Research Article

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Branches of Forced Oscillations Induced by a Delayed Periodic Force

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Abstract: We study global continuation properties of the set of T-periodic solutions of parameterized second order delay differential equations with constant time lag on smooth manifolds. We apply our results to get multiplicity of T-periodic solutions. Our topological approach is mainly based on the notion of degree of a tangent vector field.

Keywords: Periodic Solutions, Forced Motion, Delay Differential Equations, Multiplicity of Periodic Solutions, Forced Motion on Manifolds, Degree of a Tangent Vector Field

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1 Introduction

Let M be a (possibly noncompact) boundaryless smooth manifold embedded in $\mathbb{R}^k$. Consider the following parameterized *delayed motion equation* on M:

$$\ddot{x}_\pi(t) = h\big(x(t), \dot{x}(t)\big) + \lambda f\big(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)\big), \tag{1.1}$$

where $\ddot{x}_\pi(t)$ stands for the tangent part of the acceleration $\ddot{x}(t) \in \mathbb{R}^k$ at the point $x(t) \in M$, λ is a nonnegative real parameter, $\tau > 0$ is a constant time lag, and h, f are tangent vector fields on M. More precisely, the maps $h\colon TM \to \mathbb{R}^k$ and $f\colon \mathbb{R} \times TM \times TM \to \mathbb{R}^k$ are assumed to be continuous and such that $h(p, v)$ and $f(t, p, v, q, w)$ belong to T_pM for any $(t, p, v, q, w) \in \mathbb{R} \times TM \times TM$. Here, given $p \in M$, $T_pM \subseteq \mathbb{R}^k$ is the tangent space of M at p, and

$$TM = \{(p, v) \in \mathbb{R}^k \times \mathbb{R}^k\colon p \in M,\ v \in T_pM\}$$

is the tangent bundle of M. A second order delay differential equation (DDE for short) like (1.1) can be interpreted as a parameterized motion equation of a constrained mechanical system with configuration space M, acted on by a force which may as well depend on the state and velocity at a previous time instant. Indeed, for any fixed $\lambda \geq 0$, the right-hand side of (1.1) represents a *delayed active force* on M.

Assuming that the nonautonomuos vector field f is T-periodic in the first variable, we look for T-periodic solutions of (1.1). Firstly, we study the topological properties of the set of the so-called *T-periodic pairs*

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of (1.1) obtaining a *global continuation* result, Theorem 3.7 below. Then, in Section 4, we apply our result to get multiplicity of T-periodic solutions of (1.1) under additional assumptions.

To obtain our main result, the strategy essentially consists in writing (1.1) as a first order system on the tangent bundle TM which, in turn, is a boundaryless smooth submanifold of $\mathbb{R}^k \times \mathbb{R}^k$. Thus, we take advantage of global continuation results for first order DDEs obtained by M. Furi and the third author in [19].

In the undelayed case, continuation results as well as multiplicity of T-periodic solutions for a second order ODE on M of the form

$$\ddot{x}_\pi(t) = h(x(t), \dot{x}(t)) + \lambda\Phi(t, x(t), \dot{x}(t)), \tag{1.2}$$

where h is as in (1.1), and $\Phi\colon \mathbb{R} \times TM \to \mathbb{R}^k$ is T-periodic in the first variable and such that $\Phi(t, p, v)$ belongs to T_pM for any $(t, p, v) \in \mathbb{R} \times TM$, have been obtained in [18] and [15]. In the present paper, we aim to generalize all these results to the DDE case.

Recently, in collaboration with Benevieri and Furi, we devoted a series of papers to the study of first and second order DDEs on manifolds, focusing on continuation properties and existence of T-periodic solutions (see, e.g., [4] and references therein). In [4], we proved that, for any $\varepsilon > 0$, the equation

$$\ddot{x}_\pi(t) = \Psi(t, x(t), x(t-\tau)) - \varepsilon\dot{x}(t) \tag{1.3}$$

admits at least one *forced oscillation* (i.e., a T-periodic solution) for any continuous map $\Psi\colon \mathbb{R} \times M \times M \to \mathbb{R}^k$ which is T-periodic in the first variable and such that $\Psi(t, p, q)$ belongs to T_pM for any $(t, p, q) \in \mathbb{R} \times M \times M$, provided that the constraint M is compact with nonzero Euler–Poincaré characteristic. As far as we know, when the frictional coefficient ε is zero, the problem of the existence of forced oscillations of (1.3) is still open, even in the undelayed case. For another approach to the study of T-periodic solutions of DDEs on manifolds see, e.g., [1, 2].

More recently, we investigated a different and related class of equations, called *retarded functional differential equations* (RFDEs for short), in which the vector field(s) may depend on the whole history of the process. In [6], we studied, on a manifold M, a second order RFDE with infinite delay of the type

$$x''_\pi(t) = F(t, x_t), \tag{1.4}$$

where, with the standard notation in functional equations, the functional field F is a map

$$F\colon \mathbb{R} \times BU((-\infty, 0], M) \to \mathbb{R}^k$$

such that $F(t, \varphi) \in T_{\varphi(0)}M$ for all (t, φ) (here $BU((-\infty, 0], M)$ denotes the set of bounded and uniformly continuous maps from $(-\infty, 0]$ into M). In the particular case of the *retarded spherical pendulum*, i.e., when $M = S^2$, in [6], we proved that (1.4) has a forced oscillation under the assumptions that F is bounded and locally Lipschitz in the second variable.

In principle, RFDEs represent a generalization of DDEs (as well as of ODEs). Nevertheless, due also to their broad generality, RFDEs require technical assumptions to guarantee basic properties of solutions (such as uniqueness and continuation, see, e.g., [5]), and we do not know how to overcome this difficulty. For instance, in [6], the functional field F is assumed to be locally Lipschitz in the second variable, unlike here, where the main results require the sole continuity assumption. Another characteristic of equations like (1.4) is that the right-hand side is not allowed to depend on the velocity (notice, however, that in [7] a different approach has been proposed to tackle this kind of problems). This can be a weak point in view of applications. The same drawbacks plague also the situation when, unlike in (1.4), the functional field F perturbs a nonzero vector field. In fact, in [10] for the spherical pendulum (i.e., when $M = S^2$), as well as in [11] for the scalar case (i.e., $M = \mathbb{R}$), multiplicity results for T-periodic solutions of functional perturbations of autonomous vector fields have been obtained by the authors, even in the possible presence of friction, at the cost of imposing the same limitations as above on the forcing perturbations. This does not come as a surprise as also for first order RFDEs, like in, e.g., [16], it seems necessary to impose similar restrictions on the forcing term.

By contrast, continuation results as well as multiplicity of T-periodic solutions for periodically forced second order ODEs on manifolds can be obtained, under the sole continuity of the involved vector fields.

Roughly speaking, the problem that we address here could be considered as an "intermediate" situation between the ODE case of equation (1.2) and the RFDE one, where an equation similar to (1.4) is considered. In this middle ground, only minimal regularity assumptions are required, and the active forces may be allowed to depend explicitly on the velocity (which is not the case with (1.4)), thus providing greater flexibility. Besides filling a theoretical gap, our investigation is also justified by applications, since delay differential equations with constant time lag emerge naturally in applied mathematics (see, e.g., [3, 9, 13]).

To illustrate our results, we provide an application to the spherical pendulum, similar in spirit to that of [10], that besides permitting a more general form of friction, can involve a merely continuous periodic forcing term that depends on time and on present and past position and velocity (with a fixed time lag).

