Parabolic Minkowski convolutions of solutions to parabolic boundary value problems

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Abstract

We introduce a new kind of convolution, which is a sort of parabolic version of the classical supremal convolution of convex analysis. This operation allow us to compare solutions of different parabolic problems in different domains. As examples of applications of our main result, we study the parabolic concavity of solutions to parabolic boundary value problems, analyzing in particular the case of heat equation with an inhomogeneous term and with a nonlinear reaction term. We also apply our technique to the study of the dead core problem obtaining new results about necessary conditions for the existence of a dead core and estimates of the dead core time, proving some optimality of the ball.

1 Introduction

We are interested in comparing the solutions of different parabolic equations in different cylinders. Let $\{\Omega_i\}_{i=1}^m$ be bounded smooth domains in $\mathbf{R}^n$, where $n \geq 1$ and $m \geq 2$. For any

$$\lambda \in \Lambda_m := \left\{ (\lambda_1, \dots, \lambda_m) \in (0,1)^m \ : \ \sum_{i=1}^m \lambda_i = 1 \right\},$$

we denote by Ω_λ the Minkowski combination (with coefficient λ) of $\{\Omega_i\}_{i=1}^m$, that is

$$\Omega_\lambda := \sum_{i=1}^m \lambda_i \Omega_i = \left\{ \sum_{i=1}^m \lambda_i x_i \ : \ x_i \in \Omega_i, \ i = 1, \dots, m \right\}. \tag{1.1}$$

Let u_μ $(\mu = 1, \dots, m, \lambda)$ be a solution of the parabolic boundary value problem

$$\partial_t u = \Delta u + f_\mu(x, t, u, \nabla u) \quad \text{in} \quad D_\mu, \qquad u(x,t) = 0 \quad \text{on} \quad \partial D_\mu, \tag{1.2}$$

where $D_\mu := \Omega_\mu \times (0, \infty)$. The purpose of this paper is to find a relationship between the solutions $u_1, \dots, u_m$ in $D_1, \dots, D_m$ and the solution u_λ in D_λ by introducing the *parabolic Minkowski convolution* of $\{u_i\}_{i=1}^m$. For instance, we obtain the following result as a corollary of the main result of this paper.

Theorem 1.1 *Let $\{\Omega_i\}_{i=1}^m$ be bounded smooth domains in $\mathbf{R}^n$ and Ω_λ the domain defined by* (1.1)*, where $n \geq 1$, $m \geq 2$ and $\lambda \in \Lambda_m$. Let $u_\mu \in C^{2,1}(D_\mu) \cap C(\overline{D_\mu})$ satisfy*

$$\partial_t u_\mu = \Delta u_\mu + 1 \quad \textit{in} \quad D_\mu, \qquad u_\mu(x,t) = 0 \quad \textit{on} \quad \partial D_\mu, \tag{1.3}$$

where $D_\mu := \Omega_\mu \times (0, \infty)$ and $\mu = 1, \dots, m, \lambda$. Then, for any $\alpha \geq 1/2$,

$$u_\lambda \left(\sum_{i=1}^m \lambda_i x_i, t \right)^{1/2} \geq \sum_{i=1}^m \lambda_i u_i(x_i, t_i)^{1/2} \tag{1.4}$$

holds for all $(x_i, t_i) \in D_i$ $(i = 1, \dots, m)$ and $t \geq \left(\sum_{i=1}^m \lambda_i t_i^\alpha \right)^{1/\alpha}$.

Inequalities such as (1.4) imply interesting qualitative properties of the solutions $\{u_\mu\}$. Here we just give a result on the level sets of the solutions $\{u_\mu\}$. For this and further use, we introduce first the following notation: we set

$$\mathcal{L}\left(u_\mu(t)\,;\,\ell\right) := \{x \in \Omega_\mu \ : \ u_\mu(x,t) > \ell\}, \qquad \mu = 1, \dots, m, \lambda,$$

for $\ell \geq 0$.

Corollary 1.1 *Assume the same conditions as in Theorem* 1.1*. Then*

$$\sum_{i=1}^m \lambda_i \mathcal{L}\left(u_i(t_i)\,;\,\ell_i\right) \subseteq \mathcal{L}\left(u_\lambda(t)\,;\, \left(\sum_{i=1}^m \lambda_i \ell_i^{1/2} \right)^2 \right) \tag{1.5}$$

holds for all $\ell_i > 0$, $t_i > 0$ $(i = 1, \dots, m)$ *and* $t \geq \left(\sum_{i=1}^{m} \lambda_i t_i^{\alpha}\right)^{1/\alpha}$. *In particular,*

$$\sum_{i=1}^{m} \lambda_i \mathcal{L}(u_i(t); \ell) \subseteq \mathcal{L}(u_\lambda(t); \ell)$$

for all $\ell > 0$ *and* $t > 0$.

For further examples and applications, see Corollary 5.2 and Remark 5.1.

We are also interested in the parabolic concavity properties of solutions to parabolic boundary value problems. Let us recall the notion of α-parabolic p-concavity for nonnegative functions, which was introduced in [21] and [22]. For a, $b > 0$, $\lambda \in (0, 1)$ and $p \in [-\infty, \infty]$, we define

$$M_p(a, b; \lambda) := \begin{cases} [(1-\lambda)a^p + \lambda b^p]^{1/p} & \text{if} \quad p \notin \{-\infty, 0, \infty\}, \\ a^{1-\lambda} b^{\lambda} & \text{if} \quad p = 0, \\ \max\{a, b\} & \text{if} \quad p = \infty, \\ \min\{a, b\} & \text{if} \quad p = -\infty, \end{cases}$$

which is the (λ-weighted) p-mean of a and b. Furthermore, for a, $b \geq 0$, we define $M_p(a, b; \lambda)$ as above if $p \geq 0$ and $M_p(a, b; \lambda) = 0$ if $p < 0$ and $a \cdot b = 0$.

Definition 1.1 *Let* K *be a convex set in* $\mathbf{R}^n$, $Q := K \times (0, \infty)$ *and* $\alpha, p \in [-\infty, \infty]$. *A non-negative function* v *defined in* Q *is said* α-parabolically p-concave *if*

$$v\big((1-\lambda)x_1 + \lambda x_2, M_\alpha(t_1, t_2; \lambda)\big) \geq M_p\big(v(x_1, t_1), v(x_2, t_2); \lambda\big) \tag{1.6}$$

for all (x_1, t_1), $(x_2, t_2) \in Q$ *and* $\lambda \in (0, 1)$. *If* v *is* $(1/2)$ *-parabolically* p *-concave in* Q*, then it is simply said parabolically* p *-concave in* Q.

Roughly speaking, for $\alpha \in \mathbf{R} \setminus \{0\}$, v is α-parabolically p-concave in Q if

- v is a constant function in Q for $p = \infty$;
- $v(x, t^{1/\alpha})^p$ is concave in Q for $p > 0$;
- $\log v(x, t^{1/\alpha})$ is concave in Q for $p = 0$;
- $v(x, t^{1/\alpha})^p$ is convex in Q for $p < 0$;
- the level sets $\{(x, t) \in Q : v(x, t^{1/\alpha}) > \mu\}$ are convex for every $\mu \geq 0$ for $p = -\infty$.

The study of geometric properties (in particular concavity properties) of solutions to partial differential equations is a classical subject and it has been largely investigated in the framework of elliptic equations. Also the literature treating parabolic problems is now quite large, most of the results however concern concavity properties with respect to the spatial variable only (see for instance [1], [3], [4], [19], [20], [25], [30], [13], [31]–[33] and references therein), while the concavity properties involving the space and time variables jointly were studied for instance in [7], [18], [21]–[24] and [28]. Among others, the authors of this paper considered in [23] the parabolic boundary value problem

$$\partial_t u = \Delta u + f(x,t,u,\nabla u) \quad \text{in} \quad D, \qquad u(x,t) = 0 \quad \text{on} \quad \partial D, \tag{1.7}$$

where Ω is a bounded convex domain in $\mathbf{R}^n$ and $D := \Omega \times (0,\infty)$, and they studied the α-parabolic p-concavity of the solution u, where $1/2 \le \alpha \le 1$ and $0 < p < 1$, under the following structure condition:

$$\begin{aligned}&\text{the function } g_{\alpha,p,\theta}(x,t,v) := v^{3-1/p} f(x, t^{1/\alpha}, v^{1/p}, v^{1/p-1}\theta) \text{ is concave}\\ &\text{with respect to } (x,t,v) \in D \times (0,\infty) \text{ for any fixed } \theta \in \mathbf{R}^n.\end{aligned} \tag{1.8}$$

In this paper we weaken the above structure condition and study more delicate parabolic concavity properties of the solutions of (1.7) (see Section 4). This enables us to study the parabolic concavity of the solution of

$$\partial_t u = \Delta u - u_+^\gamma \quad \text{in} \quad D, \qquad u(x,t) = 0 \quad \text{on} \quad \partial D, \tag{1.9}$$

where $0 < \gamma < 1$ and $u_+ := \max\{u, 0\}$. If Ω is sufficiently large, then the solution u of (1.9) reaches zero in finite time, and a *dead-core*

$$Z(t) := \{x \in \Omega \, : \, u(x,t) = 0\}$$

appears at a time $T_\Omega = \inf\{t \, : \, Z_\Omega(t) \neq \emptyset\}$, called the *dead core time* of Ω. In Section 7, as an application of the main result of this paper, we study qualitative properties of the level sets of u, in particular the convexity of the dead-core with respect to the space and the time variables, and we find new necessary conditions of the onset of a dead core. We also prove some optimality of the ball with respect to the dead core time; in particular, in the plane we prove that *the disk has the smallest dead core time among sets with given perimeter.*

This paper is motivated by [23, 35] and the results are based on a refinement of the technique developed there. We introduce the notion of the α-parabolic Minkowski convolution $U_{\alpha,\lambda}$ of $\{u_i\}_{i=1}^m$, which can be regarded as a generalization of the parabolic concave envelope defined in [23]. We prove that $U_{\alpha,\lambda}$ is a viscosity subsolution of (1.2) with $\mu = \lambda$ by modifying the arguments in [23], and this is the main result of this paper (see Theorem 3.1). Then we can compare the solution u_λ and $U_{\alpha,\lambda}$ with the aid of the comparison principle, and obtain inequalities such as (1.4). This also enables us to study the parabolic concavity of the solutions of parabolic boundary value problems, see Section 4.

The rest of this paper is organized as follows. In Section 2 we introduce some notation and recall basic properties of α-parabolically p-concave functions. Furthermore, we recall the notion of viscosity solutions. In Section 3 we define the α-parabolic Minkowski convolution $U_{\alpha,\lambda}$ of the solutions $\{u_i\}_{i=1}^m$ ($-\infty \le \alpha \le \infty$), and prove Theorem 3.1. In Section 4 we apply the results of Section 3 to study the parabolic concavity properties of (1.7). In Sections 5 and 6 we apply the results in Sections 3 and 4 to the heat equation with an inhomogeneous term and to the heat equation with a nonlinear reaction term, respectively. In Section 7 we focus on problem (1.9) and study qualitative properties of the dead-core, giving also estimates of the *dead-core time*, that is the time of onset of the dead core.

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2 Preliminaries

For $x \in \mathbf{R}^n$ and $r > 0$, let $B(x,r) := \{z \in \mathbf{R}^n \, : \, |z-x| < r\}$.

