) = \delta x \in \mathbb{R}^n$ where

$$604 \quad k = -2\widehat{S}_{\widehat{\tau}_1}^{-1} f_1 \circ \widehat{S}_{\widehat{\tau}_1}, \quad g_t := \widehat{S}_{t^*}^{-1} f_1 \circ \widehat{S}_t, \quad \dot{g}_t := \widehat{S}_{t^*}^{-1} [f_s, f_1] \circ \widehat{S}_t.$$

605

606 **REMARK 6.1.** *As a consequence a necessary condition for the coercivity of \widetilde{J} is*

$$607 \quad (50) \quad \widetilde{J}_0 := L_k^2 \widetilde{c}(\widehat{x}_T) + H_{12}(\widehat{\ell}_1) > 0.$$

608 *In [25], for a problem of this class, the author shows that the trajectory is optimal*
609 *with respect to trajectories associated to controls with the same bang-bang-singular*
610 *structure if the 2×2 matrix associated to the problem obtained by moving both the*
611 *switching times is positive definite. We point out that \widetilde{J}_0 is the $(1, 1)$ -entry of such*
612 *matrix.*

613 The regularity condition along the singular arc is trivially satisfied; the other ones
614 read as follows:

- 615 • Regularity along the bang arcs:

$$616 \quad F_1(\widehat{\lambda}(t)) < 0, \quad t \in [0, \widehat{\tau}_1), \quad F_1(\widehat{\lambda}(t)) > 0, \quad t \in (\widehat{\tau}_1, \widehat{\tau}_2).$$

- 617 • Regularity at the switching points: $F_{01}(\widehat{\ell}_1) > 0$, $(F_{001} + F_{101})(\widehat{\ell}_2) > 0$.

- 618 • Strong generalized Legendre condition (SGLC):

$$619 \quad F_{1s1}(\widehat{\lambda}(t)) = F_{101}(\widehat{\lambda}(t)) > 0 \quad \forall t \in [\widehat{\tau}_2, T].$$

620 **6.1. Van der Pol Oscillator.** As an example consider Van der Pol Oscillator
621 with final time $T = 4$, in the form studied in [25] where the author numerically shows
622 the existence of a bang-bang-singular extremal and studies its optimality with respect
623 to trajectories with the same control structure.

624 (51a) minimize $\xi_3(T)$ subject to

$$\dot{\xi}_1(t) = \xi_2(t),$$

$$625 \quad (51b) \quad \dot{\xi}_2(t) = -\xi_1(t) + \xi_2(t)(1 - \xi_1^2(t)) + u(t), \quad \text{a.e. } t \in [0, T],$$

$$\dot{\xi}_3(t) = \frac{1}{2} (\xi_1^2(t) + \xi_2^2(t)),$$

$$626 \quad (51c) \quad \xi(0) = (0, 1, 0), \quad u \in [-1, 1].$$

627

628 More precisely the author numerically shows that there are two bang arcs

629
$$\widehat{u}(t) \equiv -1 \quad \forall t \in [0, \widehat{\tau}_1), \quad \widehat{u}(t) \equiv 1 \quad \forall t \in (\widehat{\tau}_1, \widehat{\tau}_2)$$

 630 where $\widehat{\tau}_1 \simeq 1.3667$, $\widehat{\tau}_2 \simeq 2.4601$ and a singular arc characterized by a state-feedback
 631 control where

632
$$f_s := x_2 \partial_{x_1} + x_1 \partial_{x_2} + \frac{x_1^2 + x_2^2}{2} \partial_{x_3}, \quad f_{s1} = -\partial_{x_1} - x_2 \partial_{x_3}.$$

 633 Let us notice that the flow of f_s can be computed explicitly:

634
$$\exp(t - T)f_s(x) = \begin{pmatrix} x_1 \cosh(t - T) + x_2 \sinh(t - T) \\ x_1 \sinh(t - T) + x_2 \cosh(t - T) \\ x_3 + \frac{x_1 x_2}{2} (\cosh(2(t - T)) - 1) + \frac{x_1^2 + x_2^2}{2} \sinh(2(t - T)) \end{pmatrix}$$

