Generalized geometry of pseudo Riemannian manifolds and generalized debar operator

Generalized geometry of pseudo-Riemannian manifolds and generalized $\overline{\partial}-$operator

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ABSTRACT. Let (M, g, ∇) be a pseudo-Riemannian manifold with a torsion free linear connection and let J^g be the generalized complex structure on M defined by g, [13], [14]. We prove that in the case J^g is $\nabla-$integrable the $\pm i-$eigenbundles of J^g, $E^{1,0}_{J^g}$, $E^{0,1}_{J^g}$, are complex Lie algebroids. Moreover $E^{0,1}_{J^g}$ and $\left(E^{1,0}_{J^g}\right)^*$ are canonically isomorphic thus we define the concept of generalized $\overline{\partial} - operator$ of (M, g, ∇) and we describe a class of generalized holomorphic sections of $T(M) \oplus T^*(M)$. Also we relate Lie bialgebroid property of $\left(E^{1,0}_{J^g}, \left(E^{1,0}_{J^g}\right)^*\right)$ to conditions on the metric g in the case of affine Hessian manifolds. [1]

1 Introduction

Let (M, g) be a smooth pseudo-Riemannian manifold, let $T(M)$ be the tangent bundle, let $T^*(M)$ be the cotangent bundle and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of M. Generalized complex structures were introduced by Nigel Hitchin in [6], and further investigated by Marco Gualtieri in [8], in order to unify symplectic and complex geometry. In this paper we consider a more general concept of generalized complex structure introduced in [13], [14] and also studied in [15], [1]. In the previous papers [13], [14], we defined a generalized complex structure of M as a complex structure on E and we studied some classes of such structures, in particular calibrated complex structures with respect to the canonical symplectic structure, $(\, , \,)$, of E. Using a torsion free linear connection, ∇, on M we introduced a bracket, $[\, , \,]_\nabla$, on sections of E, the corresponding concept of $\nabla-$integrability for complex structures and we studied integrability conditions. Moreover in [14] we described a large class of almost complex structures on cotangent bundles of manifolds endowed with a torsion free linear connection, induced by generalized complex structures and we proved that, in the case ∇ has zero curvature, a $\nabla-$integrable generalized complex structure on M defines a complex structure on $T^*(M)$. In this paper we concentrate on the canonical generalized complex structure defined by

[1]Mathematics Subject Classifications (2010): 53C15, 53D18 53C80 53C38 53D05
Key words: General geometric structures on manifolds, Hessian manifolds, Generalized geometry, Lie algebroids, Lie bialgebroids.

$g,\ J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}$. We prove that in the case J^g is $\nabla-$integrable the $\pm i-$eigenbundles of J^g, $E^{1,0}_{J^g}$, $E^{0,1}_{J^g}$, are complex Lie algebroids. Then we observe that the natural symplectic structure of $T(M) \oplus T^*(M)$ defines a canonical isomorphism between $E^{0,1}_{J^g}$ and $\left(E^{1,0}_{J^g}\right)^*$ and this allows us to define the *generalized* $\overline{\partial}_{J^g}-$*operator* on M. We prove that in the case J^g is $\nabla-$integrable we get $\left(\overline{\partial}_{J^g}\right)^2 = 0$, moreover $\overline{\partial}_{J^g}$ is the exterior derivative of the Lie algebroid $E^{1,0}_{J^g}$, in particular $\left(C^\infty\left(\wedge^\bullet\left(E^{1,0}_{J^g}\right)\right), \wedge, \overline{\partial}_{J^g}, [\ ,\ ]_\nabla\right)$, where $\wedge$ is the Schouten bracket, is a differential Gerstenhaber algebra, [9], [19]. We also study generalized holomorphic sections and we prove that, for any $X \in T(M) \otimes \mathbb{C}$, the section $\sigma = X + ig(X) \in C^\infty\left(\left(E^{1,0}_{J^g}\right)^*\right)$ is generalized holomorphic if and only if $g(X)$ is a Lagrangian submanifold of $T^*(M)$, in particular in the case (M, g, ∇) is an affine Hessian manifold we describe a large class of local generalized holomorphic sections of $\left(E^{1,0}_{J^g}\right)^*$. Finally we study the Lie bialgebroid condition on the two algebroids in duality $\left(E^{1,0}_{J^g}, \left(E^{1,0}_{J^g}\right)^*\right)$ and we prove that, unlike the case of generalized complex structures in the sense of Hitchin, this gives restrictions on g. The paper is organized as in the following. In section 2 we introduce preliminary material: we describe the main geometrical properties of the generalized tangent bundle and of generalized complex structures; moreover we recall the basic definitions in the setting of complex Lie algebroids and Lie bialgebroids. Original results are concentrated in section 3: the definition of the generalized $\overline{\partial}_{J^g}-$ *operator* and its properties; in particular we relate generalized holomorphic sections of $E^{0,1}_{J^g}$ to Lagrangian submanifolds of $T^*(M)$. Section 4 is devoted to Hessian manifold because they occur as interesting examples in our context.