For $a = (a_1, \dots, a_m) \in (0,\infty)^m$, $\lambda \in \Lambda_m$ and $p \in [-\infty, +\infty]$, we set

$$\mathbf{M}_p(a;\lambda) := \begin{cases} [\lambda_1 a_1^p + \lambda_2 a_2^p + \cdots + \lambda_m a_m^p]^{1/p} & \text{if} \quad p \neq -\infty,\, 0,\, +\infty, \\ \max\{a_1, \dots, a_m\} & \text{if} \quad p = +\infty, \\ a_1^{\lambda_1} \cdots a_m^{\lambda_m} & \text{if} \quad p = 0, \\ \min\{a_1, a_2, \dots, a_m\} & \text{if} \quad p = -\infty, \end{cases} \tag{2.1}$$

which is the (λ-weighted) p-mean of a. For $a = (a_1, \dots, a_m) \in [0,\infty)^m$, we define $\mathbf{M}_p(a;\lambda)$ as above if $p \geq 0$ and $\mathbf{M}_p(a;\lambda) = 0$ if $p < 0$ and $\prod_{i=1}^m a_i = 0$. Notice that $\mathbf{M}_p(a;\lambda)$ is a generalization of $M_p(a,b;\lambda)$ defined in Section 1 and it is a continuous function of the argument a. Due to the Jensen inequality, we have

$$\mathbf{M}_p(a;\lambda) \leq \mathbf{M}_q(a;\lambda) \quad \text{if} \quad -\infty \leq p \leq q \leq \infty, \tag{2.2}$$

for any $a \in [0,\infty)^m$ and $\lambda \in \Lambda_m$. Moreover, it easily follows that

$$\lim_{p \to +\infty} \mathbf{M}_p(a;\lambda) = \max\{a_1, \dots, a_m\}, \qquad \lim_{p \to -\infty} \mathbf{M}_p(a;\lambda) = \min\{a_1, \dots, a_m\}.$$

For further details, see e.g. [17].

Next we recall some properties of α-parabolically p-concave functions. Let K be a convex set in $\mathbf{R}^n$, $Q := K \times (0,\infty)$, $-\infty \leq p \leq \infty$ and $\alpha \in \mathbf{R}$. Due to Definition 1.1 and (2.2), if v is α-parabolically p-concave in Q, then the following holds:

(a) v is α-parabolically q-concave in Q for any $-\infty \leq q \leq p$;

(b) v is β-parabolically p-concave in Q for any $\beta \geq \alpha$ provided that v is non-decreasing with respect to the time variable t.

In addition, similarly to [27, Section 2], we have:

(c) Let $\{v_j\}$ be nonnegative functions in Q such that, for every $j \in \mathbf{N}$, v_j is α_j-parabolically p_j-concave in Q for some $\alpha_j \in \mathbf{R}$ and $p_j \in [-\infty,\infty]$. Let v be the pointwise limit of a sequence $\{v_j\}$ in Q, $\lim_{j\to\infty} \alpha_j = \alpha \in \mathbf{R}$ and $\lim_{j\to\infty} p_j = p \in [-\infty,\infty]$. If v is continuos with respect to the time variable, then v is α-parabolically p-concave in Q;

(d) Let $\alpha \in \mathbf{R}$ and p, $q \in [0,\infty]$. If v and w are α-parabolically p-concave and q-concave in Q, respectively, then $v \cdot w$ is α-parabolically r-concave in Q, where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

We recall the notion of viscosity subsolutions, supersolutions and solutions of (3.1). Let Ω be a domain in $\mathbf{R}^n$, $D = \Omega \times (0,\infty)$, and $f = f(x,t,v,\theta)$ a continuous function on $D \times \mathbf{R} \times \mathbf{R}^n$.

An upper semicontinuous function u in D is said to be a *viscosity subsolution* of (3.1) if, for any $(x_0, t_0) \in D$, the inequality

$$\partial_t \phi(x_0, t_0) \leq \Delta\phi(x_0, t_0) + f(x_0, t_0, \phi(x_0, t_0), \nabla\phi(x_0, t_0)) \tag{2.3}$$

holds for every $C^{2,1}(D)$ test function ϕ *touching* u *from above at* (x_0, t_0), i.e. satisfying

$$\phi(x_0, t_0) = u(x_0, t_0) \text{ and } \phi(x, t) \geq u(x, t) \text{ in a neighborhood of } (x_0, t_0).$$

Analogously, a lower semicontinuous function u in D is said to be a *viscosity supersolution* of (3.1) if, for any $(x_0, t_0) \in D$, the inequality

$$\partial_t \phi(x_0, t_0) \geq \Delta\phi(x_0, t_0) + f(x_0, t_0, \phi(x_0, t_0), \nabla\phi(x_0, t_0))$$

holds for every $C^{2,1}$ test function ϕ *touching* u *from below at* (x_0, t_0), i.e. satisfying

$$\phi(x_0, t_0) = u(x_0, t_0) \text{ and } \phi(x, t) \leq u(x, t) \text{ in a neighborhood of } (x_0, t_0).$$

A continuous function u in D is said to be a *viscosity solution* of (3.1) if u is a viscosity subsolution and a viscosity supersolution of (3.1) at the same time.

The technique proposed in this paper uses the following (weak) comparison principle for viscosity solutions:

$$\begin{cases} \text{Let } u \in C(\overline{D}) \cap C^{2,1}(D) \text{ and } v \in C(\overline{D}) \text{ be a classical solution} \\ \text{and a viscosity subsolution of (3.1), respectively,} \\ \text{such that } u \geq v \text{ on } \partial D. \text{ Then } u \geq v \text{ in } \overline{D}. \end{cases} \tag{WCP}$$

For sufficient conditions for (WCP), see e.g., [10, Section 8].

Also the following easy lemma will be fundamental in the sequel.

Lemma 2.1 *Let Ω be a domain in $\mathbf{R}^n$, $D = \Omega \times (0, \infty)$, $f = f(x, t, v, \theta)$ a continuous function on $D \times \mathbf{R} \times \mathbf{R}^n$ and u an upper semicontinuous function in D. Assume that for every $(x_0, t_0) \in D$, there exists a $C^{2,1}$ test function φ touching u from below at (x_0, t_0) such that*

$$\partial_t \varphi(x_0, t_0) \leq \Delta\varphi(x_0, t_0) + f(x_0, t_0, \varphi(x_0, t_0), \nabla\varphi(x_0, t_0)) \tag{2.4}$$

holds. Then u is a viscosity subsolution of (3.1).

Proof. Let ϕ be any $C^{2,1}(D)$ test function touching u from above at (x_0, t_0). Then ϕ touches also φ from above at (x_0, t_0), whence

$$\partial_t \phi(x_0, t_0) = \partial_t \varphi(x_0, t_0)\,, \quad \nabla\phi(x_0, t_0) = \nabla\varphi(x_0, t_0)\,, \quad \Delta\phi(x_0, t_0) \geq \Delta\varphi(x_0, t_0)\,.$$

Then (2.3) follows from (2.4). $\square$

3 Main theorem

Let us fix $\lambda \in \Lambda_m$ and let $\{\Omega_i\}_{i=1}^m$ be bounded smooth domains in $\mathbf{R}^n$, where $n \geq 1$ and $m \geq 2$. Define the Minkowski combination Ω_λ of $\{\Omega_i\}_{i=1}^m$ as in (1.1). For $\mu \in \{1, \ldots, m, \lambda\}$, set $D_\mu := \Omega_\mu \times (0, \infty)$ $(\mu = 1, \ldots, m, \lambda)$ and let $u_\mu \in C^{2,1}(D_i) \cap C(\overline{D_i})$ satisfy

$$\begin{cases} \partial_t u_i = \Delta u_i + f_i(x, t, u_i, \nabla u_i) \geq 0 & \text{in} \quad D_\mu, \\ u_i = 0 & \text{on} \quad \partial D_\mu, \end{cases} \tag{3.1}$$

where f_μ is a nonnegative continuous function in $D_\mu \times [0, \infty) \times \mathbf{R}^n$.

For any $\alpha \in [-\infty, \infty]$ and $p \in [-\infty, \infty]$, we define the *α-parabolic Minkowski p-convolution* $U_{\alpha,p,\lambda}$ of $\{u_i\}_{i=1}^m$ as follows:

$$\begin{aligned} U_{\alpha,p,\lambda}(x,t) := \sup \Bigg\{ & \mathbf{M}_p(u(y_1, \tau_1), \ldots, u_m(y_m, \tau_m); \lambda) \\ & : (y_i, \tau_i) \in \overline{D_i} \ (i = 1, \ldots, m) \quad \text{with} \quad x = \sum_{i=1}^m \lambda_i y_i, \ t = \mathbf{M}_\alpha(\tau_1, \ldots, \tau_m; \lambda) \Bigg\} \end{aligned} \tag{3.2}$$

for $(x, t) \in \overline{D_\lambda}$. In the case $p = 1$, we write $U_{\alpha,\lambda} = U_{\alpha,p,\lambda}$; moreover, in the case $p = 1$ and $\alpha = 1/2$, we simply write $U_\lambda = U_{\alpha,p,\lambda}$.

Notice that, from $u_i \in C(\overline{D_i})$ and the fact that $u_i > 0$ in D_i and vanishes on ∂D_i for $i = 1, \ldots, m$, it follows that

$$U_{\alpha,\lambda} \in C(\overline{D_\lambda}), \qquad U_{\alpha,\lambda} > 0 \quad \text{in} \quad D_\lambda \qquad \text{and} \qquad U_{\alpha,\lambda} = 0 \quad \text{on} \quad \partial D_\lambda. \tag{3.3}$$

In this section we prove the main result of this paper, which gives a sufficient condition for $U_{\alpha,\lambda}$ to be a viscosity subsolution of

$$\begin{cases} \partial_t u = \Delta u + f_\lambda(x, t, u, \nabla u) & \text{in} \quad D_\lambda, \\ u = 0 & \text{on} \quad \partial D_\lambda, \end{cases} \tag{3.4}$$

where $f_\lambda \in C(D_\lambda \times [0, \infty) \times \mathbf{R}^n)$.

For further use, we denote by $\nu_i = \nu_i(x)$ $(i = 1, \ldots, m)$ the inner unit normal vector to $\partial\Omega_i$ at $x \in \partial\Omega_i$ and set

$$\tilde{\nu}_i(x) := \begin{cases} \nu_i(x) & \text{if} \quad x \in \partial\Omega_i, \\ 0 & \text{if} \quad x \in \Omega_i, \end{cases} \qquad \text{and} \qquad \mu(t) := \begin{cases} 1 & \text{if} \quad t = 0, \\ 0 & \text{if} \quad t > 0. \end{cases} \tag{3.5}$$

Theorem 3.1 *Let $1/2 \leq \alpha \leq 1$. Assume the following conditions:*

(i) *for any $(x, t) \in \partial D_i$ $(i = 1, \ldots, m)$,*

$$\lim_{\rho \to 0+} \rho^{-1} u_i \left(x + \tilde{\nu}_i(x)\rho, t + \mu(t)\rho^{1/\alpha} \right) = \infty; \tag{3.6}$$

(ii) *for any fixed* $\theta \in \mathbf{R}^n$,

$$\begin{aligned} & f_\lambda\left(\sum_{i=1}^m \lambda_i y_i, \mathbf{M}_\alpha(\tau_1,\dots,t_m;\lambda), \sum_{i=1}^m \lambda_i v_i, \theta\right) \\ & \geq \mathbf{M}_{-1}\left(f_1(y_1,\tau_1,v_1,\theta),\dots,f_m(y_m,\tau_m,v_m,\theta):\lambda\right) \end{aligned} \tag{3.7}$$

holds for every

$$(y_i,\tau_i,v_i) \in R_i := \{(x,t,u_i(x,t)) \,:\, (x,t)\in D_i\}\,, \quad i=1,\dots,m.$$

Then $U_{\alpha,\lambda}$ *is a viscosity subsolution of* (3.4) *such that* $U_{\alpha,\lambda}=0$ *on* ∂D_λ.

As a straightforward corollary of the above theorem, we obtain the following.

Corollary 3.1 *In the same conditions as in Theorem* 3.1, *if the comparison principle* (WCP) *holds for* (3.1) *with* $\mu=\lambda$, *then*

$$u_\lambda\left(\sum_{i=1}^m \lambda x_i, t\right) \geq \sum_{i=1}^m \lambda_i u_i(x_i,t_i) \tag{3.8}$$

for $(x_i,t_i)\in D_i$, $i=1,\dots,m$, *and* $t \geq \mathbf{M}_\alpha(t_1,\dots,t_m;\lambda)$.

We prepare the following lemma for the proof of Theorem 3.1.

Lemma 3.1 *Assume the same conditions as in Theorem* 3.1. *Then for every* $(x_*,t_*)\in D_\lambda$, *there exist* $(x_1,t_1)\in D_1,\dots,(x_m,t_m)\in D_m$ *such that*

$$x_* = \sum_{i=1}^m \lambda_i x_i, \qquad t_*^\alpha = \sum_{i=1}^m \lambda_i t_i^\alpha, \qquad U_{\alpha,\lambda}(x_*,t_*) = \sum_{i=1}^m \lambda_i u_i(x_i,t_i), \tag{3.9}$$

$$\nabla u_1(x_1,t_1) = \cdots = \nabla u_m(x_m,t_m), \tag{3.10}$$

$$t_1^{1-\alpha}\partial_t u_1(x_1,t_1) = \cdots = t_m^{1-\alpha}\partial_t u_m(x_m,t_m). \tag{3.11}$$

Proof. The proof is similar to the proof of [23, Lemma 3.2]. We give it for completeness.