2 Preliminaries

2.1 The Degree of a Tangent Vector Field

Let M be a boundaryless smooth manifold embedded in $\mathbb{R}^k$, and let $w\colon M \to \mathbb{R}^k$ be a continuous tangent vector field on M, that is, $w(p)$ belongs to T_pM, the tangent space of M at p. Given an open subset U of M, we say that the pair (w, U) is *admissible* if $w^{-1}(0) \cap U$ is compact. In this case, one can associate to the pair (w, U) an integer, $\deg(w, U)$, called the *degree* (or *Euler characteristic*) of w in U, which, roughly speaking, counts algebraically the number of zeros of w in U (for general reference see, e.g., [20, 21, 25]).

If w is (Fréchet) differentiable at $p \in M$ and $w(p) = 0$, then the differential $dw_p\colon T_pM \to \mathbb{R}^k$ maps T_pM into itself, so that the determinant $\det dw_p$ of dw_p is defined. If, in addition, p is a nondegenerate zero (i.e., $dw_p\colon T_pM \to \mathbb{R}^k$ is injective), then p is an isolated zero and $\det dw_p \neq 0$. In fact, if w is admissible for the degree in U, when the zeros of w are all nondegenerate, then the set $w^{-1}(0) \cap U$ is finite and

$$\deg(w, U) = \sum_{p \in w^{-1}(0) \cap U} \operatorname{sign} \det dw_p. \tag{2.1}$$

Observe that in the flat case, i.e., when $M = \mathbb{R}^k$, $\deg(w, U)$ coincides with the Brouwer degree $\deg_B(w, V, 0)$, where V is any bounded open neighborhood of $w^{-1}(0) \cap U$ whose closure is contained in U. All the standard properties of the Brouwer degree on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, are still valid in the more general context of differentiable manifolds (see, e.g., [25]).

The excision property allows the introduction of the notion of index of an isolated zero of a tangent vector field. Indeed, let $p \in M$ be an isolated zero of w. Clearly, $\deg(w, V)$ is well-defined for each open $V \subseteq M$ such that $V \cap w^{-1}(0) = \{p\}$. By the excision property, $\deg(w, V)$ is constant with respect to such V's. This common value of $\deg(w, V)$ is, by definition, the *index of w at p*, and is denoted by $\mathrm{i}(w, p)$. With this notation, if (w, U) is admissible and all the zeros of w in U are isolated, the additivity property yields that

$$\deg(w, U) = \sum_{p \in w^{-1}(0) \cap U} \mathrm{i}(w, p). \tag{2.2}$$

By formula (2.1), we have that if p is a nondegenerate zero of w, then

$$\mathrm{i}(w, p) = \operatorname{sign} \det dw_p.$$

Notice that (2.1) and (2.2) differ in the fact that, in the latter, the zeros of w are not necessarily nondegenerate as they have to be in the former. In fact, in (2.2), w need not be differentiable at its zeros.

In the case when M is a compact manifold, the celebrated Poincaré–Hopf theorem states that $\deg(w, M)$ coincides with the Euler–Poincaré characteristic $\chi(M)$ of M and, therefore, is independent of w. In particular, if all the zeros of w are isolated, it follows that

$$\chi(M) = \sum_{p \in w^{-1}(0)} \mathrm{i}(w, p).$$

2.2 Delay Differential Equations on Manifolds

In this paper, we will mainly be concerned with second order delay differential equations (DDEs for short) on a boundaryless smooth manifold $M \subseteq \mathbb{R}^k$. Observe that M need not be compact, nor closed as a subset of $\mathbb{R}^k$. Sometimes it will be convenient to write a second order DDE, in an equivalent way, as a first order DDE on the tangent bundle $TM \subseteq \mathbb{R}^{2k} := \mathbb{R}^k \times \mathbb{R}^k$ of M. For this reason, in this section, we also introduce a "general" first order DDE on a manifold $N \subseteq \mathbb{R}^s$. Then, in the sequel of the paper, we will consider the case in which $N = TM \subseteq \mathbb{R}^{2k}$ and, as we will show, the first order equation will descend from a second order one.

Let $N \subseteq \mathbb{R}^s$ be a boundaryless smooth manifold. A continuous map $G\colon \mathbb{R} \times N \times N \to \mathbb{R}^s$ is said to be a *(time-dependent) vector field tangent to N in the second variable* if

$$G(t, \xi, \hat{\xi}) \in T_\xi N \quad \text{for all } (t, \xi, \hat{\xi}) \in \mathbb{R} \times N \times N.$$

Given $\tau \geq 0$ and G as above, consider the delay differential equation

$$\dot{x}(t) = G\big(t, x(t), x(t-\tau)\big). \tag{2.3}$$

By a *solution* of (2.3) we mean a continuous function $x\colon J \to N$, defined on a (possibly unbounded) interval with length greater than τ, which is of class C^1 on the subinterval $(\inf J + \tau, \sup J)$ of J and verifies $\dot{x}(t) = G(t, x(t), x(t-\tau))$ for all $t \in J$ with $t > \inf J + \tau$.

Let us now go back to the manifold $M \subseteq \mathbb{R}^k$ and introduce a second order DDE on M. A continuous map $\varphi\colon \mathbb{R} \times TM \times TM \to \mathbb{R}^k$ such that $\varphi(t, p, v, q, w) \in T_pM$ for all $(t, p, v, q, w) \in \mathbb{R} \times TM \times TM$ will be called a *(time-dependent) delayed active force on M*.

Given $\tau \geq 0$ and a delayed active force φ on M, the motion equation associated with φ can be written in the form

$$\ddot{x}_\pi(t) = \varphi\big(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)\big), \tag{2.4}$$

where $\ddot{x}_\pi(t) \in T_{x(t)}M$ stands for the tangent component of the acceleration $\ddot{x}(t) \in \mathbb{R}^k$ at the point $x(t)$. In fact, equation (2.4) can be interpreted as the motion equation of a constrained mechanical system with configuration space M, acted on by a force which may depend as well on the state and velocity at a previous time instant.

By a *solution* of (2.4), we mean a continuous function $x\colon J \to M$, defined on a (possibly unbounded) interval, with length greater than τ, which is of class C^2 on the subinterval $(\inf J + \tau, \sup J)$ of J and verifies

$$\ddot{x}_\pi(t) = \varphi\big(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)\big)$$

for all $t \in J$ with $t > \inf J + \tau$.

2.3 Parameterized DDEs and the Degree of a Second Order Vector Field

From now on, we will deal with a special case of (2.4), namely, we will consider a parameterized second order DDE of the form

$$\ddot{x}_\pi(t) = h\big(x(t), \dot{x}(t)\big) + \lambda f\big(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)\big), \tag{E$_\lambda$}$$

where $\lambda \geq 0$, $h\colon TM \to \mathbb{R}^k$ is continuous and such that $h(p, v) \in T_pM$ for any $(p, v) \in TM$, and

$$f\colon \mathbb{R} \times TM \times TM \to \mathbb{R}^k$$

is a time-dependent delayed active force.

