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Title: On the regularity of solutions to the Moore-Gibson-Thompson equation: a perspective via wave equations with memory[1]

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Abstract:

We undertake a study of the initial-boundary value problem for the (third-order in time) Moore-Gibson-Thompson (MGT) equation. The key to the present investigation is that the MGT equation falls within a large class of systems with memory, with affine term depending on a parameter. For this model equation a regularity theory is provided, which is also of independent interest; it is shown in particular that the effect of boundary data that are square integrable (in time and space) is the same displayed by the wave equation. Then, a general picture of the (interior) regularity of solutions corresponding to homogeneous boundary conditions is specifically derived for the MGT equation in various functional settings. This confirms the gain of one unity in space regularity for the time derivative of the unknown, a feature that sets the MGT equation apart from other Partial Differential Equations models for wave propagation. The adopted perspective and method of proof enables us to attain as well boundary regularity results for both the integro-differential equation and the MGT equation.

Keywords: interior regularity, boundary regularity, Moore-Gibson-Thompson equation, wave equations with memory, Volterra integro-differential equations, ultrasound propagation

MSC: 35L35, 35B65, 45K05, 47D09

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ON THE REGULARITY OF SOLUTIONS TO THE MOORE-GIBSON-THOMPSON EQUATION: A PERSPECTIVE VIA WAVE EQUATIONS WITH MEMORY

FRANCESCA BUCCI AND LUCIANO PANDOLFI

Abstract. We undertake a study of the initial-boundary value problem for the (third-order in time) Moore-Gibson-Thompson (MGT) equation. The key to the present investigation is that the MGT equation falls within a large class of systems with memory, with affine term depending on a parameter. For this model equation a regularity theory is provided, which is also of independent interest; it is shown in particular that the effect of boundary data that are square integrable (in time and space) is the same displayed by the wave equation. Then, a general picture of the (interior) regularity of solutions corresponding to homogeneous boundary conditions is specifically derived for the MGT equation in various functional settings. This confirms the gain of one unity in space regularity for the time derivative of the unknown, a feature that sets the MGT equation apart from other Partial Differential Equations models for wave propagation. The adopted perspective and method of proof enables us to attain as well boundary regularity results for both the integro-differential equation and the MGT equation.

1. Introduction

The Jordan-Moore-Gibson-Thompson equation is the following quasilinear partial differential equation (PDE) describing the acoustic velocity potential in ultrasound wave propagation:

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \frac{\partial}{\partial t}\Big(\frac{1}{c^2}\frac{B}{2A}\psi_t^2 + |\nabla\psi|^2\Big) \tag{1.1}$$

($\psi = \psi(t,x)$ is the acoustic velocity potential and so $-\nabla\psi$ is the acoustic particle velocity), A and B being suitable constants; *cf.* Moore & Gibson [30], Thompson [40], Jordan [10].

To give some insight into the specific features of Eq. (1.1), we note that the classical model with the relaxation time $\tau = 0$ has infinite signal speed due to the use of the Fourier law for the flux, that is $\mathbf{q} = -\kappa\nabla\theta$, in its derivation (see [14], [13]). When $\tau = 0$, the linearized approximation is the strongly damped wave equation, whose solutions are described by an analytic semigroup, *cf.* [4]. In order to amend the aforesaid unphysical behaviour (known as *paradox of heat conduction*), the paper [10] proposes to replace the Fourier law with the generalized constitutive law

$$\mathbf{q} + \tau\big(\mathbf{q}_t + \epsilon\psi\nabla\mathbf{q}\big) = -\kappa\nabla\theta$$

(which in the linearized case $\epsilon = 0$ reduces to the Maxwell-Cattaneo law) and this accounts for the third derivative in time in Eq. (1.1). The equation so obtained is hyperbolic and the velocity of signal propagation is finite.
The reader is referred to [11] for more details on the derivation of Eq. (1.1) and to [13] for a nice overview of established PDE models of nonlinear sound propagation.

Aiming at the understanding of the quasilinear equation (1.1), a great deal of attention has been recently devoted to its linearization—which is the Eq. (1.2) below, referred to in the literature as the Moore-Gibson-Thompson (MGT) equation—whose mathematical analysis is also of independent interest and poses already several questions and challenges. Thus, the MGT equation is

$$\tau u_{ttt} + \alpha u_{tt} - c^2\Delta u - b\Delta u_t = 0 \qquad \text{in } (0,T)\times\Omega \tag{1.2}$$

in the unknown $u = u(t,x)$, $t \geq 0$, $x \in \Omega$, representing the acoustic velocity potential or alternatively, the acoustic pressure (see [16] for a discussion on this issue). The coefficients c, b, α are constant and positive; they represent the speed and diffusivity of sound (c, b), and, respectively, a viscosity parameter (α).

We assume here that $\Omega \subset \mathbb{R}^n$ is a bounded region with C^2 boundary $\Gamma := \partial\Omega$ (it is a natural conjecture that existence results for wave equations in non-smooth domains ([8]) might be extended to wave equations with memory and to the MGT equation by using the methods we present in this paper).
Equation (1.2) is supplemented with initial and boundary conditions (BC):

$$u(0,\cdot) = u_0\,,\ u_t(0,\cdot) = u_1\,,\ u_{tt}(0,\cdot) = u_2(x) \qquad \text{in } (0,T)\times\Omega \tag{1.3}$$

$$\mathcal{T}u(t,\cdot) = g(t,\cdot) \qquad \text{on } (0,T)\times\Gamma. \tag{1.4}$$

$\mathcal{T}$ denotes here a boundary operator, which associates to a function u either its trace $\gamma_0 u := u|_\Gamma$ on Γ, or the outward normal derivative $\gamma_1 u := \frac{\partial}{\partial\nu}u\big|_\Gamma$ (it would be the *conormal* derivative, in the case of a more general elliptic operator than the Laplacian).

The original studies of the MGT equation subject to *homogenous* (Dirichlet or Neumann) boundary conditions carried out in Kaltenbacher *et al.* [15] and Marchand *et al.* [29] establish appropriate functional settings for semigroup well-posedenss, as well as stability and spectral properties of the dynamics, depending on the parameters values. They obtain, in particular:

i) assuming $b > 0$ the linear dynamics is governed by a strongly continuous *group* in the function space $H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ (in the case of Dirichlet BC), or $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ (Neumann BC);
ii) in the case $b = 0$ the associated initial-boundary value problems are ill-posed; see Remark 3.1;
iii) the parameter $\gamma = \alpha - \tau \frac{c^2}{b}$ is a threshold of stability/instability: it must be positive, if the property of uniform stability is required.

The critical role of γ for a dissipative behaviour was recently pointed out also in Dell'Oro and Pata [6], within the framework of viscoelasticity. (We add that linear and nonlinear variants of the MGT equation including an *additional* memory term have been the object of recent investigation; see [18] and references therein.)

Here we seek to determine a range of settings in which the MGT equation is well posed. Recall that an equation in a normed space $\mathcal{C}$ and which depends on a parameter $p \in \mathcal{P}$ (a second normed space) is well posed when a solution exists for every p, is unique and depends continuously on p. Thus, focus is on the map

$$(u_0, u_1, u_2, g) \longmapsto u$$

that associates to initial and boundary data the corresponding solution $u = u(t, x)$ of problem (1.2)-(1.3)-(1.4); we aim to identifying appropriate settings which guarantee that a unique solution does exist, and it is regular, i.e. it depends continuously on the said data. (We note that the time and more often the space variable x will generally not be explicit, unless when needed for the sake of clarity. In addition, in order to simplify the notation we shall put the relaxation time τ equal to 1 throughout.)

As it will be shown in the paper, it will be the embedding of equation (1.2) in a general class of integro-differential equations (depending on a parameter) to spark our method of proof for the regularity analysis of the associated initial-boundary value problem. Indeed, the MGT equation is a special instance of the following wave equation with persistent memory,

$$u_{tt} - b\Delta u = -b\gamma \int_0^t N(t-s)\Delta u(s)\, ds + F(t)\xi\,, \tag{1.5}$$

which displays an affine term depending on a suitable ξ, and that will be supplemented with (initial and boundary) data

$$u(0) = u_0\,,\ u_t(0) = u_1\,, \qquad \mathcal{T}u = g\,. \tag{1.6}$$

The assumptions on the real valued functions $N(t)$, $F(t)$ and on ξ in (1.5) are specified later; see Theorem 1.2. The initial-boundary value problem (IBVP) (1.5)-(1.6) reduces to the IBVP (1.2)-(1.3)-(1.4), when

$$N(t) = F(t) = e^{-\alpha t}\,, \qquad \xi = u_2 - b\Delta u_0\,. \tag{1.7}$$

The obtained regularity results will follow by combining the (interior and trace) regularity theory for wave equations with non-homogenous boundary data—the Neumann case being the most challenging (see [22], [23], and the optimal result of

[39])—with the methods developed in [32] for equations with persistent memory. In order to carry out a regularity analysis of the model equation with memory (1.5), we shall use the trick of MacCamy [28] and the theory of Volterra equations.
A study of the quadratic regulator problem for Eq. (1.2) with Neumann-absorbing boundary conditions has been pursued via a semigroup approach in [2]. Existence of a unique optimal control and its closed-loop expression is estabilished via appropriate Riccati operators, by using semigroup theory and variational arguments.

For equations with memory of the form (1.5) the reader is referred, e.g., to [32, Chapter 2]; see also [5, Chapter 5]. A classical reference on evolutionary integral equations is [36]. A novelty in the equation (1.5) is brought about by the presence of the (vectorial) parameter ξ.

It is important to emphasize at the outset that the adopted perspective and approach allow for a relatively straightforward derivation of regularity estimates for the normal derivative on $\partial\Omega$ of the solutions of to wave equations with memory, and then to the MGT equation, in light of the corresponding literature on wave equations without memory. Early results pertain to the case of Dirichlet boundary conditions; see [20, 21] and [25, p. 195]. A proof for the model equation with memory (1.5) (depending on the parameter ξ), supplemented with Dirichlet BC, is given in Theorem 6.2; this brings about a boundary regularity result for the MGT equation (with the same BC), that is Corollary 6.3. The reader is referred to the Remarks 6.4 for a discussion and overview of pertinent references on this matter of topical interest.

1.1. **Main results: synopsis.** The outcome of the *interior* regularity analysis carried out in this paper is stated in Theorem 4.2, pertaining to the general model equation with memory (1.5), and Theorem 5.3 for the MGT equation itself. Beside being instrumental in achieving the subsequent ones, the former results are also of independent interest. The aforeasid results are presented by means of elaborate tables: thus, aiming at rendering explicit the major achievements on the regularity of solutions to equations (1.5) and (1.2)—the latter linked and complementing those in our key reference [15]—we highlight them in Theorem 1.2 below. Theorem 1.2 includes as well a *boundary* regularity result for the solutions (more precisely, for the normal derivative of the solutions on the boundary) in the case of homogeneous Dirichlet BC—an issue which is dealt with in Section 6, see Theorem 6.2 and Corollary 6.3.