Let $i=1,\dots,m$. It follows from the strong maximum principle that $u_i\equiv 0$ in D_i or $u_i>0$ in D_i; then by assumption (i) we have

$$u_i(x,t) > 0 \quad \text{in} \quad D_i. \tag{3.12}$$

Let $(x_*,t_*)\in D_\lambda$. It follows from (3.2) and (3.12) that

$$U_{\alpha,\lambda}(x_*,t_*) > 0 \qquad \text{in} \quad D_\lambda. \tag{3.13}$$

Since

$$\left\{(y_1,s_1,y_2,s_2,\dots,y_m,s_m)\in \overline{D_1}\times\cdots\times\overline{D_m} \,:\, x_*=\sum_{i=1}^m \lambda_i y_i,\ t_*^\alpha = \sum_{i=1}^m \lambda_i s_i^\alpha\right\}$$

is closed and bounded in $\mathbf{R}^{mn}$, thanks to the continuity of M_p and of the $u_i's$, we can find $(x_i,t_i)\in\overline{D_i}$ $(i=1,\dots,m)$ realizing the maximum in (3.2) and then satisfying (3.9). Since the Lagrange multiplier theorem implies (3.10) and (3.11) provided that $(x_i,t_i)\in D_i$ for $i=1,\dots,m$, it suffices to prove the latter.

The proof is by contradiction. Assume that $(x_k,t_k)\in\partial D_k$ for some $k\in\{1,\dots,m\}$. It follows from (3.13) that $u_\ell(x_\ell,t_\ell)>0$ for some $\ell\in\{1,\dots,m\}$ with $k\neq\ell$. Here we can assume, without loss of generality, that $k=1$ and $\ell=2$, that is

$$(x_1,t_1)\in\partial D_1,\qquad (x_2,t_2)\in D_2. \tag{3.14}$$

Set $v_i(x,\tau):=u_i(x,\tau^{1/\alpha})$, $\tau_*=t_*^\alpha$ and $\tau_i=t_i^\alpha$. Then (3.9) reads

$$U_{\alpha,\lambda}(x_*,t_*)=\sum_{i=1}^m\lambda_i v(x_i,\tau_i),\quad x_*=\sum_{i=1}^m\lambda_i x_i,\quad \tau_*=\sum_{i=1}^m\lambda_i\tau_i\,, \tag{3.15}$$

and, jointly with the definition of $u_{\alpha,\lambda}$, it yields

$$U_{\alpha,\lambda}(x_*,t_*)\ge\sum_{i=1}^m\lambda_i v(y_i,\eta_i) \tag{3.16}$$

for every $(y_i,\eta_i)\in\overline{D_i}$, $i=1,\dots,m$, such that $x_*=\sum_{i=1}^m\lambda_i y_i$ and $\tau_*=\sum_{i=1}^m\lambda_i\eta_i$.

For any $\rho\in(0,1)$, set

$$\begin{aligned}&\tilde{x}_1:=x_1+\nu_1(x_1)\frac{\rho}{\lambda_1},\quad \tilde{x}_2:=x_2-\nu_1(x_1)\frac{\rho}{\lambda_2},\quad \tilde{x}_i=x_i\ \ (i=3,\dots,m),\\ &\tilde{\tau}_1:=\tau_1+\mu(t_1)\frac{\rho}{\lambda_1},\qquad \tilde{\tau}_2:=\tau_2-\mu(t_1)\frac{\rho}{\lambda_2},\qquad \tilde{\tau}_i:=\tau_i\quad (i=3,\dots,m).\end{aligned} \tag{3.17}$$

Thanks to (3.5), (3.14) and (3.15), we can take a sufficiently small $\rho>0$ such that

$$\tilde{x}_i\in\Omega_i\quad (i=1,\dots,m),\qquad \tilde{\tau}_1>0,\qquad \tilde{\tau}_2>0\,, \tag{3.18}$$

and notice that

$$\sum_{i=1}^m\lambda_i\tilde{x}_i=\sum_{i=1}^m\lambda_i x_i=x_*,\qquad \sum_{i=1}^m\lambda_i\tilde{\tau}_i=\sum_{i=1}^m\lambda_i\tau_i=\tau_*. \tag{3.19}$$

By (3.12) and (3.18) we see that $v(\tilde{x}_2,\tilde{\tau}_2)>0$. Moreover we can find positive constants M and R such that

$$|(\nabla v)(x,\tau)|+|(\partial_t v)(x,\tau)|\le M\quad\text{in}\quad B(x_2,R)\times(\tau_2-R,\tau_2+R)\subset D_2.$$

Applying the mean value theorem, we obtain

$$\lambda_2[v(\tilde{x}_2,\tilde{\tau}_2)-v(x_2,\tau_2)]\ge-\lambda_2M|(\tilde{x}_2,\tilde{\tau}_2)-(x_2,\tau_2)|\ge-2M\rho. \tag{3.20}$$

On the other hand, by (3.6) and (3.17), we see that

$$\lambda_1[v(\tilde{x}_1, \tilde{\tau}_1) - v(x_1, t_1)] = \lambda_1 v(\tilde{x}_1, \tilde{\tau}_1) > 2M\rho \tag{3.21}$$

for a sufficiently small ρ. Combining (3.20) with (3.21), we obtain

$$\begin{aligned}\sum_{i=1}^m \lambda_i v(\tilde{x}_i, \tilde{\tau}_i) &> \lambda_1 v(x_1, \tau_1) + 2M\rho + \lambda_2 v(x_2, \tau_2) - 2M\rho + \sum_{i=3}^m \lambda_i v(x_i, \tau_i) \\ &= \sum_{i=1}^m \lambda_i v(x_i, \tau_i) = U_{\alpha,\lambda}(x_*, t_*).\end{aligned}$$

This together with (3.19) contradicts (3.16) and the proof is complete. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Fix any $(x_*, t_*) \in D$ and let $\{a_i\}_{i=1}^m \subseteq [0, \infty)$ be such that

$$\sum_{i=1}^m \lambda_i a_i = 1. \tag{3.22}$$

By (3.6) we apply Lemma 3.1 to find $(x_i, t_i) \in D_i$ $(i = 1, \ldots, m)$ satisfying (3.9)–(3.11). Set

$$\begin{aligned}&U_* := U_{\alpha,\lambda}(x_*, t_*), \qquad U_i := u_i(x_i, t_i), \\ &y_i(x) = x_i + a_i(x - x_*), \qquad \tau_i(t) = \left[t_i^\alpha + a_i(t^\alpha - t_*^\alpha)\right]^{1/\alpha}.\end{aligned} \tag{3.23}$$

It follows that

$$U_* = \sum_{i=1}^m \lambda_i U_i, \qquad x = \sum_{i=1}^m \lambda_i y_i(x), \qquad t^\alpha = \sum_{i=1}^m \lambda_i \tau_i(t)^\alpha. \tag{3.24}$$

Define

$$\varphi(x, t) := \sum_{i=1}^m \lambda_i u_i(y_i(x), \tau_i(t)), \tag{3.25}$$

and notice that it is a $C^{2,1}$-function in a neighborhood of $(x_*, t_*) \in D$ and satisfies

$$\varphi(x_*, t_*) = \sum_{i=1}^m \lambda_i U_i = U_*. \tag{3.26}$$

Moreover, we deduce from the definition of $U_{\alpha,\lambda}$ and (3.24) that

$$U_{\alpha,\lambda}(x, t) \geq \varphi(x, t) \tag{3.27}$$

in a neighborhood of (x_*, t_*).

Next we prove

$$\partial_t \varphi(x_*, t_*) \leq \Delta\varphi(x_*, t_*) + f(x_*, t_*, \varphi(x_*, t_*), \nabla\varphi(x_*, t_*)). \tag{3.28}$$

By (3.23) and (3.25) we calculate

$$\nabla\varphi(x,t) = \sum_{i=1}^m \lambda_i a_i \, \nabla u_i(y_i(x), \tau_i(t)), \tag{3.29}$$

$$\nabla^2\varphi(x,t) = \sum_{i=1}^m \lambda_i a_i^2 \, \nabla^2 u_i(y_i(x), \tau_i(t)), \tag{3.30}$$

in a neighborhood of (x_*, t_*). Since $y_i(x_*) = x_i$ and $\tau_i(t_*) = t_i$, by (3.10), (3.22) and (3.29) we have

$$\nabla\varphi(x_*, t_*) = \sum_{j=1}^m \lambda_j a_j \nabla u_j(x_j, t_j) = \nabla u_i(x_i, t_i) \tag{3.31}$$

for $i = 1, \dots, m$. Similarly, it follows from (3.1), (3.22) and (3.11) that

$$\partial_t\varphi(x_*, t_*) = \sum_{j=1}^m \lambda_j a_j \left(\frac{t_j}{t_*}\right)^{1-\alpha} \partial_t u_j(x_j, t_j) = \left(\frac{t_i^\alpha}{t_*^\alpha}\right)^{\frac{1}{\alpha}-1} \partial_t u_i(x_i, t_i) \geq 0 \tag{3.32}$$

for $i = 1, \dots, m$. Then, by (3.1), (3.23), (3.30), (3.31) and (3.32) we obtain

$$\begin{aligned}
&\partial_t\varphi(x_*, t_*) - \Delta\varphi(x_*, t_*) \\
&= \partial_t\varphi(x_*, t_*) - \sum_{i=1}^m \lambda_i a_i^2 \Delta u_i(x_i, t_i) \\
&= \partial_t\varphi(x_*, t_*) - \sum_{i=1}^m \lambda_i a_i^2 \Big[\partial_t u_i(x_i, t_i) - f_i(x_i, t_i, u_i(x_i, t_i), \nabla u_i(x_i, t_i))\Big] \\
&= \partial_t\varphi(x_*, t_*) \left[1 - \sum_{i=1}^m \lambda_i a_i^2 \left(\frac{t_i^\alpha}{t_*^\alpha}\right)^{1-\frac{1}{\alpha}}\right] + \sum_{i=1}^m \lambda_i a_i^2 f_i(x_i, t_i, U_i, \theta),
\end{aligned} \tag{3.33}$$

where $\theta := \nabla\varphi(x_*, t_*)$. On the other hand, since $1/2 \leq \alpha \leq 1$, $h(\eta, \tau) := \eta^2 \tau^{1-1/\alpha}$ is a convex function in $\mathbf{R} \times (0, \infty)$. Then, by (3.9) and (3.22) we have

$$\begin{aligned}
\sum_{i=1}^m \lambda_i a_i^2 \left(\frac{t_i^\alpha}{t_*^\alpha}\right)^{1-1/\alpha} &= \sum_{i=1}^m \lambda_i h\left(a_i, \frac{t_i^\alpha}{t_*^\alpha}\right) \\
&\geq h\left(\sum_{i=1}^m \lambda_i a_i, \sum_{i=1}^m \lambda_i \frac{t_i^\alpha}{t_*^\alpha}\right) = h(1,1) = 1.
\end{aligned} \tag{3.34}$$

If $f_i(x_i, t_i, U_i, \theta) \neq 0$ for all $i = 1, \dots, m$, then, setting

$$a_i := \frac{1}{f_i(x_i, t_i, U_i, \theta)} \Bigg/ \sum_{i=1}^m \frac{\lambda_i}{f_i(x_i, t_i, U_i, \theta)}, \qquad i = 1, \dots, m,$$

we have (3.22). Furthermore, it follows from (3.7) and (3.9) that

$$
\begin{aligned}
&\sum_{i=1}^{m}\lambda_i a_i^2 f_i(x_i,t_i,U_i,\theta)\\
&=\sum_{i=1}^{m}\frac{\lambda_i}{f_i(x_i,t_i,U_i,\theta)}\bigg/\left(\sum_{i=1}^{m}\frac{\lambda_i}{f_i(x_i,t_i,U_i,\theta)}\right)^2=\left[\sum_{i=1}^{m}\frac{\lambda_i}{f_i(x_i,t_i,U_i,\theta)}\right]^{-1}\\
&=\mathbf{M}_{-1}\left(f_1(x_1,t_1,U_1,\theta),\dots,f_1(x_m,t_m,U_m,\theta);\lambda\right)\\
&\le f_\lambda\left(\sum_{i=1}^{m}\lambda x_i,\mathbf{M}_\alpha(t_1,\dots,t_m;\lambda),\sum_{i=1}^{m}\lambda U_i,\theta\right)=f_\lambda(x_*,t_*,U_*,\theta)\\
&=f_\lambda(x_*,t_*,\varphi(x_*,t_*),\nabla\varphi(x_*,t_*)).
\end{aligned}
\tag{3.35}
$$

If $f_j(x_j,t_j,U_j,\theta)=0$ for some $j\in\{1,\dots,m\}$, then, setting

$$a_i=1/\lambda_i \quad \text{if} \quad i=j \qquad \text{and} \qquad a_i=0 \quad \text{if} \quad i\neq j,$$

we have

$$0=\sum_{i=1}^{m}\lambda_i a_i^2 f_i(x_i,t_i,U_i,\theta)\le f_\lambda(x_*,t_*,\varphi(x_*,t_*),\nabla\varphi(x_*,t_*)). \tag{3.36}$$

Since $\partial_t\varphi(x_*,t_*)\ge 0$ (see (3.32)), by (3.33)–(3.36) we obtain (3.28). Since (x_*,t_*) is arbitrary, by (3.3) and Lemma 2.1 we see that $U_{\alpha,\lambda}$ is a viscosity subsolution of (3.1), and the proof is complete. □

Similarly to Theorem 3.1, we give a sufficient condition for $U_{\alpha,p,\lambda}$ to be a viscosity subsolution of (3.4) in the case $p\neq 1$.

