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Short-time behavior for game-theoretic p-caloric functions

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ABSTRACT

We consider the solution u of $u_t - \Delta_p^G u = 0$ in a (not necessarily bounded) domain Ω, such that $u = 0$ in Ω at time $t = 0$ and $u = 1$ on the boundary of Ω at all times. Here, Δ_p^G is the *game-theoretic* or *normalized* p-laplacian. We derive new precise asymptotic formulas for $t \to 0$, that generalize that of S.R.S. Varadhan [39] for large deviations and that of the second author and S. Sakaguchi [26] for the heat content of a ball touching the boundary. We also determine the behavior for $t \to 0$ of the q-mean of u on such a ball. Applications to time-invariant level surfaces of u are then obtained.

RÉSUMÉ

Dans un domaine Ω (pas nécessairement borné), on considère la solution u de l'équation $u_t - \Delta_p^G u = 0$, telle que $u = 0$ dans Ω au temps $t = 0$ et $u = 1$ sur la frontière de Ω pour tout temps $t > 0$. Ici, Δ_p^G dénote le p-laplacien de la théorie des jeux ou p-laplacien normalisé. On dérive des nouvelles formules asymptotiques précises pour $t \to 0$, qui généralisent celle de S.R.S. Varadhan [39] pour les grandes déviations et celle du deuxième auteur et S. Sakaguchi [26] pour le contenu calorique d'une boule qui touche la frontière du domaine. On détermine aussi le comportement pour $t \to 0$ de la q-moyenne de u sur cette boule. On obtient ainsi des applications aux surfaces de niveau de u invariantes dans le temps.

1. Introduction

The *game-theoretic p-laplacian* is formally defined by

$$\Delta_p^G u = \frac{1}{p}\, |\nabla u|^{2-p} \operatorname{div}\left\{ |\nabla u|^{p-2} \nabla u \right\} \tag{1.1}$$

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or

$$\Delta_p^G u = \frac{1}{p}\left\{\Delta u + (p-2)\frac{\langle \nabla^2 u \nabla u, \nabla u\rangle}{|\nabla u|^2}\right\}$$

and can be written as

$$\Delta_p^G u = \frac{1}{p}\,|\nabla u|^{2-p}\Delta_p u,$$

where

$$\Delta_p u = \operatorname{div}\left\{|\nabla u|^{p-2}\nabla u\right\}$$

is the classical p-laplacian. We can suppose that $p \in (1,\infty]$, if we agree that for Δ_∞^G we intend the limit of Δ_p^G as $p \to \infty$. Notice that $2\Delta_2^G = \Delta$, the classical Laplace operator.

The use of Δ_p^G finds applications in several fields. Certainly in game theory, in the study of the so-called *tug-of-war* games (for $p = \infty$, [35]) and their variants *with noise* (for $1 < p < \infty$, [36], [34]). The extremal case $p = 1$ is related to the motion of hypersuperfaces by the *mean curvature flow* ([14]). The 1-homogeneity of Δ_p^G also makes the corresponding parabolic operator $\partial_t - \Delta_p^G$ scaling invariant, and that is useful for some techniques of *image processing*, where the brightness of an initial image does not affect the evolution process. As shown in [12], the choice of the parameter p affects in which direction the brightness evolves.

It is evident that, for $p \neq 2$, Δ_p^G and Δ_p are both nonlinear. However, Δ_p^G is somewhat reminiscent of the lost linearity of the Laplace operator, since it is 1-homogeneous, differently from Δ_p, which is instead $(p-1)$-homogeneous. The nonlinearity of Δ_p^G is indeed due to its non-additivity. Nevertheless, Δ_p^G acts additively if one of the relevant summands is constant and, more importantly, on functions of one variable and on radially symmetric functions. We shall see in the sequel that these last properties are decisive for the purposes of this paper.

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, and let Γ be its boundary. In this paper, we shall consider properly generalized solutions of the following problem:

$$u_t = \Delta_p^G u \quad \text{in} \quad \Omega \times (0,\infty), \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.3}$$

$$u = 1 \quad \text{on} \quad \Gamma \times (0,\infty). \tag{1.4}$$

We point out that equation (1.2) is not variational and hence, since Δ_p^G is not even defined at the spatial critical points of u, a convenient way to define a solution of (1.2) is in the *viscosity sense*. Besides the seminal work of Crandall, Ishii and Lions ([10]), where one can find the foundations of that theory, an appropriate list of references for the particular case of Δ_p^G includes [3], [5,6], [12], [14], [19], [20], [18], [34]. The two conditions in (1.3) and (1.4) should be intended in a classical sense, that is u should be continuous on $\overline{\Omega} \times [0,\infty)$ away from $\Gamma \times \{0\}$. The existence of a solution of (1.2)–(1.4) in this sense can be obtained by adapting arguments contained in [6] and [34] (see Remark 7).

The concern of this paper is the study of the behavior of the solution of (1.2)–(1.4) when $t \to 0^+$. When $p = 2$, it is well-known that the asymptotics of the solution $u(x,t)$ is controlled by the distance of the point $x \in \Omega$ from Γ; in fact we know that

$$\lim_{t\to 0^+} 2t \log u(x,t) = -d_\Gamma(x)^2 \quad \text{uniformly for} \quad x \in \overline{\Omega}, \tag{1.5}$$

where

$$d_\Gamma(x) = \min_{y\in\Gamma} |x-y| \quad \text{for} \quad x \in \Omega.$$

Formula (1.5) was proved in [39], by classical methods, in [13], by using arguments pertaining the theory of viscosity solutions, and in [15, Section 10.1], by probabilistic methods.

The second author and S. Sakaguchi used (1.5) to prove another formula that links even more the short-time behavior of the solution u to the geometric features of the domain. Indeed, in [28], for a sufficiently smooth domain, it is showed that, if $B_R(x) \subset \Omega$ is a ball centered at $x \in \Omega$ and touching Γ at a unique point $y_x \in \Gamma$ (hence $R = d_\Gamma(x) = |x - y_x|$), then

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int\limits_{B_R(x)} u(z,t)\,dz = \frac{c_N\, R^{\frac{N-1}{2}}}{\sqrt{\Pi_\Gamma(y_x)}}, \tag{1.6}$$

where

$$\Pi_\Gamma(y_x) = \prod_{j=1}^{N-1} \Big[1 - R\,\kappa_j(y_x)\Big]. \tag{1.7}$$

Here, $\kappa_j(y_x), j = 1, \dots, N-1$, are the principal curvatures of Γ at y_x, and c_N is a constant only depending on the dimension. In [26], the integral in (1.6) is referred to as the *heat content* of the ball $B_R(x)$.

Generalizations of the two formulas (1.5) and (1.6) to nonlinear settings can be found in [28], [38], for the evolutionary p-Laplace equation, and in [30], for a class of non-degenerate fast diffusion equations. In order to obtain (1.5), one can merely assume that Ω, bounded or not, is such that $\Gamma = \partial\left(\mathbb{R}^N \setminus \overline{\Omega}\right)$ (see [39], for the linear case; [32], [33], for the various nonlinear cases; [29], [31], [32], for the extension to unbounded domains).

The aim of this article is to extend (1.5) and (1.6) to the case $p \neq 2$, that is when the problem (1.2)–(1.4) is considered. In Subsections 2.4 and 3.1, we shall in fact prove for $1 < p \le \infty$ the following companions to formulas (1.5) and (1.6):

$$\lim_{t\to 0^+} 4t \log u(x,t) = -p'\, d_\Gamma(x)^2, \tag{1.8}$$

uniformly on every compact subset of $\overline{\Omega}$,

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int\limits_{B_R(x)} u(z,t)\,dz = \frac{c_N\, R^{\frac{N-1}{2}}}{(p')^{\frac{N+1}{4}} \sqrt{\Pi_\Gamma(y_x)}}. \tag{1.9}$$

Here $p' = p/(p-1)$ is, as usual, the conjugate exponent of p, and the constant c_N can be explicitly computed (see Theorem 3.3). It is worth noticing that (1.8) and (1.9) differ from (1.5) and (1.6) only in the constants at the right-hand sides. In Section 2, we shall also see how to extend the validity (1.8) to the case in which the constant in (1.4) is replaced by a function $h : \Gamma \times (0,\infty) \to \mathbb{R}$ bounded below and above by positive constants (see Corollary 2.11).

The integral at the left-hand side of (1.9) is just a multiple of the mean value of $u(\cdot,t)$ on $B_R(x)$. This can be replaced by other significant statistical quantities related to u. One that seems to be particularly appropriate in this context is the so-called *q-mean* $\mu_q^u(x,t)$ of $u(\cdot,t)$ on $B_R(x)$, that can be defined for $1 < q \le \infty$ as

$$\text{the unique } \mu = \mu_q^u(x,t) \;\text{ such that } \|u(\cdot,t) - \mu\|_{L^q(B_R(x))} \le \|u(\cdot,t) - \lambda\|_{L^q(B_R(x))} \;\text{ for all } \;\lambda \in \mathbb{R}. \tag{1.10}$$

We shall show the following scale invariant formula for $\mu_q^u(x,t)$:

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$$\lim_{t\to 0^+}\left(\frac{R^2}{t}\right)^{\frac{N+1}{4(q-1)}}\mu_q^u(x,t)=\frac{c_{N,q}}{(p')^{\frac{N+1}{4(q-1)}}\,\Pi_\Gamma(y_x)^{\frac{1}{2(q-1)}}}\tag{1.11}$$

(see Theorem 3.5 for the value of $c_{N,q}$).