It is known (see, e.g., [14]) that, by setting $\xi(t) = (x(t), \dot{x}(t))$, (E$_\lambda$) can be written as a first order equation on the tangent bundle TM in the form

$$\dot{\xi}(t) = \hat{h}\big(\xi(t)\big) + \lambda \tilde{f}\big(t, \xi(t), \xi(t-\tau)\big), \tag{2.5}$$

where $\hat{h}\colon TM \to \mathbb{R}^k \times \mathbb{R}^k$ and $\tilde{f}\colon \mathbb{R} \times TM \times TM \to \mathbb{R}^k \times \mathbb{R}^k$ are defined by

$$\hat{h}(p, v) = (v, r(p, v) + h(p, v)),$$
$$\tilde{f}(t, p, v, q, w) = (0, f(t, p, v, q, w)),$$

and $r\colon TM \to \mathbb{R}^k$ is a smooth map, quadratic in the second variable $v \in T_pM$ for any $p \in M$, with values in $(T_pM)^\perp$. Such a map r may be interpreted as the reactive force due to the constraint M. Actually, $r(p, v)$ is the unique vector in $\mathbb{R}^k$ which makes $(v, r(p, v))$ tangent to TM at (p, v). The map $\hat{h}$, called the *second order vector field associated to* h, is a tangent vector field on TM. It is readily verified that also $\tilde{f}$ is a vector field tangent to TM, even if it is not a second order vector field. Hence, for any fixed $\lambda > 0$, the map

$$G(t, \xi, \hat{\xi}) := \hat{h}(\xi) + \lambda\tilde{f}(t, \xi, \hat{\xi}), \quad \xi = (p, v),\ \hat{\xi} = (q, w),$$

is a time-dependent vector field tangent to TM in the second variable, and (2.5) is a first order DDE on TM of the form (2.3). According to the definition of solution of first and second order DDEs given in Section 2.2, one has that equations (E_λ) and (2.5) are equivalent, in the sense that $x\colon J \to M$ is a solution of (E_λ) if and only if $\xi = (x, \dot{x})\colon J \to TM$ is a solution of (2.5).

We close this section with a formula which expresses the degree of $\hat{h}$, which is a tangent vector field on TM, in terms of the degree of $h|_M := h(\cdot\,, 0)$, which is a tangent vector field on M. This notation, as well as the one $W \cap M$ in Lemma 2.1 below, is motivated by the identification of M with the zero section of TM. Indeed, [18, Lemma 3.2 and Corollary 3.3] yield the following:

Lemma 2.1. *Let h and $\hat{h}$ be as above, and let W be an open subset of TM. Then $h|_M$ is admissible on $W \cap M$ if and only if $\hat{h}$ is admissible on W, and* $\deg(\hat{h}, W) = \deg(-h|_M, W \cap M)$. *Furthermore, if M is compact, and W is such that $W \cap M = M$, then* $\deg(\hat{h}, W) = \chi(M)$.

3 Branches of *T*-Periodic Pairs

In this section, assuming that the delayed active force f is periodic of period $T > 0$ in t, we study the set of T-periodic solutions of the parameterized second order DDE (E_λ). Roughly speaking, we will consider the set of pairs (λ, x), with x a T-periodic solution of (E_λ) corresponding to $\lambda \geq 0$, which will be called *T-periodic pairs* (see Definition 3.4 below). We will prove a global continuation property of the set of T-periodic pairs. Namely, we will show that, under appropriate assumptions, there are connected sets of T-periodic pairs “emanating”, so to speak, from the set of zeros of $h(\cdot\,, 0)$. In order to carry out this program, we will first specify the ambient space for the set of T-periodic pairs and then define a corresponding notion for general first order equations on manifolds for which a similar result holds. The latter, applied to equation (2.5), will be a major step in view of the main result of this section, Theorem 3.7.

Let us begin with some notation. Given a subset A of $\mathbb{R}^n$, we will denote by $C^j_T(A)$, $j \in \{0, 1\}$, the metric subspace of the Banach space $(C^j_T(\mathbb{R}^n), \|\cdot\|_j)$ of all the T-periodic C^j maps $x\colon \mathbb{R} \to A$ with the usual C^j norm (when $j = 0$ we will simply write $C_T(A)$). Observe that $C^j_T(A)$ is not complete unless A is complete (i.e. closed in $\mathbb{R}^n$). Nevertheless, if A is locally compact, then $C^j_T(A)$ is locally complete.

As pointed out, it is convenient to preliminarily define the notion of T-periodic pair for parameterized first order DDEs. This is important because, as already mentioned, there is a very strict correlation between the set of T-periodic pairs of (E_λ) and the corresponding set for the associated first order equation (2.5).

Consider the following parameterized first order DDE on a (boundaryless smooth) manifold $N \subseteq \mathbb{R}^s$:

$$\dot{x}(t) = \gamma(x(t)) + \lambda\psi(t, x(t), x(t - \tau)), \tag{3.1}$$

where $\gamma\colon N \to \mathbb{R}^s$ is a tangent vector field on N, and $\psi\colon \mathbb{R} \times N \times N \to \mathbb{R}^s$ is a time-dependent vector field, T-periodic in the first variable and tangent to N in the second one.

Equation (3.1) is clearly a special case of (2.3). We say that $(\lambda, x) \in [0, \infty) \times C_T(N)$ is a *T-periodic pair for* (3.1) if x is a T-periodic solution of (3.1) corresponding to λ. If $\lambda = 0$ and x is constant, then (λ, x)

is said to be *trivial*. Indeed, it is not hard to see that trivial T-periodic pairs of (3.1) correspond to zeros of γ. More precisely, for any $p \in \gamma^{-1}(0)$, we have a trivial T-periodic pair $(0, \overline{p})$ of (3.1), and, conversely, if $(0, x) \in [0, \infty) \times C_T(N)$ is a trivial T-periodic pair, then, by local uniqueness of the solutions of Cauchy problem, $x(t) \equiv p$ for some $p \in \gamma^{-1}(0)$. Notice that (3.1) may have nonconstant T-periodic solutions for $\lambda = 0$, as it happens, for example, in $N = \mathbb{R}^2$, for $T = 2\pi$ and $\gamma(x_1, x_2) = (-x_2, x_1)$.

Denote by Y the set of all the T-periodic pairs of (3.1). Known properties of the set of solutions of differential equations imply that Y is closed, hence it is locally complete, as a closed subset of a locally complete space. We will use the following fact, see [17, Lemma 3.1].

Lemma 3.1. *The set Y is locally compact. Moreover, if N is closed in $\mathbb{R}^s$, any bounded subset of Y is actually totally bounded. As a consequence, in this case, closed and bounded sets of T-periodic pairs are compact.*

The following result (see [19, Theorem 5.1]) concerns the structure of the set of T-periodic pairs of (3.1).

Let us introduce some further notation. Given $N \subseteq \mathbb{R}^s$, denote by $i\colon N \to [0, \infty) \times C_T(N)$ the map that to any point $\xi \in N$ associates the pair $(0, \overline{\xi})$, where $\overline{\xi}$ denotes the function constantly equal to ξ. It is not difficult to show that i is a closed map. Given an open subset $\mathcal{O}$ of $[0, \infty) \times C_T(N)$, we introduce the handy notation $\mathcal{O} \cap N$ for the open subset $i^{-1}(\mathcal{O})$ of N.

Theorem 3.2 (see [19]). *Let $\gamma\colon N \to \mathbb{R}^s$ and $\psi\colon \mathbb{R} \times N \times N \to \mathbb{R}^s$ be as in* (3.1). *Let $\mathcal{O}$ be an open subset of $[0, \infty) \times C_T(N)$, and assume that* $\deg(\gamma, \mathcal{O} \cap N)$ *is defined and nonzero. Then $\mathcal{O}$ contains a connected set of nontrivial T-periodic pairs for* (3.1) *whose closure in $\mathcal{O}$ meets the set* $\{(0, \overline{\xi}) \in \mathcal{O} : \gamma(\xi) = 0\}$ *and is not compact.*

Our aim is to specialize Theorem 3.2 to the case of the first order DDE (2.5), i.e., when $N = TM$ is the tangent bundle of $M \subseteq \mathbb{R}^k$, and the vector fields descend from a second order equation. In this situation, it is convenient to consider the map $\hat{\imath}\colon M \to [0, \infty) \times C_T(TM)$ that associates $p \in M$ to $(0, \overline{(p, 0)})$, where $\overline{(p, 0)}$ is the map in $C_T(TM)$ constantly equal to $(p, 0)$. Similarly to what we have done above, given an open subset $\hat{\Omega}$ of $[0, \infty) \times C_T(TM)$, we denote by $\hat{\Omega} \cap M$ the open set $\hat{\imath}^{-1}(\hat{\Omega})$.

