THE INFINITESIMAL FORM OF BRUNN-MINKOWSKI TYPE INEQUALITIES

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ABSTRACT. Log-Brunn-Minkowski inequality was conjectured by Boröczky, Lutwak, Yang and Zhang [7], and it states that a certain strengthening of the classical Brunn-Minkowski inequality is admissible in the case of symmetric convex sets. It was recently shown by Nayar, Zvavitch, the second and the third authors [27], that Log-Brunn-Minkowski inequality implies a certain dimensional Brunn-Minkowski inequality for log-concave measures, which in the case of Gaussian measure was conjectured by Gardner and Zvavitch [17].

In this note, we obtain local statements for both Log-Brunn-Minkowski and dimensional Brunn-Minkowski inequalities for rotation invariant log-conave measures near a ball. Remarkably, the assumption of symmetry is only necessary for Log-Brunn-Minkowski stability, which emphasizes an important difference between the two conjectured inequalities.

Also, we determine the infinitesimal version of the log-Brunn-Minkowski inequality. As a consequence, we obtain a strong Poincaré-type inequality in the case of unconditional convex sets, as well as for symmetric convex sets on the plane.

Additionally, we derive an infinitesimal equivalent version of the B-conjecture for an arbitrary measure.

1. INTRODUCTION

1.1. **History and background.** The classical Brunn-Minkowski inequality states that for a scalar $\lambda \in [0,1]$ and for Borel measurable sets A and B in $\mathbb{R}^n$, such that $(1-\lambda)A+\lambda B$ is measurable as well,

$$|\lambda A + (1-\lambda)B|^{\frac{1}{n}} \geq \lambda |A|^{\frac{1}{n}} + (1-\lambda)|B|^{\frac{1}{n}}. \tag{1}$$

Here $|\cdot|$ denotes the Lebesgue measure, the addition between sets is the standard vector addition, and multiplication of sets by non-negative reals is the usual dilation. This inequality has found many important applications in Geometry and Analysis (see *e.g.* Gardner [16] for an exhaustive survey on this subject).

For example, the classical isoperimetric inequality, which states that Euclidean balls maximize the volume at fixed perimeter, can be deduced in a few lines from (1). Also, Maurey [29] deduced from this inequality the Poincaré inequality for the Gaussian measure and Gaussian concentration properties. Based on Maurey's results, Bobkov and Ledoux proved that the Brunn-Minkowski inequality implies Brascamp-Lieb and log-Sobolev inequalities [3]; they also deduced sharp Sobolev and Gagliardo-Nirenberg inequalities [4]. A different argument was developed by the first named author in [11] to deduce Poincaré type inequalities on the boundary of convex bodies from the Brunn-Minkowski inequality.

Recall that a set in $\mathbb{R}^n$ is called convex if together with any two points it contains an interval containing them. A convex body is a convex compact set with non-empty interior. The family of convex bodies of $\mathbb{R}^n$ will be denoted by $\mathcal{K}^n$.

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A measure γ on $\mathbb{R}^n$ is called log-concave if for any pair of sets A and B and for any scalar $\lambda \in [0,1]$,

$$\gamma(\lambda A + (1-\lambda)B) \geq \gamma(A)^\lambda \gamma(B)^{1-\lambda}. \tag{2}$$

Borell showed [6] that a measure is log-concave if it has a density (with respect to the Lebesgue measure) which is log-concave (see also Prékopa [34], Leindler [24]). In particular, Lebesgue measure on $\mathbb{R}^n$ is log-concave:

$$|\lambda A + (1-\lambda)B| \geq |A|^\lambda |B|^{1-\lambda}. \tag{3}$$

Note that (1) implies (3) by the arithmetic-geometric mean inequality. Conversely, a simple argument shows that (3) implies (1) (see, for example, Gardner [16]). This argument is based on the homogeneity of the Lebesgue measure, therefore, a property analogous to (1) may not hold for log-concave measures which are not homogeneous. The transposition of (1) to a measure γ,