Formula (1.8) thus extends to a game-theoretic realm the information given by (1.5), that was originally conceived in the context of *large deviations theory*, where it is related to the asymptotic probability that a sample path in a stochastic process remains in a domain for a given amount of time (see [15], [13]). Besides this kind of application, (1.8) can be used to derive topological and geometric information about the solution of (1.2)–(1.4). For instance, it can help us to study the set of minimum points of u that, due to (1.5), indeed approaches that of d_Γ, when $t\to 0^+$ (as seen in [27] or [7] for the case $p=2$).

Another application is to the study of the so-called *stationary* or *time-invariant* level surfaces of solutions of evolutionary partial differential equations. A surface $\Sigma\subset\Omega$ (of co-dimension 1) is said to be a *time-invariant level surface* for a solution u of (1.2)–(1.4), if it is a level surface for $u(\cdot,t)$ for any time $t>0$, that is, if there is a function $a_\Sigma:(0,\infty)\to\mathbb{R}$ such that

$$u(x,t)=a_\Sigma(t)\ \text{ for any }\ (x,t)\in\Sigma\times(0,\infty).$$

Thus, it is not difficult to infer that, under suitable sufficient assumptions on Σ, (1.5) implies that a time-invariant surface Σ is always *parallel* to Γ.

Time-invariant level surfaces are known to enjoy remarkable symmetry properties (see [26–33]), [24], [25], [9], [23]). For instance, in [26], [29] and [24], (1.6) was used to prove that stationary isothermic surfaces must be spherical, if Γ is compact, or planar or cylindrical, if Γ is not compact and satisfies certain global assumptions. In fact, it was shown that Σ is time-invariant if and only if the heat content of any ball $B_r(x)\subset\Omega$ satisfies a *balance law*, that is it does not depend on x for $x\in\Sigma$ (but only on r and t). By using this fact for $r=R$ and (1.6), one can easily infer that

$$\Pi_\Gamma=\text{constant on }\Gamma\tag{1.12}$$

— that is, it turns out that Γ is a *Weingarten-type* surface. The above mentioned sufficient assumptions on Γ and (1.12) then help to obtain the desired symmetry results.

When $p\neq 2$, due to the lack of linearity, it is not known that, if the solution of (1.2)–(1.4) admits a time-invariant level surface Σ, then (1.12) holds — nevertheless, we believe that it should hold true in all the cases $1<p\le\infty$. For now, by using formulas (1.9) and (1.11), we can only claim that, if for $1<q<\infty$ the q-mean $\mu_q^u(x,t)$ defined in (1.10) does not depend on x for $x\in\Sigma$, then (1.12) holds and hence some symmetry results can be inferred (see Subsection 3.3).

The proof of (1.8) follows the tracks of Varadhan's argument and is presented in Section 2. In [39], by taking advantage of the linearity of the heat equation, (1.5) is obtained by proving the formula

$$\lim_{\varepsilon\to 0^+}\varepsilon\,\log u^\varepsilon(x)=-d_\Gamma(x),\ \ x\in\overline{\Omega},$$

for the modified Laplace transform of u,

$$u^\varepsilon(x)=\varepsilon^{-2}\int_0^\infty u(x,t)\,e^{-t/\varepsilon^2}dt,\ \ x\in\overline{\Omega},$$

and then by employing an inverse-Laplace-transform argument.

When $p\neq 2$, we bypass the lack of linearity in two steps. First, by the same Varadhan's argument, we prove (1.8) when Ω is a half-space and a ball — here, we take advantage of the linearity of Δ_p^G on one-dimensional and radially symmetric functions. Secondly, the validity of (1.8) is obtained by employing

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as barriers the global solution of (1.2) (see [5]), that generalizes the fundamental solution of case $p = 2$, and the solution of (1.2)–(1.4) in the ball.

The proofs of (1.9) and (1.11) are carried out in Section 3 and are based on *new sharper* short-time asymptotic estimates. In fact, in Theorems 2.9 and 2.10, under the assumption that Γ is of class C^2, we show that

$$4t \log u(x,t) + p'\, d_\Gamma(x)^2 = O(t \log t) \quad \text{as} \quad t \to 0^+,$$

uniformly for $x \in \overline{\Omega}$. We then construct lower and upper barriers for $u(x,t)$ that depend on d_Γ and t. Finally, for those barriers we obtain suitable versions of (1.9) and (1.11), by applying a geometrical lemma ([28, Lemma 2.1]). To obtain (1.11), the monotonicity and continuity properties of the q-mean $\mu_q^u(x,t)$ with respect to u (see [17]) turn out to be very useful.

2. First-order asymptotics

In this section, we shall prove (1.8). We start with an asymptotic formula for the solutions of a family of elliptic problems related to (1.2)–(1.4).

2.1. An auxiliary elliptic problem

For $\varepsilon > 0$, we consider the following auxiliary elliptic problem:

$$u - \varepsilon^2 \Delta_p^G u = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

$$u = 1 \quad \text{on} \quad \Gamma. \tag{2.2}$$

This problem is somewhat related to (1.2)–(1.4) in the sense that it is easy to show that, when $p = 2$, due to the linearity of the Laplace operator, the function defined by

$$u^\varepsilon(x) = \frac{1}{\varepsilon^2} \int_0^\infty u(x,\tau)\, e^{-\tau/\varepsilon^2}\, d\tau, \quad \text{for} \quad x \in \overline{\Omega}, \tag{2.3}$$

satisfies (2.1)–(2.2), if $u = u(x,t)$ is a solution of (1.2)–(1.4) (see [26]). When $p \neq 2$, Δ_p^G is no longer linear in general. Nevertheless, it is linear on radially symmetric functions and this fact will be useful in the sequel. We shall denote by B_R the ball with radius R centered at the origin.

Lemma 2.1. *Set $1 < p \le \infty$ and $\Omega = B_R$. Then, the (viscosity) solution of (2.1) and (2.2) is given by*

$$u^\varepsilon(x) = \begin{cases} \dfrac{\int_0^\pi e^{\sqrt{p'}\frac{|x|}{\varepsilon}\cos\theta} (\sin\theta)^\alpha\, d\theta}{\int_0^\pi e^{\sqrt{p'}\frac{R}{\varepsilon}\cos\theta} (\sin\theta)^\alpha\, d\theta} & \text{if } 1 < p < \infty, \\[2ex] \dfrac{\cosh(|x|/\varepsilon)}{\cosh(R/\varepsilon)} & \text{if } p = \infty, \end{cases} \tag{2.4}$$

for $x \in \overline{B}_R$; here $\alpha = \frac{N-p}{p-1}$.

Proof. We observe that the function u^ε is twice-differentiable in B_R and $x = 0$ is its unique critical point. By a direct computation, u^ε satisfies (2.1), away from its critical point. Lemma A.3 ensures that u^ε is a (viscosity) solution of (2.1) in B_R. It is evident that $u^\varepsilon(x)$ also satisfies (2.2). Thus, by the uniqueness of viscosity solutions of problem (2.1)–(2.2) (see Lemma A.2), we get our claim. □

Theorem 2.2 *(Asymptotics as $\varepsilon \to 0^+$). Set $1 < p \le \infty$, $\Omega = B_R$, and let u^ε be the solution of (2.1)–(2.2). Then, for every $x \in \overline{B}_R$, it holds that*

$$\lim_{\varepsilon\to 0^+} \varepsilon \log [u^\varepsilon(x)] = -\sqrt{p'}\, d_\Gamma(x),$$

and the limit is uniform on $\overline{B}_R$.

Proof. We present the proof of the more difficult case $1 < p < \infty$. We observe that (2.4) implies that

$$u^\varepsilon(x)\, e^{\sqrt{p'}\frac{R-|x|}{\varepsilon}} = \frac{\int_0^\pi e^{-\sqrt{p'}\frac{|x|}{\varepsilon}(1-\cos\theta)}(\sin\theta)^\alpha\, d\theta}{\int_0^\pi e^{-\sqrt{p'}\frac{R}{\varepsilon}(1-\cos\theta)}(\sin\theta)^\alpha\, d\theta},$$

and hence

$$\varepsilon \log u^\varepsilon(x) + \sqrt{p'}\,(R-|x|) = \log\left[\frac{\int_0^\pi e^{-\sqrt{p'}\frac{|x|}{\varepsilon}(1-\cos\theta)}(\sin\theta)^\alpha\, d\theta}{\int_0^\pi e^{-\sqrt{p'}\frac{R}{\varepsilon}(1-\cos\theta)}(\sin\theta)^\alpha\, d\theta}\right]^\varepsilon. \tag{2.5}$$

Now, each value of the function

$$f_\varepsilon(x) = \left\{\frac{1}{\int_0^\pi(\sin\theta)^\alpha\, d\theta}\int_0^\pi e^{-\sqrt{p'}\frac{|x|}{\varepsilon}(1-\cos\theta)}(\sin\theta)^\alpha\, d\theta\right\}^\varepsilon$$

can be viewed as the norm of the function (of θ) $e^{-\sqrt{p'}|x|(1-\cos\theta)}$ in the Lebesgue space $L^{\frac{1}{\varepsilon}}(0,\pi)$ equipped with the measure $(\sin\theta)^\alpha d\theta/\int_0^\pi(\sin\theta)^\alpha d\theta$, that is unitary. It is well known that such a norm increases as $\varepsilon \to 0^+$ and tends to the function

$$f_0(x) = \max\left\{e^{-\sqrt{p'}|x|(1-\cos\theta)} : \theta\in[0,\pi]\right\} = 1,\ x \in \overline{\Omega},$$

as seen, for example, in [22, Section 2.1].