Theorem 3.2 combined with Lemma 2.1 yields the following fact concerning the T-periodic pairs of (2.5).

Lemma 3.3. *Let $h, f, \hat{h}$ and $\tilde{f}$ be as in Section 2.3. Assume in addition that f, are T-periodic in t. Let $\hat{\Omega} \subseteq [0, \infty) \times C_T(TM)$ be open. Assume that* $\deg(h|_M, \hat{\Omega} \cap M)$ *is defined and nonzero. Then $\hat{\Omega}$ contains a connected set of nontrivial T-periodic pairs for* (2.5) *whose closure in $\hat{\Omega}$ meets the set* $\{(0, \overline{(p, 0)}) \in \hat{\Omega} : h|_M(p) = 0\}$ *and is not compact.*

Proof. Notice that with our notation, we have

$$\hat{h}^{-1}(0) \cap (\hat{\Omega} \cap TM) = \{(p, 0) \in \hat{\Omega} \cap TM : h(p, 0) = 0\} = \{p \in \hat{\Omega} \cap M : h|_M(p) = 0\} = h|_M^{-1}(0) \cap (\hat{\Omega} \cap M).$$

Thus, by Lemma 2.1, we have $\deg(\hat{h}, \hat{\Omega} \cap TM) = \deg(h|_M, \hat{\Omega} \cap M)$. The assertion now follows from Theorem 3.2. □

We finally wish to apply Lemma 3.3 in order to obtain, for the second order DDE (E_λ), a global result in the spirit of Theorem 3.2. To this end, we need to introduce the notion of T-periodic pair for (E_λ), where we assume that f is T-periodic in the first variable.

Definition 3.4. A pair $(\lambda, x) \in [0, \infty) \times C_T^1(M)$ is called a *T-periodic pair for the second order DDE* (E_λ) if x is a T-periodic solution of (E_λ) corresponding to λ. In particular, we will say that (λ, x) is *trivial* if $\lambda = 0$, and x is constant.

Observe that for $\lambda = 0$, (E_λ) may also have nonconstant T-periodic solutions. This may happen when $h = 0$, as in the case of the inertial motion on a sphere, but also for tangent vector fields with compact $h^{-1}(0)$ as, for instance, in $M = \mathbb{R}$ and $h(p) = -p$. Thus one might have nontrivial T-periodic pairs also for $\lambda = 0$. Denote by $X \subseteq [0, \infty) \times C_T^1(M)$ and by $\hat{X} \subseteq [0, \infty) \times C_T(TM)$ the set of all the T-periodic pairs of (E_λ) and (2.5), respectively.

As pointed out in Section 2.3, the equations (E_λ) and (2.5) are in some sense equivalent. So, also their T-periodic pairs correspond each other in the sense of the following remark.

Remark 3.5. The map

$$[0,\infty)\times C^1_T(M)\ni(\lambda,x)\overset{j}{\longmapsto}(\lambda,x,\dot{x})\in[0,\infty)\times C_T(TM)$$

establishes a correspondence between the sets X and $\hat{X}$ that "preserves" the notion of triviality for a T-periodic pair. More precisely, if $X_*\subseteq X$ and $\hat{X}_*\subseteq\hat{X}$ denote the sets of the trivial T-periodic pairs of (E_λ) and (2.5), respectively, the map j, when restricted to X, is a homeomorphism of X onto $\hat{X}$ under which X_* corresponds to $\hat{X}_*$. Furthermore, j is a homeomorphism of $[0,\infty)\times C^1_T(M)$ onto its image; thus, as a restriction of a linear map defined on $\mathbb{R}\times C^1_T(\mathbb{R}^k)$, it is actually a Lipschitz map with Lipschitz inverse. Consequently, under this homeomorphism, bounded sets correspond to bounded sets, and totally bounded sets correspond to totally bounded sets.

By this remark and Lemma 3.1, we get that X is locally compact. If, moreover, M is closed in $\mathbb{R}^k$, then the set X has in addition the following useful property.

Remark 3.6. Assume that M is closed in $\mathbb{R}^k$. If $A\subseteq X$ is bounded, by Remark 3.5, $j(A)$ is bounded as well. Since TM is closed, Lemma 3.1 implies that $j(A)$ is totally bounded, and thus, again from Remark 3.5, it follows that $A=j^{-1}(j(A))$ is totally bounded. As a consequence, $C^1_T(M)$ being complete, closed and bounded subsets of X are compact.

Consider the map $i^*: M\to[0,\infty)\times C^1_T(M)$ that associates to any $p\in M$ the constant pair $(0,\overline{p})$, $\overline{p}\in C^1_T(M)$ being the function constantly equal to p. Notice that if $N=TM$, the map $i\colon TM\to[0,\infty)\times C_T(TM)$, introduced before Theorem 3.2, is the composition of i^* with the inclusion of $[0,\infty)\times C^1_T(M)$ in $[0,\infty)\times C_T(M)$. In order to clarify the relations existing between the maps i, $\hat{\imath}$, i^* and j introduced above, consider the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\;p\mapsto(p,0)\;} & TM \\ {\scriptstyle i^*}\downarrow & \overset{\hat{\imath}}{\searrow} & \downarrow{\scriptstyle i} \\ [0,\infty)\times C^1_T(M) & \xrightarrow{\;j\;} & [0,\infty)\times C_T(TM) \end{array}$$

Let Ω be an open subset of $[0,+\infty)\times C^1_T(M)$, and denote, for simplicity, by $\Omega\cap M$ the open subset $(i^*)^{-1}(\Omega)$ of M. Theorem 3.7 gives conditions which ensure the existence in Ω of a connected subset of nontrivial T-periodic pairs of equation (E_λ) (also called an Ω-*global bifurcating branch*), whose closure in Ω is noncompact and intersects the set of trivial T-periodic pairs contained in Ω.

We are now ready to state our main result, Theorem 3.7 below.

Theorem 3.7. *Let $h\colon TM\to\mathbb{R}^k$ be continuous and such that $h(p,v)\in T_pM$ for any $(p,v)\in TM$, and let $f\colon\mathbb{R}\times TM\times TM\to\mathbb{R}^k$ be a T-periodic delayed active force on M. Let $\Omega\subseteq[0,\infty)\times C^1_T(M)$ be an open set, and assume that* $\deg(h|_M,\Omega\cap M)$ *is defined and nonzero. Then Ω contains a connected set of nontrivial T-periodic pairs for* (E_λ)*, whose closure in Ω meets the set* $\{(0,\overline{p})\in\Omega : h|_M(p)=0\}$ *and is not compact.*

Proof. Let j be the map introduced in Remark 3.5. Since $j(\Omega)$ is relatively open in

$$j([0,\infty)\times C^1_T(M))\subseteq[0,\infty)\times C_T(TM),$$

there exists an open subset $\hat{\Omega}$ of $[0,\infty)\times C_T(TM)$ such that

$$\hat{\Omega}\cap j([0,\infty)\times C^1_T(M))=j(\Omega).$$

With our notation we have that $\Omega\cap M=\hat{\Omega}\cap M$.