$$\gamma(\lambda A + (1-\lambda)B)^{\frac{1}{n}} \geq \lambda\gamma(A)^{\frac{1}{n}} + (1-\lambda)\gamma(B)^{\frac{1}{n}}, \quad \forall\, \lambda \in [0,1], \tag{4}$$

as A and B vary in some class of sets, will be called a **dimensional Brunn-Minkowski inequality**. Note that if γ is the Gaussian measure and if $A = \{p\}$ is a one-point set, while B is any measurable set (with positive measure), then the set $A + B$ is the translate of B by p. Hence, letting $|p| \to \infty$, and keeping B fixed, one may check that (4) fails. This could suggest to focus on convex sets containing the origin. On the other hand, Nayar and Tkocz [32] constructed an example in which (4) fails for the Gaussian measure while both A and B contain the origin. Gardner and Zvavitch [17] studied inequality (4) for the Gaussian measure under special assumptions on the sets A and B, and they showed that it holds if the sets A and B are convex symmetric dilates of each other. Gardner and Zvavitch [17] proposed the conjecture below in the case of the Gaussian measure; we shall state it in a more general form which is believed to be natural.

Conjecture 1.1 (Gardner, Zvavitch (generalized)). *Let $n \geq 2$ be an integer. Let γ be a log-concave symmetric measure (i.e. $\gamma(A) = \gamma(-A)$ for every measurable set A) on $\mathbb{R}^n$. Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Then*

$$\gamma(\lambda K + (1-\lambda)L)^{\frac{1}{n}} \geq \lambda\gamma(K)^{\frac{1}{n}} + (1-\lambda)\gamma(L)^{\frac{1}{n}}. \tag{5}$$

Next, we pass to describe the log-Brunn-Minkowski inequality. For a scalar $\lambda \in [0,1]$ and for convex bodies K and L containing the origin in their interior, with support functions h_K and h_L, respectively (see section 3 for the definition), define their geometric average as follows:

$$K^\lambda L^{1-\lambda} := \{x \in \mathbb{R}^n \,:\, \langle x, u\rangle \leq h_K^\lambda(u) h_L^{1-\lambda}(u)\ \forall u \in \mathbb{S}^{n-1}\}, \tag{6}$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{R}^n$ (note that the fact that the origin lies in the interior of a convex body implies that its support function is strictly positive). This set is again a convex body, whose support function is, in general, smaller than the geometric mean of the support functions of K and L. The following conjecture is widely known as log-Brunn-Minkowski conjecture (see Böröczky, Lutwak, Yang, Zhang [7]).

Conjecture 1.2 (Böröczky, Lutwak, Yang, Zhang). *Let $n \geq 2$ be an integer. Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Then*

$$|K^\lambda L^{1-\lambda}| \geq |K|^\lambda |L|^{1-\lambda}. \tag{7}$$

Important applications and motivations for Conjecture 1.2 can be found in Böröczky, Lutwak, Yang, Zhang [8], [9].

It is not difficult to see that the condition of symmetry is necessary (see Böröczky, Lutwak, Yang, Zhang [7] or Remark 1.11 below). As for the positive direction, Böröczky, Lutwak, Yang and Zhang showed that this conjecture holds for $n = 2$. Saroglou [36] and Cordero, Fradelizi, Maurey [15] proved that (7) is true when the sets K and L are unconditional (i.e. they are symmetric with respect to every coordinate hyperplane). Rotem [35] showed that Log-Brunn-Minkowski conjecture holds for complex convex bodies. Saroglou showed [37] that the validity of Conjecture 1.2 would imply the same statement for every log-concave symmetric measure γ on $\mathbb{R}^n$: for every symmetric $K, L \in \mathcal{K}^n$ and for every $\lambda \in [0, 1]$,

$$\gamma(K^\lambda L^{1-\lambda}) \geq \gamma(K)^\lambda \gamma(L)^{1-\lambda}. \tag{8}$$

By definition and by the arithmetic-geometric mean inequality, the support function of $K^\lambda L^{1-\lambda}$ is smaller than the convex linear combinations of the support functions of K and L. In other words, we have the inclusion:

$$K^\lambda L^{1-\lambda} \subset \lambda K + (1 - \lambda)L.$$

Therefore, (8) is stronger than (2), for every measure.