For the statement and understanding of all our findings, we need to introduce appropriate functional spaces, along with the related notation. Let A be the unbounded operator defined as follows:

$$Aw := (\Delta - I)w\,, \quad \mathcal{D}(A) = \big\{w \in H^2(\Omega)\colon \mathcal{T}w = 0 \text{ on } \Gamma\big\}\,; \tag{1.8}$$

namely, A is the (so called) realization of the differential operator $\Delta - I$ in $L^2(\Omega)$, with homogeneous boundary conditions (BC) defined by $\mathcal{T}$, in the present work of either Dirichlet or Neumann type; of course, the domain of A depends on $\mathcal{T}$ (we might take the realization of the Laplacian in the case of Dirichlet BC; however, translating the differential operator allows us to deal with both significant BC at one time). We further note that A is the infinitesimal generator of an exponentially stable analytic semigroup and the fractional powers of $-A$ are well defined. Thus,

we are allowed to introduce the functional spaces X_s defined as follows:

$$X_s = \begin{cases} \mathcal{D}((-A)^{s/2}) & \text{if } s \geq 0 \\ [\mathcal{D}((-A)^{s/2})]' & \text{if } s < 0\,, \end{cases} \tag{1.9}$$

endowed with the graph norm if $s \geq 0$, while the norm of a dual space is needed otherwise.

Remarks 1.1. (i) In this paper, $\mathcal{T}$ is either γ_0 (Dirichlet BC) or γ_1 (Neumann BC). We have $X_s = \{\phi \in H^s(\Omega)\colon \mathcal{T}\phi = 0\}$ when $s > 1/2$ and $\mathcal{T} = \gamma_0$ (when $s > 3/2$ and $\mathcal{T} = \gamma_1$, respectively). If instead $s < 1/2$ (in the Dirichlet case) or $s < 3/2$ (in the Neumann case, respectively), one has $X_s = H^s(\Omega)$.

(ii) The differential operator $\Delta = A + I$, originally defined in (1.8), can be extended to every space X_s. It turns out that *the said extension belongs to* $\mathcal{L}(X_s, X_{s-2})$ *for every* s*, it is surjective and boundedly invertible.* Using the fact that $A + I$ is a selfadjoint operator with compact resolvent in $L^2(\Omega)$, the extension is simply obtained via Fourier expansion.

The next Theorem highlights some of the major results that are obtained in Sections 4, 5, 6.

Theorem 1.2 (A compendium of main results)**.** *Let* $N(t) \in H^2(0,T)$ *for every* $T > 0$*. The following assertions hold:*

i) (Interior regularity for the equation with memory (1.5) with homogeneous BC). *Assume* $F(t) \in L^2(0,T)$ *for every* $T > 0$*. Let* $g \equiv 0$*. If* $u_0 \in X_0$ *and* $u_1, \xi \in X_{-1}$*, then the solution* u *to the IBVP problem* (1.5)-(1.6) *exists and it is unique in the space*

$$u \in C([0,T]; X_0) \cap C^1([0,T]; X_{-1}) \cap H^2(0,T; X_{-2})$$

and depends continuously on the data.

ii) (Interior regularity for the MGT equation (1.2) with homogeneous BC). *If* $g \equiv 0$ *and* $(u_0, u_1, u_2) \in X_1 \times X_1 \times X_0$*, then the solution* u *to the IBVP problem* (1.2)-(1.3)-(1.4) *exists and it is unique in the space*

$$u \in C^1([0,T]; X_1) \cap C^2([0,T]; X_0) \tag{1.10}$$

and depends continuously on the data.

iii) (Boundary-to-interior regularity for equations (1.5) and (1.2), with trivial initial data). *Assume* $g \in L^2(0,T; L^2(\Gamma))$ *and* $u_0 = u_1 = \xi = 0$ *(in the case of Eq.* (1.2)*:* $u_0 = u_1 = u_2 = 0$*, and hence* $u_2 - \Delta u_0 = 0$*). Then there exists* α_0 *such that a solution to the initial/boundary value problem* (1.5)-(1.6) *(*(1.2)-(1.3)–(1.4)*, respectively)—as given by the one to the integral equation* (3.13)*—exists in the space*

$$C([0,T]; X_{\alpha_0}) \cap C^1([0,T]; X_{\alpha_0 - 1}) \cap H^2(0,T; X_{\alpha_0 - 2})\,, \tag{1.11}$$

it is unique and depends continuously on g*. The values of* α_0 *depend on the boundary operator* $\mathcal{T}$ *(and partly on* Ω*); they are specified in* (1.14) *below.*

iv) (Regularity of boundary traces for the MGT equation (1.2)). *Let* $u = u(t,x)$ *be a solution to the MGT equation* (1.2) *corresponding to initial data* (u_0, u_1, u_2) *and homogeneous Dirichlet boundary data. Assume* $(u_0, u_1, u_2) \in H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$*, along with the compatibility condition*

$$u_2 - \Delta u_0 \in L^2(\Omega)\,.$$

Then, for every $T > 0$ there exists $M = M_T$ such that

$$\int_0^T \int_{\partial\Omega} \left| \frac{\partial}{\partial \nu} u(x,t) \right|^2 d\sigma \, dt \leq M \left(\|u_0\|_{H_0^1(\Omega)} + \|u_1|^2_{L^2(\Omega)} + \right. \\ \left. + \|u_2 - \Delta u_0\|^2_{L^2(\Omega)} \right).$$

The statements in i) and ii) reveal that while the equation with memory (1.5) displays a somewhat expected regularity, that is the same as most PDE models for wave propagation, the interior regularity of solutions to the MGT equation (1.2) under homogeneous boundary conditions improves. The computations performed in this paper show how the higher regularity is gained by the solutions of equation (1.2), when we particularize the formulas using (1.7); see the proof of Theorem 5.3. Instead, the regularity result in iii) that pertains to the case of non-homogeneous boundary data ($g \neq 0$), is not improved by special choices of the kernel $N(t)$, such as $N(t) = e^{-\alpha t}$.

It is worth observing that the present study, which develops from the regularity analysis of a general class of PDE systems with memory, does not disclose that the dynamics of the MGT equation (1.2)—with Dirichlet or Neumann boundary conditions—is governed by a *group*, as proved in [15] and [29].

We note that the values of α_0 which occurr in (1.11)—and which correspond to appropriate Sobolev exponents—are the ones established in the case of linear hyperbolic equations with $L^2(\Sigma) = L^2((0,T)\times\Gamma)$ boundary data (of either Dirichlet or Neumann type). We record explicitly for the IBVP

$$\begin{cases} u_{tt} = \Delta u - u + f & \text{in } (0,T)\times\Omega =: Q \\ u(0,\cdot) = u_0\,,\ u_t(0,\cdot) = u_1 & \text{in } \Omega \\ \mathcal{T} u = g & \text{on } (0,T)\times\Gamma =: \Sigma \end{cases} \tag{1.12}$$

a statement which embodies a complex of successive achievements; see the cited references. (For a chronological overview with historical and technical remarks see also [24, Notes on Chapter 8, p. 761].)

Theorem 1.3 ([21], [22], [39])**.** *Assume that $u_0, u_1 = 0$, $f = 0$, and $g \in L^2(\Sigma)$. Then, the unique solution to the IBVP* (1.12) *satisfies*

$$(u, u_t) \in C([0,T]; H^{\alpha_0}(\Omega) \times H^{\alpha_0 - 1}(\Omega)) = C([0,T]; X_{\alpha_0} \times X_{\alpha_0 - 1}) \tag{1.13}$$

with

$$\alpha_0 = \begin{cases} 0 & \text{if } \mathcal{T} \text{ is the Dirichlet trace operator} \\ \frac{2}{3} & \text{if } \mathcal{T} \text{ is the Neumann trace operator and } \Omega \text{ is a smooth domain} \\ \frac{3}{4} & \text{if } \mathcal{T} \text{ is the Neumann trace operator and } \Omega \text{ is a parallelepiped.} \end{cases} \tag{1.14}$$

Notice that the second equality in Eq. (1.13) follows from the first statement of Remarks 1.1.

We finally point out the recent progress of [41] on the regularity of wave equations, dealing with the case of boundary data g that are *not* 'smooth in space', e.g., $g \in L^2(0,T; H^{-1/2}(\Gamma))$. In view of the approach taken in the present work,

it is clear that the results obtained therein could be utilized as well in order to attain regularity results for equations with memory and for the MGT equation under boundary data that are less regular (than square integrable) in space.

1.2. **Orientation.** The plan of the paper is briefly outlined below. For the reader's convenience and since these tools will be utilized throughout, in Section 2 we provide a minimal background and references on the approach to linear wave equations via cosine operator theory.

In Section 3 we perform an analysis of the equation with memory (1.5) that encompasses the MGT equation. An equivalent equation—in fact easier, since the convolution term therein does not involve differential operators at all—is derived, which in turn results in a Volterra equation of the second kind; see Proposition 3.6. This step will play a crucial role in the proof of our first regularity result, that is Theorem 4.2, concerning the model equation with memory (1.5). Section 4 is then almost entirely devoted to the proof of Theorem 4.2.

In Section 5 we return to the third order MGT equation and show how the (interior) regularity results specifically pertaining to the MGT equation, stated in Theorem 5.3, follow as a consequence of Theorem 4.2. Finally, Section 6 is devoted to the regularity of boundary traces; see Theorem 6.2 and Corollary 6.3.
A discussion and explanation of the introduced definition of solutions to the third order (in time) equation under investigation is postponed to Appendix A.

2. Preliminaries on wave equations

The proofs in the present work rely in a crucial way on the representation of solutions to linear wave equations by means of cosine operators (introduced in [38]). We present the key formulas in this section, following [19] which initiated the use of cosine operators in the study of boundary control problems. The notations of [1]—which seems to be the first paper to deal with systems with persistent memory via cosine operators—and [7] are adopted. The IBVP (1.12) is considered with both f and g square integrable in $(0,T)\times\Omega$ and on $(0,T)\times\Gamma$, respectively, for every $T>0$. We note that the formulas and results recorded in the present Section are valid as well if $f\in L^1(0,T;L^2(\Omega))$; this fact will be used later, in particular in Section 6.