Therefore, from (2.5) we conclude that $\varepsilon \log u^\varepsilon + \sqrt{p'}\, d_\Gamma$ converges pointwise to zero as $\varepsilon \to 0^+$, since $d_\Gamma(x) = R-|x|$. Moreover, the convergence is uniform on $\overline{B}_R$ owing to *Dini's monotone convergence theorem* (see [37, Chapter 7, Theorem 7.13]). Indeed, $\overline{B}_R$ is compact, the sequence $\{f_\varepsilon\}_{\varepsilon>0}$ is monotonic, and the functions f_ε and f_0 are continuous on $\overline{B}_R$. □

2.2. The parabolic one-dimensional and radially symmetric cases

Preliminarily, we focus our attention on the case of the half-space $H \subset \mathbb{R}^N$ in which $x_1 > 0$. In this case the problem (1.2)–(1.4) has a regular one-dimensional solution, as the following proposition states. In what follows, we will use the *complementary error function* defined by

$$\operatorname{Erfc}(\sigma) = \frac{2}{\sqrt{\pi}}\int_\sigma^\infty e^{-\tau^2}\, d\tau, \quad \sigma\in\mathbb{R}.$$

Proposition 2.3. *Set $\Omega = H$; the solution of the problem (1.2)–(1.4) is the function Ψ defined by*

$$\Psi(x,t) = \sqrt{\frac{p'}{4\pi}}\int_{\frac{x_1}{\sqrt{t}}}^\infty e^{-\frac{1}{4}p'\sigma^2}\, d\sigma = \operatorname{Erfc}\left(\frac{\sqrt{p'}x_1}{2\sqrt{t}}\right) \quad \textit{for} \quad (x,t)\in\overline{H}\times(0,\infty).$$

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Moreover, it holds that

$$\lim_{t \to 0^+} 4t \log [\Psi(x,t)] = -p'\, d_\Gamma(x)^2,$$

uniformly for x in every strip $\{x \in \mathbb{R}^N : 0 \le x_1 \le \delta\}$ with $\delta > 0$.

Proof. By direct calculation, we see that Ψ is a classical solution of (1.2); an inspection tells us that it also satisfies conditions (1.3) and (1.4).

By a change of variables, we get that

$$\Psi(x,t) = e^{-\frac{p' x_1^2}{4t}} \sqrt{\frac{p'}{4\pi}} \int_0^\infty e^{-\frac{1}{2} p' \frac{x_1}{\sqrt{t}}\, \sigma - \frac{1}{4} p' \sigma^2} d\sigma,$$

and hence

$$4t \log \Psi(x,t) + p' x_1^2 = 4t \, \log \left(\sqrt{\frac{p'}{4\pi}} \int_0^\infty e^{-\frac{1}{2} p' \frac{x_1}{\sqrt{t}}\, \sigma - \frac{1}{4} p' \sigma^2} d\sigma \right).$$

Thus, for $0 \le x_1 \le \delta$, we have that

$$4t \, \log \left(\sqrt{\frac{p'}{4\pi}} \int_0^\infty e^{-\frac{1}{2} p' \frac{\delta}{\sqrt{t}}\, \sigma - \frac{1}{4} p' \sigma^2} d\sigma \right) \le 4t \log \Psi(x,t) + p' x_1^2 \le 0,$$

and this implies the desired uniform convergence. □

To determine the short-time asymptotic behavior of the solution of (1.2)–(1.4) in a ball, the following lemma, that holds for a general domain, will be useful.

Lemma 2.4. *Let Ω be a domain in $\mathbb{R}^N$ and $u(x,t)$ be the solution of problem (1.2)–(1.4). Then, for each $x \in \overline{\Omega}$, the function $(0,\infty) \ni t \mapsto u(x,t)$ is increasing.*

Proof. Fix $\tau > 0$ and define the function $v = v(x,t)$ by

$$v(x,t) = u(x, t+\tau), \ \ (x,t) \in \Omega \times (0,\infty).$$

Since (1.2) is invariant under translations in time, one can easily check that v satisfies (1.2) and (1.4), and is such that $v(x,0) = u(x,\tau)$, that is positive by the maximum principle for u. Thus, the comparison principle (see Theorem A.1) gives that $v \ge u$ on $\overline{\Omega} \times (0,\infty)$, which implies the desired monotonicity for u. □

Next, we need to be sure that the viscosity solution of (1.2)–(1.4) in a ball B_R can be transformed into the viscosity solution of (2.1)–(2.2).

Lemma 2.5 *(Solution in a ball). Set $1 < p \le \infty$ and let Ω be the ball B_R centered at the origin with radius R; the function u^ε, defined by (2.4), results from (2.3), where $u(x,t)$ is the (viscosity) solution of (1.2)–(1.4).*

Proof. Owing to the linearity of (1.2), the unique solution of $u(x,t)$ can be explicitly computed as a series expansion (see [21]).

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In fact, if $1 < p < \infty$, we have that

$$u(x,t) = 1 - 2\sum_{n=1}^{\infty} \frac{e^{-\frac{\gamma_n^2}{p' R^2}\,t}}{\gamma_n J_{\beta+1}(\gamma_n)} \left(\frac{R}{|x|}\right)^{\beta} J_\beta\left(\frac{\gamma_n}{R}|x|\right),$$

where J_β is the Bessel function of the first kind of order $\beta = (N-p)/(2p-2)$ and γ_n are the positive zeros of J_β; if $p = \infty$, we get instead:

$$u(x,t) = 1 - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} e^{-\frac{(2n-1)^2\pi^2}{4R^2}\,t} \cos\left((2n-1)\frac{\pi|x|}{2R}\right).$$

Tedious but straightforward calculations, based on formulas from [1], show that $u(x,t)$ is transformed, by means of (2.3), into the function u^ε, defined by (2.4). $\square$

Theorem 2.6 *(Short-time asymptotics in a ball). Set $1 < p \le \infty$, $\Omega = B_R$, and let $u(x,t)$ be the viscosity solution of (1.2)–(1.4).*

Then, it holds that

$$\lim_{t\to 0^+} 4t \log u(x,t) = -p'\, d_\Gamma(x)^2 \quad \textit{for} \quad x \in \overline{\Omega}. \tag{2.6}$$

Proof. Given $x \in \Omega$ there exists $y \in \Gamma$ such that $|x-y| = d_\Gamma(x)$ (y is unique unless $x = 0$). Let H be the half-space containing Ω and such that $\partial H \cap \Gamma = \{y\}$; notice that $d_\Gamma(x) = d_{\partial H}(x)$.

Let Ψ^y be the solution of (1.2)–(1.4) in $H \times (0,\infty)$; since Ω is contained in H, Ψ^y obviously satisfies (1.2) and (1.3) for Ω and, also, $\Psi^y \le 1$ on $\Gamma \times (0,\infty)$. By comparison (Theorem A.1), we get that $u \ge \Psi^y$ and hence

$$4t \log u(x,t) \ge 4t \log \Psi^y(x,t) \quad \text{for} \quad (x,t) \in \overline{\Omega} \times (0,\infty). \tag{2.7}$$

Thus, Theorem 2.3 implies that

$$\liminf_{t\to 0^+} 4t \log u(x,t) \ge -p'\, d_{\partial H}(x)^2 = -p'\, d_\Gamma(x)^2. \tag{2.8}$$

The last limit is uniform on $\overline{\Omega}$, as Proposition 2.3 (with the choice $\delta = 2R$) informs us.

Now, by Lemma 2.5, for every $\varepsilon > 0$ the function u^ε defined in (2.3) is the solution of (2.1)–(2.2) in Ω. Thus, by Lemma 2.4, we have that

$$\varepsilon^2 u(x,t)\, e^{-t/\varepsilon^2} \le \int_t^\infty u(x,\tau)\, e^{-\tau/\varepsilon^2}\, d\tau \le \int_0^\infty u(x,\tau) e^{-\tau/\varepsilon^2}\, d\tau = \varepsilon^2 u^\varepsilon(x),$$

and hence

$$u(x,t) \le u^\varepsilon(x)\, e^{t/\varepsilon^2};$$

the last inequality holds for any $t, \varepsilon > 0$. Next, we choose $\varepsilon = \lambda t$ and obtain that

$$u(x,t) \le u^{\lambda t}(x)\, e^{1/\lambda^2 t} \quad \text{for any} \quad t > 0.$$

Therefore,

$$4t\,\log u(x,t) \le \frac{4}{\lambda}\,(\lambda\,t)\,\log u^{\lambda t}(x) + \frac{4}{\lambda^2}, \tag{2.9}$$

and hence

$$\limsup_{t\to 0^+} 4t\,\log u(x,t) \le -\frac{4\sqrt{p'}d_\Gamma(x)}{\lambda} + \frac{4}{\lambda^2}.$$

If we choose $\lambda^* > 0$ such that

$$-\frac{4\sqrt{p'}d_\Gamma(x)}{\lambda^*} + \frac{4}{(\lambda^*)^2} = -p'\,d_\Gamma(x)^2, \quad \text{that is} \quad \lambda^* = \frac{2}{\sqrt{p'}\,d_\Gamma(x)},$$

we obtain:

$$\limsup_{t\to 0^+} 4t\,\log u(x,t) \le -p'\,d_\Gamma(x)^2.$$

This inequality and (2.8) imply that (2.6) holds pointwise in Ω. Since, for $x \in \Gamma$, (2.6) is trivial, this also holds pointwise in $\overline{\Omega}$. $\square$

2.3. Constructing barriers for problem (1.2)–(1.4)

In this subsection, we shall prove our first-order asymptotics for quite general (not necessarily bounded) domains. A useful tool will be the function defined by

$$\Phi(x,t) = t^{-\frac{N+p-2}{2(p-1)}}\,e^{-p'\frac{|x|^2}{4t}}, \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{2.10}$$

which is a solution of (1.2) that generalizes to the case $p \neq 2$ the fundamental solution of the heat equation, as shown in [5, Proposition 4.1].