Theorem 3.2 *Let $1/2\le\alpha\le 1$ and $0<p<1$. Define*

$$g_\mu(x,t,v,\theta):=v^{3-1/p}f_\mu(x,t^{1/\alpha},v^{1/p},v^{1/p-1}\theta)$$

for $(x,t,v,\theta)\in D_\mu\times(0,\infty)\times\mathbf{R}^n$, where $\mu=1,\dots,m,\lambda$. Assume the following conditions:

(i) *for any $x\in\overline{\Omega_i}$ and $i=1,\dots,m$,*

$$\lim_{\rho\to 0+}\rho^{-1}u_i\left(x+\tilde{\nu}(x)\rho,\rho^{1/\alpha}\right)^p=\infty; \tag{3.37}$$

(ii) *for any fixed $\theta\in\mathbf{R}^n$,*

$$g_\lambda\left(\sum_{i=1}^{m}\lambda_i y_i,\sum_{i=1}^{m}\lambda_i t_i,\sum_{i=1}^{m}\lambda_i v_i,\theta\right)\ge\sum_{i=1}^{m}\lambda_i g_i(x_i,t_i,v_i,\theta) \tag{3.38}$$

holds for all

$$(y_i,\tau_i,v_i)\in\tilde{R}_i:=\{(x,t^\alpha,u_i(x,t^\alpha)^p)\,:\,(x,t)\in D_i\}\,,\quad i=1,\dots,m.$$

Then the α-parabolic Minkowski p-convolution $U_{\alpha,q,\lambda}$ of $\{u_i\}_{i=1}^m$ is a viscosity subsolution of (3.4) such that $U_{\alpha,q,\lambda}=0$ on ∂D_λ.

Proof. Let $i = 1, \dots, m$. Set $v_i := u_i^p$. It follows from (3.1) and (3.37) that $u_i \geq 0$ and $u_i \not\equiv 0$ in D_i. Since $u_i = 0$ on ∂D_i, we deduce from the Hopf lemma that

$$\lim_{\rho \to 0} \rho^{-1} u_i(x + \tilde{\nu}_i(x)\rho, t) > 0 \tag{3.39}$$

for all $x \in \partial\Omega_i$ and $t > 0$. Since $\partial_t u_i \geq 0$ in D_i and $0 < p < 1$, by (3.39) we have

$$\lim_{\rho \to 0} \rho^{-1} v_i(x + \tilde{\nu}_i(x)\rho, t + \mu_{s,t}\rho^{1/\alpha}) \geq \lim_{\rho \to 0} \rho^{-1} u_i(x + \tilde{\nu}_i(x)\rho, t)^p = \infty \tag{3.40}$$

for all $(x, t) \in \partial\Omega_i \times (0, \infty)$. Combining (3.40) with (3.37), we see that v_i satisfies assumption (i) of Theorem 3.1 with u_i replaced by v_i. Furthermore, by (3.1) we see that v_i satisfies

$$\partial_t v_i = \Delta v_i + F_i(x, t, v_i, \nabla v_i) \quad \text{in} \quad D_i, \qquad v_i = 0 \quad \text{on} \quad \partial D_i, \tag{3.41}$$

where

$$F_i(x, t, v, \theta) := p v^{1-1/p} f_i(x, t, v^{1/p}, v^{1/p-1}\theta/p) + \frac{(1-p)}{p} v^{-1} |\theta|^2$$

for $(x, t, v, \theta) \in D_i \times [0, \infty) \times \mathbf{R}^n$.

Fix $\theta \in \mathbf{R}^n$. Since

$$H_\theta(v, w) := pw + \frac{(1-p)}{p} v |\theta|^2 \tag{3.42}$$

is concave with respect to $(v, w) \in (0, \infty)^2$, it follows from [27, Lemma A2] that

$$v^{-2} H_\theta(v, w)$$

is (-1)-concave with respect to $(v, w) \in (0, \infty)^2$. This implies that

$$V_\lambda^{-2} H_\theta(V_\lambda, W_\lambda) \geq \mathbf{M}_{-1}\left(v_1^{-2} H_\theta(v_1, w_1), \dots, v_m^{-2} H_\theta(v_m, w_m); \lambda\right) \tag{3.43}$$

for all $(v_i, w_i) \in (0, \infty)^2$ $(i = 1, \dots, m)$, where

$$V_\lambda := \sum_{i=1}^m \lambda_i v_i, \qquad W_\lambda := \sum_{i=1}^m \lambda_i w_i.$$

Setting

$$w_i := v_i^{3-1/p} f_i\left(x_i, t_i, v_i^{1/p}, v_i^{1/p-1}\theta/p\right) = g_i(x_i, t_i^\alpha, v_i, \theta/p),$$

by (3.38) we have

$$W_\lambda = \sum_{i=1}^m \lambda_i g_i(x_i, t_i^\alpha, v_i, \theta/p) \leq g_\lambda\left(\sum_{i=1}^m \lambda_i x_i, \sum_{i=1}^m \lambda_i t_i^\alpha, V_\lambda, \theta/p\right),$$

which together with (3.42) and (3.43) implies that

$$
\begin{aligned}
&\mathbf{M}_{-1}\left(F_1(x_1,t_1,v_1,\theta),\dots,F_m(x_m,t_m,v_m,\theta);\lambda\right)\\
&=\mathbf{M}_{-1}\left(v_1^{-2}H_\theta(v_1,w_1),\dots,v_m^{-2}H_\theta(v_m,w_m);\lambda\right)\\
&\le V_\lambda^{-2}H_\theta(V_\lambda,W_\lambda)=V_\lambda^{-2}\left[pW_\lambda+\frac{1-p}{p}V_\lambda|\theta|^2\right]\\
&\le V_\lambda^{-2}\left[p\,g_\lambda\left(\sum_{i=1}^m\lambda_i x_i,\sum_{i=1}^m\lambda_i t_i^\alpha,V_\lambda,\theta/p\right)+\frac{1-p}{p}V_\lambda|\theta|^2\right]\\
&=pV_\lambda^{1-1/p}f_\lambda\left(\sum_{i=1}^m\lambda_i x_i,\left(\sum_{i=1}^m\lambda_i t_i^\alpha\right)^{1/\alpha},V_\lambda^{1/p},V_\lambda^{1/p-1}\theta/p\right)+\frac{1-p}{p}V_\lambda^{-1}|\theta|^2\\
&=F_\lambda\left(\sum_{i=1}^m\lambda_i x_i,\mathbf{M}_\alpha(t_1,\dots,t_m;\lambda),V_\lambda,\theta\right),
\end{aligned}
$$

where

$$
F_\lambda(x,t,v,\theta):=pv^{1-1/p}f_\lambda(x,t,v^{1/p},v^{1/p-1}\theta/p)+\frac{1-p}{p}v^{-1}|\theta|^2
$$

This means that assumption (ii) of Theorem 3.1 holds with f_μ replaced by F_μ. Therefore, by Theorem 3.1 we see that the α-parabolic Minkowski 1-convolution $U_{\alpha,\lambda}$ of $\{v_i\}_{i=1}^m$ is a viscosity subsolution of

$$
\partial_t v=\Delta v+F_\lambda(x,t,v,\nabla v)\quad\text{in}\quad D_\lambda \tag{3.44}
$$

such that $U_{\alpha,\lambda}=0$ on ∂D_λ. This implies that the α-parabolic Minkowski p-convolution $U_{\alpha,p,\lambda}$ of $\{u_i\}_{i=1}^m$ is a viscosity subsolution of (3.4) such that $U_{\alpha,p,\lambda}=0$ on ∂D_λ. Thus Theorem 3.2 follows. □

Similarly to Corollary 3.1 after Theorem 3.1, as a straightforward corollary of Theorem 3.2 we obtain the following.

Corollary 3.2 *In the same conditions as in Theorem* 3.2*, if the comparison principle* (WCP) *holds for equation* (3.1) *with* $\mu=\lambda$*, then*

$$
u_\lambda\left(\sum_{i=1}^m\lambda x_i,\mathbf{M}_\alpha(t_1,\dots,t_m);\lambda)\right)\ge\mathbf{M}_p(u_1(x_1,t_1),\dots,u_m(x_m,t_m);\lambda) \tag{3.45}
$$

for all $(x_i,t_i)\in D_i$ *and* $i=1,\dots,m$.

In the rest of this section we collect some results deduced from inequalities such as (3.8) and (3.45). Let w_μ $(\mu=1,\dots,m,\lambda)$ be nonnegative continuous function in D_μ. Assume the following conditions:

- w_μ $(\mu=1,\dots,m,\lambda)$ is non-decreasing with respect to the time variable t for any fixed $x\in\Omega_\mu$;
- for some $p\in[-\infty,\infty]$ and $\alpha\ge[-\infty,\infty]$,

$$
w_\lambda\left(\sum_{i=1}^m\lambda x_i,\mathbf{M}_\alpha(t_1,\dots,t_m);\lambda)\right)\ge\mathbf{M}_p(w_1(x_1,t_1),\dots,w_m(x_m,t_m);\lambda) \tag{3.46}
$$

  holds for all $(x_i,t_i)\in D_i$ and $i=1,\dots,m$.

These imply

$$\|w_\lambda(\cdot,t)\|_{L^\infty(\Omega_\lambda)} \ge \mathbf{M}_p(\|w_1(\cdot,t_1)\|_{L^\infty(\Omega_1)},\dots,\|w_m(\cdot,t_m)\|_{L^\infty(\Omega_m)};\lambda)$$

and

$$\sum_{i=1}^m \lambda_i \mathcal{L}\left(w_i(t_i)\,;\,\ell_i\right) \subseteq \mathcal{L}\left(w_\lambda(t)\,;\,\mathbf{M}_p(\ell_1,\dots,\ell_m;\lambda)\right) \tag{3.47}$$

for all $\ell_i > 0$, $t_i > 0$ $(i = 1,\dots,m)$ and $t \ge \mathbf{M}_\alpha(t_1,\dots,t_m;\lambda)$. Furthermore, it follows from (3.46) that

$$w_\lambda\left(\sum_{i=1}^m \lambda x_i, t\right)^r \ge \mathbf{M}_{q/r}(w_1(x_1,t_1)^r,\dots,w_m(x_m,t_m)^r;\lambda) \tag{3.48}$$

holds for all $(x_i,t_i)\in D_i$ $(i=1,\dots,m)$ and $t \ge \mathbf{M}_\alpha(t_1,\dots,t_m;\lambda)$, where $r>0$. Then, if $-r/n \le p \le \infty$, the inequality

$$\|w_\lambda(\cdot,t)\|_{L^r(\Omega_\lambda)} \ge \mathbf{M}_{p_*}(\|w_1(\cdot,t_1)\|_{L^r(\Omega_1)},\dots,\|w_m(\cdot,t_m)\|_{L^r(\Omega_m)};\lambda) \tag{3.49}$$

holds for all $t_i > 0$ $(i=1,\dots,m)$ and $t \ge \mathbf{M}_\alpha(t_1,\dots,t_m;\lambda)$, where

$$p_* = \begin{cases} r/n & \text{if} \quad p = +\infty, \\ pr/(np+r) & \text{if} \quad p \in (-r/n,+\infty), \\ -\infty & \text{if} \quad p = -r/n. \end{cases} \tag{3.50}$$

This is deduced from the following lemma on the Borell-Brascamp-Lieb inequality, which is a generalization of the Prékopa-Leindler inequality (see [12, Theorem 10.1] for references).