We shall utilize throughout the operator A in (1.8), which is the realization of the translation $\Delta - I$ of the Laplacian in $L^2(\Omega)$, with suitable homogeneous boundary conditions, according to the (boundary) operator $\mathcal{T}$. (In the Dirichlet case A might be simply the realization of the Laplacian.) As noted already, A is boundedly invertibile, i.e. A^{-1} exists and it is bounded, in fact compact (even if $\mathcal{T}$ represents the normal derivative on Γ). In addition, A is the infinitesimal generator of an exponentially stable *analytic* semigroup, and the fractional powers of $-A$ are well defined. The scale of functions spaces X_s in (1.9)—expressed by means of the domains of these fractional powers—will be used throughout the paper. We recall once more that $A\colon X_s\to X_{s-2}$ is continuous, surjective and boundedly invertible.

Next, we introduce the Green maps $G\in\mathcal{L}(L^2(\Gamma),L^2(\Omega))$ defined as follows:

$$G\colon L^2(\Gamma)\ni\varphi\longmapsto G\varphi=:\psi \iff \begin{cases}\Delta\psi=\psi & \text{on } \Omega\\ \mathcal{T}\psi=\varphi & \text{on } \Gamma\,.\end{cases} \tag{2.1}$$

Remark 2.1. By elliptic theory, it is known that there exists an appropriate $s > 0$ such that $\operatorname{im} G \subset X_s$ so that $AG \subset X_{s-2}$. In particular, because $\operatorname{im} G \subset H^{1/2}(\Omega)$ in the case of Dirichlet BC and $\operatorname{im} G \subset H^{3/2}(\Omega)$ in the Neumann case, the following inclusions hold true: $\operatorname{im} G \subset X_{1/2-\sigma}$ in the former case, $\operatorname{im} G \subset X_{3/2-\sigma}$ in the latter, respectively, for every $\sigma > 0$.

It is known that the solution to the IBVP (1.12) is given by

$$\begin{aligned} u(t) = R_+(t)u_0 + \mathcal{A}^{-1}R_-(t)u_1 - \mathcal{A}\int_0^t R_-(t-s)Gg(s)\,ds + \\ + \mathcal{A}^{-1}\int_0^t R_-(t-s)f(s)\,ds\,, \end{aligned} \tag{2.2}$$

where the operator $\mathcal{A}$, and the families of operators $R_+(\cdot)$, $R_-(\cdot)$ are defined as follows:

$$\mathcal{A} = i(-A)^{1/2}\,, \qquad R_+(t) = \frac{e^{\mathcal{A}t} + e^{-\mathcal{A}t}}{2}\,, \qquad R_-(t) = \frac{e^{\mathcal{A}t} - e^{-\mathcal{A}t}}{2}\,, \tag{2.3}$$

$R_+(t)$ being the strongly continuous *cosine* operator generated by A in $L^2(\Omega)$; see [38], [7], [24, Vol. II]. The previous definitions make sense because $\mathcal{A}$ is the infinitesimal generator of a C_0-*group* of operators; in particular, we have as well

$$X_s = \mathcal{D}(\mathcal{A}^s) \qquad \text{if } s \geq 0,$$

and $\mathcal{A}$ is bounded and boundedly invertible from X_s to X_{s-1} for every s.

In order to compute the derivatives (in time) of $u(t)$ in (2.2) (the technical justification is in [31]) we use $(R_+(t)x)_t = \mathcal{A}R_-(t)x$, $(R_-(t)x)_t = \mathcal{A}R_+(t)x$ for every $x \in \operatorname{dom}\mathcal{A}$, thereby obtaining the following equalities:

$$\begin{aligned} u_t(t) = \mathcal{A}R_-(t)u_0 + R_+(t)u_1 - A\int_0^t R_+(t-s)Gg(s)\,ds + \\ + \int_0^t R_+(t-s)f(s)\,ds\,, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} u_{tt}(t) = AR_+(t)u_0 + \mathcal{A}R_-(t)u_1 - AGg(t) - A\Big(\mathcal{A}\int_0^t R_-(t-s)Gg(s)\,ds\Big) + \\ + f(t) + \mathcal{A}\int_0^t R_-(t-s)f(s)\,ds = \\ = Au(t) - AGg(t) + f(t)\,. \end{aligned} \tag{2.5}$$

Remarks 2.2. (i) By using the regularity of G in Remark 2.1, we see from Eq. (2.5) that $u_{tt} \in L^2(0,T;X_{\alpha_0-2})$.

(ii) If $f(\cdot)$ belongs to $C^1([0,T])$, one is allowed to integrate by parts (in time) in (2.2) to get

$$\mathcal{A}^{-1}\int_0^t R_-(t-s)f(s)\,ds = -A^{-1}\Big[f(t) - R_+(t)f(0) - \int_0^t R_+(t-s)f'(s)\,ds\Big]\,,$$

which yields a gain of one unity for space regularity. The integration by parts is rigorously justified in [31, Lemma 5].

The explicit formula (2.2), along with (2.4) and (2.5) are among the keys for the following regularity result. The statement iii) of the Theorem is by far the most challenging, as its proof is based on pseudo-differential methods and microlocal analysis.

Theorem 2.3. *Let $T > 0$ be given, and $s \in \mathbb{R}$. The following statements hold true for the solutions to the initial/boundary value problem (1.12).*

i) *Assume $g = 0$, $f = 0$. Then $(u_0, u_1) \longmapsto u$ is continuous from $X_s \times X_{s-1}$ into $C([0,T], X_s) \cap C^1([0,T], X_{s-1}) \cap C^2([0,T], X_{s-2})$.*

ii) *Assume $u_0 = 0$, $u_1 = 0$, $g = 0$. Then the map $f \longmapsto u$ is continuous from $L^2(0,T; X_s)$ into $C([0,T], X_{s+1}) \cap C^1([0,T), X_s)$, while $u_{tt} - f \in C([0,T], X_{s-1})$.*

iii) *Assume $u_0 = 0$, $u_1 = 0$, $f = 0$. Then, there exists $\alpha_0 \geq 0$—depending on $\mathcal{T}$ and possibly on the geometry of Ω—such that for every $g \in L^2((0,T) \times \Gamma)$ one has $u \in C([0,T], X_{\alpha_0}) \cap C^1([0,T]; X_{\alpha_0 - 1}) \cap C^2([0,T], X_{\alpha_0 - 2})$. The mapping $g \longmapsto u$ is continuous in the indicated spaces.*

Remarks 2.4. With reference to the assertion iii) above, we remind the reader that the proper value of the Sobolev exponent α_0 is given in (1.14).

The properties stated in the previous Theorem justify (2.2) as a formula for the solutions to the IBVP (1.12), since the following fact is easily checked: when $u_0, u_1 \in \mathcal{D}(\Omega)$ ($C^\infty(\Omega)$ functions with compact support), $f \in \mathcal{D}((0,T) \times \Omega)$, $g \in \mathcal{D}((0,T) \times \Gamma)$, then $u - Gg \in C([0,T], \mathcal{D}(A)) \cap C^1([0,T], \mathcal{D}(A)) \cap C^2([0,T]; L^2(\Omega))$ and the following equality holds:

$$u_{tt}(t) = A(u(t) - Gg(t)) + f(t)\,,$$

along with $u(0) = u_0$, $u_t(0) = u_1$. Thus, the boundary condition $\mathcal{T}u = g$ is satisfied in the sense that $u(t) - Gg(t) \in \mathcal{D}(A)$ for almost any t.

3. The MGT equation as an equation with memory

We initially proceed formally. Rewrite the left hand side of equation (1.2) as

$$\begin{aligned} u_{ttt} + \alpha u_{tt} - c^2\Delta u - b\Delta u_t = \\ &= \big(u_{tt} - b\Delta u\big)_t + \alpha\big(u_{tt} - b\Delta u\big) - c^2\Delta u + \alpha b\Delta u = \\ &= \big(u_{tt} - b\Delta u\big)_t + \alpha\big(u_{tt} - b\Delta u\big) + b\gamma\Delta u = 0 \end{aligned} \tag{3.1}$$

where we recall that $\gamma = \alpha - c^2/b$. Solving the equation

$$\big(u_{tt} - b\Delta u\big)_t = -\alpha\big(u_{tt} - b\Delta u\big) - b\gamma\Delta u$$

in the 'unknown' $u_{tt} - b\Delta u$ gives the following integral equation in the unknown (and in fact not yet defined as solution) u:

$$u_{tt} - b\Delta u = e^{-\alpha t}\xi - b\gamma\int_0^t e^{-\alpha(t-s)}\Delta u(s)\,ds\,, \tag{3.2}$$

where we set $\xi = u_2 - b\Delta u_0$.

Remark 3.1. If it happens that $\gamma = 0$, then (3.2) is nothing but a wave equation with affine term $f(t) = e^{-\alpha t}\xi$ and the regularity of the corresponding solutions follows from Theorem 2.3. Thus, we explicitly assume $\gamma \neq 0$, and recall from the Introduction that b is assumed to be positive. It is important to emphasize that

in the case $b = 0$ the problem is ill-posed, since the semigroup generation fails, as proved in [15, Theorem 1.1]; instead, if $b < 0$ then the PDE becomes a *nonlocal* elliptic equation of a kind studied by Skubacevskiĭ in [37].

In view of the obtained equation (3.2), we consider the following (more general) model equation with persistent memory, depending on the parameter ξ:

$$u_{tt} - b\Delta u = -b\gamma \int_0^t N(t-s)\Delta u(s)\, ds + F(t)\xi \tag{3.3}$$

(already appeared—as (1.5)—in the Introduction and recorded here for the reader's convenience; notice that both the functions $N(t)$ and $F(t)$ equal $e^{-\alpha t}$ in the MGT equation).

The study of the IBVP for the integro-differential equation (3.3) is carried out under the assumptions listed below.

Hypotheses 3.2. *i) The coefficient b is positive. ii) The memory kernel $N(t)$ and the function $F(t)$ are real valued; $N(t) \in H^2(0,T)$ while $F(t) \in L^2(0,T)$ for every $T > 0$.*

3.1. **An equivalent Volterra integral equation.** A first step in our analysis is to show that we can get rid of the (second order) differential operator in the convolution term of (3.3). To do so, let us preliminarly introduce the Volterra equation of the second kind

$$X(t) - \gamma \int_0^t N(t-s)X(s)\, ds = G(t)\,, \qquad t \in [0,T]\,. \tag{3.4}$$

This equation has a unique solution $X(t)$ given by the following formula:

$$X(t) = G(t) - \int_0^t R_0(t-s)G(s)\, ds\,, \tag{3.5}$$

where $R_0(\cdot)$ is the (unique) solution to the integral equation

$$R_0(t) - \gamma \int_0^t N(t-s)R_0(s)\, ds = -\gamma N(t)\,, \qquad t \in [0,T]\,. \tag{3.6}$$

The function $t \longmapsto R_0(t)$ is the *resolvent kernel* of the Volterra equation (3.4). An important observation is that $R_0 \in H^2(0,T)$ since $N \in H^2(0,T)$ and $R_0(0) = -\gamma N(0)$. We then see (either from (3.5) or from (3.4)) that if $G(t)$ is continuous then $X(t)$ is continuous; if $G(t)$ is square integrable then $X(t)$ is square integrable.