2 Preliminary Material

2.1 Geometry of the generalized tangent bundle

Let M be a smooth manifold of real dimension n and let $E = T(M) \oplus T^*(M)$ be the *generalized tangent bundle* of M, we recall the main geometric properties of E.

Smooth sections of E are elements $X + \xi \in C^\infty(E)$ where $X \in C^\infty(T(M))$ is a vector field and $\xi \in C^\infty(T^*(M))$ is a $1-$ form.

E is equipped with a natural *symplectic structure* defined by:

$$(X + \xi, Y + \eta) = -\frac{1}{2}(\xi(Y) - \eta(X)) \tag{1}$$

and a natural *indefinite metric* defined by:

$$< X + \xi, Y + \eta > \ = -\frac{1}{2}(\xi(Y) + \eta(X)) \tag{2}$$

$< \, , \, >$ is non degenerate and of signature (n, n).

A linear connection, ∇, on M, defines, in a canonical way, a bracket on $C^\infty(E)$, $[\ ,\ ]_\nabla$, as follows:

$$[X + \xi, Y + \eta]_\nabla = [X, Y] + \nabla_X \eta - \nabla_Y \xi. \tag{3}$$

The following holds:

Lemma 2. ([13]) *For all* $X, Y \in C^\infty(T(M))$, *for all* $\xi, \eta \in C^\infty(T^*(M))$ *and for all* $f \in C^\infty(M)$ *we have*:

1. $[X + \xi, Y + \eta]_\nabla = -[Y + \eta, X + \xi]_\nabla$,
2. $[f(X + \xi), Y + \eta]_\nabla = f\,[X + \xi, Y + \eta]_\nabla - Y(f)(X + \xi)$,
3. *Jacobi's identity holds for* $[\ ,\ ]_\nabla$ *if and only if* ∇ *has zero curvature.*

Moreover:

Proposition 2. ([14]) *Let* ∇ *be a connection on* M *then there is a bundle morphism*

$$\Phi^\nabla : T(M) \oplus T^*(M) \to T(T^*(M)) \tag{4}$$

which is an isomorphism on the fibres and such that:

1. Φ^∇ *identifies* $T^*(M)$ *with vertical vectors*
that is:
$$\left(\Phi^\nabla\right)^{-1}(\ker \pi_*) = T^*(M));$$

2. $\pi_* \circ \Phi^\nabla_{|T(M)} = I_{|T(M)}$,

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3. $\left(\Phi^\nabla\right)^*(\Omega) = -2(\ ,\ )$ *if and only if* ∇ *has zero torsion*;
4. $\left(\Phi^\nabla\right)([\ ,\ ]_\nabla) = \left[\Phi^\nabla\ , \Phi^\nabla\ \right]$ *if and only if* ∇ *has zero curvature.*

where, we denoted by π_* the tangent map of

$$\pi : T^*(M) \to M, \tag{5}$$

$$\pi_* : T(T^*(M)) \to T(M) \tag{6}$$

$$(\pi_*(A))(f) = A(f \circ \pi)$$

for all $A \in T(T^*(M))$ and for all $f \in C^\infty(M)$,

and $\Omega = d\theta$ where θ is the *Liouville's 1-form* defined by:

$$\theta(A) = p(A)(\pi_*(A)), \text{ for all } A \in T(T^*(M)). \tag{7}$$

In this paper we consider the following concept of generalized complex structure, introduced in [13], [14] and also studied in [15], [1] :

Definition 3. A *generalized complex structure on* M is an endomorphism $J : E \to E$ such that $J^2 = -I$.

A pseudo-Riemannian metric on M, g, defines, in a natural way, a complex structure J^g on E by:

$$J^g(X+\xi) = -g^{-1}(\xi) + g(X) \tag{8}$$

where $g : T(M) \to T^*(M)$ is identified to the bemolle musical isomorphism defined by:

$$g(X)(Y) = g(X,Y), \tag{9}$$

in block matrix form, is:

$$J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}. \tag{10}$$

Let ∇ be a connection on M and let $[\ ,\ ]_\nabla$ be the bracket on $C^\infty(E)$ defined by ∇, the following holds:

Lemma 4. ([14]) *Let $J : E \to E$ be a generalized complex structure on M and let*

$$N^\nabla(J) : C^\infty(E) \times C^\infty(E) \to C^\infty(E) \tag{11}$$

defined by:

$$N^\nabla(J)(\sigma,\tau) = [J\sigma, J\tau]_\nabla - J[J\sigma,\tau]_\nabla - J[\sigma, J\tau]_\nabla - [\sigma,\tau]_\nabla \tag{12}$$

for all $\sigma, \tau \in C^\infty(E)$; $N^\nabla(J)$ is a skew symmetric tensor.