635 so that we can also compute

636
$$\dot{g}_t(x) = -\cosh(t - T)\partial_{x_1} + \sinh(t - T)\partial_{x_2} - x_2 \cosh(t - T)\partial_{x_3}$$

 637 Using the numerical results in [25], in [21] the authors prove that the regularity
 638 assumptions are satisfied. Here we recall some features which are needed in the
 639 following:

640
$$\widehat{\mu}_3(t) \equiv 1, \quad t \in [0, T], \quad \widehat{\mu}_1(T) = \widehat{\xi}_2(T) = 0, \quad F_{1s1}(\widehat{\lambda}(t)) \equiv 1, \quad t \in [\widehat{\tau}_2, T].$$

 641 In order to prove that our theory applies to this example we prove that the second
 642 variation is coercive by proving (39) and (44). We thus need to write the Hamiltonian
 643 $H_t''(\delta p, \delta x)$ and the associated linear system. Since $L_T'' = \{(0, \delta x) : \delta x \in \mathbb{R}^3\}$ and

644
$$H_t''(\delta p, \delta x) = -\frac{1}{2} (-\delta p_1 \cosh(t - T) + \delta p_2 \cosh(t - T) - \delta x_2 \cosh(t - T))^2$$

 645 we get $\dot{\mu}_1''(t) \equiv \dot{\mu}_3''(t) \equiv \dot{\zeta}_3''(t) \equiv 0$, $\mu_1''(T) = \mu_3''(T) = 0$, $\zeta_3(T) = \delta x_3$ so that
 646 $\mu_1''(t) \equiv \mu_3''(t) \equiv 0$, $\zeta_3''(t) \equiv \delta x_3$ and

647
$$\dot{\mu}_2''(t) = -(\mu_2''(t) \sinh(t - T) - \xi_2''(t) \cosh(t - T)) \cosh(t - T),$$

648
$$\dot{\xi}_1''(t) = (\mu_2''(t) \sinh(t - T) - \xi_2''(t) \cosh(t - T)) \cosh(t - T),$$

649
$$\dot{\xi}_2''(t) = -(\mu_2''(t) \sinh(t - T) - \xi_2''(t) \cosh(t - T)) \sinh(t - T).$$

651 Thus

652
$$\mathcal{H}_t''(0, \delta x) = \begin{pmatrix} \delta x_2 \sinh(t - T) dx_2 \\ ((\delta x_1 - \delta x_2 \sinh(t - T)) \partial_{x_1} + \delta x_2 \cosh(t - T) \partial_{x_2} + \delta x_3 \partial_{x_3}) \end{pmatrix}$$

 653 hence $\pi \mathcal{H}_t''(0, \delta x) = 0$ implies $\delta x = 0$, i.e. (39) is satisfied.

 654 Inequality (44) reads $\widetilde{J}_0 + \langle \mu''(\widehat{\tau}_2), \zeta''(\widehat{\tau}_2) \rangle > 0$ when $\zeta''(\widehat{\tau}_2) = k(\widehat{x}_T)$ that is

655
$$\widetilde{J}_0 + k_2^2(\widehat{x}_T) \tanh(\widehat{\tau}_2 - T) > 0.$$

 656 $\widetilde{J}_0 \simeq 215.1022$ was computed in [25]. The push-forward $k(\widehat{x}_T) = -2\widehat{S}_{\widehat{\tau}_1}^{-1} \partial_{x_2} \circ \widehat{S}_{\widehat{\tau}_1}$ can
 657 be computed numerically, obtaining $k(\widehat{x}_T) \simeq 14.5864\partial_{x_1} - 14.6632\partial_{x_2} - 0.0005\partial_{x_3}$.
 658 Thus $k_2^2(\widehat{x}_T) \tanh(\widehat{\tau}_2 - T) \simeq -196.1122$. Hence (44) is satisfied, i.e. the second varia-
 659 tion J'' coercive. This proves that our results apply to the Van der Pol oscillator.

660 **Acknowledgments.** The authors are grateful to Dott. Francesco Mugelli from
 661 the University of Florence for the numerical computations in the example above.