We now perform several formal computations which will lead to a definition of the solutions to equation (3.3) (with appropriate initial and boundary data). Rewrite the equation (3.3) in the following different fashion,

$$\Delta u - \gamma \int_0^t N(t-s)\Delta u(s)\, ds = \frac{1}{b}\big(u_{tt} - F(t)\xi\big)\,, \tag{3.7}$$

that is a Volterra integral equation of the second kind in the unknown Δu. With reference to the general form (3.4), we have here

$$G(t) = \frac{1}{b}\big(u_{tt} - F(t)\xi\big)\,.$$

The formula (3.5) gives

$$b\Delta u = u_{tt} - F(t)\xi - \int_0^t R_0(t-s)\big(u_{ss}(s) - F(s)\xi\big)\, ds\,,$$

with $R_0(\cdot)$ defined by (3.6) above. Since $R_0 \in H^2(0,T)$ we formally integrate by parts twice, thereby obtaining

$$\begin{aligned} b\Delta u = u_{tt} - F(t)\xi - \Big\{R_0(t-s)u_t(s)\big|_{s=0}^{s=t} - \int_0^t R_0'(t-s)u_s(s)\,ds\Big\} + \\ + \int_0^t R_0(t-s)F(s)\xi\,ds = \\ = u_{tt} - F(t)\xi - R_0(0)u_t(t) + R_0(t)u_1 - R_0'(0)u(t) - R_0'(t)u_0 - \\ - \int_0^t R_0''(t-s)u(s)\,ds + \int_0^t R_0(t-s)F(s)\xi\,ds\,, \end{aligned}$$

where the memory term does not contain differential operators.

Remark 3.3. The computations carried out so far—known as MacCamy's trick (see [28])—are purely formal, since the solutions to the equation (3.3) have not yet been defined.

The obtained equation is a wave equation perturbed by a persistent memory, namely,

$$\begin{aligned} u_{tt} = b(\Delta - I)u + (R_0'(0) + b)u(t) + R_0(0)u_t(t) + \int_0^t R_0''(t-s)u(s)\,ds - \\ - R_0'(t)u_0 - R_0(t)u_1 + \Big\{F(t)\xi - \int_0^t R_0(t-s)F(s)\xi\,ds\Big\}\,. \end{aligned}$$

The introduction of the function

$$v(t) = e^{-\frac{1}{2}R_0(0)t}u(t) \tag{3.8}$$

enables us to eliminate the term $R_0(0)u_t$, and to attain the following equation in the unknown v:

$$v_{tt} = b(\Delta v - v) + \int_0^t K(t-s)v(s)\,ds + \beta v(t) + \big(h_2(t)\xi + h_1(t)u_1 + h_0(t)u_0\big)\,, \tag{3.9}$$

with the constant β and the functions $K(\cdot)$, $h_i(\cdot)$, $i = 0,1,2$ given by the formulas below:

$$\begin{aligned} &K(t) = e^{-\frac{1}{2}R_0(0)t}R_0''(t) \text{ is square integrable in } (0,T);\\ &\beta = b + \frac{1}{4}R_0^2(0) + R_0'(0)\,;\\ &h_0(t) = e^{-\frac{1}{2}R_0(0)t}R_0'(t) \in H^1(0,T)\,;\\ &h_1(t) = e^{-\frac{1}{2}R_0(0)t}R_0(t) \in H^2(0,T)\,;\\ &h_2(t) = e^{-\frac{1}{2}R_0(0)t}\big(F(t) - \int_0^t R_0(t-s)F(s)\,ds\big) \text{ is square integrable.} \end{aligned} \tag{3.10}$$

The above suggests the following Definition, which is justified in the Appendix.

Definition 3.4. *Let $\mathcal{H}$ be a Hilbert space. An $\mathcal{H}$-valued function $t \longmapsto u(t)$ is a solution of equation* (3.3) *supplemented with the initial/boundary conditions* (1.6) *if the function $t \longmapsto v(t)$ defined in* (3.8) *is an $\mathcal{H}$-valued continuous function which solves the integro-differential equation* (3.9)*, with β, $K(\cdot)$, $h_i(\cdot)$, $i = 0,1,2$ defined by* (3.10)*.*

Remark 3.5. In the case $F(t) \equiv N(t) = e^{-\alpha t}$, then the above definition yields the definition of solutions to the MGT equation (1.2), with initial/boundary conditions (1.3)-(1.4); see Definition 5.1. The present Section investigates the regularity of solutions to the model equation (3.3) in terms of the *independent* parameters u_0, u_1, ξ and the boundary input g.

On the basis of Definition 3.4 we are led to study the following IBVP:

$$\begin{cases} v_{tt} = b(\Delta v - v) + \displaystyle\int_0^t K(t-s)v(s)\,ds + \beta v(t) + \big(h_2(t)\xi + h_1(t)u_1 + h_0(t)u_0\big) \\ v(0,\cdot) = v_0\,,\ v_t(0,\cdot) = v_1 \\ \mathcal{T}v = e^{-\frac{1}{2}R_0(0)t}g\,, \end{cases} \tag{3.11}$$

where initial data are related to the ones of u via the following relations:

$$v_0 = u_0\,, \quad v_1 = u_1 - \frac{1}{2}R_0(0)u_0\,. \tag{3.12}$$

The next Proposition connects the IBVP (3.11) to a Volterra equation of the second kind, with suitable kernel and affine term.

Proposition 3.6. *Any solution to the initial/boundary value problem* (3.11) *solves the Volterra equation*

$$v(t) + \int_0^t L(t-s)v(s)\,ds = H(t)\,, \tag{3.13}$$

where $L(\cdot)$ is a strongly continuous kernel defined by

$$L(t)v = -\frac{\beta}{\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)v - \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))K(s)v\,ds \tag{3.14}$$

(and $K(\cdot)$ is defined explicitly in (3.10)*), while the affine term $H(\cdot)$ is given by*

$$\begin{aligned} H(t) = \Big[R_+(\sqrt{b}t) - \frac{R_0(0)}{2\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)\Big]u_0 + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)u_1 - \\ - \sqrt{b}\mathcal{A}\int_0^t R_-(\sqrt{b}(t-s))G\,e^{-\frac{1}{2}R_0(0)s}g(s)\,ds + \\ + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))\big[h_2(s)\xi + h_1(s)u_1 + h_0(s)u_0\big]\,ds\,. \end{aligned} \tag{3.15}$$

Proof. The proof is straightforward, in view of formula (2.2) for the solution to wave equations with initial and boundary data. We just recall the meaning of symbols introduced previously: $R_0(\cdot)$ is the resolvent kernel of the Volterra equation (3.4) and so it solves the integral equation (3.6); the abstract operator A is the realization of the differential operator $\Delta - I$ with boundary conditions driven by $\mathcal{T}$, while $R_+(\sqrt{b}t)$, the *cosine* operator generated by bA, and $R_-(\sqrt{b}t)$ are defined in (2.3). We finally note that $H(\cdot)z \in C([0,T];X_\alpha)$, for every $z \in X_\alpha$. □

4. Interior regularity for the equation (1.5)

In this Section we see how the regularity results pertaining to wave equations stated in Theorem 2.3 can be suitably extended to the general equation with memory of the form (3.3). This will eventually imply the *stronger* regularity of solutions to the third order MGT equation (1.2) (see the next Section).

The key and starting point is the Volterra integral equation (3.13) in the unknown v. Its kernel $L(\cdot)$ is now operator valued and *strongly continuous* from $[0,+\infty)$ to $\mathcal{L}(X_\alpha)$ for every α. By using Theorem 2.3 we will derive the regularity properties of the right hand side of (3.13), that will be inherited by v and then by the solutions to the wave equation with memory (3.3). These properties will be expressed in terms of the boundary datum g, as well as of ξ and the initial data u_0, u_1.

It is convenient to write explicitly the solution of a Volterra integral equation in a Hilbert space $\mathcal{H}$. Introduce the notation $*$ for the convolution, namely

$$(L*h)(t)=\int_0^t L(t-s)h(s)\,ds=\int_0^t L(s)h(t-s)\,ds\,.$$

Here $L(t)$ is a strongly continuous function of time, with values in $\mathcal{L}(\mathcal{H})$ and $h(t)$ is an integrable $\mathcal{H}$-valued function.
Moreover, let $L^{(*n)}$ denote iterated convolutions, recursively defined by the following equalities

$$L^{(*1)}=L\,,\quad L^{(*(n+1))}*h=L*\left(L^{(*n)}*h\right)$$

(for every integrable $\mathcal{H}$-valued function h). Then, the solution to the Volterra equation (3.13)—that is $v+L*v=H$, in short—is

$$v=H+\sum_{k=1}^{\infty}(-1)^k L^{(*k)}*H\,.$$

Uniform convergence of the series is easily proved. In the special case of our interest, with $\mathcal{H}=X_\alpha$ and $L(t)$ given by (3.14), the following result follows.

Lemma 4.1. *Let $T>0$ and let the kernel $L(\cdot)$ be given by* (3.14)*. If $H\in C([0,T];X_\alpha)$, then the solution v of the Volterra equation $v+L*v=H$ satisfies $v\in C([0,T];X_\alpha)$.*

Proof. It is sufficient to observe that for every t we have $L(t)\in\mathcal{L}\,(X_\alpha)$ and $L(\cdot)$ is a strongly continuous operator valued function in $\mathcal{L}\,(X_\alpha)$. Consequently, $v+L*v=H$ is a Volterra integral equation in X_α to which the results in [5, Chapter 5] apply. □

We will repeatedly use Lemma 4.1 in order to pinpoint the regularity of the solutions to the initial/boundary value problems associated with the equation (3.3).

Theorem 4.2 (Regularity for equation (3.3))**.** *Consider the Eq.* (3.3) *with initial data (u_0,u_1) and boundary data defined by* (1.4)*. Then, the linear map*

$$(u_0,u_1,\xi,g)\longmapsto(u,u_t,u_{tt})$$

is continuous among the spaces detailed in Table 1.

Proof. The proof of the several statements contained in Table 1 is structured in few major steps.