In [27] the second and third named authors, Nayar and Zvavitch showed that (8) implies (5) for every ray-decreasing measure γ on $\mathbb{R}^n$ and for every pair of convex sets K and L. Therefore, Conjecture 1.1 holds on the plane and for unconditional sets.

We conclude the overview of the open questions of this framework with the so called B-conjecture, proposed by Banaszczyk and popularized by Latała [23].

Conjecture 1.3 (B-conjecture). *Let $n \geq 2$ be an integer, and let γ be a log-concave symmetric measure on $\mathbb{R}^n$. Then for every symmetric convex body $K \subset \mathbb{R}^n$, the function $t \to \gamma(e^t K)$ is log-concave on $\mathbb{R}^+$.*

The B-conjecture was proved in the case of Gaussian measure by Cordero-Erausquin, Fradelizi and Maurey [15]. Their results were extended by Livne Bar-on [25]. Notice that applying inequality (8) to symmetric convex bodies K, L that are dilates of each other yields the B-conjecture, and therefore Conjecture 1.2 is stronger than Conjecture 1.3.

1.2. **Infinitesimal versions of inequalities.** We present the core idea of this paper, i.e. how to derive the infinitesimal version of concavity inequalities of Brunn-Minkowski type. We follow a method which has been studied by the first named author, Hug and Saorin-Gomez in [11], [12] and [14]. A similar circle of ideas was used by Kolesnikov, E. Milman in [22] to study Brunn-Minkowski type inequalities, and in [21] to obtain an infinitesimal version of Ehrhard's inequality. We illustrate this approach first in the case of the dimensional Brunn-Minkowski inequality for an arbitrary measure.

We need to introduce some notation. We say that a convex body K is $C^{2,+}$ if ∂K is of class C^2 and the Gauss curvature is strictly positive at every $x \in \partial K$.

The first key point of the method we use here, is that the property of being $C^{2,+}$ is stable under small "additive" perturbations (with respect to either the Minkowski addition, or the log-addition). This can be expressed in more precise terms using support functions (see Section 3 for the definition). Let $K \in \mathcal{K}^n$ and let $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$ be its support function. When no ambiguity is possible, we will write h instead of h_K. Then K is of class $C^{2,+}$ if and only if $h \in C^2(\mathbb{S}^{n-1})$ and the following matrix inequality is verified

$$(h_{ij}(u) + h(u)\delta_{ij}) > 0 \quad \forall\, u \in \mathbb{S}^{n-1}, \tag{9}$$

where h_{ij}, $i, j = 1, \dots, n-1$, stand for the second covariant derivatives with respect to an orthonormal coordinate frame on $\mathbb{S}^{n-1}$, and δ_{ij}, $i, j = 1, \dots, n-1$ are the usual Kronecker symbols (more details on condition (9) will be given in Section 3). We will denote by $C^{2,+}(\mathbb{S}^{n-1})$ the class of support functions of convex bodies of class $C^{2,+}$.

We will denote the family of centrally symmetric convex bodies by $\mathcal{K}_s^n$. Central symmetry of a convex body K is easily readable on its support function h: $K \in \mathcal{K}_s^n$ if and only if h is even. Notice, moreover, that $K \in \mathcal{K}_s^n$ implies that $0 \in K$ and consequently $h \geq 0$; if, moreover, K is of class $C^{2,+}$ then the origin is an interior point, and this implies $h > 0$ on $\mathbb{S}^{n-1}$. $C_e^{2,+}(\mathbb{S}^{n-1})$ will denote the set of support functions of centrally symmetric $C^{2,+}$ convex bodies, i.e. functions from $C^{2,+}(\mathbb{S}^{n-1})$ which are additionally even.

Due to the strict inequality (and to the compactness of $\mathbb{S}^{n-1}$), (9) is stable under small perturbations of h. More precisely, let h be the support function of a $C^{2,+}$ convex body K, and let $\psi \in C^2(\mathbb{S}^{n-1})$; then the function

$$h_s := h + s\psi \tag{10}$$

still verifies (9) if the parameter s is sufficiently small, say $|s| \leq a$ for some appropriate $a > 0$. Hence for every s in this range there exists a unique convex body K_s with the support function h_s.