The next two lemmas shall give two global barriers for the solution of (1.2)–(1.4).

Lemma 2.7 *(A barrier from below). Let $z \in \mathbb{R}^N \setminus \overline{\Omega}$ and U^z be the function defined by*

$$U^z(x,t) = A_{N,p}\,d_\Gamma(z)^{\frac{N+p-2}{p-1}}\,\Phi(x-z,t) \quad \textit{for} \quad (x,t) \in \overline{\Omega} \times (0,\infty), \tag{2.11}$$

with

$$A_{N,p} = \left[\frac{p\,e}{2(N+p-2)}\right]^{\frac{N+p-2}{2(p-1)}}.$$

Assume that $u(x,t)$ is the bounded (viscosity) solution of (1.2)–(1.4), then we have that

$$U^z \le u \quad \textit{on} \quad \overline{\Omega} \times (0,\infty).$$

Proof. If we consider

$$A(z) = \max\{\Phi(x-z,t) : (x,t) \in \Gamma \times (0,\infty)\},$$

then the function $U(x,t) = A(z)^{-1}\Phi(x-z,t)$ satisfies (1.2), (1.3) and is such that $U \le 1 = u$ on $\Gamma \times (0,\infty)$. Notice that, the function U^z is decreasing with respect to $|x-z|$, then it is bounded by 1 in the whole $\Omega \times (0,\infty)$. Therefore, by applying Theorem A.1 we have that $u \ge U$ on $\overline{\Omega} \times (0,\infty)$.

Since we directly compute that

$$A(z)^{-1} = A_{N,p}\, d_\Gamma(z)^{\frac{N+p-2}{p-1}},$$

we get our claim. □

Lemma 2.8 *(A barrier from above). Let u be the bounded (viscosity) solution of (1.2)–(1.4) and $V = V(x,t)$ be the function defined by*

$$V(x,t) = u_B(0, t/d_\Gamma(x)^2) \quad \textit{for} \quad (x,t) \in \overline{\Omega} \times (0,\infty), \tag{2.12}$$

where u_B is the solution of (1.2)–(1.4) in the unit ball B and we mean that $u_B(0, t/d_\Gamma(x)^2) = 1$ when $x \in \Gamma$.

Then, it holds that $u \le V$ on $\overline{\Omega} \times (0,\infty)$.

Proof. For $x \in \Gamma$, the inequality is satisfied as an equality, by definition. Let $x \in \Omega$ and let $v^x = v^x(y,t)$ be the solution of (1.2)–(1.4) in $B^x \times (0,\infty)$, where B^x is the ball centered at x with radius $d_\Gamma(x)$. The maximum principle and Theorem A.1 give that

$$u(y,t) \le v^x(y,t) \quad \text{for every} \quad (y,t) \in \overline{B^x} \times (0,\infty),$$

and hence, in particular, $u(x,t) \le v^x(x,t)$ for every $t > 0$. Since x is arbitrary in Ω, we infer that

$$u(x,t) \le v^x(x,t) \quad \text{for} \quad (x,t) \in \Omega \times (0,\infty). \tag{2.13}$$

Now, for fixed $x \in \Omega$, consider the function defined by

$$w(y,t) = v^x(x + d_\Gamma(x)\, y, d_\Gamma(x)^2 t) \quad \text{for} \quad (y,t) \in \overline{B} \times (0,\infty);$$

since (1.2) is translation and scaling invariant, we have that w satisfies the problem (1.2)–(1.4) in B, and hence equals u_B on $\overline{B} \times (0,\infty)$.

Therefore, evaluating u_B for $(0, t/d_\Gamma(x)^2)$ gives that

$$u_B(0, t/d_\Gamma(x)^2) = w(0, t/d_\Gamma(x)^2) = v^x(x,t) \ge u(x,t),$$

by (2.13), and this concludes the proof. □

2.4. First-order asymptotics for general domains

We are now ready to prove our short-time asymptotic result for the solution of (1.2)–(1.4).

Theorem 2.9 *(Pointwise convergence). Set $1 < p \le \infty$. Let Ω be a domain in $\mathbb{R}^N$, with boundary Γ such that $\Gamma = \partial(\mathbb{R}^N \setminus \overline{\Omega})$, and let u be the bounded (viscosity) solution of (1.2)–(1.4).*

Then, we have that

$$\lim_{t\to 0^+} 4t \log[u(x,t)] = -p'\, d_\Gamma(x)^2 \quad \textit{for every } x \in \overline{\Omega}. \tag{2.14}$$

Proof. It is clear that (2.14) holds for $x \in \Gamma$. Thus, for $x \in \Omega$, set $r = d_\Gamma(x)$ and let z be a point in $\Gamma \cap \partial B_r(x)$. Since $\Gamma = \partial(\mathbb{R}^N \setminus \overline{\Omega})$, there is a sequence of points in $z_n \in \mathbb{R}^N \setminus \overline{\Omega}$ that converges to z.

We first compute the limit in (2.14), by replacing u by the barriers constructed in Lemmas 2.7 and 2.8. In fact, from (2.11) and (2.10), we easily compute that

$$\lim_{t\to 0^+} 4t \log U^{z_n}(x,t) = -p'|x-z_n|^2,$$

whereas, by Theorem 2.6, we infer that

$$\lim_{t\to 0^+} 4t \log V(x,t) = -p'\, d_\Gamma(x)^2. \tag{2.15}$$

Now, for each $n \in \mathbb{N}$, Lemmas 2.7 and 2.8 tell us that

$$4t \log U^{z_n}(x,t) \le 4t \log u(x,t) \le 4t \log V(x,t),$$

for $t > 0$. Thus, we get that

$$\begin{aligned} -p'|x-z_n|^2 = \liminf_{t\to 0^+} 4t \log U^z(x,t) \le \liminf_{t\to 0^+} 4t \log u(x,t) \le \\ \le \limsup_{t\to 0^+} 4t \log u(x,t) \le \limsup_{t\to 0^+} 4t \log V(x,t) = -p'\, d_\Gamma(x)^2. \end{aligned}$$

Letting z_n tend to z gives the conclusion, since $|x-z| = d_\Gamma(x)$. $\square$

Remark 1. Notice that the convergence in (2.15) is uniform on every subset of $\overline{\Omega}$ where the d_Γ is bounded; actually, we obtain that

$$4t \log V(x,t) + p'\, d_\Gamma(x)^2 = O(t \log t),$$

for $t \to 0^+$, on such subsets. Indeed, by a comparison with the one-dimensional solution as done in Theorem 2.6, we can write that

$$4t \log \operatorname{Erfc}\left(\frac{\sqrt{p'} d_\Gamma(x)}{2\sqrt{t}}\right) \le 4t \log V(x,t) \tag{2.16}$$

while (2.9) with $\lambda = \frac{2}{\sqrt{p'}}$, (2.4) and some manipulations give for $1 < p < \infty$ that

$$4t \log V(x,t) \le -p'\, d_\Gamma(x)^2 + 4t \log\left[\frac{\int_0^\pi (\sin\theta)^\alpha\, d\theta}{\int_0^\pi e^{-p' \frac{1-\cos\theta}{2t} d_\Gamma(x)^2} (\sin\theta)^\alpha\, d\theta}\right]. \tag{2.17}$$

The explicit expressions in (2.16) and (2.17), imply the desired claim.

The case $p = \infty$ is analogous, simpler, and even yields the better behavior $O(t)$ as $t \to 0^+$.

With some extra sufficient condition on the regularity of Ω, we can obtain uniform convergence.

Let $\omega : (0,\infty) \to (0,\infty)$ be a strictly increasing continuous function such that $\omega(\tau) \to 0$ as $\tau \to 0^+$. We say that a domain Ω is of class $C^{0,\omega}$, if there exists a number $r > 0$ such that, for every point $x \in \Gamma$, there is a coordinate system $(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, and a function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$ such that

(i) $B_r(x) \cap \Omega = \{(y', y_N) \in B_r(x) : y_N < \phi(y')\}$;
(ii) $B_r(x) \cap \Gamma = \{(y', y_N) \in B_r(x) : y_N = \phi(y')\}$;
(iii) $|\phi(y') - \phi(z')| \le \omega(|y'-z'|)$ for all $(y', \phi(y')), (z', \phi(z')) \in B_r(x) \cap \Gamma$.