0. Premise and outline. Consider the Volterra equation (3.13), and note that the functions $v_t(t)$ and $v_{tt}(t)$ solve the same Volterra integral equation of the second kind in a Hilbert space, yet with different affine terms, $H_1(\cdot)$ and $H_2(\cdot)$ say, respectively, which will be computed in the next step. In view of Lemma 4.1, the (time and space) regularity of these affine terms—depending on u_0, u_1, ξ and g—will naturally bring about the (time and space) regularity for the triple (v,v_t,v_{tt}).

u_0	u_1	ξ	g	$u = u(t,x)$ *solution of* (3.3)
X_λ	0	0	0	$C([0,T];X_\lambda) \cap C^1([0,T];X_{\lambda-1}) \cap C^2([0,T];X_{\lambda-2})$
0	X_μ	0	0	$C([0,T];X_{\mu+1}) \cap C^1([0,T];X_\mu) \cap C^2([0,T];X_{\mu-1})$
0	0	X_ν	0	$\begin{cases} \textit{if } F(t) \in L^2(0,T) \textit{ then:} \\ C([0,T];X_{\nu+1}) \cap C^1([0,T];X_\nu) \cap H^2([0,T];X_{\nu-1})\,; \\ \textit{if } F(t) \in H^1(0,T) \textit{ then:} \\ C([0,T];X_{\nu+2}) \cap C^1([0,T];X_{\nu+1}) \cap H^2([0,T];X_\nu) \end{cases}$
0	0	0	$L^2(\Sigma)$	$C([0,T];X_{\alpha_0}) \cap C^1([0,T];X_{\alpha_0-1}) \cap H^2([0,T];X_{\alpha_0-2})$

TABLE 1. Regularity of solutions of the equation (1.5) (i.e. (3.3)). The spaces X_λ, X_μ and X_ν (with real indices) are defined in (1.9). The transformations are continuous between the indicated spaces.

To do so we will set to zero all data but one. Finally, the derived regularity properties will be inherited by the triple (u, u_t, u_{tt}) pertaining to the original equation with persistent memory (3.3), still depending on u_0, u_1, ξ and g.

1. The affine terms of Volterra equations. We rewrite (3.13) in the form

$$v(t) + \int_0^t L(s)v(t-s)ds = H(t)$$

and compute the derivatives of both the sides. Inserting the expressions (3.14) and (3.15) of $L(t)$ and $H(t)$, and replacing initial data v_0 and v_1 with their respective expressions in terms of u_0 and u_1 (see (3.12)), we obtain the following Volterra integral equations in the unknowns $v_t(t)$ and $v_{tt}(t)$:

$$v_t(t) + \int_0^t L(t-s)v_s(s)\,ds = H_1(t)\,, \tag{4.1}$$

$$v_{tt}(t) + \int_0^t L(t-s)v_{ss}(s)\,ds = H_2(t) \tag{4.2}$$

where

$$H_1(t) := -L(t)v_0 + H_t(t) =$$
$$= \Big[\frac{\beta}{\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)u_0 + \frac{1}{\sqrt{b}}\int_0^t R_-(\sqrt{b}(t-s))\mathcal{A}^{-1}K(s)u_0\,ds\Big] + H_t(t)\,, \quad (4.3)$$

$$H_2(t) := \Big[\frac{\beta}{\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)u_1 + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))K(s)u_1\,ds\Big] -$$
$$- \frac{R_0(0)}{2\sqrt{b}}\Big[\beta\mathcal{A}^{-1}R_-(\sqrt{b}t)u_0 + \mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))K(s)u_0\,ds\Big] +$$
$$+ \beta R_+(\sqrt{b}t)u_0 + \int_0^t R_+(\sqrt{b}(t-s))K(s)u_0\,ds + H_{tt}(t)\,,$$

while the explicit expression (3.15) of $H(t)$ is recorded here for the reader's convenience:

$$H(t) = \Big[R_+(\sqrt{b}t) - \frac{R_0(0)}{2\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)\Big]u_0 + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)u_1 -$$
$$- \sqrt{b}\mathcal{A}\int_0^t R_-(\sqrt{b}(t-s))G\,e^{-\frac{1}{2}R_0(0)s}g(s)\,ds +$$
$$+ \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))\big[h_2(s)\xi + h_1(s)u_1 + h_0(s)u_0\big]\,ds\,.$$

As it will appear clear immediately below, we neglected to write explicitly the derivatives of $H(t)$, just because their regularity is easily deduced invoking once more Theorem 2.3.

2a. Effects of boundary data action. With u_0, u_1, $\xi \equiv 0$, $g \in L^2(\Sigma)$, the affine term $H(t)$ in (3.13) (recorded above) reduces to

$$H(t) = -\sqrt{b}\mathcal{A}\int_0^t R_-(\sqrt{b}(t-s))Gg(s)\,ds\,. \quad (4.4)$$

Therefore we know from assertion iii) of Theorem 2.3 that

$$(H, H_t, H_{tt}) \in C([0,T]; X_{\alpha_0} \times X_{\alpha_0-1} \times X_{\alpha_0-2})\,.$$

Thus, Lemma 4.1 shows that the solutions of the Volterra equation (3.13) as well as those pertaining to the former equation with memory (3.3) belong to

$$C([0,T]; X_{\alpha_0}) \cap C^1([0,T]; X_{\alpha_0-1}) \cap C^2([0,T]; X_{\alpha_0-2})\,.$$

2b. Effects of the initial datum u_0. Assume u_1, $\xi = 0$, $g = 0$, and $u_0 \in X_\lambda$. The affine term of the equation (3.13) in the unknown v becomes

$$H(t) = \Big[R_+(\sqrt{b}t) - \frac{R_0(0)}{2\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)\Big]u_0 + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_0(s)u_0\,ds\,,$$

so that readily

$$H \in C([0,T]; X_\lambda) \cap C^1([0,T]; X_{\lambda-1}) \cap C^2([0,T]; X_{\lambda-2})\,,$$

which immediately implies $v(t) \in C([0,T]; X_\lambda)$. Recall now the term H_1 in (4.3) and notice that its regularity is determined by the regularity of H_t. Then, H_1—as

well as v_t, in view of Lemma 4.1—belongs to $C^1([0,T];X_{\lambda-1})$, while H_2 and then $v_{tt}(t)$ belong to $C^1([0,T];X_{\lambda-2})$, which establishes the first row of Table 1.

2c. Effect of the initial datum u_1. Assume $u_0,\ \xi=0$, and $g=0$ while $u_1\in X_\mu$. In this case

$$H(t)=\frac{1}{\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)u_1+\frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_1(s)u_1\,ds\,,$$

so that we have a slight regularization

$$(H,H_t,H_{tt})\in C([0,T];X_{\mu+1}\times X_\mu\times X_{\mu-1})\,.$$

The transformation $u_1\longmapsto H$ is continuous in the indicated spaces (*cf.* assertion *i)* of Theorem 2.3).

The obtained regularity for H and its derivatives holds for H_i, $i=1,2$, and then is inherited by the solution $v(t)$: namely,

$$v\in C([0,T];X_{\mu+1})\cap C^1([0,T];X_\mu)\cap C^2([0,T];X_{\mu-1})\,;$$

in turn, the same is valid for u, thereby confirming the second row of Table 1.

2d. Effect of the parameter ξ. We finally discuss the dependence on ξ. Assume $u_0,u_1=0$, and $g=0$ and $\xi\in X_\nu$. In this case

$$H(t)=\frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)\xi\,ds$$

and, from (3.10), $h_2(t)\in L^2(0,T)$, just like $F(t)$.

We invoke once more item *ii)* of Theorem 2.3, and ascertain again a slightly regularizing property: the transformation $\xi\longmapsto v$ is continuous from X_ν to $C([0,T];X_{\nu+1})\cap C^1([0,T];X_\nu)\cap H^2([0,T];X_{\nu-1})$ (while if in addition $F(t)$—and consequently, $h_2(t)$—is continuous, then $v\in C^2([0,T];X_{\nu-1})$).

In the case $F\in H^2(0,T)$ (as the case of the MGT equation) we have a stronger regularization, since we can integrate by parts as follows:

$$\begin{aligned}H(t)&=-\frac{1}{b}\,A^{-1}\int_0^t\frac{d}{ds}R_+(\sqrt{b}(t-s))h_2(s)\xi\,ds=\\&=-\frac{1}{b}\,A^{-1}\Big[\big(h_2(t)-R_+(\sqrt{b}t)h_2(0)\big)\xi-\int_0^t R_+(\sqrt{b}(t-s))h_2'(s)\xi\,ds\Big]\,;\end{aligned}\tag{4.5}$$

a rigorous justification is found, e.g., in [31, Lemma 5].

For a better understanding, we compute explicitly for $t\in(0,T)$:

$$\begin{aligned}H_t(t)&=-\frac{1}{b}\,A^{-1}\Big[\cancel{h_2'(t)\xi}-\sqrt{b}\mathcal{A}R_-(\sqrt{b}t)\xi-\cancel{h_2'(t)\xi}-\\&\qquad-\sqrt{b}\mathcal{A}\int_0^t R_-(\sqrt{b}(t-s))h_2'(s)\xi\,ds\Big]=\\&=\frac{1}{b}\mathcal{A}^{-1}R_-(\sqrt{b}t)\xi+\frac{1}{b}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2'(s)\xi\,ds\in X_{\nu+1}\,,\end{aligned}$$

and

$$H_{tt}(t)=R_+(\sqrt{b}t)\xi+\int_0^t R_+(\sqrt{b}(t-s))h_2'(s)\xi\,ds\in X_\nu\,,$$

that complete the membership $H\in X_{\nu+2}$.

It is important to note that the space regularity increases of one unity and we get the result in the third row of Table 1, where $H^2(0,T;X_\nu)$ is replaced by $C^2([0,T];X_\nu)$ if furthermore $F \in C^2([0,T])$, that is the case of the MGT equation. □

Remark 4.3. The noticeable outcome of the obtained regularity result is that u_1 and ξ are regularized by one and, respectively, two unities. Hence, when $g = 0$, $u_0 = 0$ while u_1 and ξ belong to X_r, then $(u(t), u_t(t), u_{tt}(t))$ evolves in $X_{r+1} \times X_r \times X_{r-1}$.

From Table 1 of Theorem 4.2 we deduce, in particular, the following regularity result.