Lemma 3.2 *Let $\lambda \in (0,1)$, f,g,h nonnegative functions in $L^1(\mathbf{R}^n)$, and $-1/n \le q \le \infty$. Assume that*

$$h\big((1-\lambda)x+\lambda y\big) \ge M_q(f(x),g(y);\lambda)$$

for all $x \in \mathrm{sprt}(f)$, $y\in \mathrm{sprt}(g)$. Then

$$\int_{\mathbf{R}^n} h\,dx \ge M_{q_*}\left(\int_{\mathbf{R}^n} f\,dx, \int_{\mathbf{R}^n} g\,dx\,;\lambda\right),$$

where

$$q_* = \begin{cases} 1/n & \textit{if} \quad q = +\infty, \\ q/(nq+1) & \textit{if} \quad q \in (-1/n,+\infty), \\ -\infty & \textit{if} \quad q = -1/n. \end{cases}$$

The Prékopa-Leindler inequality corresponds to the case $p = 0$.

4 Parabolic concavity

In this section, as an application of Theorems 3.1 and 3.2, we study the parabolic concavity properties of the solutions to parabolic boundary value problems.

Let Ω be a bounded convex domain in $\mathbf{R}^n$ and $D := \Omega \times (0,\infty)$. Let $u \in C^{2,1}(D) \cap C(\overline{D})$ satisfy

$$\begin{cases} \partial_t u = \Delta u + f(x,t,u,\nabla u) \geq 0 & \text{in} \quad D, \\ u = 0 & \text{on} \quad \partial D, \end{cases} \tag{4.1}$$

where f is a nonnegative continuous function in $D \times [0,\infty) \times \mathbf{R}^n$. Let $\alpha \in [-\infty,\infty]$, $p \in [-\infty,\infty]$ and $\lambda \in \Lambda_{n+1}$. Following [21] and [22], we set

$$\begin{aligned} u_{\alpha,p,\lambda}(x,t) := \sup \Bigg\{ & \mathbf{M}_p(u_1(y_1,\tau_1),\dots,u_m(y_m,t_m);\lambda) \\ & : (y_i,\tau_i) \in \overline{D_i} \ (i=1,\dots,m) \quad \text{with} \quad x = \sum_{i=1}^m \lambda_i y_i,\ t = \mathbf{M}_\alpha(\tau_1,\dots,\tau_m;\lambda) \Bigg\} \end{aligned} \tag{4.2}$$

for $(x,t) \in \overline{D}$. Furthermore, we define the α-parabolically p-concave envelope $u_{\alpha,p}$ of u by

$$u_{\alpha,p}(x,t) := \sup_{\lambda \in \Lambda_{n+1}} u_{\alpha,p,\lambda}(x,t). \tag{4.3}$$

Similarly to Section 3, we write $u_{\alpha,\lambda}$ for $u_{\alpha,1,\lambda}$ and u_α for $u_{\alpha,1}$ for simplicity.

It follows from (4.2) that $u_{\alpha,p}(x,t) \geq u(x,t)$ in D, and we see that $u_{\alpha,p}$ is the smallest α-parabolically p-concave function greater than or equal to u. Therefore u is α-parabolically p-concave in D if and only if

$$u(x,t) \geq u_{\alpha,p}(x,t) \quad \text{in} \quad D. \tag{4.4}$$

On the other hand, $u_{\alpha,p,\lambda}$ coincides with the α-parabolic Minkowski p-convolution $U_{\alpha,p,\lambda}$ of $\{u_i\}_{i=1}^m$ in the case where $m = n+1$, $\Omega_i = \Omega$ and $u_i = u$ for $i = 1,\dots,n+1$ since Ω is convex. Then we have the following results on the parabolic concavity properties for the solution of (4.1). Let ν be the inner unit normal vector to $\partial\Omega$ and set

$$\tilde{\nu}(x) = \nu(x) \quad \text{if} \quad x \in \partial\Omega, \qquad \tilde{\nu}(x) = 0 \quad \text{if} \quad x \in \Omega. \tag{4.5}$$

Theorem 4.1 *Let Ω be a bounded convex smooth domain in $\mathbf{R}^n$, $D := \Omega \times (0,\infty)$, $1/2 \leq \alpha \leq 1$ and $f = f(x,t,v,\theta)$ a nonnegative continuous function in $D \times \mathbf{R} \times \mathbf{R}^n$. Let $u \in C^{2,1}(D) \cap C(\overline{D})$ satisfy* (4.1)*. Assume the following conditions:*

(i) $\lim_{\rho \to 0+} \rho^{-1} u\left(x + \tilde{\nu}(x)\rho, t + \mu(t)\rho^{1/\alpha}\right) = \infty$ *for* $(x,t) \in \partial D$*;*

(ii) *the function*

$$g_{\alpha,\theta}(x,t,v) := f(x,t^{1/\alpha},v,\theta)$$

is (-1)*-concave with respect to*

$$(x,t,v) \in R_\alpha := \{(x,t^\alpha,u(x,t^\alpha)) \,:\, (x,t) \in D\}$$

for any fixed $\theta \in \mathbf{R}^n$*.*

Then, for any $\lambda \in \Lambda_{n+1}$*,* $u_{\alpha,\lambda}$ *is a viscosity subsolution of* (4.1) *such that* $u_{\alpha,\lambda} = 0$ *on* ∂D*. Furthermore, if the comparison principle* (WCP) *holds for equation* (4.1)*, then* u *is* α*-parabolically concave in* D*.*

Proof. It follows from Theorem 3.1 that $u_{\alpha,\lambda}$ is a viscosity subsolution of (4.1) such that $u_{\alpha,\lambda} = 0$ on ∂D. Furthermore, if the comparison principle (WCP) holds for equation (4.1), then

$$u(x,t) \geq u_{\alpha,p,\lambda}(x,t) \quad \text{in} \quad D$$

for any $\lambda \in \Lambda_{n+1}$. This implies (4.4), which means that u is α-parabolically p-concave in D. Thus Theorem 4.1 follows. □

Similarly, by Theorem 3.2 we have:

Theorem 4.2 *Let* Ω *be a bounded convex smooth domain in* $\mathbf{R}^n$*,* $D := \Omega \times (0,\infty)$*,* $0 < p < 1$*,* $1/2 \leq \alpha \leq 1$ *and* $f = f(x,t,v,\theta)$ *a nonnegative continuous function in* $D \times \mathbf{R} \times \mathbf{R}^n$*. Let* $u \in C^{2,1}(D) \cap C(\overline{D})$ *satisfy* (4.1)*. Assume the following conditions:*

(i) $\displaystyle\lim_{\rho \to 0+} \rho^{-1} u \left(x + \tilde{\nu}(x)\rho, \rho^{1/\alpha}\right)^p = \infty$ *for* $x \in \overline{\Omega}$*;*

(iii) *the function*

$$g_{\alpha,p,\theta}(x,t,v) := v^{3-1/p} f(x, t^{1/\alpha}, v^{1/p}, v^{1/p-1}\theta)$$

is concave with respect to $(x,t,v) \in D \times (0,\infty)$ *for any fixed* $\theta \in \mathbf{R}^n$*.*

Then, for any $\lambda \in \Lambda_{n+1}$*,* $u_{\alpha,p,\lambda}$ *is a viscosity subsolution of* (1.2) *such that* $u_{\alpha,p,\lambda} = 0$ *on* ∂D*. Furthermore, if the comparison principle* (WCP) *holds for equation* (1.2)*,* u *is* α*-parabolically* p*-concave in* D*.*

5 Heat equation with an inhomogeneous term

In this section we apply the results of the previous sections to the heat equation with an inhomogeneous term, and prove the following theorem.

Theorem 5.1 *Let* $n \geq 1$*,* $m \geq 2$ *and* $\lambda \in \Lambda_m$*. Let* $\{\Omega_i\}_{i=1}^m$ *be bounded smooth domains in* $\mathbf{R}^n$ *and* Ω_λ *the domain defined by* (1.1)*. For any* $\mu \in \{1,\dots,m,\lambda\}$*, let* $u_\mu \in C^{2,1}(D_\mu) \cap C(\overline{D_\mu})$ *satisfy*

$$\partial_t u_\mu = \Delta u_\mu + f_\mu(x,t) \quad \text{in} \quad D_\mu, \qquad u_\mu = 0 \quad \text{on} \quad \partial D_\mu, \tag{5.1}$$

where f_μ *is a positive continuous function in* D_μ*. Assume the following conditions:*

(1) *for any* $\mu \in \{1,\dots,m,\lambda\}$*,* $f_\mu = f_\mu(x,t)$ *is nondecreasing with resect to* $t > 0$ *for every fixed* $x \in \Omega_i$*;*

(2) *there exist* $q \in [1,\infty]$ *and* $\alpha \in [1/2,1]$ *such that*

$$f_\lambda \left(\sum_{i=1}^m \lambda_i x_i, \mathbf{M}_\alpha(t_1,\dots,t_m;\lambda) \right) \geq \mathbf{M}_q\Big(f_1(x_1,t_1), \dots, f_m(x_m,t_m); \lambda \Big)$$

holds for all $(x_i,t_i) \in \Omega_i$ *and* $i = 1,\dots,m$*.*

Then

$$u_\lambda\left(\sum_{i=1}^m \lambda_i x_i, t\right) \ge \mathbf{M}_p(u_1(x_1,t_1),\dots,u_m(x_m,t_m);\lambda) \tag{5.2}$$

holds for all $(x_i,t_i)\in\Omega_i$ $i=1,\dots,m$, $t\ge \mathbf{M}_\beta(t_1,\dots,t_m;\lambda)$ *and* $\beta\ge\alpha$*, where* $p:=q/(1+2q)$ *if* $1\le q<\infty$ *and* $p:=1/2$ *if* $q=\infty$*.*

We prepare the following lemma, which follows from [34, Theorem 5].

Lemma 5.1 *Let* Ω *be a bounded convex smooth domain in* $\mathbf{R}^n$ *and* $D:=\Omega\times(0,\infty)$*. Let* $w\in C^2(D)\cap C(\overline{D})$ *satisfy*

$$\partial_t w=\Delta w+1 \quad in \quad D, \qquad w=0 \quad on \quad \partial D. \tag{5.3}$$

Let $x\in\overline{\Omega}$ *and* $y\in\Omega$*. Then*

$$\begin{aligned} &\lim_{\rho\to 0}\rho^{-1}w(x+\rho\tilde\nu(x),\rho^{1/\alpha})^{1/2}=\infty \qquad if \quad \alpha>1/2,\\ &\liminf_{\rho\to 0}\rho^{-1}w(x+\rho\tilde\nu(x),\rho^{1/\alpha})^{1/2}>0 \qquad if \quad \alpha=1/2, \end{aligned} \tag{5.4}$$

where $\tilde\nu$ *is as in* (4.5)*.*

Proof of Theorem 5.1. We consider the case where $1\le q<\infty$ and $1/2<\alpha\le 1$. For any $\epsilon>0$ and $\delta>0$, it follows from assumption (2) of Theorem 5.1 that

$$\begin{aligned} \sum_{i=1}^m \lambda_i[f_i(x_i,t_i)+\epsilon]^q &\le \sum_{i=1}^m \lambda_i(1+\delta)f_i(x_i,t_i)^q + C_\delta\,\epsilon^q\\ &\le (1+\delta)f_\lambda\left(\sum_{i=1}^m\lambda_i x_i,\mathbf{M}_\alpha(t_1,\dots,t_m);\lambda\right)^q + C_\delta\,\epsilon^q\\ &\le \left[(1+\delta)^{1/q}f_\lambda\left(\sum_{i=1}^m\lambda_i x_i,\mathbf{M}_\alpha(t_1,\dots,t_m);\lambda\right)+C_\delta^{1/q}\epsilon\right]^q \end{aligned} \tag{5.5}$$

for all $(x_i,t_i)\in\Omega_i$ and $i=1,\dots,m$, where C_δ is a constant depending only on δ. Let $\delta_\epsilon\in(0,1)$ be such that

$$\delta_\epsilon\to 0 \quad \text{and} \quad C_{\delta_\epsilon}^{1/q}\epsilon\to 0 \quad \text{as} \quad \epsilon\to 0. \tag{5.6}$$

Set

$$f_i^\epsilon(x,t):=f_i(x,t)+\epsilon \quad \text{and} \quad f_\lambda^\epsilon(x,t):=(1+\delta_\epsilon)^{1/q}f_\lambda(x,t)+C_{\delta_\epsilon}^{1/q}\epsilon. \tag{5.7}$$

This together with (5.5) implies that

$$f_\lambda^\epsilon\left(\sum_{i=1}^m\lambda_i x_i,\mathbf{M}_\alpha(t_1,\dots,t_m);\lambda\right)\ge \mathbf{M}_q(f_1^\epsilon(x_1,t_1),\dots,f_m^\epsilon(x_m,t_m);\lambda) \tag{5.8}$$

for all $(x_i,t_i)\in\Omega_i$ and $i=1,\dots,m$.