Definition 5. $N^\nabla(J)$ is called the *Nijenhuis tensor of J with respect to ∇.*

Definition 6. Let $J : E \to E$ be a generalized complex structure on M, J is said to be $\nabla-$*integrable* if $N^\nabla(J) = 0$.

Proposition 7. ([14]) *Let ∇ be a torsion free connection on M and let $J^g = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$ be the generalized complex structure on M defined by a pseudo-Riemannian metric g, J^g is $\nabla-$integrable if and only if g is a Codazzi tensor, that is for all $X, Y \in C^\infty(T(M))$ we have:*

$$(\nabla_X g)Y = (\nabla_Y g)X. \tag{13}$$

A direct computation gives the following:

Proposition 8. *Let $\{x_1, ..., x_n\}$ be local coordinates on M, let $\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right\}$ be the corresponding local frame for $T(M)$, let $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ and $\{\Gamma^l_{ik}\}$ be the Cristoffel symbols of ∇, then g is a Codazzi tensor if and only if for all $i, j, k = 1, ..., n$:*

$$\frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_j} = \sum_{l=1}^{n} \left(\Gamma^l_{ik} g_{lj} - \Gamma^l_{jk} g_{li}\right). \tag{14}$$

Examples with non parallel g can be found in the context of Hessian manifolds, their description will be the object of section 4.

2.2 Complex Lie algebroids and bialgebroids

The concept of Lie algebroid was introduced by Pradines in [16]; Lie bialgebroids were introduced by Mackenzie and Xu in [11] to encode the compatibility condition of a pair of two Lie algebroids in duality.

Here we first recall the definition of complex Lie algebroid:

Definition 9. A *complex Lie algebroid* is a complex vector bundle L over a smooth real manifold M such that: a Lie bracket $[\ ,\ ]$ is defined on $C^{\infty}(L)$, a smooth bundle map $\rho : L \to T(M)$, called *anchor*, is defined and, for all $\sigma, \tau \in C^{\infty}(L)$, for all $f \in C^{\infty}(M)$ the following conditions hold:

1. $\rho([\sigma, \tau]) = [\rho(\sigma), \rho(\tau)]$
2. $[f\sigma, \tau] = f([\sigma, \tau]) - (\rho(\tau)(f))\,\sigma$.

We now recall the definition of complex Lie bialgebroid.

Let L and its dual vector bundle L^* be Lie algebroids; on sections of $\wedge L$, respectively $\wedge L^*$, the *Schouten bracket* is defined by:

$$[\ ,\ ]_L : C^{\infty}(\wedge^p L) \times C^{\infty}(\wedge^q L) \longrightarrow C^{\infty}\left(\wedge^{p+q-1} L\right)$$

$$[X_1 \wedge \dots \wedge X_p, Y_1 \wedge \dots \wedge Y_q]_L =$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [X_i, Y_j]_L \wedge X_1 \wedge ..\overset{\widehat{i}}{.}.. \wedge X_p \wedge Y_1 \wedge ..\overset{\widehat{j}}{.}.. \wedge Y_q$$

and, for $f \in C^{\infty}(M)$, $X \in C^{\infty}(L)$

$$[X, f]_L = -[f, X]_L = \rho(X)(f);$$

respectively, by:

$$[\ ,\ ]_{L^*} : C^{\infty}(\wedge^p L^*) \times C^{\infty}(\wedge^q L^*) \longrightarrow C^{\infty}\left(\wedge^{p+q-1} L^*\right)$$

$$\left[X_1^* \wedge \dots \wedge X_p^*, Y_1^* \wedge \dots \wedge Y_q^*\right]_{L^*} =$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} \left[X_i^*, Y_j^*\right]_{L^*} \wedge X_1^* \wedge ..\overset{\widehat{i}}{.}.. \wedge X_p^* \Lambda Y_1^* \wedge ..\overset{\widehat{j}}{.}.. \wedge Y_q^*$$

and, for $f \in C^{\infty}(M)$, $X \in C^{\infty}(L^*)$

$$[X, f]_{L^*} = -[f, X]_{L^*} = \rho(X)(f).$$

Moreover the *exterior derivatives* d and d_* associated with the Lie algebroid structure of L and L^* are defined respectively by:

$$d : C^\infty (\wedge^p L^*) \longrightarrow C^\infty \left(\wedge^{p+1} L^*\right)$$

$$(d\alpha)(\sigma_0, ..., \sigma_p) =$$

$$= \sum_{i=0}^{p} (-1)^i \rho(\sigma_i) \alpha \left(\sigma_0, ..\overset{\widehat{i}}{.}.., \sigma_p\right) + \sum_{i<j} (-1)^{i+j} \alpha \left([\sigma_i, \sigma_j]_L, \sigma_0, ..\overset{\widehat{i}}{.}.\overset{\widehat{j}}{.}.., \sigma_p\right)$$