Corollary 4.4. *Consider equation* (3.3) *with initial data* (u_0, u_1)*, and trivial boundary data, namely,* $g \equiv 0$ *in* (1.4)*. If* $F(\cdot) \in C^1$ *and if* $(u_0, u_1, \xi) \in X_r \times X_{r-1} \times X_{r-2}$ *(for every* r *the spaces* X_r *are defined in (1.9)), then the corresponding weak solution satisfies*

$$(u, u_t, u_{tt}) \in C([0,T]; X_r \times X_{r-1} \times X_{r-2})\,.$$

5. Interior regularity for the MGT equation

In this Section we utilize the analysis performed for the general class of equations with memory (3.3), in order to derive a result pertaining to the MGT equation, that is Theorem 5.3 below. This theorem establishes, in particular, the statements of Theorem 1.2 detailing the regularity from the boundary to the interior for the MGT equation (i.e. item *iii)*), as well as the one pertaining to the interior regularity, under homogeneous boundary data (i.e. item *ii)*). The latter result is consistent with the analyis formerly carried out in [15], that brought about semigroup well-posedness of the MGT equation in the space $\mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega)$, A being the proper realization of the Laplacian in $L^2(\Omega)$; see [15, Theorem 1.2]. The peculiar regularity of the MGT equation is here (re)confirmed in a wealth of functional settings.

Recall that for the special case of the MGT equation we have in particular

$$N(t) = F(t) = e^{-\alpha t}\,, \qquad \xi = u_2 - b\Delta u_0$$

in (3.3). The meaning given to solutions is still the one in Definition 3.4.

We restart from the integral equation which defines the resolvent associated with the convolution kernel $-\gamma N(t)$ of (3.3), that is equation (3.6) and that in the present case—with $N(t) = e^{-\alpha t}$—reads as

$$R_0(t) - \gamma \int_0^t e^{-(t-s)}\, R_0(s)\, ds = -\gamma e^{-\alpha t}\,. \tag{5.1}$$

It is then easily verified that the solution to (5.1) is given by

$$R_0(t) = -\gamma e^{(\gamma-\alpha)t} = -\gamma e^{-\frac{c^2}{b}t}\,, \tag{5.2}$$

which gives $R_0(0) = -\gamma$ and hence

$$v(t) = e^{\frac{\gamma}{2}t} u(t)$$

for v defined in (3.8).

In view of Definition 3.4, and taking into account the actual expression of $R_0(t)$ in (5.2), the following instance of Definition 3.4 comes into the picture.

Definition 5.1 (Instance of Definition 3.4)**.** *A function $u = u(t,x)$ is a solution of the IBVP* (1.2)-(1.3)-(1.4) *for the Moore-Gibson-Thompson equation if and only if the function $v(t,x) = e^{\frac{\gamma}{2}t}u(t,x)$ solves*

$$v_{tt} = b(\Delta v - v) + \int_0^t K(t-s)v(s)\,ds + \beta v(t) + h_2(t)\big(u_2 - b\Delta u_0\big) + h_1(t)u_1 + h_0(t)u_0\,,$$

where

$$K(t) = -\gamma(\gamma-\alpha)^2 e^{(\frac{3}{2}\gamma-\alpha)t}\,, \qquad \beta = b - \gamma\Big(\frac{3}{4}\gamma - \alpha\Big)\,,$$
$$h_0(t) = -\gamma(\gamma-\alpha)e^{(\frac{3}{2}\gamma-\alpha)t}\,, \quad h_1(t) = -\gamma e^{(\frac{3}{2}\gamma-\alpha)t}\,, \quad h_2(t) = e^{(\frac{3}{2}\gamma-\alpha)t}\,.$$

Remark 5.2. The definition of $\xi = u_2 - \Delta u_0$ makes sense in $L^2(\Omega)$ when $u_0 \in H^2(\Omega)$. More broadly, we will also consider $u_0 \in X_s$ for any s, in which case Δ will mean $A - I$, where A is the extension of the operator in (1.8) described in Remark 1.1.

We note that C^∞-functions with compact support are not dense in $H^2(\Omega) \cap H_0^1(\Omega)$, which is the domain of the operator A (defined in (1.8)) in the case of Dirichlet BC. Hence, its dual is not a space of distributions. Consequently, for instance when u is smooth (say H^2) and Dirichlet BC are in place, the trace of u on the boundary $\partial\Omega$—that we denote $\gamma_0 u$—enters Au (A is the extended operator), and then

$$\xi = u_2 - b\Delta u_0 = u_2 - b\Delta\big(u_0 - G\left(\gamma_0 u\right)\big) = u_2 - b(A+I)u_0 + b(A+I)G\left(\gamma_0 u\right)\,.$$

When particularizing to the MGT equation the results in Theorem 4.2, we find the following result.

Theorem 5.3 (Regularity for the MGT equation)**.** *Let $u = u(t,x)$ be the solution to the MGT equation* (1.2)*, defined as the solution to the Volterra integral equation* (3.13)*, with L and H as in* (3.14)-(3.15)*, where $F(t) = N(t) = e^{-\alpha t}$. The maps $(u_0,u_1,u_2) \longmapsto u$ (when $g = 0$) and $g \longmapsto u$ (when $u_0 = u_1 = u_2 = 0$, so that $\xi = u_2 - \Delta u_0 = 0$) are linear and continuous in the spaces described by the Table 2.*

Proof. Along the lines of the first steps of the proof of Theorem 4.2, we return to the Volterra equation (3.13) and once again appeal to Lemma 4.1; the affine term $H(t)$ in (3.15) must be rewritten taking into account that in the present case we have $\xi = u_2 - b\Delta u_0$.

1. The case of trivial initial data, namely $u_0 = u_1 = u_2 = 0$, is easily discussed: indeed, in this case $H(t)$ simply reduces to

$$H(t) = -\sqrt{b}\mathcal{A}\int_0^t R_-(\sqrt{b}(t-s))G\,e^{\frac{\gamma}{2}s}g(s)\,ds\,,$$

which is nothing but the solution to a linear wave equation with trivial initial data, trivial affine term, and $\tilde{g}(s) = e^{\frac{\gamma}{2}s}g(s)$ as a boundary datum. Consequently, because $\tilde{g} \in L^2(\Sigma)$, the results in Theorem 1.3 and Remark 2.2(i) apply, bringing about the last row of Table 2.

u_0	u_1	u_2	g	$u = u(t,x)$ *solution of the MGT equation*
X_λ	0	0	0	$C^1([0,T];X_\lambda) \cap C^2([0,T];X_{\lambda-1})$
0	X_μ	0	0	$C([0,T];X_{\mu+1}) \cap C^1([0,T];X_\mu) \cap C^2([0,T];X_{\mu-1})$
0	0	X_ν	0	$C([0,T];X_{\nu+2}) \cap C^1([0,T];X_{\nu+1}) \cap C^2([0,T];X_\nu)$
0	0	0	$L^2(\Sigma)$	$C([0,T];X_{\alpha_0}) \cap C^1([0,T];X_{\alpha_0-1}) \cap H^2([0,T];X_{\alpha_0-2})$

TABLE 2. Regularity of solutions to the MGT equation (1.2). The spaces X_λ, X_μ and X_ν (with real indices) are defined in (1.9). The transformations are continuous between the indicated spaces.

2. Assume next that the boundary datum instead is trivial, i.e. $g \equiv 0$. Return to $H(t)$ in (3.15), and focus on its last summand, that is

$$\frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))\big[h_2(s)\big(u_2 - b\Delta u_0\big) + h_1(s)u_1 + h_0(s)u_0\big]\,ds\,,$$

and more specifically on the most tricky term

$$T(t) = \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)[u_2 - b\Delta u_0]\,ds\,.$$

We rewrite

$$\begin{aligned} T(t) = &\underbrace{\frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)u_2\,ds}_{T_1(t)} - \\ &\underbrace{-\sqrt{b}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)\Delta u_0\,ds}_{T_2(t)} \end{aligned} \tag{5.3}$$

and compute

$$\begin{aligned} T_2(t) &= -\sqrt{b}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)\big(\Delta u_0 - u_0 + u_0\big)\,ds = \\ &= \underbrace{-\sqrt{b}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)Au_0\,ds}_{T_{21}(t)} - \\ &\qquad \underbrace{-\sqrt{b}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)u_0\,ds}_{T_{22}(t)}\,. \end{aligned} \tag{5.4}$$

Assuming $u_0 \in X_\lambda$, then $Au_0 \in X_{\lambda-2}$; moreover, recall that $A = \mathcal{A}^2$, and the relation between the operators $R_-(\cdot)$ and $R_+(\cdot)$. Then proceed with the computations, integrating by parts to get

$$\begin{aligned} T_{21}(t) &= -\sqrt{b}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)Au_0\,ds \\ &= -\sqrt{b}\mathcal{A}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)u_0\,ds = \\ &= \int_0^t \frac{d}{ds}\big\{R_+(\sqrt{b}(t-s))h_2(s)u_0\big\}\,ds - \int_0^t R_+(\sqrt{b}(t-s))h_2'(s)u_0\,ds = \\ &= h_2(t)u_0 - R_+(\sqrt{b}(t))h_2(0)u_0 - \int_0^t R_+(\sqrt{b}(t-s))h_2'(s)u_0\,ds\,. \end{aligned} \tag{5.5}$$

Combine (5.5) with (5.4) and (5.3), insert the resulting expression of $T(t)$ in $H(t)$, to obtain

$$\begin{aligned} H(t) = {} & \cancel{R_+(\sqrt{b}t)u_0} - \frac{R_0(0)}{2\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)u_0 + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)u_1 - \\ & - \sqrt{b}\mathcal{A}\int_0^t R_-(\sqrt{b}(t-s))G\,e^{-\frac{1}{2}R_0(0)s}g(s)\,ds + \\ & + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)u_2\,ds + \\ & + h_2(t)u_0 \cancel{- R_+(\sqrt{b}t)u_0} - \int_0^t R_+(\sqrt{b}(t-s))h_2'(s)u_0\,ds - \\ & - \sqrt{b}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)u_0\,ds + \\ & + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))\big[h_1(s)u_1 + h_0(s)u_0\big]\,ds\,, \end{aligned}$$

where the term $R_+(\sqrt{b}t)u_0$ appears twice with opposite signs, and hence cancel.

Rearranging the summands and replacing $t-s$ with s in the integrals we attain

$$\begin{aligned} H(t) = \Big(h_2(t) - \frac{R_0(0)}{2\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)\Big)u_0 - \int_0^t R_+(\sqrt{b}s)h_2'(t-s)u_0\,ds + \\ + \mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}s)\Big(\frac{1}{\sqrt{b}}h_0(t-s) - \sqrt{b}h_2(t-s)\Big)u_0\,ds + \\ + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}R_-(\sqrt{b}t)u_1 + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}s)h_1(t-s)u_1\,ds - \\ + \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}s)h_2(t-s)u_2\,ds\,, \end{aligned} \tag{5.6}$$

which allows the understanding of the regularity of $H(t)$, along with the sought regularity properties of solutions to the MGT equation.