Theorem 2.10 *(Uniform convergence). Let $1 < p \le \infty$, suppose that Ω is a domain of class $C^{0,\omega}$, and set*

$$\psi(t) = \min_{0 \le s \le R} \sqrt{s^2 + [t - \omega(s)]^2}.$$

Let u be the bounded (viscosity) solution of (1.2)–(1.4).

Then, it holds that

$$4t \, \log u(x,t) + p' \, d_\Gamma(x)^2 = O(t \log \psi(t)) \quad \text{as} \quad t \to 0^+, \tag{2.18}$$

uniformly on every compact subset of $\overline{\Omega}$. In particular, if $t \log \psi(t) \to 0$ as $t \to 0^+$, then the solution u of (1.2)–(1.4) satisfies (2.14) uniformly on every compact subset of $\overline{\Omega}$.

Proof. In Remark 1, we already observed that in (2.14) the convergence is uniform from above. We shall now modify the argument for the barrier $U^{z_n}(x,t)$ in such a way that it becomes uniform in x. In fact, we will choose the points z_n along a suitable curve parametrized upon the time t.

For every $x \in \Omega$, we choose a coordinate system $(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, with its origin at a point in Γ at minimal distance $d_\Gamma(x)$ from x. In this coordinate system, we choose $z(t) = (0', t)$ that, if t is small enough is by construction a point in $\mathbb{R}^N \setminus \overline{\Omega}$, since $t > \phi(0')$. Also, by our assumptions on Ω, $d_\Gamma(z(t))$ is bounded from below by the distance of $z(t)$ from the graph of the function $y' \mapsto \omega(|y'|)$ defined for $y' \in \{y \in B_r(0) : y_N = 0\}$, that is

$$d_\Gamma(z(t)) \ge \min_{0 \le s \le r} \sqrt{s^2 + [\omega(s) - t]^2}.$$

It is clear that this construction does not depend on the particular point $x \in \overline{\Omega}$ chosen, but only on the regularity assumptions on Ω. Thus, we define our uniform barrier from below by $U^{z(t)}(x,t)$ and hence, from the definition of $U^z(x,t)$, we have that

$$4t \, \log U^{z(t)}(x,t) = 4t \, \log \left[A_{N,p} \, t^{-\frac{N+p-2}{2(p-1)}} \right] + 4t \, \frac{N+p-2}{p-1} \, \log d_\Gamma(z(t)) - p' \, |z(t) - x|^2,$$

and hence

$$4t \, \log U^{z(t)}(x,t) \ge 4t \, \log \left[A_{N,p} \, t^{-\frac{N+p-2}{2(p-1)}} \right] + 4t \, \frac{N+p-2}{p-1} \, \log \psi(t) - p' \, [d_\Gamma(x) + t]^2, \tag{2.19}$$

since $|z(t) - x| \le |z(t)| + |x| = t + d_\Gamma(x)$. The desired estimate (2.18) follows from an inspection of (2.19) and Remark 1. □

Remark 2. Under sufficient assumptions on ω, we can replace ψ by $a \, \omega^{-1}$, for some positive constant a, where ω^{-1} is the inverse function of ω. For instance, if Ω is of class C^α, with $0 < \alpha < 1$ — that means that Γ is locally a graph of an α-Hölder continuous function — then the assumptions of Theorem 2.10 are fulfilled, since $\psi(t) \ge a \, t^{1/\alpha}$ as $t \to 0^+$.

The same assertion of Theorem 2.10 holds true even if we replace 1 in (1.4) by a bounded time-dependent non-constant boundary data, provided that this is bounded away from zero.

Corollary 2.11. *Let w be the bounded solution of (1.2), (1.3) satisfying*

$$w = h \quad \text{on} \quad \Gamma \times (0, \infty),$$

where the function $h : \Gamma \times (0, \infty) \to \mathbb{R}$ *is such that*

$$\underline{h} \le h \le \overline{h} \quad on \quad \Gamma \times (0, \infty),$$

for some positive numbers $\underline{h}, \overline{h}$.

Then, we have that

$$4t \log w(x, t) = -p' d_\Gamma(x)^2 + O(t \log \psi(t)) \quad as \quad t \to 0^+,$$

uniformly on every compact subset of $\overline{\Omega}$.

Proof. Since $\underline{h}\, u \le w \le \overline{h}\, u$ on $\Gamma \times (0, \infty)$, we can apply Theorem A.1 to get:

$$\underline{h}\, u(x, t) \le w(x, t) \le \overline{h}\, u(x, t) \quad \text{on} \quad \overline{\Omega} \times (0, \infty).$$

This implies that, for every $x \in \overline{\Omega}$ and $t > 0$,

$$4t \log \underline{h} + 4t \log u(x, t) \le 4t \log w(x, t) \le 4t \log \overline{h} + 4t \log u(x, t).$$

The conclusion then easily follows from Theorem 2.10. □

The next corollary of Theorem 2.10 will be useful in Section 3.

Corollary 2.12. *Let* $v : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ *be defined by*

$$\operatorname{Erfc}\left(\frac{\sqrt{p'}\, v(x, t)}{2\sqrt{t}}\right) = u(x, t) \quad for \quad (x, t) \in \overline{\Omega} \times (0, \infty).$$

Then

$$v(x, t) = d_\Gamma(x) + O(t \log \psi(t)) \quad as \quad t \to 0^+,$$

uniformly on every compact subset of $\overline{\Omega}$.

Proof. From the definition of $v(x, t)$, operating as in the proof of Proposition 2.3 yields that

$$4t\, \log u(x, t) + p'\, v(x, t)^2 = 4t\, \log \left(\sqrt{\frac{p'}{4\pi}} \int_0^\infty e^{-\frac{1}{2} p' \frac{v(x,t)}{\sqrt{t}}\, \sigma - \frac{1}{4} p' \sigma^2} d\sigma \right) \le 0.$$

By this inequality, since the first summand at the left-hand side converges uniformly on every compact $K \subset \overline{\Omega}$ as $t \to 0^+$, we can infer that there exist $\overline{t} > 0$ and $\delta > 0$ such that $0 \le v(x, t) \le \delta$ for any $x \in K$ and $0 < t < \overline{t}$.

Thus, for $x \in K$ we have that

$$- \left[4t\, \log u(x, t) + p'\, d_\Gamma(x)^2\right] + 4t\, \log \left(\sqrt{\frac{p'}{\pi}} \int_0^\infty e^{-\frac{1}{2} p' \frac{\delta}{\sqrt{t}}\, \sigma - \frac{1}{4} p' \sigma^2} d\sigma \right) \le$$
$$p'\, \left[v(x, t)^2 - d_\Gamma(x)^2\right] \le - \left[4t\, \log u(x, t) + p'\, d_\Gamma(x)^2\right],$$

which implies the desired uniform estimate, by means of (2.18). □

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3. Second-order asymptotics

In this section, we shall suppose that Ω is a domain of class C^2 (not necessarily bounded) and, for any point $y \in \Gamma$, denote by $\kappa_1(y), \dots, \kappa_{N-1}(y)$ the principal curvatures of Γ at y with respect to the interior normal direction to Γ. Moreover, we let Π_Γ be the function defined in (1.7):

$$\Pi_\Gamma(y) = \prod_{j=1}^{N-1} [1 - R\,\kappa_j(y)] \quad \text{for} \quad y \in \Gamma.$$

We then recall a useful geometrical lemma ([28, Lemma 2.1]).

Lemma 3.1. *Let $x \in \Omega$ and assume that, for $R > 0$, there exists $y_x \in \Gamma$ such that $\overline{B_R(x)} \cap (\mathbb{R}^N \setminus \Omega) = \{y_x\}$ and that $\kappa_j(y_x) < 1/R$ for $j = 1, \dots, N-1$. Set $\Gamma_s = \{y \in \Omega : d_\Gamma(y) = s\}$, for $s > 0$.*

Then, it holds that

$$\lim_{s \to 0^+} s^{-\frac{N-1}{2}} \mathcal{H}_{N-1}(\Gamma_s \cap B_R(x)) = \frac{\omega_{N-1}\,(2R)^{\frac{N-1}{2}}}{(N-1)\sqrt{\Pi_\Gamma(y_x)}},$$

where $\mathcal{H}_{N-1}$ denotes $(N-1)$-dimensional Hausdorff measure and ω_{N-1} is the surface area of a unit sphere in $\mathbb{R}^{N-1}$.

3.1. Short-time asymptotics for heat content

Our asymptotic result for the heat content of a ball $B_R(x)$ is based on the following lemma.