Let u^ϵ_μ be a solution of

$$\partial_t u_\mu = \Delta u_\mu + f^\epsilon_\mu(x,t) \quad \text{in} \quad D_\mu, \qquad u_\mu = 0 \quad \text{on} \quad \partial D_\mu. \tag{5.9}$$

By (5.6) and (5.7) we apply the standard arguments for parabolic equations to see that

$$\lim_{\epsilon\to 0} u^\epsilon_\mu(x,t) = u_\mu(x,t) \quad \text{in} \quad D_\mu. \tag{5.10}$$

Furthermore, for any $i = 1, \dots, m$, by (5.7) and (5.9), we apply the comparison principle to obtain

$$u^\epsilon_i(x,t) \geq \epsilon w(x,t) \quad \text{in} \quad D_i,$$

where w is a solution of (5.3). This together with Lemma 5.1 implies that

$$\lim_{\rho\to 0} \rho^{-1} u^\epsilon_i(x + \rho\tilde{\nu}(x), \rho^{1/\alpha})^p = \infty \tag{5.11}$$

for every $x \in \overline{\Omega_i}$ and for every $\epsilon > 0$. This means that assumption (i) of Theorem 3.2 is satisfied by u^ϵ_i, $i = 1, \dots, m$. Furthermore, for any $h > 0$, by (5.9) we see that the function

$$w_\mu(x,t) := \frac{1}{h}(u^\epsilon_i(x,t+h) - u^\epsilon_i(x,t))$$

satisfies

$$\partial_t w_\mu = \Delta w_\mu + \frac{1}{h}(f^\epsilon_\mu(x,t+h) - f^\epsilon_\mu(x,t)) \geq \Delta w_i \quad \text{in} \quad D_\mu, \qquad w_\mu \geq 0 \quad \text{on} \quad \partial D_\mu.$$

It follows from the comparison principle that

$$w_\mu(x,t) = \frac{1}{h}(u^\epsilon_i(x,t+h) - u^\epsilon_i(x,t)) \geq 0 \quad \text{in} \quad D_\mu.$$

Passing the limit as $h \to 0$, we obtain

$$\partial_t u^\epsilon_\mu \geq 0 \quad \text{in} \quad D_\mu. \tag{5.12}$$

Set

$$h(v,w) := v^{3-1/p} w^{1/q}, \qquad g_\mu(x,t,v) := v^{3-1/p} f_\mu(x, t^{1/\alpha}).$$

It follows from property (d) in the Preliminaries and $p = q/(1+2q)$ that $h(v,w)$ is concave with respect to $(v,w) \in (0,\infty) \times [0,\infty)$. This together with (5.8) implies that

$$\begin{aligned}
\sum_{i=1}^m \lambda_i g_i(x_i,t_i,v_i) &= \sum_{i=1}^m \lambda_i h\left(v_i, (f^\epsilon_i(x_i, t_i^{1/\alpha}))^q\right) \leq h\left(\sum_{i=1}^m \lambda_i v_i, \sum_{i=1}^m \lambda_i (f^\epsilon_i(x_i, t_i^{1/\alpha}))^q\right) \\
&= \left(\sum_{i=1}^m \lambda_i v_i\right)^{3-1/p} \mathbf{M}_q(f^\epsilon_1(x_1, t_1^{1/\alpha}), \dots, f^\epsilon_m(x_m, t_m^{1/\alpha}); \lambda) \\
&\leq \left(\sum_{i=1}^m \lambda_i v_i\right)^{3-1/p} f^\epsilon_\lambda\left(\sum_{i=1}^m \lambda_i x_i, \left(\sum_{i=1}^m \lambda_i t_i\right)^{1/\alpha}\right) \\
&= g_\lambda\left(\sum_{i=1}^m \lambda_i x_i, \sum_{i=1}^m \lambda_i t_i, \sum_{i=1}^m \lambda_i v_i\right)
\end{aligned} \tag{5.13}$$

for all $(x_i, t_i) \in \Omega_i$ and $i = 1, \dots, m$. This means that assumption (ii) of Theorem 3.2 is satisfied. Furthermore, the comparison principle (WCP) holds for (5.9). Therefore, thanks to (5.11) and (5.12), we apply Theorem 3.2 to problem (5.9) to obtain

$$u_\lambda^\epsilon\left(\sum_{i=1}^m \lambda_i x_i, \mathbf{M}_\alpha(t_1, \dots, t_m; \lambda)\right) \geq \mathbf{M}_p\left(u_1^\epsilon(x_1, t_1), \dots, u_m^\epsilon(x_m, t_m); \lambda\right)$$

for all $x_i \in \overline{\Omega_i}$ $(i = 1, \dots, m)$. This together with (2.2) and (5.12) implies that

$$u_\lambda^\epsilon\left(\sum_{i=1}^m \lambda_i x_i, t\right) \geq \mathbf{M}_p\left(u_1^\epsilon(x_1, t_1), \dots, u_m^\epsilon(x_m, t_m); \lambda\right) \tag{5.14}$$

for all $x_i \in \overline{\Omega_i}$ $(i = 1, \dots, m)$, $t \geq \mathbf{M}_\beta(t_1, \dots, t_m; \lambda)$ and $\beta \geq \alpha$. Passing the limit as $\epsilon \to 0$, by (5.10) and (5.14) we obtain (5.2) in the case where $1 \leq p < \infty$ and $1/2 < \alpha \leq 1$. Then we deduce that (5.2) holds in the case $1 \leq p \leq \infty$ and $1/2 \leq \alpha \leq 1$, and Theorem 5.1 follows. □

Combining Theorem 5.1 with (3.47) and (3.49), we have:

Corollary 5.1 *Assume the same conditions as in Theorem* 5.1. *Let* $p := q/(1 + 2q)$ *if* $1 < q < \infty$ *and* $p := 1/2$ *if* $q = \infty$. *Then*

$$\sum_{i=1}^m \lambda_i \mathcal{L}\left(u_i(t_i)\,;\, \ell_i\right) \subseteq \mathcal{L}\left(\, u_\lambda(t)\,;\, \mathbf{M}_p(\ell_1, \dots, \ell_m; \lambda)\right) \tag{5.15}$$

for all $\ell_i > 0$, $t_i > 0$ $(i = 1, \dots, m)$ *and* $t \geq \mathbf{M}_\alpha(t_1, \dots, t_m; \lambda)$. *Furthermore,*

$$\|u_\lambda(\cdot, t)\|_{L^r(\Omega_\lambda)} \geq \mathbf{M}_{p_*}(\|u_1(\cdot, t_1)\|_{L^r(\Omega_1)}, \dots, \|u_m(\cdot, t_m)\|_{L^r(\Omega_m)}; \lambda) \tag{5.16}$$

for all $t_i > 0$ $(i = 1, \dots, m)$ *and* $t \geq \mathbf{M}_\alpha(t_1, \dots, t_m; \lambda)$, *where* $r > 0$ *and*

$$p_* = \begin{cases} \dfrac{qr}{(n + 2r)q + 1} & \text{if} \quad 1 < q < \infty, \\ \dfrac{r}{n + 2r} & \text{if} \quad q = \infty. \end{cases}$$

Furthermore, applying Theorem 5.1 and Corollary 5.1 with $q = \infty$ to problem (1.3), we have:

Corollary 5.2 *Assume the same conditions as in Theorem* 1.1. *Then inequalities* (1.4) *and* (1.5) *hold. Furthermore, the inequality*

$$\|u_\lambda(\cdot, t)\|_{L^r(\Omega_\lambda)} \geq \mathbf{M}_{r/(n+2r)}(\|u_1(\cdot, t_1)\|_{L^r(\Omega_1)}, \dots, \|u_m(\cdot, t_m)\|_{L^r(\Omega_m)}; \lambda) \tag{5.17}$$

holds for all $t_i > 0$ $(i = 1, \dots, m)$ *and* $t \geq \mathbf{M}_\alpha(t_1, \dots, t_m; \lambda)$, *where* $r > 0$.

Theorem 1.1 and Corollary 1.1 follow from Corollary 5.2.

Remark 5.1 *Assume the same conditions as in Theorem* 1.1*. Let* τ_μ *be the torsional rigidity of* $\overline{\Omega_\mu}$*, that is*

$$\frac{1}{\tau_\mu} := \inf\left\{ \int_{\Omega_\mu} |\nabla v|^2\, dx \Big/ \left(\int_{\Omega_\mu} |v|\, dx \right)^2 \, : \, v \in W^{1,2}(\Omega_\mu) \setminus \{0\} \right\}.$$

Let v_μ $(\mu = 1, \dots, m, \lambda)$ *be a unique solution of*

$$\Delta v + 1 = 0 \quad in \quad \Omega_\mu, \qquad v = 0 \quad on \quad \partial\Omega_\mu.$$

Then

$$\tau_\mu = 4 \int_\Omega |\nabla v_\mu|^2\, dx = 4 \int_\Omega v_\mu\, dx. \tag{5.18}$$

On the other hand, the solution u_μ *of* (1.3) *converges to* v_μ *uniformly on* Ω_μ*. Therefore it follows from* (5.17) *with* $r = 1$ *and* (5.18) *that*

$$\tau_\mu^{1/(n+2)} \ge \sum_{i=1}^m \lambda_i \tau_i^{1/(n+2)},$$

which coincides with the inequality obtained by [5]*. See also* [9]*.*

In addition, as another application of Theorem 5.1, we use a similar argument as in Section 4 to obtain the following results.