Notice first that in comparison with the general model equation with memory (1.5) the space regularity of $H(t)$ is not improved, owing to the presence of the term $h_2(t)u_0$. Instead, the regularity of $H_t(t)$ is improved thanks to the cancellation of the term $R_+(\sqrt{b}t)u_0$: in fact, if $g\equiv 0$, $u_1 = u_2 = 0$, $u_0 \in X_\lambda$, then $H_t \in C([0,T];X_\lambda)$, while $H_{tt} \in C([0,T];X_{\lambda-1})$. However, the said cancellation (of a term depending only on u_0) has no effect on the remaining terms: the dependence on u_1 and u_2 is subject to the smoothing effect already described in Table 1 (in terms of u_1 and ξ). Thus, the results displayed in Table 2 follow. (The cancellation has also another significant effect: the summand $h_2(t)u_0$ decays in time, but does not propagate in space, as explained in the second item of Remarks 5.4.) Observe that in the term

$$\frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)u_2\,ds$$

one may integrate by parts, thereby obtaining

$$\begin{aligned} \frac{1}{\sqrt{b}}\mathcal{A}^{-1}\int_0^t R_-(\sqrt{b}(t-s))h_2(s)u_2\,ds = -\frac{1}{b}\mathcal{A}^{-2}\Big\{h_2(t)u_2 - R_+(\sqrt{b}(t)h_2(0)u_2 \\ + \int_0^t R_+(\sqrt{b}(t-s))h_2'(s)u_2\,ds\Big\} \end{aligned}$$

that confirms the said smoothing effect.

Using once more that the functions $h_i(t)$, $i=0,1,2$ are twice differentiable, it is easily seen that when $u_0 \in X_\lambda$, $u_1 = u_2 = 0$, $g = 0$, then

$$H(t) \in C^1([0,T];X_\lambda) \cap C^2([0,T];X_{\lambda-1})\,; \tag{5.7}$$

the regularity of v and the one of u follow accordingly. In conclusion, the representation (5.6) of $H(t)$ shows that all the regularity results summarized in the rows of Table 1 remain valid, with the exception of those in the first row, that are improved consistently with (5.7). □

Remarks 5.4. (i) The first line of Table 2 asserts that both the solution u to the MGT equation *and its derivative* u_t belong to $C([0,T];X_\lambda)$, whereas a unity of regularity (in space) is lost in the second derivative: $u_{tt} \in C([0,T];X_{\lambda-1})$. This feature (to be contrasted with the general property in the first line of Table 1) is peculiar of the MGT equation, as seen in the proof of Theorem 5.3.

(ii) We further note that $R_-(\sqrt{b}t)u_0$ and $R_+(\sqrt{b}t)u_0$ solve the wave equation, and so the 'shape' of u_0 is propagated in space, as in the wave equation. Instead, the term $h_2(t)u_0$ (which decays exponentially in time) is a stationary wave and does not propagate in the space variables.
Thanks to the formulas for the solutions of the Volterra integral equations, this stationary wave appears also in the solution of the MGT equation.

6. Boundary regularity

In this Section we establish a sharp regularity result for the normal trace on $\Gamma = \partial\Omega$ of solutions to the MGT equation (1.2), supplemented with (homogeneous) Dirichlet boundary condition. This result, presented as Corollary 6.3, follows from a boundary regularity result pertaining to the family of wave equations with memory (1.5), depending on $\xi \in L^2(\Omega)$, that is Theorem 6.2 below. In doing so we re-obtain, in the case $\xi = 0$, a result established under distinct assumptions (on the memory kernel) in previous papers; see the discussion at the end of the section.

We point out that the present approach to the analysis of wave equations with memory enables us to establish the boundary (beside the interior) regularity of solutions in a direct and straightforward way: the existing results on the trace regularity of the wave equation are lifted to the equation with memory by simply using the properties of linear operators. Thus, our method of proof paves the way for the derivation of appropriate boundary regularity results for the model equation with memory (1.5), as well as for the MGT equation (1.2), when supplemented with Neumann boundary conditions (a case which, for the wave equation, is drastically more difficult).

Let the operator $\mathcal{T}$ be the Dirichlet trace on $\Gamma = \partial\Omega$, and let G be the Green map defined by (2.1) accordingly, i.e. with $\mathcal{T} = \gamma_0$. Then, an elementary computation which utilizes the (second) Green Theorem yields, for $\phi \in D(A)$, the following trace result:

$$G^* A\phi = -\frac{\partial \phi}{\partial \nu}\Big|_\Gamma \qquad \forall \phi \in D(A)\,; \tag{6.1}$$

see, e.g., [24, Vol. I, p. 181].

The following result is by now well known (*cf.* [20], [25], [21], [26]).

Theorem 6.1. *Let $u = u(t,x)$ be a solution to the initial/boundary value problem (1.12) for the wave equation with homogeneous Dirichlet boundary data (i.e. $g = 0$). Then, for every $T > 0$ there exists $M = M_T$ such that*

$$\int_0^T \int_{\partial\Omega} \Big|\frac{\partial}{\partial \nu} u(x,t)\Big|^2 d\sigma\, dt \le M \left(\|u_0\|_{H_0^1(\Omega)} + \|u_1|^2_{L^2(\Omega)} + \|f\|^2_{L^1(0,T;L^2(\Omega))} \right).$$

We now show that this property is inherited by the solutions to the equation with memory (depending on ξ) (1.5), and next by the solutions to the MGT equation (1.2), provided a suitable compatibility condition for initial data is satisfied; see (6.6) below. The first statement is as follows.

Theorem 6.2. *Under the standing Hypotheses 3.2 and assuming $\xi \in L^2(\Omega)$, let $u = u(t,x)$ be a solution to the equation with memory (3.3), with initial data (u_0, u_1) and homogeneous Dirichlet boundary data. Then, for every $T > 0$ there exists*

$M = M_T$ *such that*

$$\int_0^T \int_{\partial\Omega} \left| \frac{\partial}{\partial \nu} u(x,t) \right|^2 d\sigma \, dt \leq M \left(\|u_0\|_{H_0^1(\Omega)} + \|u_1|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)} \right). \tag{6.2}$$

Proof. Since the equation is supplemented with Dirichlet boundary conditions, we may take $A = \Delta$, with domain $H^2(\Omega) \cap H_0^1(\Omega)$. The estimate (6.2) is obtained as a simple consequence of the boundary regularity of solutions to the equation with memory in (3.11), that reads as

$$v_{tt} = \Delta v + k_0 v + \int_0^t K(t-s) v(s) ds + \mathcal{F}(t)\,, \tag{6.3}$$

where we have set $b = 1$ for the sake of simplicity, $k_0 = \beta - 1$, while $\mathcal{F}(t)$ is now

$$\mathcal{F}(t) := \left(h_2(t)\xi + h_1(t) u_1 + h_0(t) u_0 \right),$$

with the scalar functions $h_i(\cdot)$, $i = 0, 1, 2$ (as well as β) introduced in (3.10).
(Note that $h_2 \in L^2(0,T)$, along with $\|h_2\|_{L^2(0,T)} \leq \sqrt{T} \|h_2\|_{L^1(0,T)}$.)

It then follows that

$$\begin{aligned} v(t) = \underbrace{R_+(t) v_0 + \mathcal{A}^{-1} R_-(t) v_1}_{=:w(t)} + \\ + \mathcal{A}^{-1} \int_0^t R_-(t-s) \Big[k_0 v(s) + \int_0^s K(s-r) v(r) dr \Big] ds + \\ + \underbrace{\mathcal{A}^{-1} \int_0^t R_-(t-s) \mathcal{F}(s)\, ds}_{=:T(t)}\,. \end{aligned}$$

We note first of all that the estimate (6.2) holds true for the function $w(t)$, since it represents the solution of a classical wave equation (without the memory term). Next, we observe that the integral term $T(t)$ depends on u_0, u_1 and ξ, as the function $\mathcal{F}(t)$ does. We examine the dependence on ξ (the dependence on u_0 and u_1 is studied analogously).

Recall the following version of Young inequality: if $h \in L^1(0,T;\mathbb{R})$ and $X \in L^2(0,T;\mathbb{R})$, then the convolution $X * h$ satisfies

$$\|X * h\|_{L^2(0,T;\mathbb{R})} \leq \|X\|_{L^2(0,T;\mathbb{R})} \, \|h\|_{L^1(0,T;\mathbb{R})}\,.$$

The desired regularity estimate is established first when $\xi \in D(A)$, and then extended to $\xi \in L^2(\Omega)$ by continuity.
If $\xi \in D(A)$, then for a.e. $t \in [0,T]$ we have:

$$\left| \frac{\partial}{\partial \nu} \mathcal{A}^{-1} \int_0^t R_-(t-s) h_2(s) \xi \, ds \right|_{L^2(\Gamma)} = \left| D^* A \left[\int_0^t \mathcal{A}^{-1} R_-(t-s) h_2(s) \xi \, ds \right] \right|_{L^2(\Gamma)} =$$

$$= \left| \left[\int_0^t D^* A \left(\mathcal{A}^{-1} R_-(t-s) \xi \right) h_2(s)\, ds \right] \right|_{L^2(\Gamma)} \leq \int_0^t |h_2(t-s)| \, \|X(s)\|_{L^2(\Gamma)}\, ds\,,$$

where we set

$$X(t) := D^* A \left[\mathcal{A}^{-1} R_-(t) \xi \right] = \frac{\partial}{\partial \nu} \left[\mathcal{A}^{-1} R_-(t) \xi \right]\,.$$

The inequality pertaining to the wave equation establishes

$$\left\| D^*A\left[\mathcal{A}^{-1}R_-(t)\xi\right]\right\|_{L^2(0,T;L^2(\Gamma))} = \left\|\frac{\partial}{\partial\nu}\left[\mathcal{A}^{-1}R_-(t)\xi\right]\right\|_{L^2(0,T;L^2(\Gamma))} \le M\|\xi\|_{L^2(\Omega)},$$

which is extended by continuity to every $\xi \in L^2(\Omega)$. Young inequality then gives

$$\left\| \int_0^{\cdot} h_2(\cdot - s)X(s)\,ds\right\|_{L^2(0,T;L^2(\Gamma))} \le M\|\xi\|_{L^2(\Omega)}.$$

Analogous estimates hold true for the remaining summands (which depend on u_0 and u_1) in $T(t)$.