Lemma 3.2 *(Short-time asymptotics for a barrier). Let $x \in \Omega$ and assume that, for $R > 0$, there exists $y_x \in \Gamma$ such that $\overline{B_R(x)} \cap (\mathbb{R}^N \setminus \Omega) = \{y_x\}$ and that $\kappa_j(y_x) < 1/R$ for $j = 1, \dots, N-1$.*

Let f be a continuous function on $\mathbb{R}$ such that

$$\lim_{s \to \infty} f(s) = 0 \qquad \text{and} \qquad \int_0^\infty s^{\frac{N-1}{2}} f(s)\, ds < \infty.$$

Let $\xi, \eta : (0, \infty) \to (0, \infty)$ be two functions of time such that $\xi(t)$ is positive in $(0, \infty)$, and

$$\lim_{t \to 0^+} \xi(t) = \lim_{t \to 0^+} \eta(t) = 0.$$

Then it holds that

$$\lim_{t \to 0^+} \xi(t)^{-\frac{N+1}{2}} \int_{B_R(x)} f\left(\frac{d_\Gamma(z)}{\xi(t)} + \eta(t)\right) dz = \frac{\omega_{N-1}\,(2R)^{\frac{N-1}{2}}}{(N-1)\sqrt{\Pi_\Gamma(y_x)}} \int_0^\infty s^{\frac{N-1}{2}} f(s) ds. \tag{3.1}$$

Proof. By the co-area formula and a simple change of variables, we have that

$$\int_{B_R(x)} f\left(\frac{d_\Gamma(z)}{\xi(t)} + \eta(t)\right) dz = \int_0^{2R} f\left(\frac{s}{\xi(t)} + \eta(t)\right) \mathcal{H}_{N-1}(\Gamma_s \cap B_R(x))\, ds =$$

$$\xi(t) \int_{\eta(t)}^{\frac{2R}{\xi(t)} + \eta(t)} f(\sigma)\, \mathcal{H}_{N-1}(\Gamma_{\xi(t)[\sigma - \eta(t)]} \cap B_R(x))\, d\sigma.$$

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Thus,

$$\xi(t)^{-\frac{N+1}{2}} \int\limits_{B_R(x)} f\left(\frac{d_\Gamma(z)}{\xi(t)} + \eta(t)\right) dz = \int\limits_{\eta(t)}^{\frac{2R}{\xi(t)}+\eta(t)} [\sigma - \eta(t)]^{\frac{N-1}{2}} f(\sigma) \frac{\mathcal{H}_{N-1}(\Gamma_{\xi(t)[\sigma-\eta(t)]} \cap B_R(x))}{\xi(t)^{\frac{N-1}{2}} [\sigma - \eta(t)]^{\frac{N-1}{2}}} d\sigma.$$

Therefore, by taking the limit as $t \to 0^+$, we obtain (3.1) by Lemma 3.1 and the dominated convergence theorem, after observing that there are constants $c, \bar{t} > 0$ such that

$$[\sigma - \eta(t)]^{\frac{N-1}{2}} \le c\,(\sigma^{\frac{N-1}{2}} + 1) \quad \text{for} \quad \eta(t) \le \sigma < \infty$$

and $0 < t < \bar{t}$. $\square$

We are now ready to prove our formula for the heat content of u.

Theorem 3.3 *(Short-time asymptotics for heat content). Set $1 < p \le \infty$. Let $x \in \Omega$ and assume that, for $R > 0$, there exists $y_x \in \Gamma$ such that $\overline{B_R(x)} \cap (\mathbb{R}^N \setminus \Omega) = \{y_x\}$ and that $\kappa_j(y_x) < 1/R$ for $j = 1, \dots, N-1$.*

If u is the bounded (viscosity) solution of (1.2)–(1.4), then it holds that

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int\limits_{B_R(x)} u(z,t)\, dz = \frac{c_N}{(p')^{\frac{N+1}{4}}} \frac{R^{\frac{N-1}{2}}}{\sqrt{\Pi_\Gamma(y_x)}}, \tag{3.2}$$

where

$$c_N = \frac{2^{\frac{N+3}{2}} \pi^{\frac{N-1}{2}}}{(N+1)\Gamma\left(\frac{N+1}{4}\right)}$$

and $\Gamma(\cdot)$ denotes Euler's gamma function.

Proof. Let $x \in \Omega$ and $B = B_R(x)$. We set

$$\eta_B(t) = \frac{1}{\sqrt{t}} \max_{z\in\overline{B}} |v(z,t) - d_\Gamma(z)| \quad \text{for} \quad t > 0. \tag{3.3}$$

Since we are assuming that Γ is of class C^2, Corollary 2.12 implies the estimate

$$d_\Gamma(z) - \sqrt{t}\,\eta_B(t) \le v(z,t) \le d_\Gamma(z) + \sqrt{t}\,\eta_B(t), \;\; x \in \overline{B},$$

with $\eta_B(t) = O(\sqrt{t}\log t)$ as $t \to 0^+$. Thus, we infer that

$$\operatorname{Erfc}\left(\sqrt{\frac{p'}{4t}}\, d_\Gamma(z) + \eta_B(t)\right) < u(z,t) < \operatorname{Erfc}\left(\sqrt{\frac{p'}{4t}}\, d_\Gamma(z) - \eta_B(t)\right),$$

for any $(z,t) \in \overline{B} \times (0,\infty)$. We then choose $\xi(t) = 2\sqrt{t}/\sqrt{p'}$ and $f(s) = \operatorname{Erfc}(s)$, and check that the assumptions of Lemma 3.2 are satisfied.

Thus, we compute that

$$\int\limits_0^\infty s^{\frac{N-1}{2}} \operatorname{Erfc}(s)\, ds = \frac{N-1}{2\sqrt{\pi}(N+1)}\, \Gamma\left(\frac{N-1}{4}\right),$$

and hence formula (3.2) then follows from Lemma 3.2, after some straightforward computation. $\square$

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3.2. Asymptotics for q-means

Formula (3.2) can also be seen as an asymptotic formula for the *mean value* of u on the ball $B_R(x)$. In fact, the following scale invariant formula follows:

$$\lim_{t\to 0^+}\left(\frac{R^2}{t}\right)^{\frac{N+1}{4}} \fint_{B_R(x)} u(y,t)\,dy = \frac{c_N}{(p')^{\frac{N+1}{4}}\sqrt{\Pi_\Gamma(y_x)}},$$

with

$$c_N = \frac{2^{\frac{N+1}{2}}}{\sqrt{\pi}}\,\frac{N}{N+1}\,\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+1}{4}\right)}.$$

Other statistical quantities that seem to be particularly appropriate and interesting in a game-theoretic context are the so-called q-*means*. We shall consider for $1 < q \le \infty$ the q-mean $\mu_q^u(x,t)$ of $u(\cdot,t)$ on $B_R(x)$, as defined in (1.10); this coincides with the mean value when $q = 2$.

Lemma 3.4 *(Asymptotics for the q-mean of a barrier). Set $1 < q < \infty$, let $x \in \Omega$, and assume that, for $R > 0$, there exists a point $y_x \in \Gamma$ such that $\overline{B_R(x)} \cap (\mathbb{R}^N \setminus \Omega) = \{y_x\}$ and $\kappa_j(y_x) < 1/R$ for $j = 1,\dots,N-1$.*

Let ξ and η be functions satisfying the assumptions of Lemma 3.2. For a non-negative, decreasing and continuous function f on $\mathbb{R}$ such that

$$\int_0^\infty f(\sigma)^{q-1}\sigma^{\frac{N-1}{2}}\,d\sigma < \infty,$$

set

$$w(y,t) = f\left(\frac{d_\Gamma(y)}{\xi(t)} + \eta(t)\right) \quad \text{for} \quad (y,t) \in \overline{\Omega}\times(0,\infty).$$

If $\mu_q^w(x,t)$ is the q-mean of w on $B_R(x)$, then the following formula holds:

$$\lim_{t\to 0^+}\left(\frac{R}{\xi(t)}\right)^{\frac{N+1}{2(q-1)}} \mu_q^w(x,t) = \left\{\frac{c_N\int_0^\infty f(\sigma)^{q-1}\sigma^{\frac{N-1}{2}}\,d\sigma}{\sqrt{\Pi_\Gamma(y_x)}}\right\}^{\frac{1}{q-1}} \tag{3.4}$$

where $c_N = 2^{-\frac{N+1}{2}} N!\,\Gamma\left(\frac{N+1}{2}\right)^{-2}$.

Proof. We know from [17] that $\mu(t) = \mu_q^w(x,t)$ is the unique root of the following equation

$$\int_{B_R(x)} |w(y,t) - \mu(t)|^{q-2}[w(y,t) - \mu(t)]\,dy = 0,$$

or

$$\int_{B_R(x)} [w(y,t) - \mu(t)]_+^{q-1} dy = \int_{B_R(x)} [\mu(t) - w(y,t)]_+^{q-1} dy, \tag{3.5}$$

where $[\sigma]_+ = \max\{0,\sigma\}$.

Firstly, we compute the short-time behavior of the left-hand side of (3.5). Let $\Gamma_s = \{z \in B_R(x) : d_\Gamma(z) = s\}$. By the co-area formula, we get that

$$\int\limits_{B_R(x)} [w(y,t)-\mu(t)]_+^{q-1} dy = \int\limits_0^{2R} \left[f\left(\frac{s}{\xi(t)} + \eta(t)\right) - \mu(t)\right]_+^{q-1} \mathcal{H}_{N-1}(\Gamma_s)\, ds.$$

By the change of variable $s = \xi(t)\,[\sigma - \eta(t)]$, we obtain that

$$\int\limits_{B_R(x)} [w(y,t)-\mu(t)]_+^{q-1} dy = \xi(t) \int\limits_{\eta(t)}^{\beta(t)} [f(\sigma) - \mu(t)]_+^{q-1} \mathcal{H}_{N-1}\left(\Gamma_{\xi(t)[\sigma-\eta(t)]}\right) d\sigma,$$

where we set $\beta(t) = \frac{2R}{\xi(t)} + \eta(t)$.