Theorem 5.2 *Let* Ω *be a bounded convex smooth domain in* $\mathbf{R}^n$ *and* $D := \Omega \times (0, \infty)$*. Let* $u \in C^{2,1}(D) \cap C(\overline{D})$ *satisfy*

$$\partial_t u = \Delta u + f(x,t) \quad in \quad D, \qquad u = 0 \quad on \quad \partial D,$$

where f *is a nonnegative continuous function in* D*. Assume the following conditions:*

- $f = f(x,t)$ *is nondecreasing with respect to* $t > 0$ *for any fixed* $x \in \Omega$*;*
- f *is* α*-parabolically* q*-concave in* D *for some* $1/2 \le \alpha \le 1$ *and* $1 \le q \le \infty$*.*

Then u *is* β*-parabolically* p*-concave in* D *for any* $\beta \ge \alpha$*, where* $p = q/(1+2q)$ *if* $q < \infty$ *and* $p = 1/2$ *if* $q = \infty$*.*

Theorem 5.3 *Let* Ω *be a bounded convex smooth domain in* $\mathbf{R}^n$ *and* $D := \Omega \times (0, \infty)$*. Let* $u \in C^{2,1}(D) \cap C(\overline{D})$ *satisfy*

$$\partial_t u = \Delta u + t^\gamma h(x) \quad in \quad D, \qquad u = 0 \quad on \quad \partial D,$$

where $0 \le \gamma \le 1$ *and* h *is a nonnegative continuous function in* Ω*. Let* $\alpha \in [\gamma, 1]$ *be such that* $\alpha \ge 1/2$*. Assume that* h *is* q*-concave in* Ω *with*

$$\frac{\alpha}{\alpha - \gamma} \le q \le \infty \quad if \quad \alpha > \gamma \qquad and \qquad q = \infty \quad if \quad \alpha = \gamma. \tag{5.19}$$

Then u *is* β*-parabolically* p*-concave in* D *for any* $\beta \ge \alpha$*, where*

$$p = \frac{\alpha q}{(\gamma + 2\alpha) q + \alpha} \quad if \quad q < \infty \quad and \quad p = \frac{\alpha}{\gamma + 2\alpha} \quad if \quad q = \infty.$$

Proof. Set $f(x,t) := t^\gamma h(x)$. Since $f(x,t^{1/\alpha}) = t^{\gamma/\alpha}h(x)$, f is α-parabolically r-concave in D, where $1/r = \gamma/\alpha + 1/q$, that is

$$r = \frac{\alpha q}{\gamma q + \alpha} \quad \text{if} \quad q < \infty \quad \text{and} \quad r = \frac{\alpha}{\gamma} \quad \text{if} \quad q = \infty.$$

Then it follows from $\gamma \le \alpha$ and (5.19) that $r \ge 1$. Furthermore, we have

$$\frac{r}{1+2r} = \frac{\alpha q}{(\gamma + 2\alpha)q + \alpha}.$$

Therefore, by Theorem 5.2 we obtain the desired conclusion, and the proof is complete. □

6 Heat equation with a nonlinear reaction term

In this section we apply the results in Sections 3 and 4 to the heat equation with a nonlinear reaction term. For $\mu = 1, \dots, m, \lambda$, let Ω_μ and D_μ be as in Section 3 and let $u_\mu \in C^{2,1}(D_\mu) \cap C(\overline{D_\mu})$ satisfy

$$\begin{cases} \partial_t u_\mu = \Delta u_\mu + f(u_\mu) \ge 0 & \text{in} \quad D_\mu, \\ u_\mu > 0 & \text{in} \quad D_\mu, \\ u_\mu = 0 & \text{on} \quad \partial D_\mu, \end{cases} \tag{6.1}$$

where $f = f(s)$ is a nonnegative continuous function in $[0,\infty)$. Put

$$M := \max_\mu \|u_\mu\|_{L^\infty(D_\mu)} \quad \text{and} \quad F(s) := \int_0^s f(\tau)d\tau.$$

Inspired by [26], we assume the following conditions on the nonlinear term f:

$$f(s) > 0 \quad \text{for} \quad s \in (0,M), \tag{6.2}$$

$$f \in C^2((0,M)); \tag{6.3}$$

$$2(f'(s))^2 - f(s)f''(s) - \frac{(f(s))^2 f'(s)}{F(s)} \ge 0 \quad \text{for} \quad s \in (0,M). \tag{6.4}$$

Furthermore, we assume

$$h(s) := [F(s)]^{-1/2} f(s) \to \infty \quad \text{as} \quad s \to +0. \tag{6.5}$$

Then we have the following result as an application of Theorem 3.1.

Theorem 6.1 *Let $n \ge 1$, $m \ge 2$ and $\lambda \in \Lambda_m$. Let $\{\Omega_i\}_{i=1}^m$ be bounded smooth domains in $\mathbf{R}^n$ and Ω_λ the domain defined by* (1.1)*. For any $\mu \in \{1, \dots, m, \lambda\}$, let $u_\mu \in C^{2,1}(D_\mu) \cap C(\overline{D_\mu})$ satisfy* (6.1)*. Assume* (6.2)*–*(6.4) *and the following condition:*

(A) *if $\tilde{u}$ is a viscosity subsolution of*

$$\partial_t u = \Delta u + f(u) \quad in \quad D_\lambda$$

such that $\tilde{u} = 0$ on ∂D_λ, then $u_\lambda \ge \tilde{u}$ in D_λ.

Define the function v_μ $(\mu = 1, \dots, m, \lambda)$ *by*

$$v_\mu(x,t) := g(u_\mu(x,t)), \quad where \quad g(\xi) := \int_0^\xi [F(s)]^{-1/2} ds\,. \tag{6.6}$$

Assume

$$\lim_{\rho\to 0} \rho^{-1} v_i \left(x + \tilde{\nu}(x)\rho, \rho^{1/\alpha} \right) = \infty \tag{6.7}$$

for all $x \in \overline{\Omega_i}$ *and* $i = 1, \dots, m$. *Then*

$$v_\lambda \left(\sum_{i=1}^m \lambda x_i, t \right) \geq \sum_{i=1}^m \lambda_i v_i(x_i, t_i) \tag{6.8}$$

holds for all $(x_i, t_i) \in D_i$ $(i = 1, \dots, m)$, $t \geq \mathbf{M}_\alpha(t_1, \dots, t_m; \lambda)$ *and* $\alpha \geq 1/2$. *Moreover,*

$$u_\lambda \left(\sum_{i=1}^m \lambda x_i, t \right) \geq \min\{u_1(x_1, t_1), \dots, u_m(x_m, t_m)\} \tag{6.9}$$

and

$$\sum_{i=1}^m \lambda_i \mathcal{L}\left(u_i(t_i)\,;\, \ell_i\right) \subseteq \mathcal{L}\left(\, u_\lambda(t)\,; \min\{\ell_1, \dots, \ell_m; \}\,\right) \tag{6.10}$$

hold for all $(x_i, t_i) \in D_i$, $\ell_i > 0$ $(i = 1, \dots, m)$, $t \geq \mathbf{M}_\alpha(t_1, \dots, t_m; \lambda)$ *and* $\alpha \geq 1/2$.

Proof. By (6.1) and (6.6) we see that v_μ satisfies

$$\begin{cases} \partial_t v_\mu = \Delta v_\mu + \dfrac{1}{2} h(v_\mu)(2 + |\nabla v_\mu|^2)) \geq 0 & \text{in} \quad D_\mu, \\ v_\mu = 0 & \text{on} \quad \partial D_\mu, \end{cases} \tag{6.11}$$

where $h(v_\mu)$ is implicitly defined by the relation $h(v_\mu) = [F(u_\mu)]^{-1/2} f(u_\mu)$. By the Hopf lemma we have (3.39), that is

$$\lim_{\rho\to +0} \rho^{-1} u_i(x + \tilde{\nu}(x)\rho, t) > 0 \tag{6.12}$$

for all $x \in \partial\Omega_i$ and $t > 0$. On the other hand, it follows from (6.6) that

$$g(\xi) \geq C\xi^{1/2}, \quad s > 0,$$

for some constant $C > 0$. This together with (6.12) yields

$$\liminf_{\rho\to +0} \rho^{-1} v_i(x + \rho\tilde{\nu}(x), t + \mu(t)\rho^2) \geq C \liminf_{\rho\to +0} \rho^{-1} u_i(x + \rho\tilde{\nu}(x), t)^{1/2} = \infty \tag{6.13}$$

for all $(x,t) \in \partial\Omega \times (0, \infty)$. Therefore we deduce from (6.7) and (6.13) that assumption (i) of Theorem 4.1 is satisfied with $\alpha = 1/2$ and u replaced by v_μ. Furthermore, it follows from (6.4) that

$$\frac{\partial^2}{\partial v^2} \frac{1}{h(v)} \geq 0 \quad \text{for} \quad v \in (0, g(M)), \tag{6.14}$$

which implies that $h(v)$ is (-1)-concave with respect to $v \in [0, g(M)]$. Therefore, by Theorem 4.1 we obtain (6.8) with $\alpha = 1/2$, which together with (3.46) and (3.47) implies (6.8) for all $\alpha \geq 1/2$.

Now observe that g is strictly increasing, hence it is invertible and its inverse g^{-1} is also strictly increasing. Then (6.8) yields

$$\begin{aligned} u_\lambda \left(\sum_{i=1}^m \lambda x_i, t \right) &= g^{-1} \left(v_\lambda \left(\sum_{i=1}^m \lambda x_i, t \right) \right) \geq g^{-1} \left(\sum_{i=1}^m \lambda_i v_i(x_i, t_i) \right) \\ &\geq g^{-1}(\min\{v_1(x_1, t_1), \dots, v_m(x_m, t_m)\}) \\ &= \min\{g^{-1}(v_1(x_1, t_1)), \dots, g^{-1}(v_m(x_m, t_m))\} \\ &= \min\{u_1(x_1, t_1), \dots, u_m(x_m, t_m)\} \end{aligned}$$

for all $(x_i, t_i) \in D_i$ $(i = 1, \dots, m)$, $t \geq \mathbf{M}_\alpha(t_1, \dots, t_m; \lambda)$ and $\alpha \geq 1/2$, that is (6.9), which in turn implies (6.10). Thus Theorem 6.1 is proved. □

Similarly to Section 4, by Theorem 6.1 we have:

Theorem 6.2 *Let Ω be a bounded smooth convex domain in $\mathbf{R}^n$ and $D := \Omega \times (0, \infty)$, where $n \geq 1$. Let $u \in C^{2,1}(D) \cap C(\overline{D})$ satisfy* (6.1) *with D_μ replaced by D. Assume* (6.2)–(6.4) *with M replaced by $\|u\|_{L^\infty(D)}$ and condition (A) with D_μ replaced by D. Define the function v defined by*

$$v(x, t) := g(u(x, t)), \quad \text{where} \quad g(\xi) := \int_0^\xi [F(s)]^{-1/2} ds.$$

Assume

$$\lim_{\rho \to 0} \rho^{-1} v_i \left(x + \tilde{\nu}(x)\rho, \rho^{1/\alpha} \right) = \infty \tag{6.15}$$

for all $x \in \overline{\Omega_i}$. Then v is α-parabolic concave in D for any $\alpha \geq 1/2$. In particular, u is α-parabolic quasiconcave in D for any $\alpha \geq 1/2$.

Remark 6.1 *Assume the same conditions as in Theorem* 6.1 *and $f(0) > 0$. Let $i \in \{1, \dots, m\}$ and $x \in \overline{\Omega_i}$. By* (6.1) *we can find a smooth domain E_i in Ω_i and $T > 0$ such that $x \in \overline{E_i}$ and*

$$f(u_i(y, s)) \geq \frac{f(0)}{2}, \qquad (y, s) \in E_i \times (0, T).$$

Then it follows from the comparison principle that

$$u_i \geq \frac{f(0)}{2} w(x, t) \quad \text{in} \quad E_i \times (0, T),$$

where w be a solution of (5.3) *in D replaced by $E_i \times (0, \infty)$. We remark that, if $x \in \partial\Omega_i$, then $x \in \partial E_i$ and the inner unit normal vector to $\partial\Omega_i$ at x coincides with that of E_i at x. Then, by Lemma 5.1 we have*

$$\liminf_{\rho \to 0} \rho^{-1} u_i(x + \rho\tilde{\nu}(x), \rho^2)^{\frac{1}{2}} \geq \frac{f(0)}{2} \liminf_{\rho \to +0} \rho^{-1} w(x + \rho\tilde{\nu}(x), \rho^2)^{1/2} > 0, \tag{6.16}$$

which means that v_i satisfies condition (6.7). *Similarly, in the same conditions as in Theorem* 6.2 *and $f(0) > 0$, v satisfies* (6.15).

As a corollary of Theorem 6.1 we have:

Corollary 6.1 *Let $n \geq 1$, $m \geq 2$ and $\lambda \in \Lambda_m$. Let $\{\Omega_i\}_{i=1}^m$ be bounded smooth domains in $\mathbf{R}^n$ and Ω_λ the domain defined by (1.1). Let $u_\mu \in C^{2,1}(D_\mu) \cap C(\overline{D_\mu})$ be a solution of*

$$\partial_t u = \Delta u + u^\gamma \quad in \quad D_\mu, \quad u > 0 \quad in \quad D_\mu, \quad u = 0 \quad on \quad \partial D_\mu, \tag{6.17}$$

where $0 < \gamma < 1$. Then the same conclusion as in Theorem 6.1 hold with $g(s) = s^{(1-\gamma)/2}$.