As previously, denote by X and $\hat{X}$ the set of T-periodic pairs of equations (E_λ) and (2.5), respectively. Let $\Gamma\subseteq\hat{X}$ be a connected component of T-periodic pairs for (2.5) as in Lemma 3.3. Then $(j|_{\hat{X}})^{-1}(\Gamma)$ is a connected set as in the assertion. □

We give now some consequences of Theorem 3.7 in the particular case when M is closed as a subset of $\mathbb{R}^k$, so that Remark 3.6 applies. The first corollary helps visualizing the notion of branch of T-periodic pairs, under this additional assumption.

Corollary 3.8. *Let M, h and f be as in Theorem 3.7, and assume in addition that M is closed in $\mathbb{R}^k$. Let $\Omega \subseteq [0,\infty) \times C^1_T(M)$ be open, and assume that $\deg(h|_M, \Omega \cap M)$ is defined and nonzero. Then a connected subset Γ of Ω, as in Theorem 3.7, is either unbounded or, if bounded, its closure $\mathrm{cl}(\Gamma)$ in $\mathrm{cl}(\Omega)$ reaches the boundary $\partial\Omega$ of Ω.*

Proof. Assume that Γ is bounded, then so is its closure $\mathrm{cl}(\Gamma)$. By Remark 3.6, we get that $\mathrm{cl}(\Gamma)$ is compact. But Γ is not contained in any compact subset of Ω, hence $\mathrm{cl}(\Gamma) \cap \Omega \neq \emptyset$. □

Another consequence of Theorem 3.7 is the following "geometric" property of the set of T-periodic pairs of (E_λ), which is in the spirit of a celebrated result due to P. H. Rabinowitz [26]. Its proof, that we provide here for completeness, is taken – almost verbatim – from [18, Corollary 4.4]

Corollary 3.9. *Let M, h and f be as in Theorem 3.7, and assume in addition that M is closed in $\mathbb{R}^k$. Let U be an open subset of M such that $\deg(h|_M, U) \neq 0$. Then equation (E_λ) admits a connected set Γ of nontrivial T-periodic pairs whose closure $\mathrm{cl}(\Gamma)$ contains some $(0, \overline{p}_0)$ with $p_0 \in U$ and is either unbounded or goes back to some $(0, \overline{p})$, where $p \notin U$.*

Proof. Consider the following open subset of $[0,\infty) \times C^1_T(M)$:

$$\Omega = \big([0,\infty) \times C^1_T(M)\big) \setminus \big(h|_M^{-1}(0) \setminus U\big).$$

Since $\Omega \cap h|_M^{-1}(0) = U \cap h|_M^{-1}(0)$, the excision property of the degree yields

$$\deg(h|_M, \Omega \cap M) = \deg(h|_M, U) \neq 0.$$

Thus, by Theorem 3.7, we get the existence of a connected set Γ of nontrivial T-periodic pairs whose closure $\mathrm{cl}(\Gamma)$ in $[0,\infty) \times C^1_T(M)$ is not contained in any compact subset of Ω.

Assume that

$$\mathrm{cl}(\Gamma) \cap \big(h|_M^{-1}(0) \setminus U\big) = \emptyset,$$

in this case $\mathrm{cl}(\Gamma) \subseteq \Omega$. Since $M \subseteq \mathbb{R}^k$ is closed, Remark 3.6 implies that $\mathrm{cl}(\Gamma)$ cannot be both bounded and complete, otherwise it would be compact. Thus $\mathrm{cl}(\Gamma)$, being a closed subset of the complete metric space $[0,\infty) \times C^1_T(M)$, must be unbounded. □

Finally, if M is compact with nonzero Euler–Poincaré characteristic, we get the following immediate consequence of Theorem 3.7.

Corollary 3.10. *Let M, h and f be as in Theorem 3.7. Assume in addition that M is compact with $\chi(M) \neq 0$. Then there exists an unbounded connected set of nontrivial T-periodic pairs for (E_λ) whose closure in $[0,\infty) \times C^1_T(M)$ intersects the set of trivial T-periodic pairs $\{(0, \overline{p}) : h|_M(p) = 0\}$.*

4 Multiplicity Results

In this section, we apply our global continuation theorem to obtain multiplicity results for T-periodic solutions of (E_λ). Our results are related to those of [15], where the ODE case is considered, and on the other hand, to those of [10], in which we studied the equation of the retarded spherical pendulum. Key notions for our investigation will be those of *T-resonance* and of *ejecting set* (see, e.g., [8, 10, 15]).

Let us start by recalling the definition of ejecting set. Let Z be a metric space and X a subset of $[0,+\infty) \times Z$. Given $\lambda \geq 0$, we denote by X_λ the slice $\{z \in Z : (\lambda, z) \in X\}$. We recall the following definition from [15].

Definition 4.1. We say that $E \subseteq X_0$ is *ejecting* (for X) if it is relatively open in X_0, and there exists a connected subset of X which meets $\{0\} \times E$ and is not contained in $\{0\} \times X_0$.

By inspection of the proof of [15, Theorem 3.3] one immediately sees that the following result holds.

Theorem 4.2 (see [15]). *Let Z be a metric space, and let X be a locally compact subset of $[0, +\infty) \times Z$. Assume that X_0 contains n pairwise disjoint ejecting subsets $E_1, \dots, E_n$. Suppose that $n-1$ of them are compact. Then there are open neighborhoods $U_1, \dots, U_n$ in Z of $E_1, \dots, E_n$, respectively, with pairwise disjoint closure, and a positive number λ_* such that for $\lambda \in [0, \lambda_*)$*

$$X_\lambda \cap U_i \neq \emptyset, \quad i = 1, \dots, n.$$

In particular, we have that the cardinality of X_λ is greater than or equal to n for any $\lambda \in [0, \lambda_*)$.

A notion that is useful for the investigation of the set of T-periodic pairs in a neighborhood of a stationary point is that of T-resonance ([12], see also [8, 15]). Usually, this notion is introduced in the context of first order equations; however, for the sake of simplicity and given the focus of this paper, we confine ourselves to the case of equation (E_λ) for $\lambda = 0$ (for this approach compare [10, Appendix]). Namely,

$$\ddot{x}_\pi(t) = h(x(t), \dot{x}(t)), \tag{4.1}$$

where $h\colon TM \to \mathbb{R}^k$ is as in (E_λ). Here, h is further assumed to be C^1.

Definition 4.3. Given $T > 0$, a point $p \in h|_M^{-1}(0)$ is said to be (second order) *T-resonant* for h if the linearized equation on T_pM,

$$z'' = \partial_1 h(p, 0)z + \partial_2 h(p, 0)z',$$

admits T-periodic solutions other than the trivial one $z(t) \equiv 0$. If this is not true, it is also customary to say informally that a point $p \in h|_M^{-1}(0)$ is *non-T-resonant* for equation (4.1).

Remark 4.4. The resonance of a zero p of $h|_M$ can be decided with a simple criterion. As already pointed out in Section 2.1, $p \in h|_M^{-1}(0)$ implies that the differential $d(h|_M)_p\colon T_pM \to \mathbb{R}^k$ maps T_pM into itself. As shown, e.g., in [22] by a direct computation, one has that p is (second order) *T-resonant* for h if there exists $\ell \in \mathbb{Z}$ such that

$$\det\Big(\partial_1 h(p, 0) + \frac{2\pi\ell i}{T}\partial_2 h(p, 0) + \Big(\frac{2\pi\ell}{T}\Big)^2 I\Big) = 0, \tag{4.2}$$

where I is the identity on T_pM. In particular, if p is *not T-resonant*, taking $\ell = 0$, we have that

$$\det(\partial_1 h(p, 0)) \neq 0.$$

The connection between the notions of ejecting set and of T-resonant point is underlined by the following lemma that shows that, for the set of T-periodic pairs of the parameterized equation (E_λ), a zero of $h|_M$ which is not T-resonant corresponds to an ejecting set (or rather, ejecting point).