for $\alpha \in C^\infty (\wedge^p L^*)$, $\sigma_0, ..., \sigma_p \in C^\infty (L)$,

and:

$$d_* : C^\infty (\wedge^p L) \longrightarrow C^\infty \left(\wedge^{p+1} L\right)$$

$$(d_*\alpha)(\sigma_0, ..., \sigma_p) =$$

$$= \sum_{i=0}^{p} (-1)^i \rho(\sigma_i) \alpha \left(\sigma_0, ..\overset{\widehat{i}}{.}.., \sigma_p\right) + \sum_{i<j} (-1)^{i+j} \alpha \left([\sigma_i, \sigma_j]_{L^*}, \sigma_0, ..\overset{\widehat{i}}{.}.\overset{\widehat{j}}{.}.., \sigma_p\right)$$

for $\alpha \in C^\infty (\wedge^p L)$, $\sigma_0, ..., \sigma_p \in C^\infty (L^*)$.

Definition 10. A *complex Lie bialgebroid* is a pair of complex dual Lie algebroids (L, L^*) such that the differential d_* is a derivation of $(C^\infty (\wedge L), [\ ,\ ]_L)$, that is the following compatibility condition is satisfied:

$$d_* [\sigma, \tau]_L = [d_*\sigma, \tau]_L + [\sigma, d_*\tau]_L \tag{15}$$

for $\sigma, \tau \in C^\infty (L)$.

The following facts are well known:

Proposition 11. ([9]) *In a Lie bialgebroid (L, L^*), d_* is a derivation of the graded Lie algebra $(C^\infty (\wedge L), [\ ,\ ]_L)$, and d is a derivation of $(C^\infty (\wedge L^*), [\ ,\ ]_{L^*})$.*

Proposition 12. ([9]) *Let (L, L^*) be a pair of Lie algebroids in duality; the following properties are equivalent*:

1. *(L, L^*) is a Lie bialgebroid,*
2. *d_* is a derivation of $(C^\infty (\wedge L), [\ ,\ ]_L)$,*
3. *d is a derivation of $(C^\infty (\wedge L^*), [\ ,\ ]_{L^*})$,*
4. *(L^*, L) is a Lie bialgebroid.*

In the following section we will define natural Lie algebroids and bialgebroids in the context of generalized geometry.

3 Generalized $\overline{\partial}$-operator associated to $\mathbf{J}^g$

Let (M,g) be a pseudo-Riemannian manifold and let J^g be the generalized complex structure on M defined by g, let

$$E^{\mathbb{C}} = (T(M) \oplus T^*(M)) \otimes \mathbb{C}$$

be the complexified generalized tangent bundle. The splitting in $\pm i$ eigenspaces of J^g is denoted by:

$$E^{\mathbb{C}} = E^{1,0}_{J^g} \oplus E^{0,1}_{J^g} \tag{16}$$

with

$$E^{0,1}_{J^g} = \overline{E^{1,0}_{J^g}}. \tag{17}$$

A direct computation gives:

$$E^{1,0}_{J^g} = \{Z - ig(Z) \mid Z \in T(M) \otimes \mathbb{C}\}, \tag{18}$$

and, for any linear connection ∇, the following holds:

Lemma 13. *$E^{1,0}_{J^g}$ and $E^{0,1}_{J^g}$ are $[\ ,\ ]_\nabla$ $-$involutive if and only if $N^\nabla(J^g) = 0$.*

Proof. Let $Z, W \in T(M) \otimes \mathbb{C}$, then we have:

$$[Z \pm ig(Z), W \pm ig(W)]_\nabla \mp iJ^g [Z \pm ig(Z), W \pm ig(W)]_\nabla =$$

$$= -(I \mp J^g) N^\nabla(J^g)(Z,W).$$

□

Moreover:

Lemma 14. *If J^g is ∇-integrable then Jacobi identity holds for $[\ ,\ ]_\nabla$ on $E^{1,0}_{J^g}$ and $E^{0,1}_{J^g}$.*

Proof. Let $Z, W, V \in T(M) \otimes \mathbb{C}$, then we have:

$$Jac\,[[Z \pm ig(Z), W \pm ig(W)]_\nabla, V \pm ig(V)]_\nabla =$$

$$= Jac\,[[Z,W], V] \pm ig\,(Jac\,[[Z,W], V]) = 0$$

where Jac denotes *Jacobiator*, that is:

$$Jac\,[[\alpha,\beta]_\nabla, \gamma]_\nabla = [[\alpha,\beta]_\nabla, \gamma]_\nabla + [[\beta,\gamma]_\nabla, \alpha]_\nabla + [[\gamma,\alpha]_\nabla, \beta]_\nabla. \ \square$$

In particular we get:

Proposition 15. *If J^g is ∇-integrable then $E^{1,0}_{J^g}$ and $E^{0,1}_{J^g}$ are complex Lie algebroids.*

The following holds:

Proposition 16. *The natural symplectic structure on E defines a canonical isomorphism between $E^{0,1}_{J^g}$ and the dual bundle of $E^{1,0}_{J^g}$, $\left(E^{1,0}_{J^g}\right)^*$.*

Proof. Let $Z, W \in T(M) \otimes \mathbb{C}$, we define:

$$(W + ig(W))(Z - ig(Z)) = (W + ig(W), Z - ig(Z)).$$

We get:

$$(W + ig(W))(Z - ig(Z)) = -ig(W, Z).$$

$\square$

The canonical isomorphism between $E^{0,1}_{J^g}$ and the dual bundle $\left(E^{1,0}_{J^g}\right)^*$ allows us to define the $\overline{\partial}_{J^g} - operator$ associated to the complex structure J^g as in the following:

let $f \in C^\infty(M)$ and let $df \in C^\infty(T^*(M)) \hookrightarrow C^\infty(T(M) \oplus T^*(M))$, we pose

$$\overline{\partial}_{J^g} f = 2\,(df)^{0,1} = df + iJ^g df$$

or:

$$\overline{\partial}_{J^g} f = df - ig^{-1}(df);$$

moreover we define:

$$\overline{\partial}_{J^g} : C^\infty\left(E^{0,1}_{J^g}\right) \to C^\infty\left(\wedge^2\left(E^{0,1}_{J^g}\right)\right)$$

via the natural isomorphism

$$\left(E^{1,0}_{J^g}\right)^* \simeq E^{0,1}_{J^g}$$

as:

$$\overline{\partial}_{J^g} : C^\infty\left(\left(E^{1,0}_{J^g}\right)^*\right) \to C^\infty\left(\wedge^2\left(E^{1,0}_{J^g}\right)^*\right)$$

$$\left(\overline{\partial}_{J^g}\alpha\right)(\sigma, \tau) = \rho(\sigma)\,\alpha(\tau) - \rho(\tau)\,\alpha(\sigma) - \alpha\left([\sigma, \tau]_\nabla\right)$$

for $\alpha \in C^\infty\left(\left(E^{1,0}_{J^g}\right)^*\right)$, $\sigma, \tau \in C^\infty\left(E^{1,0}_{J^g}\right)$.

In general:

$$\overline{\partial}_{J^g}: C^\infty\left(\wedge^p\left(E^{1,0}_{J^g}\right)^*\right) \to C^\infty\left(\wedge^{p+1}\left(E^{1,0}_{J^g}\right)^*\right)$$

is defined by:

$$\left(\overline{\partial}_{J^g}\alpha\right)(\sigma_0, ..., \sigma_p) =$$

$$= \sum_{i=0}^{p}(-1)^i \rho(\sigma_i)\, \alpha\left(\sigma_0, ..\overset{\widehat{i}}{.}.., \sigma_p\right) + \sum_{i<j}(-1)^{i+j}\alpha\left([\sigma_i, \sigma_j]_\nabla, \sigma_0, ..\overset{\widehat{i}}{.}..\overset{\widehat{j}}{.}.., \sigma_p\right)$$

for $\alpha \in C^\infty\left(\wedge^p\left(E^{1,0}_{J^g}\right)^*\right)$, $\sigma_0, ..., \sigma_p \in C^\infty\left(E^{1,0}_{J^g}\right)$.

Definition 17. $\overline{\partial}_{J^g}$ is called *generalized* $\overline{\partial}$*−operator* of (M, g, ∇) or *generalized* $\overline{\partial}_{J^g}$ *− operator.*

We have immediately that $\overline{\partial}_{J^g}$ is the exterior derivative, d_L, of the Lie algebroid $L = E^{1,0}_{J^g}$. Moreover the exterior derivative d_{L^*} of $L^* = \left(E^{1,0}_{J^g}\right)^*$ is given by the operator ∂_{J^g} defined by:

$$\partial_{J^g}: C^\infty\left(\wedge^p\left(E^{1,0}_{J^g}\right)\right) \to C^\infty\left(\wedge^{p+1}\left(E^{1,0}_{J^g}\right)\right)$$

$$\left(\partial_{J^g}\sigma\right)(\alpha^*_0, ..., \alpha^*_p) =$$

$$= \sum_{i=0}^{p}(-1)^i \rho(\alpha^*_i)\, \sigma\left(\alpha^*_0, ..\overset{\widehat{i}}{.}.., \alpha^*_p\right) + \sum_{i<j}(-1)^{i+j}\sigma\left([\alpha^*_i, \alpha^*_j]_\nabla, \alpha^*_0, ..\overset{\widehat{i}}{.}..\overset{\widehat{j}}{.}.., \alpha^*_p\right)$$

for $\sigma \in C^\infty\left(\wedge^p\left(E^{1,0}_{J^g}\right)\right)$, $\alpha^*_0, ..., \alpha^*_p \in C^\infty\left(\left(E^{1,0}_{J^g}\right)^*\right)$.