Proof. In the case $f(s) = s^\gamma$ with $0 < \gamma < 1$, f satisfies conditions (6.2)–(6.5). It follows from [11, Lemma 3.1] that u_μ is a unique solution of (6.17). By (6.6) we set

$$v_\mu = \frac{2(1+\gamma)^{1/2}}{1-\gamma} u_\mu^{\frac{1-\gamma}{2}}.$$

By the same argument as in the proof of [23, Theorem 6], we see that conditions (A) and (6.7) are satisfied. Therefore Corollary 6.1 follows Theorem 6.1. □

Similarly, by Theorem 6.2 we have a result on the parabolic $(1-\gamma)/2$-concavity properties of the solution of (6.17). We omit to state this result here since it is the same result as in [23, Theorem 6].

7 Dead-core problem

Consider the following parabolic boundary value problem for the heat equation with strong absorption

$$\partial_t u = \Delta u - u_+^\gamma \quad \text{in} \quad D, \qquad u = 1 \quad \text{on} \quad \partial D, \tag{7.1}$$

where Ω is a bounded smooth domain in $\mathbf{R}^n$, $D := \Omega \times (0, \infty)$, $0 < \gamma < 1$ and $u_+ := \max\{u, 0\}$. Problem (7.1) arises in the modeling of an isothermal reaction-diffusion process (see [2] and [37]) and it has been studied intensively for past years (see e.g., [8], [14]–[16] [26], [28], [38] and references therein). If the domain Ω is sufficiently large, then the solution u reaches zero in finite time and the set

$$Z_\Omega(t) := \{x \in \Omega \,:\, u(x,t) = 0\}$$

is called the *dead-core* at time t in the literature. We denote by T_Ω the *dead-core time* of u, that is the first time when u reaches zero, i.e.

$$T_\Omega = \inf\{t \,:\, Z_\Omega(t) \neq \emptyset\}.$$

It follows from the maximum principle that

$$0 \leq u(x,t) < 1 \quad \text{and} \quad \partial_t u \leq 0 \quad \text{in} \quad D, \tag{7.2}$$

which implies

$$Z_\Omega(t) \subseteq Z_\Omega(t') \quad \text{if} \quad 0 < t \leq t'. \tag{7.3}$$

Notice also that the dead core is monotone increasing with respect to inclusion, that is

$$Z_\Omega(t) \subseteq Z_{\Omega'}(t) \qquad \text{if } \Omega \subseteq \Omega',$$

whence we get the reverse monotonicity property for the dead core time:

$$T_\Omega \geq T_{\Omega'} \qquad \text{if } \Omega \subseteq \Omega' . \tag{7.4}$$

It was shown in [28] that the dead-core $Z(t)$ is convex for all $t > 0$ if Ω is convex (see also [26]).

Set $v = 1 - u$. By (7.1) and (7.2) we see that v satisfies

$$\partial_t v = \Delta v + (1-v)^\gamma \geq 0 \quad \text{in} \quad D, \qquad v = 0 \quad \text{on} \quad \partial D. \tag{7.5}$$

On the other hand, $f(s) = (1-s)^\gamma$ with $M = 1$ satisfies (6.2)–(6.4) and the comparison principle holds for problem (7.5). Therefore, applying Theorems 6.1 and 6.2 (see also Remark 6.1) to the solution v of (7.5), we have:

Theorem 7.1 *Let $n \geq 1$, $m \geq 2$ and $\lambda \in \Lambda_m$. Let $\{\Omega_i\}_{i=1}^m$ be bounded smooth domains in $\mathbf{R}^n$ and Ω_λ the domain defined by (1.1). For any $\mu \in \{1, \ldots, m, \lambda\}$, let u_μ be a solution of (7.1) with D replaced by $D_\mu := \Omega_\mu \times (0, \infty)$. Then*

$$\sum_{i=1}^m \lambda_i \{x \in \Omega_i \,:\, u_i(x, t_i) \leq \ell\} \subseteq \{x \in \Omega_\lambda \,:\, u_\lambda(x,t) \leq \ell\}$$

for all $t_i > 0$ $(i = 1, \ldots, m)$, $\ell > 0$ and $t \geq M_{1/2}(t_1, \ldots, t_m; \lambda)$. In particular,

$$\sum_{i=1}^m \lambda_i Z_{\Omega_i}(t_i) \subseteq Z_{\Omega_\lambda}(t)$$

holds for all $t_i > 0$ $(i = 1, \ldots, m)$, $\ell > 0$ and $t \geq M_{1/2}(t_1, \ldots, t_m; \lambda)$, and

$$T_{\Omega_\lambda}^{1/2} \leq \sum_{i=1}^m \lambda_i T_{\Omega_i}^{1/2}.$$

Theorem 7.2 *Let Ω be a bounded convex domain in $\mathbf{R}^n$, where $n \geq 1$. Then*

$$Z := \{(x,t) \in D \,:\, u(x,t) = 0\}$$

is parabolically convex, that is

$$\left((1-\eta)x + \eta y, \left((1-\eta)t^{1/2} + \eta s^{1/2}\right)^2\right) \in Z \quad \text{if} \quad (x,t),\, (y,s) \in Z,$$

for any $\eta \in (0,1)$. In particular, $Z(t)$ is convex for any $t > 0$.

Next we study the dead-core in view of the mean width of bounded convex domains. For any bounded convex domain Ω, we define

$$h_\Omega(x) := \max_{y \in \overline{\Omega}} (x, y), \qquad x \in \mathbf{R}^n,$$

which is called the *support function* of Ω and has the following properties for convex domains Ω and Ω':

- $h_{\ell\Omega}(x) = \ell h_\Omega(x)$ for $x \in \mathbf{R}^n$ and $\ell \geq 0$;
- $h_{\Omega+\Omega'}(x) = h_\Omega(x) + h_{\Omega'}(x)$ for $x \in \mathbf{R}^n$.

Then for every direction ξ we define the *width of* Ω *in direction* ξ as

$$w(\Omega, \xi) = h_\Omega(\xi) + h_\Omega(-\xi)\,,$$

and we notice that the *minimal width* $d(\Omega) = \min\{w(\Omega, \xi) \,:\, \xi \in \mathbf{S}^{n-1}\}$ is the thickness of the thinnest slab containing Ω, while the *maximal width* $D(\Omega) = \max\{w(\Omega, \xi) \,:\, \xi \in \mathbf{S}^{n-1}\}$ coincides with the diameter of Ω.

Furthermore, we denote by $w(\Omega)$ the *mean width* of Ω, that is

$$w(\Omega) := \frac{1}{n\omega_n} \int_{\mathbf{S}^{n-1}} w(\Omega, \xi)\, d\xi = \frac{2}{n\omega_n} \int_{\mathbf{S}^{n-1}} h_\Omega(\xi)\, d\xi.$$

In the case where Ω is a ball, $w(\Omega)$ coincides with the diameter of Ω. In $\mathbf{R}^2$ the mean width of a convex set coincides with the perimeter up to a factor π, i.e. $|\partial\Omega| = \pi w(\Omega)$.

The function h_Ω and the mean width $w(\Omega)$ can be in fact defined also for *non-convex sets*, precisely in the same way. In this case they respectively coincide with the support function and the mean width of the convex hull $conv(\Omega)$ of the set Ω. Notice that, without convexity, in the plane the mean width is no more proportional to the perimeter, but still we have

$$\pi w(\Omega) \leq |\partial\Omega|\,,$$

since the perimeter of $conv(\Omega)$ is shorter than the perimeter Ω.

For any bounded convex domain Ω and any $\Theta = \{\theta_i\}_{i=1}^m \in SO(n)$ ($m \in \mathbf{N}$), we define

$$\Omega^\sharp(\Theta) := \frac{1}{m}(\theta_1\Omega + \cdots + \theta_m\Omega),$$

which is called a rotation mean of Ω. Since

$$T_{\theta\Omega} = T_\Omega$$

for any $\theta \in SO(n)$, by Theorem 7.1 we see that

$$T^{1/2}_{\Omega^\sharp(\Theta)} \leq \frac{1}{m}\left(T^{1/2}_{\theta_1\Omega} + \cdots + T^{1/2}_{\theta_m\Omega}\right) = T^{1/2}_\Omega. \tag{7.6}$$

We recall the following lemma on rotation means of bounded convex domains (see [36, Theorem 3.3.5]).

Lemma 7.1 *Let Ω be a bounded convex domain in $\mathbf{R}^n$. Then there exists a sequence of rotation means of Ω converging in Hausdroff metric to a ball $\Omega^\sharp$ with the same mean width of Ω.*

Notice that, possibly up to a translation, $\Omega^\sharp$ is the ball centered at the origin with radius $w(\Omega)/2$, i.e.

$$\Omega^\sharp = B(0, w(\Omega)/2)\,.$$

Let Ω be a bounded convex domain in $\mathbf{R}^n$ and let $\{\Omega_m^\sharp\}_{m=1}^\infty$ be a sequence of rotation means associated to Ω by Lemma 7.1 such that $\Omega_m^\sharp$ converges to $\Omega^\sharp$ as $m \to \infty$. Then it follows from (7.6) that

$$T_{\Omega^\sharp} \leq T_\Omega. \tag{7.7}$$

The latter in fact holds also for non-convex sets, when $\Omega^\sharp$ is regarded as the ball associated to the convex hull $conv(\Omega)$ of Ω by Lemma 7.1. Then, thanks to (7.4) and in view of the considerations made few lines above about the mean width of non-convex sets, (7.7) can be rephrased in the following intriguing way.

Theorem 7.3 *Among sets with given mean width, the ball has the smallest dead core time. In the plane: among sets with given perimeter, the disk has the smallest dead core time.*

Consider the elliptic boundary value problem,

$$\Delta U - U_+^\gamma = 0 \quad \text{in} \quad \Omega, \qquad U = 1 \quad \text{on} \quad \partial\Omega. \tag{7.8}$$

Then problem (7.8) has a unique solution satisfying $0 \leq U < 1$ in Ω and the solution u of (7.1) decreases monotonically in time to the solution U of (7.8). Then, if U has a dead core Z_∞ then $Z_\Omega(t) \subseteq Z_\infty$ for every $t > 0$ and if U has not a dead core, then $Z_\Omega(t) = \emptyset$ for every $t > 0$. Furthermore, the following holds (see [37]).

(i) If $\Omega = B(0, r)$, then the solution U of (7.8) has a dead-core if and only if

$$r^2 \geq P := \frac{2n(1-\gamma) + 4\gamma}{(1-\gamma)^2}. \tag{7.9}$$

In particular, in the case $r^2 = P$, the dead-core of w consists of the origin.

(ii) For any interior point x_0 of the dead core Z_∞ of U, x_0 belongs to the dead-core $Z_\Omega(t)$ of u after a finite time. On the other hand, the points on the boundary of Z_∞ do not belong to $Z_\Omega(t)$ for any finite time.

(iii) The solution U does not have a dead core if $d(\Omega) < 2\sqrt{2(1+\gamma)}/(1-\gamma)^2$.

(iv) The solution u does not have a dead-core if $r_*(\Omega) < P^{1/2}$, where $r_*(\Omega)$ is the radius of the ball whose volume coincides with that of Ω, i.e. $r_*(\Omega) = (|\Omega|/\omega_n)^{1/n}$. Furthermore, the solution u of (7.1) does not reach zero in finite time if $r_*(\Omega) \leq P^{1/2}$, i.e. if $|\Omega| \leq \omega_n P^{n/2}$.

Let us recall that $w(\Omega) = 2r$ if $\Omega = B(0, r)$. Then, by (7.7) and properties (i) and (ii) we have the following result.

Theorem 7.4 *Let Ω be a bounded smooth domain in $\mathbf{R}^n$. Then the solution u of* (7.1) *does not reach zero in finite time if*

$$w(\Omega) \leq 2P^{1/2}.$$

Furthermore, the solution w of (7.8) *does not have a dead-core if*

$$w(\Omega) < 2P^{1/2}.$$

It follows from the Brunn-Minkowski inequality that

$$|\Omega^\sharp(\Theta)|^{1/n} \geq \frac{1}{m}\left(|\theta_1\Omega|^{1/n} + \cdots + |\theta_m\Omega|^{1/n}\right) = |\Omega|^{1/n},$$

which implies that $|\Omega^\sharp| \geq |\Omega|$, that is $w(\Omega)/2 > r_*(\Omega)$. Therefore Theorem 7.4 improves property (iv).

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