662

REFERENCES

- 663 [1] A. A. AGRACHEV AND Y. L. SACHKOV, *Control Theory from the Geometric Viewpoint*, Springer-
 664 Verlag, 2004.
- 665 [2] A. A. AGRACHEV, G. STEFANI, AND P. ZEZZA, *An invariant second variation in optimal control*,
 666 *Internat. J. Control*, 71 (1998), pp. 689–715.
- 667 [3] A. A. AGRACHEV, G. STEFANI, AND P. ZEZZA, *Strong optimality for a bang-bang trajectory*,
 668 *SIAM J. Control Optimization*, 41 (2002), pp. 991–1014.
- 669 [4] M. S. ARONNA, J. F. BONNANS, A. V. DMITRUK, AND P. A. LOTITO, *Quadratic order condi-*
 670 *tions for bang-singular extremals*, *NUMERICAL ALGEBRA, CONTROL AND OPTI-*
 671 *MIZATION*, 2 (2012), pp. 511–546. doi:10.3934/naco.2012.2.511.
- 672 [5] Z. CHEN, J.-B. CAILLAU, AND Y. CHITOUR, *L1-minimization for mechanical systems*, *SIAM*
 673 *Journal on Control and Optimization*, 54 (2016), pp. 1245–1265, [https://doi.org/10.1137/](https://doi.org/10.1137/15M1013274)
 674 [15M1013274](https://doi.org/10.1137/15M1013274).
- 675 [6] F. C. CHITTARO AND G. STEFANI, *Minimum-time strong optimality of a singular arc: The*
 676 *multi-input non involutive case*, *ESAIM: COCV*, 22 (2016), pp. 786–810, [http://dx.doi.](http://dx.doi.org/10.1051/cocv/2015026)
 677 [org/10.1051/cocv/2015026](http://dx.doi.org/10.1051/cocv/2015026).
- 678 [7] U. FELGENHAUER, L. POGGIOLINI, AND G. STEFANI, *Optimality and stability result for bang-*
 679 *bang optimal controls with simple and double switch behaviour*, *CONTROL AND CYBER-*
 680 *NETICS*, 38 (2009), pp. 1305 – 1325.
- 681 [8] M. R. HESTENES, *Applications of the theory of quadratic forms in Hilbert space to calculus of*
 682 *variations*, *Pacific J. Math*, 1 (1951), pp. 525–581.
- 683 [9] U. LEDZEWICZ AND H. SCHTTLER, *Optimal bang-bang controls for a two-compartment model*
 684 *in cancer chemotherapy*, *Journal of Optimization Theory and Applications*, 114 (2002),
 685 pp. 609–637, <https://doi.org/10.1023/A:1016027113579>.
- 686 [10] A. A. MILYUTIN AND N. P. OSMOLOVSKII, *Calculus of Variations and Optimal Control*, vol. 180
 687 of *Translations of Mathematical Monographs*, American Mathematical Society, 1998.
- 688 [11] N. OSMOLOVSKII AND H. MAURER, *Applications to Regular and Bang-Bang Control*, Society
 689 for Industrial and Applied Mathematics, Philadelphia, PA, 2012, [https://doi.org/10.1137/](https://doi.org/10.1137/1.9781611972368)
 690 [1.9781611972368](https://doi.org/10.1137/1.9781611972368), <http://epubs.siam.org/doi/abs/10.1137/1.9781611972368>.
- 691 [12] L. POGGIOLINI, *On local state optimality of bang-bang extremals in a free horizon Bolza problem*,
 692 *Rendiconti del Seminario Matematico dell’Università e del Politecnico di Torino*, (2006).
- 693 [13] L. POGGIOLINI, *Structural stability of bang–bang trajectories with a double switching time in*
 694 *the minimum time problem*, *SIAM JOURNAL ON CONTROL AND OPTIMIZATION*,
 695 55 (2017), pp. 3779–3798, <https://doi.org/10.1137/16M1083761>.
- 696 [14] L. POGGIOLINI AND M. SPADINI, *Sufficient optimality conditions for a bang-bang trajectory in*
 697 *a Bolza problem*, in *Mathematical Control Theory and Finance*, A. Sarychev, A. Shiryaev,