We get the following:

Proposition 18. *If* J^g *is* $\nabla -$ *integrable then* $\left(\overline{\partial}_{J^g}\right)^2 = 0$ *and* $\left(\partial_{J^g}\right)^2 = 0$.

Proof. It follows from the fact that Jacobi identity holds on $E^{1,0}_{J^g}$ and $\left(E^{1,0}_{J^g}\right)^*$.
□

Definition 19. $\alpha \in C^\infty\left(\wedge^p\left(E^{1,0}_{J^g}\right)^*\right)$ is called *generalized holomorphic* if $\overline{\partial}_{J^g}\alpha = 0$.

We remark that $\overline{\partial}_{J^g} f = 0 \Longleftrightarrow df = 0$, so the generalized holomorphic condition for functions gives only constant functions on connected components of M.

Proposition 20. *Let* $\sigma = X + ig(X) \in E^{0,1}_{J^g}$ *then* $\overline{\partial}_{J^g}\sigma = 0$ *if and only if* $g(X)$ *is a d-closed 1-form.*

Proof. Let $X, Y \in T(M) \otimes \mathbb{C}$, we have:

$\left(\overline{\partial}_{J^g}\sigma\right)(Y - ig(Y), Z - ig(Z))$
$= Y(X + ig(X), Z - ig(Z)) - Z(X + ig(X), Y - ig(Y)) +$
$-(X + ig(X), [Y, Z] - ig([Y, Z]))$
$= i\{Yg(X, Z) + Zg(X, Y) + g(X, [Y, Z])\}$
$= -i(dg(X))(Y, Z)$. □

In particular, by using a classical result in symplectic geometry, [12], we get:

Proposition 21. *Let $X \in T(M)\otimes\mathbb{C}$, then $\sigma = X + ig(X) \in E^{0,1}_{J^g}$ is $\overline{\partial}_{J^g}$−cloded if and only if $g(X)$ is a Lagrangian submanifold of $T^*(M)$ with the standard symplectic structure.*

4 Examples

As we remarked in section 2. Hessian manifolds appear naturally in this context and provide interesting examples. Their introduction was inspired by the Bergmann metric on bounded domains in $\mathbb{C}^n$ and are a very interesting topic, related to many other fields in mathematics and theoretical physics as, for example: Kähler and symplectic geometry, affine differential geometry, special manifolds, string theory and mirror symmetry, [2], [4], [5], [10], [17], [18].

We now recall the general definition of Hessian metric:

Definition 22. Let (M, g, ∇) be a pseudo-Riemannian manifold with a torsion free linear connection, g is called of *Hessian type* if there exists $u \in C^\infty(M)$ such that $g = Hess(u) = \nabla^2 u$. (M, g, ∇) is called *Hessian (pseudo-Riemannian) manifold* if g is of Hessian type.

We prove the following:

Proposition 23. *Let (M, g, ∇) be a Hessian (pseudo-Riemannian) manifold then g is a Codazzi tensor if and only if ∇ is flat.*

Proof. Let $\{x_1, ..., x_n\}$ be local coordinates on M, let $g = \nabla^2 u$, then:

$$g_{jk} = \frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{l=1}^n \Gamma^l_{jk}\frac{\partial u}{\partial x_l}$$

in particular g is a Codazzi tensor if and only if for all $i, j, k = 1, ..., n$:

$$\begin{aligned}
&\frac{\partial^3 u}{\partial x_j \partial x_k \partial x_i} - \sum_{l=1}^n \frac{\partial \Gamma^l_{jk}}{\partial x_i}\frac{\partial u}{\partial x_l} - \sum_{l=1}^n \Gamma^l_{jk}\frac{\partial^2 u}{\partial x_l \partial x_i} + \\
&-\frac{\partial^3 u}{\partial x_i \partial x_k \partial x_j} + \sum_{l=1}^n \frac{\partial \Gamma^l_{ik}}{\partial x_j}\frac{\partial u}{\partial x_l} + \sum_{l=1}^n \Gamma^l_{ik}\frac{\partial^2 u}{\partial x_l \partial x_j} \\
= \quad &\sum_{l=1}^n \left(\Gamma^l_{ik}\left(\frac{\partial^2 u}{\partial x_l \partial x_j} - \sum_{r=1}^n \Gamma^r_{lj}\frac{\partial u}{\partial x_r}\right) - \Gamma^l_{jk}\left(\frac{\partial^2 u}{\partial x_l \partial x_i} - \sum_{r=1}^n \Gamma^r_{li}\frac{\partial u}{\partial x_r}\right)\right),
\end{aligned}$$

or:

$$\sum_{l=1}^{n}\left(\frac{\partial\Gamma_{ik}^{l}}{\partial x_j}-\frac{\partial\Gamma_{jk}^{l}}{\partial x_i}+\sum_{r=1}^{n}\left(\Gamma_{ik}^{r}\Gamma_{rj}^{l}-\Gamma_{jk}^{r}\Gamma_{ri}^{l}\right)\right)\frac{\partial u}{\partial x_l}=0$$

and thus the statement. □

We remark that under the hypothesis of Proposition 23 g is Codazzi if and only if (M, g, ∇) is an affine Hessian manifold. Hessian manifolds are more general than affine manifolds, [10], however in the following we will consider affine Hessian manifolds.