Hence,

$$\xi(t)^{-\frac{N-1}{2}} \int\limits_{B_R(x)} [w(y,t)-\mu(t)]_+^{q-1} dy = \int\limits_{\eta(t)}^{\beta(t)} \frac{\mathcal{H}_{N-1}\left(\Gamma_{\xi(t)[\sigma-\eta(t)]}\right)}{\{\xi(t)\,[\sigma-\eta(t)]\}^{\frac{N-1}{2}}} [\sigma-\eta(t)]^{\frac{N-1}{2}} \{f(\sigma)-\mu(t)\}^{q-1}\, d\sigma.$$

Now, as $t \to 0^+$ we have that $\eta(t), \xi(t), \mu(t) \to 0$, $\beta(t) \to \infty$ and that $\xi(t)\,[\sigma - \eta(t)] \to 0$ for almost every $\sigma \ge 0$. Thus, we can infer that

$$\lim_{t\to 0^+} \xi(t) \int\limits_{B_R(x)} [w(y,t)-\mu(t)]_+^{q-1} dy = \frac{\omega_{N-1}(2R)^{\frac{N-1}{2}}}{(N-1)\sqrt{\Pi_\Gamma(y_x)}} \int\limits_0^\infty f(\sigma)^{q-1} \sigma^{\frac{N-1}{2}}\, d\sigma, \tag{3.6}$$

by Lemma 3.1 and an application of the dominated convergence theorem, as an inspection of the integrand function reveals.

Secondly, we treat the short-time behavior of the right-hand side of (3.5). By again performing the co-area formula and after some manipulations, we have that

$$\int\limits_{B_R(x)} [\mu(t) - w(y,t)]_+^{q-1} = \mu(t)^{q-1} \int\limits_0^{2R} \left[1 - f\left(\frac{s}{\xi(t)} + \eta(t)\right) \Big/ \mu(t)\right]_+^{q-1} \mathcal{H}_{N-1}(\Gamma_s)\, ds \tag{3.7}$$

which, on one hand, leads to

$$\int\limits_{B_R(x)} [\mu(t) - w(y,t)]_+^{q-1} \le \mu(t)^{q-1} |B_R(x)|.$$

Notice in particular that, by using both (3.5) and (3.6), the last inequality informs us that

$$\mu(t) \ge c\, \xi(t)^{\frac{N+1}{2(q-1)}},$$

for some positive constant c. Hence, after setting $\beta(s,t) = \frac{s}{\xi(t)} + \eta(t)$, the assumptions on f give the following chain of inequalities:

$$\int_{\beta(s,t)/2}^{\infty} f(\sigma)^{q-1}\sigma^{\frac{N-1}{2}}\,d\sigma \geq \int_{\beta(s,t)/2}^{\beta(s,t)} f(\sigma)^{q-1}\sigma^{\frac{N-1}{2}}\,d\sigma \geq$$

$$\frac{2\left(1-2^{-\frac{N+1}{2}}\right)}{N+1}\,\frac{f\left(\beta(s,t)\right)^{q-1}}{\xi(t)^{\frac{N+1}{2}}}\,[s+\eta(t)\,\xi(t)]^{\frac{N+1}{2}} \geq$$

$$\frac{2\left(1-2^{-\frac{N+1}{2}}\right)}{c(N+1)}\left[f\left(\frac{s}{\xi(t)}+\eta(t)\right)\Big/\mu(t)\right]^{q-1}[s+\eta(t)\,\xi(t)]^{\frac{N+1}{2}}.$$

Since, for almost every $s \geq 0$, the first term of the chain vanishes as $t \to 0^+$, we have that

$$\lim_{t\to 0^+}\frac{f\left(\frac{s}{\xi(t)}+\eta(t)\right)}{\mu(t)} = 0,$$

for almost every $s \geq 0$. Thus, (3.7) gives at once that

$$\lim_{t\to 0^+}\mu(t)^{1-q}\int_{B_R(x)}[\mu(t)-w(y,t)]_+^{q-1} = |B_R(x)|. \tag{3.8}$$

Finally, (3.5), (3.6) and (3.8) tell us that

$$\mu(t)^{q-1} = \xi(t)^{\frac{N+1}{2}}\,\frac{\omega_{N-1}\,(2R)^{\frac{N-1}{2}}}{(N-1)\sqrt{\Pi_\Gamma(y_x)}}\,\frac{\int_0^\infty f(\sigma)^{q-1}\,\sigma^{\frac{N-1}{2}}d\sigma+o(1)}{|B_R(x)|+o(1)},$$

that gives (3.4), after straightforward calculations involving Euler's gamma function. □

Remark 3. If $q = \infty$, we know that

$$\mu_\infty^w(x,t) = \frac{1}{2}\left\{\min_{B_R(x)} w(\cdot,t)+\max_{B_R(x)} w(\cdot,t)\right\} = \frac{1}{2}\left[f\left(\frac{\overline{d}}{\xi(t)}+\eta(t)\right)+f(\eta(t))\right],$$

where $\overline{d}$ is positive, being the maximum of d_Γ on $\overline{B_R(x)}$. Hence, it is easy to compute:

$$\lim_{t\to 0^+}\mu_\infty^w(x,t) = \frac{1}{2}\,f(0).$$

Thus, formula (3.4) does not extend continuously to the case $q = \infty$.

Theorem 3.5 *(Short-time asymptotics for q-means). Let $x \in \Omega$, and assume that, for $R > 0$, there exists a point $y_x \in \Gamma$ such that $\overline{B_R(x)} \cap (\mathbb{R}^N \setminus \Omega) = \{y_x\}$ and $\kappa_j(y_x) < 1/R$ for $j = 1, \dots, N-1$.*

Set $1 < p \leq \infty$ and suppose that u is the bounded (viscosity) solution of (1.2)–(1.4) and, for $1 < q \leq \infty$, $\mu_q^u(x,t)$ is its q-mean on $B_R(x)$.

Then, the following formulas hold:

$$\lim_{t\to 0^+}\left(\frac{R^2}{t}\right)^{\frac{N+1}{4(q-1)}}\mu_q^u(x,t) = \left\{\frac{c_N\int_0^\infty \operatorname{Erfc}(\sigma)^{q-1}\sigma^{\frac{N-1}{2}}d\sigma}{(p')^{\frac{N+1}{4}}\sqrt{\Pi_\Gamma(y_x)}}\right\}^{\frac{1}{q-1}}, \tag{3.9}$$

where $c_N = N!\,\Gamma\left(\frac{N+1}{2}\right)^{-2}$, and

$$\lim_{t\to 0^+} \mu_\infty^u(x,t) = \frac{1}{2}.$$

Proof. From [17] we know that the functional $u \mapsto \mu_q^u(x,t)$ is monotonically increasing, that is $\mu_q^u(x,t) \le \mu_q^w(x,t)$ if $u \le w$ almost everywhere in $B_R(x)$. Thus, as done in the proof of Theorem 3.3, the limit in (3.9) will result from Lemma 3.4, where we choose:

$$w(x,t) = \mathrm{Erfc}\left(\sqrt{\frac{p'}{4t}}\, d_\Gamma(y) \pm \eta(t)\right),$$

that is we choose $\xi(t) = \sqrt{4t/p'}$ and $\eta(t)$ is still given by (3.3). Thus, (3.9) will follow at once from (3.4), where $f(\sigma) = \mathrm{Erfc}(\sigma)$.

By the same argument, we also get the case $q = \infty$, since $f(0) = 1$. $\square$

Remark 4. Notice that

$$\left\{\int_0^\infty \mathrm{Erfc}(\sigma)^{q-1} \sigma^{\frac{N-1}{2}}\, d\sigma\right\}^{\frac{1}{q-1}}$$

can be seen as the $(q-1)$-norm of Erfc in $(0,\infty)$ with respect to the weighed measure $\sigma^{\frac{N+1}{2}} d\sigma$.

3.3. Symmetry results

In this subsection, we present two applications with geometric flavor of our asymptotic formulas (2.14) and (3.9).

An immediate result is that, even in the case (1.2)–(1.4), a time-invariant level surface is parallel to Γ.

Theorem 3.6. *Let Ω be a domain in $\mathbb{R}^N$ satisfying $\Gamma = \partial\left(\mathbb{R}^N \setminus \overline{\Omega}\right)$ and suppose that, for $1 < p \le \infty$, u is the solution of (1.2)–(1.4).*

If $\Sigma \subset \Omega$ is a time-invariant level surface for u, then there exists $R > 0$ such that

$$d_\Gamma(x) = R \quad \textit{for every} \quad x \in \Sigma. \tag{3.10}$$

Proof. For any choice of x_1 and $x_2 \in \Sigma$, we have that $u(x_1,t) = u(x_2,t)$ and hence $4t \log u(x_1,t) = 4t \log u(x_2,t)$ for every $t > 0$. By Theorem 2.6, we infer that $d_\Gamma(x_1) = d_\Gamma(x_2)$ and hence we obtain our claim. $\square$

As an application of (3.9), we give a characterization of spheres, based on q-means, in the spirit of [28, Theorem 1.2]. This result is new, even for the case $p = 2$.