 698 M. Guerra, and M. d. R. Grossinho, eds., Springer Berlin Heidelberg, 2008, pp. 337–357,
 699 https://doi.org/10.1007/978-3-540-69532-5_19. 10.1007/978-3-540-69532-5_19.
- 700 [15] L. POGGIOLINI AND M. SPADINI, *Strong local optimality for a bang-bang trajectory in a Mayer*
 701 *problem*, *SIAM Journal on Control and Optimization*, 49 (2011), pp. 140–161, [https://doi.](https://doi.org/10.1137/090771405)
 702 [org/10.1137/090771405](https://doi.org/10.1137/090771405), <http://epubs.siam.org/doi/abs/10.1137/090771405>.
- 703 [16] L. POGGIOLINI AND M. SPADINI, *Bang–bang trajectories with a double switching time in the*
 704 *minimum time problem*, *ESAIM: COCV*, (2015), <https://doi.org/10.1051/cocv/2015021>.
- 705 [17] L. POGGIOLINI AND G. STEFANI, *Strong local optimality for a bang-bang-singular extremal*.
 706 Work in progress.
- 707 [18] L. POGGIOLINI AND G. STEFANI, *State-local optimality of a bang-bang trajectory: a Hamiltonian*
 708 *approach*, *Systems & Control Letters*, 53 (2004), pp. 269–279.
- 709 [19] L. POGGIOLINI AND G. STEFANI, *Bang-singular-bang extremals: sufficient optimality conditions*,
 710 *Journal of Dynamical and Control Systems*, 17 (2011), pp. 469–514, [http://dx.doi.org/10.](http://dx.doi.org/10.1007/s10883-011-9127-y)
 711 [1007/s10883-011-9127-y](http://dx.doi.org/10.1007/s10883-011-9127-y). 10.1007/s10883-011-9127-y.
- 712 [20] L. POGGIOLINI AND G. STEFANI, *Structural stability for bang–singular–bang extremals in the*
 713 *minimum time problem*, *SIAM J. Control Optim.*, 51 (2013), pp. 3511–3531, [http://dx.](http://dx.doi.org/10.1137/120895421)
 714 [doi.org/10.1137/120895421](http://dx.doi.org/10.1137/120895421).
- 715 [21] L. POGGIOLINI AND G. STEFANI, *Strong local optimality for bang-bang-singular extremals in*
 716 *single input control problems*, in *20th IFAC World Congress PROCEEDINGS*, vol. 50 of
 717 *IFAC-PAPERSONLINE*, IFAC, Elsevier, 2017, pp. 6128–6133, [https://doi.org/10.1016/j.](https://doi.org/10.1016/j.ifacol.2017.08.2022)
 718 [ifacol.2017.08.2022](https://doi.org/10.1016/j.ifacol.2017.08.2022).

- 719 [22] G. STEFANI, *Strong optimality of singular trajectories*, in Geometric Control and Nonsmooth
720 Analysis, F. Ancona, A. Bressan, P. Cannarsa, F. Clarke, and P. R. Wolenski, eds., vol. 76 of
721 Series on Advances in Mathematics for Applied Sciences, Hackensack, NJ, 2008, World Sci-
722 entific Publishing Co. Pte. Ltd., pp. 300–326. pp. 361 ISBN: 978-981-277-606-8.
- 723 [23] G. STEFANI AND P. ZEZZA, *Constrained regular LQ-control problems*, SIAM J. Control Optim.,
724 35 (1997), pp. 876–900.
- 725 [24] G. STEFANI AND P. ZEZZA, *Variational Methods in Imaging and Geometric Control*, De
726 Gruyter, 2016, ch. A Hamiltonian approach to sufficiency in optimal control with minimal
727 regularity conditions: Part I.
- 728 [25] G. VOSSEN, *Switching time optimization for bang-bang and singular controls*, Journal of Op-
729 timization Theory and Applications, 144 (2010), pp. 409–429, [https://doi.org/10.1007/
730 s10957-009-9594-4](https://doi.org/10.1007/s10957-009-9594-4).