Moreover the following is well known:

Proposition 24. ([3]) *Let $(\mathbb{R}^n, \nabla)$ be the euclidean n-dimensional space with ∇ Levi Civita connection of the standard flat metric and let g be a symmetric tensor of type $(2,0)$, then g is a Codazzi tensor if and only if there exists $u \in C^\infty(\mathbb{R}^n)$ such that $g = Hess(u)$.*

We have the following:

Proposition 25. *Let (M, g, ∇) be an affine Hessian (pseudo-Riemannian) manifold, let $\{x_1, ..., x_n\}$ be affine local coordinates, let $\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right\}$ and $\{dx_1, ..., dx_n\}$ be corresponding local frames for $T(M)$ and $T^*(M)$ respectively, let $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ then for all $k = 1, ..., n$ the local section*

$$\sigma_k=\frac{\partial}{\partial x_k}+i\sum_{l=1}^{n}g_{kl}dx_l\in C^\infty\left(E_{J^g}^{0,1}\right)$$

is $\overline{\partial}_{J^g}-$cloded.

Proof. Let $u \in C^\infty(M)$ such that $g = \nabla^2 u$, as $dx_1, ..., dx_n$ are $\nabla-$parallel, we have $\Gamma_{jk}^l = 0$, then:

$$g_{jk}=\frac{\partial^2 u}{\partial x_j\partial x_k},$$

in particular:

$$g\left(\frac{\partial}{\partial x_k}\right)=\sum_{l=1}^{n}g_{kl}dx_l=\sum_{l=1}^{n}\frac{\partial^2 u}{\partial x_k\partial x_l}dx_l$$

and:

$$d\left(\sum_{l=1}^{n}g_{kl}dx_{_l}\right)=\sum_{l,j=1}^{n}\frac{\partial g_{kl}}{\partial x_j}dx_j\wedge dx_l=\sum_{j<l}\left(\frac{\partial g_{kl}}{\partial x_j}-\frac{\partial g_{kj}}{\partial x_l}\right)dx_j\wedge dx_l=0$$

then the statement. □

Moreover we can prove the following:

Proposition 26. *Let (M, g, ∇) be an affine Hessian manifold then the pair of complex dual Lie algebroids $\left(E^{1,0}_{J^g}, \left(E^{1,0}_{J^g}\right)^*\right)$ is a Lie bialgebroid if and only if g is constant.*

Proof. Let $\{x_1, ..., x_n\}$ be affine local coordinates, let $\left\{\dfrac{\partial}{\partial x_1}, ..., \dfrac{\partial}{\partial x_n}\right\}$ and $\{dx_1, ..., dx_n\}$ be corresponding local frames for $T(M)$ and $T^*(M)$ respectively, let $g_{ij} = g\left(\dfrac{\partial}{\partial x_i}, \dfrac{\partial}{\partial x_j}\right)$ and, for all $k = 1, ..., n$, let $\sigma_k = \dfrac{\partial}{\partial x_k} - i\displaystyle\sum_{l=1}^{n} g_{kl}dx_l \in C^\infty\left(E^{1,0}_{J^g}\right)$, we have:

$$[\sigma_k, \sigma_r]_\nabla = -i\sum_{s=1}^{n} \nabla_{\frac{\partial}{\partial x_k}} g_{rs}dx_s + i\sum_{l=1}^{n} \nabla_{\frac{\partial}{\partial x_r}} g_{kl}dx_l$$

$$= -i\left(\sum_{s=1}^{n} \frac{\partial g_{rs}}{\partial x_k}dx_s + \sum_{s=1}^{n} g_{rs}\left(\nabla_{\frac{\partial}{\partial x_k}} dx_s\right) - \sum_{l=1}^{n} \frac{\partial g_{kl}}{\partial x_r}dx_l - \sum_{l=1}^{n} g_{kl}\left(\nabla_{\frac{\partial}{\partial x_r}} dx_l\right)\right)$$

$$= -i\sum_{s=1}^{n}\left(\frac{\partial g_{rs}}{\partial x_k} - \frac{\partial g_{ks}}{\partial x_r}\right)dx_s = 0.$$

On the other hand:

$\sigma_k = \dfrac{\partial}{\partial x_k} - i\displaystyle\sum_{l=1}^{n} g_{kl}dx_l$ *is* $\partial_{J^g}-$closed, then we have immediately:

$$\partial_{J^g}[\sigma_k, \sigma_r]_\nabla = [\partial_{J^g}\sigma_k, \sigma_r]_\nabla + [\sigma_k, \partial_{J^g}\sigma_r]_\nabla = 0;$$

moreover, for all $f \in \wedge^0\left(E^{1,0}_{J^g}\right)$, we have:

$$\partial_{J^g}[f, \sigma_k]_\nabla - [\partial_{J^g}f, \sigma_k]_\nabla + [f, \partial_{J^g}\sigma_k]_\nabla$$

$$= \partial_{J^g}[f, \sigma_k]_\nabla - [\partial_{J^g}f, \sigma_k]_\nabla$$

$$= \partial_{J^g}\left(-\rho\left(\sigma_k\right)f\right) - \left[df + ig^{-1}df, \rho\left(\sigma_k\right) - ig\rho\left(\sigma_k\right)\right]_\nabla = 0$$

if and only if:

$$\begin{cases} d\left(\dfrac{\partial f}{\partial x_k}\right) & = & \nabla_{\frac{\partial}{\partial x_k}} df - \nabla_{g^{-1}df}\, g\left(\dfrac{\partial}{\partial x_k}\right) \\ g^{-1}d\left(\dfrac{\partial}{\partial x_k}\right) & = & \nabla_{\frac{\partial}{\partial x_k}} g^{-1}df - \nabla_{g^{-1}df}\dfrac{\partial}{\partial x_k} \end{cases}$$

or:

$$\begin{cases} d\left(\dfrac{\partial f}{\partial x_k}\right) & = & \nabla_{\frac{\partial}{\partial x_k}} df - \nabla_{g^{-1}df}\, g\left(\dfrac{\partial}{\partial x_k}\right) \\ g^{-1}\left(\nabla_{\frac{\partial}{\partial x_k}} df - \nabla_{g^{-1}df}\, g\left(\dfrac{\partial}{\partial x_k}\right)\right) & = & \nabla_{\frac{\partial}{\partial x_k}} g^{-1}df - \nabla_{g^{-1}df}\dfrac{\partial}{\partial x_k} \end{cases}. \quad (19)$$

The second condition in (19) is a consequence of ∇-integrability of J^g, then the Lie bialgebroid condition (15) is reduced to:

$$d\left(\frac{\partial f}{\partial x_k}\right) = \nabla_{\frac{\partial}{\partial x_k}} df - \nabla_{g^{-1}df}\, g\left(\frac{\partial}{\partial x_k}\right) \tag{20}$$

or, by using Einstein's convention on repeated indices:

$$g^{hs}\frac{\partial g_{ki}}{\partial x_s} = 0$$

then, for all i, k, s:

$$\frac{\partial g_{ki}}{\partial x_s} = 0 \tag{21}$$

and thus the statement. $\square$

In particular we can reformulate Proposition 26 as the following:

Proposition 27. *Let (M, g, ∇) be an affine Hessian manifold then the pair of complex dual Lie algebroids $\left(E^{1,0}_{J^g}, \left(E^{1,0}_{J^g}\right)^*\right)$ is a Lie bialgebroid if and only if ∇ is the Levi Civita connection of g.*

We remark that the generalized $\overline{\partial}_{J^g} -$ *operator* introduced in this paper,

$$\overline{\partial}_{J^g} : C^\infty\left(E^{0,1}_{J^g}\right) \to C^\infty\left(\wedge^2\left(E^{0,1}_{J^g}\right)\right),$$

and the $\overline{\partial}_J -$ *operator* for Hitchin's generalized complex structures,

$$\overline{\partial}_J : C^\infty\left(E^{0,1}_J\right) \to C^\infty\left(\wedge^2\left(E^{0,1}_J\right)\right),$$

are defined formally in the same way, (see ([7]), Section 3.3). Here we use $[\ ,\ ]_\nabla$, restricted to sections of $E^{0,1}_{J^g}$, instead of the Courant bracket, restricted to sections of $E^{0,1}_J$, and the standard symplectic form instead of the standard pseudo-Euclidean metric on $T(M) \oplus T^*(M)$, in the identifications $E^{0,1}_{J^g} \cong \left(E^{0,1}_{J^g}\right)^*$ and $E^{0,1}_J \cong \left(E^{0,1}_J\right)^*$ respectively. However Proposition 27 shows different behavior of the two operators regarding Lie bialgebroid structure of $\left(E^{1,0}_{J^g}, \left(E^{1,0}_{J^g}\right)^*\right)$ and $\left(E^{1,0}_J, \left(E^{1,0}_J\right)^*\right)$, since a generalized complex structure in Hitchin's sense always induces a Lie bialgebroid structure.

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