Lemma 4.5. *Let p be a zero of $h|_M$ which is not T-resonant. Then $\{\overline{p}\}$ is an ejecting set for the set $X \subseteq [0, \infty) \times C^1_T(M)$ of T-periodic pairs of* (E_λ).

Although there are important differences, the proof of this lemma broadly follows the same strategy as that of [10, Lemma 3.8]. For this reason, we only sketch it here.

Sketch of the proof of Lemma 4.5. The proof is performed in three steps.

(1) A similar argument to [10, Lemma 3.6] shows that since p is not T-resonant then, for any sufficiently small neighborhood V of $\overline{p}$ in $C^1_T(M)$, there exists a real number $\delta_V > 0$ such that $[0, \delta_V] \times \partial V$ does not contain any T-periodic pair of (E_λ). In particular, $\{\overline{p}\}$ is open in the slice X_0.
(2) By step (1), we have that $h|_M$ is admissible for the degree in any sufficiently small neighborhood of p. Furthermore, by Remark 4.4, we have that $\det(\partial_1 h(p, 0)) \neq 0$. Hence, the degree of $h|_M$ in a sufficiently small neighborhood of p is ± 1.
(3) By step (2), we have that in any sufficiently small neighborhood U of p the degree of $h|_M$ is nonzero. Take any neighborhood V as in step (1), and let $\Omega = [0, \delta_V) \times V$. Restricting V if necessary, we can assume $\Omega \cap M \subseteq U$. Theorem 3.7 yields a connected set Γ of nontrivial T-periodic pairs for (E_λ), whose closure in Ω meets the set $\{(0, \overline{p}) \in \Omega : h|_M(p) = 0\}$ and is not compact. By step (1), this set Γ cannot be contained in $\{0\} \times X_0$. Hence, taking into account that $\{\overline{p}\}$ is open in X_0 by step (1), we conclude that $\{\overline{p}\}$ is an ejecting set for X. □

We are now in a position to prove the main result of this section, Theorem 4.6. This multiplicity result provides a generalization of [15, Theorem 3.7] to the delay differential equation (E_λ), see also [10, Theorem 3.9].

Theorem 4.6. *Let $h\colon TM \to \mathbb{R}^k$ be continuous and such that $h(p,v) \in T_pM$ for any $(p,v) \in TM$, and let $f\colon \mathbb{R} \times TM \times TM \to \mathbb{R}^k$ be a T-periodic delayed active force on M. Assume further that M is closed in $\mathbb{R}^k$ and that h is C^1 with $h|_M^{-1}(0)$ compact. Let $p_1, \dots, p_{n-1}$ be zeros of $h|_M$ which are not T-resonant and such that*

$$\sum_{i=1}^{n-1} \mathrm{i}(h|_M, p_i) \neq \deg(h|_M, M). \tag{4.3}$$

Assume that the unperturbed equation

$$\ddot{x}_\pi = h(x, \dot{x})$$

does not admit, in $C^1_T(M)$, unbounded connected sets of T-periodic solutions that meet $(h|_M)^{-1}(0)$. Then, for $\lambda > 0$ sufficiently small, equation (E_λ) admits at least n solutions of period T.

Proof. Observe first that the points $p_1, \dots, p_{n-1}$, being zeros of $h|_M$ which are not T-resonant, are isolated in $h|_M^{-1}(0)$. As above, let X be the set of T-periodic pairs for (E_λ).

By Lemma 4.5, the sets $\{\overline{p_i}\}$, $i = 1, \dots, n-1$, are ejecting for X; in particular, the constant functions $\overline{p_i}$, $i = 1, \dots, n-1$, are isolated points of X_0. Our aim is to prove the existence in the slice X_0 of a further (possibly not compact) ejecting set. This will enable us to apply Theorem 4.2.

Let $W_1, \dots, W_{n-1}$ be pairwise disjoint open neighborhoods of $\overline{p_1}, \dots, \overline{p_{n-1}}$ in $C^1_T(M)$, respectively, with the property that

$$\mathrm{cl}(W_i) \cap X_0 = \{\overline{p_i}\} \quad \text{for } i = 1, \dots, n-1.$$

Set

$$\Omega = [0,\infty) \times \Big(C^1_T(M) \setminus \bigcup_{i=1}^{n-1} \mathrm{cl}(W_i) \Big).$$

By the additivity property of the degree, we get

$$\begin{aligned}\deg(h|_M, \Omega \cap M) &= \deg(h|_M, M) - \deg\Big(h|_M, \bigcup_{i=1}^{n-1} (W_i \cap M)\Big)\\ &= \deg(h|_M, M) - \sum_{i=1}^{n-1} \mathrm{i}(h, p_i) \neq 0.\end{aligned}$$

Thus, Theorem 3.7 yields the existence of a connected set $\Gamma \subseteq \Omega$ of nontrivial T-periodic pairs for (E_λ) whose closure in Ω is not compact and meets the set of trivial T-periodic pairs. Let Υ be the connected component of X containing Γ.

We claim that Υ_0, which is relatively open in X_0, is an ejecting set. To see this, it is sufficient to show that Υ is not contained in $\{0\} \times X_0$. Assume by contradiction that this is not the case, that is $\Upsilon = \{0\} \times \Upsilon_0$. Since $[0,\infty) \times C^1_T(M)$ is complete, the closure $\mathrm{cl}(\Upsilon)$ is complete as well. Hence, Υ is contained in the bounded complete subset $\mathrm{cl}(\Upsilon)$ of Ω, which is a contradiction.

So far, we have proved that the subsets of X_0,

$$E_1 := \{\overline{p_1}\}, \quad \dots, \quad E_{n-1} := \{\overline{p_{n-1}}\}, \quad E_n := \Upsilon_0,$$

are indeed n ejecting sets, the first $n-1$ of which are compact. The assertion now follows from Theorem 4.2. □

Remark 4.7. Note that, when M is compact, by the Poincaré–Hopf theorem one gets $\deg(h|_M, M) = \chi(M)$, so that (4.3) becomes

$$\sum_{i=1}^{n-1} \mathrm{i}(h|_M, p_i) \neq \chi(M).$$

Observe that the multiple T-periodic solutions of (E_λ) given by Theorem 4.6 are distinct as elements of $C^1_T(M)$. Nothing is said about their images. However, if we assume more about the manifold M or about the equation, a more precise statement is possible. An example of this fact is when M is a sphere in $\mathbb{R}^3$, and h is a positional field with possible friction, see Section 5 below.

Another situation, where a more precise statement is possible, occurs when the map h of (E_λ) has a suitable form, as shown in the following proposition.

Proposition 4.8. *Let M be compact and $g\colon M \to \mathbb{R}^k$ a tangent vector field on M. Let $\nu\colon [0,\infty) \to [0,\infty)$ be continuous and such that*
(1) $\nu(s) > 0$ *for all* $s > 0$;
(2) $\liminf_{s\to\infty} \nu(s) = \ell > 0$.
Let f be as in Theorem 4.6 and suppose, in addition, that f is bounded. Let $p_1, \dots, p_{n-1} \in g^{-1}(0)$ be non-T-resonant and such that

$$\sum_{i=1}^{n-1} \mathrm{i}(g, p_i) \neq \chi(M).$$

Then, for $\lambda > 0$ sufficiently small, the equation

$$\ddot{x}_\pi(t) = g(x(t)) - \nu(|\dot{x}(t)|)\dot{x}(t) + \lambda f(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)) \tag{4.4}$$

admits n solutions of period T, whose images are pairwise not coincident.