Therefore, the normal trace of v reads as

$$\begin{aligned}\frac{\partial}{\partial\nu}v(t)\Big|_\Gamma = -G^*Av(t) = -G^*A\Big[w(t) + \mathcal{A}^{-1}\int_0^t R_-(t-s)\mathcal{F}(s)\,ds\Big]\,ds - \\ - G^*A\Big[\mathcal{A}^{-1}\int_0^t R_-(t-s)\Big(k_0v(s) + \int_0^s K(s-r)v(r)dr\Big)\,ds\Big],\end{aligned}$$

and we showed so far that there exists a constant $M = M_T$ (possibly depending on T) such that

$$\begin{aligned}\left\|-G^*A\Big[w(t) + \mathcal{A}^{-1}\int_0^t R_-(t-s)\mathcal{F}(s)\,ds\Big]\,ds\right\|^2_{L^2(0,T;L^2(\Gamma))} \le \\ \le M\left(\|u_0\|^2_{H_0^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}\right). \qquad (6.4)\end{aligned}$$

We claim that a similar inequality is valid for the second summand. Indeed, we know (*cf.* the second statement of Theorem 1.2) that $v \in C([0,T];H_0^1(\Omega))$, with continuous dependence on initial data. Therefore, the second summand satisfies

$$G^*A\Big[\mathcal{A}^{-1}\int_0^t R_-(t-s)\Big(k_0v(s) + \int_0^s K(s-r)v(r)dr\Big)\,ds\Big] \in C(0,T;L^2(\Gamma)),$$

which combined with (6.4) implies that there exists $M = M_T$ such that

$$\int_0^T\int_{\partial\Omega}\Big|\frac{\partial}{\partial\nu}v(x,t)\Big|^2 d\sigma\,dt \le M\left(\|u_0\|_{H_0^1(\Omega)} + \|u_1|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}\right). \qquad (6.5)$$

This in turn establishes the sought estimate (6.2), thereby concluding the proof. □

For the MGT equation, one obtains readily the following result.

Corollary 6.3. *Let $u = u(t,x)$ be a solution to the Moore-Gibson-Thompson equation (1.2), with initial data (u_0,u_1,u_2) and homogeneous boundary data. Assume $(u_0,u_1,u_2) \in H_0^1(\Omega)\times L^2(\Omega)\times H^{-1}(\Omega)$, along with the compatibility condition*

$$u_2 - \Delta u_0 \in L^2(\Omega). \qquad (6.6)$$

Then, for every $T>0$ there exists $M = M_T$ such that

$$\int_0^T\int_{\partial\Omega}\Big|\frac{\partial}{\partial\nu}u(x,t)\Big|^2 d\sigma\,dt \le M\left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)} + \|u_2 - \Delta u_0\|^2_{L^2(\Omega)}\right).$$

Remarks 6.4. The issue of extending the trace regularity result in Theorem 6.1 to wave equations with memory has been the object of investigation since at least the beginning of the nineties. Distinct perspectives and techniques have been adopted, along with distinct assumptions on the memory kernel.

We note that the starting point for the proof of Theorem 6.2 is the (equivalent) integro-differential equation (6.3) in the unknown v, where—in contrast with (3.3)—the convolution term does not display the differential operator. The regularity estimate (6.5), which in the proof of Theorem 6.2 establishes the estimate (6.2), seems to appear in the literature for the first time in the papers by Yan [42] and Kim [17], focused on the same model equation, in the absence of the affine term $\mathcal{F}(t)$. Both papers [42] and [17] deal with the non-convolution case, namely, with more general kernels $K = K(t, s)$ which are assumed to satisfy suitable regularity assumptions. The sought inequality for the normal trace of the solutions to the equation with memory is obtained as a consequence of the known inequality for solutions to the wave equation. See also [31], where the proof is based on the Fubini Theorem.

The recent proof given by Loreti and Sforza [27] for the equation with memory (3.3) with $\xi = 0$, under Dirichlet BC, is carried out via energy methods, by using an *ad hoc* multiplier. The convolution kernel $k(\cdot)$ (that is $\gamma N(\cdot)$ in the notation of the present paper) is subject to various constraints, which are satisfied in the case of relevant kernels of exponential type.

An abstract approach to *admissibility*—a commonly used term, in the context of systems and control theory, beside to a *direct* inequality—via harmonic analysis is found in the papers by Yung [12] and Jacob and Partington [9]. In these papers the space of observations is finite dimensional.

The proof given in the present section is consistent with the chosen perspective of systems with memory. See also the proofs of corresponding inequalities in [33], [34], [35], where waves, plates and three-dimensional viscoelasticity are studied. A somewhat similar approach is found in Cavalcanti *et al.* [3], under stronger assumptions on the memory kernel (in particular, smallness of the kernel as well as of its derivatives).

Appendix A. Justification of Definition 3.4

Let us recall that in order to give a Definition of solutions to the MGT Equation (1.2) we proceeded as follows: formal calculations were used to reduce equation (1.2) to the integro-differential equation (3.3) and then to the Volterra integral equation (3.13) in the unknown v. By definition, u solves (1.2) when $v(t) = e^{-(R(0)/2)t}u(t)$ solves the Volterra integral equation (3.13) (with g replaced by $e^{-(R(0)/2)t}g(t)$). In this Appendix we provide a formal justification of the said Definition.

The argument is similar to the one used in Section 2 in the case of wave equations: we prove that the solution u is smooth and can be replaced in both the sides of (1.2) when the initial data and the control are "smooth" and then we use continuous dependence as stated in Table 2 to justify the definition in general. This procedure is a bit more elaborated than the one pertaining to the wave equation, since the third derivative (in time) comes into the picture, which requires more information on the solutions of the wave equation.

In order to distinguish the memoryless wave equation from the equation with memory and the MGT equation, we will denote by u_3 the solution to equation (1.12) (this is because we use suitable results from [32, § 2.2], where u_3 solves the wave equation when the initial data and the affine term are zero). We assume $u_3(0) \in \mathcal{D}(\Omega)$, $\frac{\partial}{\partial t} u_3\big|_{t=0} \in \mathcal{D}(\Omega)$, $g \in \mathcal{D}((0,T) \times \partial\Omega)$, where $\mathcal{D}$ denotes the space of C^∞ functions with compact support in the indicated open set (which should not be confused with the domain of an operator), while $\partial\Omega$ is relatively open respect to itself. The assumptions on the affine term $F(t)$ are made explicit below. For the sake of simplicity, in the sequel the time derivative will be denoted by $'$.

It is known that u_3 is given by formula (2.2): it is also clear that if $g, f \equiv 0$, then in view of the Sobolev embedding theorems one has $u_3 \in C^\infty((0,T) \times \Omega)$, for every $T > 0$. Our aim is to show that a similar property holds true when $g \neq 0$, $f \neq 0$.

Let us study separately the effects of g and f: accordingly, we assume first $f = 0$, so that

$$u_3(t) = -\mathcal{A} \int_0^t R_-(s) G g(t-s)\, ds = Gg(t) + \int_0^t R_+(s) G g'(t-s)\, ds\,.$$

As already noted we have $u_3(t) - Gg(t) \in \mathcal{D}(A)$ and the boundary condition is satisfied; moreover,

$$A\left(u_3(t) - Gg(t)\right) = -Gg''(t) + \int_0^t R_+(s) G g'''(t-s)\, ds \in C^\infty\left([0,T]; L^2(\Omega)\right)\,.$$

Observe that, by definition,

$$A\left(u_3(t) - Gg(t)\right) = (\Delta - I) u_3(t) \in C^\infty\left([0,T]; L^2(\Omega)\right)$$

that is $u_3(t) \in C^\infty\left([0,T]; H^2(\Omega)\right)$ with suitable homogeneous boundary condition. Analogously,

$$\mathcal{A}\Big\{A\big[u_3(t) - Gg(t)\big] + Gg''(t)\Big\} = \int_0^t R_-(s) G g^{(4)}(t-s)\, ds$$

which again is of class $C^\infty([0,T]; L^2(\Omega))$. So we have

$$\mathcal{A}\Big\{A\big[u_3(t) - Gg(t)\big] + Gg''(t)\Big\} \in C^\infty([0,T]; X_1)\,,$$

that is $u_3 \in C^\infty\big([0,T]; H^3(\Omega)\big)$.

By iteration we see that in the interior of $(0,T) \times \Omega$ the solution u_3 is of class C^∞ and hence, when computing the derivatives, the order can be interchanged.

Let us consider now the effect of the affine term $f(t)$. We assume $f \in C^\infty\big([0,T] \times \Omega\big)$ and that for every fixed $t \geq 0$ $f(t,\cdot) \in \mathcal{D}(\Omega)$, and yet possibly $f(0,\cdot) \neq 0$.

The contribution of this affine term is

$$u_3(t) = \mathcal{A}^{-1} \int_0^t R_-(s) f(t-s)\, ds \in C^\infty\big([0,T] \times X_1\big)$$

since $f^{(n)}(0) \in \mathcal{D}(A^k)$ for every couple of integers n and k, so that

$$u_3(t) \in C^\infty\big([0,T]; X_k\big) \quad \text{for every } k.$$

In particular, $u_3 \in C^\infty([0,T] \times \Omega)$ as above.

We now extend the obtained properties to the solutions v to the Volterra integral equation (3.13) so that it is possible to track back the computation and to see that

equality (1.2) holds pointwise (when the boundary control and the initial conditions have the stated regularity, $u_2 \in \mathcal{D}(\Omega)$ included).

We confine ourselves to examine the effect of the boundary data g (the effect of initial data can be examined in a similar way). Moreover, multiplication by $e^{-R(0)t/2}$ does not affect the desidered results and hence is ignored; i.e. we assume $v(t) \equiv u(t)$.

Because equation (3.13) has the form of equation (2.25) in [32, § 2] (the notations are easily adapted, in particular b is substituted by c^2 in [32]) following the proof of [32, Theorem 2.4, item 2] we see that $y(t) = v(t) - Gg(t)$ solves

$$y(t) = (u_3(t) - Gg(t)) + \int_0^t L(s)Gg(t-s)\,ds + \int_0^t L(s)y(t-s)\,ds$$

so that

$$\mathcal{A}y(t) = \int_0^t \mathcal{A}L(s)Gg(t-s)\,ds + \int_0^t L(s)\mathcal{A}y(t-s)\,ds$$

(note that $\mathcal{A}L(t)$ is a continuous operator for every t). It then follows that $y(t) \in C^\infty([0,T];X_1)$.

Exploiting the definition of $L(t)$ and integrating by parts the integral which contains $g(t)$ we see that $y(t) \in C^\infty([0,T];X_2)$. Iterating this procedure, we obtain $u \in C^\infty([0,T]\times\Omega)$. Using this regularity result we can track back the computation leading to the fact that $u(t)$ solves the MGT equation, including the fact that the Laplacian and the time derivative can be interchanged.

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