Theorem 3.7. *Set $1 < p \le \infty$ and let Ω be a domain of class C^2 with bounded and connected boundary Γ. Let u be the bounded (viscosity) solution of (1.2)–(1.4).*

Suppose that Σ is a C^2-regular surface in Ω, that is a parallel surface to Γ at distance $R > 0$.

If, for some $1 < q < \infty$ and every $t > 0$, the function

$$\Sigma \ni x \mapsto \mu_q^u(x,t)$$

is constant, then Γ must be a sphere.

Proof. Since Σ is of class C^2 and is parallel to Γ, for every $y \in \Gamma$, there is a unique $x \in \Sigma$ at distance R from y. Thus, owing to Theorem 3.5, we can infer that

$$\Pi_\Gamma = \text{constant on } \Gamma.$$

Our claim then follows from a variant of Alexandrov's Soap Bubble Theorem (see [2]), [28, Theorem 1.2], or [26, Theorem 1.1]. $\square$

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Appendix A. Viscosity solutions

For the reader's convenience, in this appendix, we collect the salient results on the viscosity solutions of (1.2) and (2.1) that we use in this paper. The main references to the existing literature that we use are the classical treatises [10] and [8], the papers [11] and [16], concerning useful versions of comparison and maximum principles for nonlinear parabolic equations, in addition to the more recent and specific works [3], [4], [5], [6], and [12] on Δ_p^G.

Equation (1.2) can be formally written as

$$u_t = F(\nabla u, \nabla^2 u),$$

where

$$F(\xi, X) = \operatorname{tr}[A(\xi)X], \quad \text{with} \quad A(\xi) = \frac{1}{p}\, I + (1 - 2/p)\, \frac{\xi \otimes \xi}{|\xi|^2}$$

for $\xi \in \mathbb{R}^N \setminus \{0\}$. Here, I and X denote the $N \times N$ identity and a real symmetric matrix.

Since F has a bounded discontinuity at $\xi = 0$, by following [10], we need to consider the so-called lower and upper semi-continuous relaxations F_* and F^* of F, respectively defined as

$$F_*(\xi, X) = \lim_{r \to 0^+} \inf\{F(\eta, Y) : 0 < |\eta - \xi|, |Y - X| < r\},$$

and $F^* = -(-F)_*$.

If $\xi \neq 0$, $F = F_* = F^*$ while, if $\xi = 0$, we have that

$$p\, F_*(0, X) = \operatorname{tr}(X) + \max(p - 2, 0)\, \lambda(X) + \min(p - 2, 0)\, \Lambda(X),$$
$$p\, F^*(0, X) = \operatorname{tr}(X) + \min(p - 2, 0)\, \lambda(X) + \max(p - 2, 0)\, \Lambda(X),$$

where $\lambda(X)$ and $\Lambda(X)$ are the maximum and minimum eigenvalue of X (see [12], [3]). For $\xi \neq 0$, F is a linear operator in the variable X that is *uniformly elliptic*, since

$$\min(1/p', 1/p)\, I \leq A(\xi) \leq \max(1/p', 1/p)\, I.$$

A.1. Parabolic case

We say that $u \in C(\Omega \times (0,\infty))$ is a viscosity solution of (1.2) if, for every $(x,t) \in \Omega \times (0,\infty)$, both of the following requirements are fulfilled:

(i) for every $\varphi \in C^2(\Omega \times (0,\infty))$ such that $u - \varphi$ attains its maximum at (x,t), then

$$\varphi_t(x,t) - F^*\left(\nabla\varphi(x,t), \nabla^2\varphi(x,t)\right) \leq 0;$$

(ii) for every $\varphi \in C^2(\Omega \times (0,\infty))$ such that $u - \varphi$ attains its minimum at (x,t), then

$$\varphi_t(x,t) - F_*\left(\nabla\varphi(x,t), \nabla^2\varphi(x,t)\right) \geq 0.$$

A quite general comparison principle holds for equation (1.2) (even for unbounded domains).

Theorem A.1 *([16], Theorem 2.1). Let u and v be a viscosity subsolution and supersolution for (1.2), that are bounded in $\Omega \times (0,\infty)$.*

If $u \leq v$ on $\Gamma \times (0,\infty)$ and $\overline{\Omega} \times \{0\}$, then

$$u \leq v \quad on \quad \overline{\Omega} \times [0,\infty).$$

Remark 5. Let u be the (bounded) viscosity solution of (1.2)–(1.4) in $\Omega \times (0,\infty)$. Theorem A.1 and an application of the strong maximum principle for u (see [11, Corollary 2.4]), then implies that

$$0 < u < 1, \quad \text{for} \quad (x,t) \in \Omega \times (0,\infty).$$

Indeed, the constant functions $v \equiv 0$ and $w \equiv 1$ on $\Omega \times (0,\infty)$ are solutions of (1.2) and $v \leq u \leq w$ on $\partial\Omega \times (0,\infty)$ and $\overline{\Omega} \times \{0\}$.

Remark 6. Since F, F^*, and F_* are monotonic in X (in the sense of [10]), we have that any smooth function u that satisfies (i) and (ii), in the classical sense, is a viscosity solution of (1.2).

Remark 7. The existence of a solution of (1.2)–(1.4) can be obtained as follows. Suppose that Γ is smooth enough, say C^2, and let S be the parabolic boundary of $\Omega \times (0,\infty)$, that is

$$S = (\Gamma \times (0,\infty)) \cup (\overline{\Omega} \times \{0\}).$$

For any $g \in C^0(S)$, there exists a unique viscosity solution u of (1.2), satisfying $u = g$ on S, as shown in [34, Theorem 1.9] or [6, Theorem 2.6].

Now, for any $n \in \mathbb{N}$, let g_n be a continuous function on S such that $0 \leq g_n \leq 1$ and

$$g_n = 1 \quad \text{on} \quad \{(x,t) \in S : t > 1/n\},$$
$$g_n = 0 \quad \text{on} \quad \{(x,0) : d_\Gamma(x) > 1/n\}.$$

The solution u_n of (1.2) satisfying $u_n = g_n$ on S is such that $0 \leq u_n \leq 1$ on $\overline{\Omega} \times [0,\infty)$, by Theorem A.1. Thus, by virtue of uniform Hölder estimates (see [6, Lemma 2.11]), up to a subsequence, u_n converges (locally uniformly) in $\Omega \times (0,\infty)$ to a solution u of (1.2). Moreover, by the choice of g_n, we have that u satisfies the initial-boundary conditions (1.3) and (1.4).

By a result in [3], we also know that $u \in C_{loc}^{1+\tau, \frac{1+\tau}{2}}(\Omega \times (0,\infty))$, for some $\tau > 0$.

A.2. Elliptic case

In the (non-homogeneous) elliptic case, the definition of viscosity solution is analogous (see [10], [4]). In fact, we say that a continuous function u in Ω is a viscosity solution of (2.1), if for every $x \in \Omega$, both of the following properties are satisfied:

(iii) for every $\varphi \in C^2(\Omega)$ such that $u - \varphi$ attains its maximum at x, then

$$\varphi(x) - \varepsilon^2 F^* \left(\nabla\varphi(x), \nabla^2\varphi(x)\right) \le 0;$$

(iv) for every $\varphi \in C^2(\Omega)$ such that $u - \varphi$ attains its minimum at x, then

$$\varphi(x) - \varepsilon^2 F_* \left(\nabla\varphi(x), \nabla^2\varphi(x)\right) \ge 0.$$

A weak comparison principle is available for (2.1).

Lemma A.2 *([4], Appendix D). Let u and v be a viscosity sub-solution and a viscosity super-solution for* (2.1).

If $u \le v$ on $\partial\Omega$, then $u \le v$ on $\overline{\Omega}$.

We conclude this appendix with the following useful lemma.

Lemma A.3 *(Extension lemma). Let Ω be an open set and let $u \in C^2(\Omega)$. Suppose that*

(i) *$x_0 \in \Omega$ is the unique critical point for u in Ω;*
(ii) *u is solution of* (2.1) *in $\Omega \setminus \{x_0\}$.*

Then u is a solution of (2.1) *in Ω.*

Proof. We can always assume that $p \ge 2$ since, otherwise, we can switch the roles of λ and Λ.

Let us now take a sequence of points $y_n \in \Omega \setminus \{x_0\}$ such that $y_n \to x_0$ as $n \to \infty$. From our assumption on u, we have that

$$u - \frac{\varepsilon^2}{p} \left\{ \operatorname{tr}(\nabla^2 u) + (p-2)\, \frac{\langle \nabla^2 u \nabla u, \nabla u\rangle}{|\nabla u|^2} \right\} = 0$$

at each y_n. Thus, both of the following inequalities hold at y_n:

$$u - \frac{1}{p}\varepsilon^2 \left\{ \operatorname{tr}(\nabla^2 u) + (p-2)\, \Lambda(\nabla^2 u) \right\} \le 0,$$

and

$$u - \frac{1}{p}\varepsilon^2 \left\{ \operatorname{tr}(\nabla^2 u) + (p-2)\, \lambda(\nabla^2 u) \right\} \ge 0.$$

By the continuity of the functions $\lambda(\nabla^2 u)$ and $\Lambda(\nabla^2 u)$ in Ω, we then infer that the last two inequalities also hold at x_0. The claim follows, by virtue of Remark 6, applied to the elliptic case. □

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