Observe that the term $-\nu(|\dot{x}(t)|)\dot{x}(t)$ in equation (4.4) grants a quite generalized form of friction. In particular, for $\nu(s) = \kappa$, κ a positive constant, we find the viscous friction formula $-\kappa\dot{x}(t)$ so that equation (4.4) becomes

$$\ddot{x}_\pi(t) = g(x(t)) - \kappa\dot{x}(t) + \lambda f(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)).$$

The proof of Proposition 4.8 is based on the following elementary technical lemma.

Lemma 4.9. *Let ν be a function as in Proposition 4.8, and let $\{a_k\}_{k\in\mathbb{N}} \subseteq [0,\infty)$ be a sequence such that $a_k\nu(a_k) \to 0$. Then $a_k \to 0$.*

Proof. Observe first that, for any $r > 0$, $\inf_{s\geq r} \nu(s) > 0$, otherwise we could find an unbounded sequence $\{s_k\}_{k\in\mathbb{N}} \subseteq [r,\infty)$ with $\nu(s_k) \to 0$, against the assumption that $\liminf_{s\to\infty} \nu(s) > 0$.

Assume by contradiction that $a_k \not\to 0$ as $k \to \infty$. Then, on one hand, $\nu(a_k) \to 0$. On the other hand, $a_k \not\to 0$ means that there exists $\varepsilon_0 > 0$ such that $a_k > \varepsilon_0$ for infinitely many k's. By the first part of the proof, this means that

$$\nu(a_k) \geq \inf_{s\geq\varepsilon_0} \nu(s) > 0,$$

against $\nu(a_k) \to 0$. □

Proof of Proposition 4.8. We claim that the unperturbed equation

$$\ddot{x}_\pi = g(x) - \nu(|\dot{x}|)\dot{x}$$

does not admit (in $C^1_T(M)$) unbounded connected sets of T-periodic solutions that meet $g^{-1}(0)$. We proceed by contradiction. Since $\liminf_{s\to\infty} \nu(s) = \ell > 0$, there exists $\zeta > 0$ such that $\nu(s) > \ell/2$ for all $s > \zeta$. By our contradictory assumption, as M is compact, there exists a T-periodic solution ξ of (4.4) such that

$$\max_{t\in[0,T]} |\dot{\xi}(t)| > \max\Big\{\zeta, \frac{2G}{\ell}\Big\},$$

where $G = \max_{p\in M}|g(p)|$. Put $\theta(t) := |\dot{\xi}(t)|^2$, and let $\sigma \in \mathbb{R}$ be such that $\theta(\sigma) = \max\{\theta(t) : t \in \mathbb{R}\}$. Then

$$\begin{aligned} 0 = \dot{\theta}(\sigma) &= 2\langle\dot{\xi}(\sigma), \ddot{\xi}(\sigma)\rangle = 2\langle\dot{\xi}(\sigma), \ddot{\xi}_\pi(\sigma)\rangle \\ &= 2\langle\dot{\xi}(\sigma), g(\xi(\sigma))\rangle - 2\nu(|\dot{\xi}(\sigma)|)|\dot{\xi}(\sigma)|^2 \\ &\leq 2|\dot{\xi}(\sigma)|G - \ell|\dot{\xi}(\sigma)|^2. \end{aligned}$$

Hence $\max_{t\in\mathbb{R}}|\dot{\xi}(t)| = |\dot{\xi}(\sigma)| \leq 2G/\ell$. This contradiction proves the claim. Therefore, Theorem 4.6 with $h(p,v) = g(p) - \nu(|v|)v$ yields that (4.4) has, for $\lambda > 0$ sufficiently small, at least n solutions $x_1^\lambda, \dots, x_n^\lambda$ of period T.

We only need to prove that their images are pairwise not coincident. Let $W_1, \dots, W_{n-1}$ be as in the proof of Theorem 4.6. By an argument as in step (1) of the proof of Lemma 4.5, reducing λ_* if necessary, we can assume that, for $i = 1, \dots, n-1$, W_i is the open ball in $C^1_T(M)$ centered at $\overline{p_i}$ with radius equal to

$$\mathfrak{r} := \frac{1}{2} \min_{1 \le i < j \le n-1} |p_i - p_j|.$$

Thus, for all $j = 1, \dots, n-1$, we have $\max_{t\in\mathbb{R}} |x_j^\lambda(t) - p_j| < \mathfrak{r}$. That is, the image of x_j^λ is contained in the ball W_j. Hence, for $i, j = 1, \dots, n-1$, $i \ne j$, the images of x_i^λ and x_j^λ are confined into disjoint balls, hence they are disjoint.

To conclude the proof, it is sufficient to show that, reducing $\lambda_* > 0$ if necessary, we have that for $\lambda \in [0, \lambda_*)$, the image of x_n^λ cannot coincide with that of any of the other solutions $x_1^\lambda, \dots, x_{n-1}^\lambda$. Assume the contrary. Then there exists a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$, with $\lambda_k \searrow 0$, such that the image of $x_n^{\lambda_k}$ coincides with at least one of the images of $x_1^{\lambda_k}, \dots, x_{n-1}^{\lambda_k}$ (not necessarily the same for all λ_k's). Since the images of these solutions are disjoint, selecting a subsequence and reordering the solutions, we can assume that $x_n^{\lambda_k}([0,T]) = x_1^{\lambda_k}([0,T])$ for all k. Letting $k \to \infty$, we have that the solution $t \mapsto x_n^{\lambda_k}(t)$ converges uniformly on $[0,T]$ to the constant function $t \mapsto p_1$.

We claim that $\dot{x}_n^{\lambda_k}(t)$ converges to zero uniformly on $[0,T]$. If we prove the claim, we get a contradiction (thus proving the assertion). To see this, let d_k be the diameter of the orbit of $x_n^{\lambda_k}$, that is,

$$d_k := \max_{t_1, t_2 \in [0,T]} |x_n^{\lambda_k}(t_1) - x_n^{\lambda_k}(t_2)|.$$

Thus, $d_k \to 0$ as $k \to \infty$.

Since $g(p_1) = 0$, there exist positive constants ρ and K such that $|g(p)| \le K|p - p_1|$ for all $p \in M$ with $|p - p_1| < \rho$. Thus, for sufficiently large k,

$$|g(x_n^{\lambda_k}(t))| \le K|x_n^{\lambda_k}(t) - p_1| \tag{4.5}$$

for all $t \in \mathbb{R}$.

Now, for any sufficiently large k, let $\varphi_k(t) := |\dot{x}_n^{\lambda_k}(t)|^2$, and let σ_k be such that $\varphi(\sigma_k) = \max\{\varphi_k(t) : t \in \mathbb{R}\}$. Taking the derivative of φ_k with respect to t at σ_k, we get, similarly to the first part of the proof,

$$0 \le 2\langle \dot{x}_n^{\lambda_k}(\sigma_k), g(x_n^{\lambda_k}(\sigma_k))\rangle - 2\nu(|\dot{x}_n^{\lambda_k}(\sigma_k)|)|\dot{x}_n^{\lambda_k}(\sigma_k)|^2 + \lambda_k H|\dot{x}_n^{\lambda_k}(\sigma_k)|, \tag{4.6}$$

where H is an upper bound for the norm of the vector field f. Combining (4.6) with (4.5), we get

$$|\dot{x}_n^{\lambda_k}(\sigma_k)|^2 \nu(|\dot{x}_n^{\lambda_k}(\sigma_k)|) \le K|x_n^{\lambda_k}(\sigma_k) - p_1|\,|\dot{x}_n^{\lambda_k}(\sigma_k)| + \lambda_k H|\dot{x}_n^{\lambda_k}(\sigma_k)|. \tag{4.7}$$

We wish to prove that $|\dot{x}_n^{\lambda_k}(\sigma_k)| \to 0$ as $k \to \infty$. If $|\dot{x}_n^{\lambda_k}(\sigma_k)| = 0$, there is nothing to prove. Therefore, we may assume $|\dot{x}_n^{\lambda_k}(\sigma_k)| \ne 0$. Inequality (4.7) becomes

$$|\dot{x}_n^{\lambda_k}(\sigma_k)|\nu(|\dot{x}_n^{\lambda_k}(\sigma_k)|) \le K|x_n^{\lambda_k}(\sigma_k) - p_1| + \lambda_k H = K d_k + \lambda_k H.$$

Hence $|\dot{x}_n^{\lambda_k}(\sigma_k)|\nu(|\dot{x}_n^{\lambda_k}(\sigma_k)|) \to 0$ as $k \to \infty$. Lemma 4.9 implies that $|\dot{x}_n^{\lambda_k}(\sigma_k)| \to 0$, whence the claim.

We conclude the proof by showing that the claim leads to a contradiction. Let $\Omega \subseteq [0,\infty) \times C^1_T(M)$ be as in the proof of Theorem 4.6, and put $W_n = \Omega_0$. We have $W_j \cap W_n = \emptyset$, $j = 1, \dots, n-1$, and $x_n^{\lambda_k} \in W_n$ for sufficiently large k. The claim, together with the fact that $t \mapsto x_n^{\lambda_k}(t)$ converges uniformly on $[0,T]$ to the constant function $t \mapsto p_1$, implies that, eventually, $x_n^{\lambda_k} \in W_1$. Hence $x_n^{\lambda_k} \in W_1 \cap W_n$ contradicting $W_1 \cap W_n = \emptyset$. □

Remark 4.10. An inspection of the first part of the proof of Theorem 4.8 reveals that, in the hypotheses of Theorem 4.6, even without the additional assumptions of Theorem 4.8, the images of at least $n-1$ of the T-periodic solutions are actually disjoint (a stronger assertion than that of the theorem) for sufficiently small $\lambda > 0$. Roughly speaking, these are the solutions branching away from the zeros of $h|_M$ which are not T-resonant.

5 An Application to the Spherical Pendulum

We illustrate here the results obtained in the previous sections by an application to multiplicity of forced oscillations for the spherical pendulum. Existence and multiplicity results for periodic orbits of pendulum and pendulum-like equations have been successfully pursued with a variety of techniques. We cannot provide here an exhaustive list of references, we only mention the survey papers [23, 24] for an historical perspective.

Let $\boldsymbol{S} \subseteq \mathbb{R}^3$ be the sphere centered at the origin with radius $r > 0$, that is, $\boldsymbol{S} = \{x \in \mathbb{R}^3 : |x| = r\}$. Consider the following equation on $\boldsymbol{S}$:

$$m\ddot{x}(t) = -m\frac{|\dot{x}(t)|^2}{r^2}x(t) + g(x(t)) - \eta\nu(|\dot{x}(t)|)\dot{x}(t) + \lambda f(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)), \tag{5.1}$$

where

- $m > 0$;
- $g\colon \boldsymbol{S} \to \mathbb{R}^3$ is a C^1 tangent vector field on $\boldsymbol{S}$;
- $\eta \geq 0$;
- $\nu\colon [0, \infty) \to [0, \infty)$ is as in Proposition 4.8;
- $f\colon \mathbb{R} \times T\boldsymbol{S} \times T\boldsymbol{S} \to \mathbb{R}^3$, as in Theorem 4.6, is a T-periodic delayed active force on $\boldsymbol{S}$.

Equation (5.1) represents the motion equation of a particle of mass m constrained to $\boldsymbol{S}$ and acted on by three forces: the positional tangent vector field g, a possible friction and a T-periodic forcing term λf which depends on a past state of the system.

The coefficient η in equation (5.1) permits us to include, in contrast to Proposition 4.8, also the case of absence of friction. In fact, $\nu(s)$ is allowed to be zero only for $s = 0$.

Observe that with an appropriate choice of units, we can assume without loss of generality that $m = 1$. Moreover, the term $R(p, v) = -m(|v|^2/r^2)p$ in equation (5.1) represents the reactive force due to the constraint. Hence, (5.1) can be rewritten, using our notation, as

$$\ddot{x}_\pi(t) = g(x(t)) - \eta\nu(|\dot{x}(t)|)\dot{x}(t) + \lambda f(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)). \tag{5.2}$$

Remark 5.1. A physically relevant example is obtained when g is the tangential component of the gravitational force. That is, when g is the tangent vector field

$$g_{\mathfrak{g}}(x_1, x_2, x_3) := \frac{\mathfrak{g}}{r^2}(x_3x_1, x_3x_2, -(r^2 - x_3^2)),$$

where $p = (x_1, x_2, x_3) \in \boldsymbol{S}$, and $\mathfrak{g}$ is a constant expressing the intensity of the gravitational field (considered uniform in this model).

As an application of Theorem 4.6 and Proposition 4.8, we obtain Proposition 5.2 below, yielding a multiplicity result for the T-periodic solutions of equation (5.2) (or, equivalently, (5.1)) when $\lambda > 0$ is small. An analogous result for an RFDE (equation of the retarded spherical pendulum) has been obtained in [10, Theorem 3.1] (see also [16]). However, as pointed out in the introduction, Proposition 5.2 cannot be deduced as a direct consequence of [10, Theorem 3.1].

Proposition 5.2. *Let η, ν, g, λ and f be as above. Assume that g admits $n-1$ zeros, $p_1, \dots, p_{n-1}$, that are not T-resonant and such that*

$$\sum_{i=1}^{n-1} \mathrm{i}(g, p_i) \neq 2.$$

Then, for $\lambda > 0$ sufficiently small, equation (5.2) *(or* (5.1)*, equivalently) admits n solutions of period T whose images are pairwise not coincident.*

Proof. The case $\eta > 0$ follows immediately from Proposition 4.8 recalling that $\chi(\boldsymbol{S}) = 2$.

Consider the case $\eta = 0$, that is consider on $\boldsymbol{S}$ the equation

$$\ddot{x}_\pi(t) = g(x(t)) + \lambda f(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)) \tag{5.3}$$

that coincides with (5.2) when $\eta = 0$.

Lemma 4.1 in [17] shows that any connected set of solutions in $C^1_T(\boldsymbol{S})$ of the second order equation (5.3) for $\lambda = 0$ is unbounded. Thus, by Theorem 4.6 and Remark 4.7 (again recall that $\chi(\boldsymbol{S}) = 2$), we have that (5.3) admits at least n solutions of period T for $\lambda > 0$ sufficiently small. The same argument of the second part of the proof of [10, Theorem 3.9] shows that the images of these solutions are pairwise not coincident. □

In the case when g represents the gravitational attraction, i.e., when $g = g_{\mathfrak{g}}$, there are two zeros of $g_{\mathfrak{g}}$ at the poles $(0, 0, r)$ and $(0, 0, -r)$, the first of which is necessarily not T-resonant, as it follows from (4.2). Hence, Proposition 5.2 yields the following consequence (compare [10, Theorem 3.2]).

Corollary 5.3. *When $g = g_{\mathfrak{g}}$, for $\lambda > 0$ sufficiently small, equation* (5.1) *admits at least two T-periodic solutions whose images are not coincident.*

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