Definition 1.4. *Let $h \in C^{2,+}(\mathbb{S}^{n-1})$, $\psi \in C^2(\mathbb{S}^{n-1})$ and let $I \subset \mathbb{R}$ be an interval containing the origin, such that $h + s\psi \in C^{2,+}(\mathbb{S}^{n-1})$ for every $s \in I$. We define the one-parameter family of convex bodies:*

$$\boldsymbol{K}(h, \psi, I) := \{K_s \,:\, h_{K_s} = h + s\psi,\ s \in I\}.$$

The next step is, given a sufficiently regular measure γ on $\mathbb{R}^n$, to express $\gamma(K)$, for every K of class $C^{2,+}$, in terms of the support function of h. In section 3.3 we derive the equality

$$\gamma(K) = \int_{\mathbb{S}^{n-1}} h(y) \det Q(h; y) \int_0^1 t^{n-1} F\left(t\nabla H(y)\right) dt dy. \tag{11}$$

Here F is the density of γ, $Q(h, \cdot)$ is the matrix involved in condition (9) and H is the 1-homogeneous extension of h.

Hence $\gamma(K)$ can be seen as a functional depending on h, and the same can be said for the functional $\gamma(K)^{1/n}$. Now let K be a $C^{2,+}$ centrally symmetric convex body, and let $\psi \in C^2(\mathbb{S}^{n-1})$ be even. Let $\mathbf{K}(h, \psi, I)$ be the corresponding one-parameter family; note that K_s is centrally symmetric for every s. If the measure γ verifies Conjecture 1.1, then the function

$$s \,\to\, [\gamma(K_s)]^{\frac{1}{n}}$$

is concave. Hence (if the following derivative exists):

$$\frac{d^2}{ds^2} [\gamma(K_s)]^{\frac{1}{n}} \bigg|_{s=0} \leq 0. \tag{12}$$

The previous inequality is what we call infinitesimal form of the Brunn-Minkowski type inequality (5). In particular this is the infinitesimal form *at the body* K: the inequality (12) means that the second variation of $\gamma^{1/n}$ is negative semi-definite at K. The way we achieved it shows that it is a consequence of (5); on the other hand we will prove that (12) is in fact equivalent to (5), under fairly reasonable assumptions on γ.

Lemma 1.5. *Assume that γ is a symmetric log-concave measure with continuously differentiable density. Conjecture 1.1 holds for γ if and only if for every one-parameter family*

$\boldsymbol{K}(h,\psi,I)$, with h and ψ even, condition (12) *is verified, that is*

$$\frac{d^2}{ds^2}\left[\gamma(K_s)\right]\bigg|_{s=0}\cdot\gamma(K_0)\leq\frac{n-1}{n}\left(\frac{d}{ds}\left[\gamma(K_s)\right]\bigg|_{s=0}\right)^2.$$

Using the representation formula (11), the left hand-side of (12) can be explicitly computed, and this inequality turns out to be an integral inequality, depending on F, h and ψ; the details are carried out in Section 7. If we fix the measure, we are left with a family of inequalities, parametrized by h, involving the test function ψ, along with its first and second covariant derivatives. A reference model for these inequalities is the Poincaré inequality ([1]) on $\mathbb{S}^{n-1}$ (see Groemer [18] for the details).

The fact that the infinitesimal forms of concavity inequalities are inequalities of Poincaré type is a general phenomenon; we refer for instance to the first named author's paper [13] for a brief discussion on this subject, and for references to related literature. This approach gives a new point of view to the problem, and it can be fruitfully used in some cases. For instance, when the convex body K is the ball of radius R centered at the origin, (12) is equivalent to

$$\alpha\int_{\mathbb{S}^{n-1}}\psi^2du-\beta\left(\int_{\mathbb{S}^{n-1}}\psi du\right)^2\leq\int_{\mathbb{S}^{n-1}}|\nabla_\sigma\psi|^2du, \tag{13}$$

where α and β are constants depending on the density of γ, R, and n. The validity of this inequality is proved in Section 7 via classical harmonic analysis. More specifically, we show:

Theorem 1.6 (Dimensional Brunn-Minkowski near a ball)**.** *Let γ be a rotation invariant log-concave measure on $\mathbb{R}^n$. Let $R\in(0,\infty)$. Let $\psi\in C^2(\mathbb{S}^{n-1})$. Then there exists a sufficiently small $a>0$ such that for every $\epsilon_1,\epsilon_2\in(0,a)$ and for every $\lambda\in[0,1]$, one has*

$$\gamma(\lambda K_1+(1-\lambda)K_2)^{\frac{1}{n}}\geq\lambda\gamma(K_1)^{\frac{1}{n}}+(1-\lambda)\gamma(K_2)^{\frac{1}{n}},$$

where K_1 is the convex set with the support function $h_1=R+\epsilon_1\psi$ and K_2 is the convex set with the support function $h_2=R+\epsilon_2\psi$.

A similar approach can be used for the log-Brunn-Minkowski inequality. In order to do this we introduce a corresponding type of one-parameter families of convex bodies. In this case, additive perturbations (10) are replaced by multiplicative perturbations.

Remark 1.7. *Let $h\in C^{2,+}(\mathbb{S}^{n-1})$ and $\varphi\in C^2(\mathbb{S}^{n-1})$, with $\varphi>0$ on $\mathbb{S}^{n-1}$. Then there exists $a>0$ such that*

$$h_s:=h\,\varphi^s\in C^{2,+}(\mathbb{S}^{n-1})\quad\forall\,s\in[-a,a].$$

In particular for every $s\in[-a,a]$ there exists a $C^{2,+}$ convex body Q_s whose support function is h_s. This follows again from condition (9)*.*

On the base of the previous remark we define the corresponding 1-dimensional systems.

Definition 1.8. *Let $h\in C^{2,+}(\mathbb{S}^{n-1})$ and $\varphi\in C^2(\mathbb{S}^{n-1})$ be strictly positive on $\mathbb{S}^{n-1}$. Let $I\subset\mathbb{R}$ be an interval containing the origin, such that $h\varphi^s\in C^{2,+}(\mathbb{S}^{n-1})$ for every $s\in I$. We define the one-parameter system of convex bodies:*

$$\boldsymbol{Q}(h,\varphi,I):=\{Q_s\in\mathcal{K}^n\,:\,h_{Q_s}=h\varphi^s,\;s\in I\}.$$

[1]The Poincaré inequality on $\mathbb{S}^{n-1}$ provides an optimal upper bound for the $L^2(\mathbb{S}^{n-1})$-norm of a function in terms of the $L^2(\mathbb{S}^{n-1})$-norm of its (spherical) gradient, under a zero-mean type condition.

As before, we assume that a measure γ is given in $\mathbb{R}^n$ such that for every one-parameter family $\mathbf{Q}(h, \varphi, I)$ the function $s \to \gamma(Q_s)$, $s \in I$, is twice differentiable in I.

Lemma 1.9. *Assume that Conjecture 1.2 holds for a measure γ, i.e. for every pair of symmetric convex sets K and L and for every $\lambda \in [0,1]$,*

$$\gamma(K^\lambda L^{1-\lambda}) \geq \gamma(K)^\lambda \gamma(L)^{1-\lambda}. \tag{14}$$

Then for every one-parameter family $Q_s \in \boldsymbol{Q}(h, \varphi, I)$, with h and φ even, the function $\gamma(Q_s)$ is log-concave in I, and more precisely

$$\frac{d^2}{ds^2} \log(\gamma(Q_s)) \bigg|_{s=0} \leq 0. \tag{15}$$

We check the validity of the infinitesimal form of the log-Brunn-Minkowski inequality when $h \equiv R$, $R > 0$, for arbitrary log-concave and rotation invariant measures (hence including the Lebesgue measure).

Theorem 1.10 (Log-Brunn-Minkowski near a ball)**.** *Let γ be a rotation invariant log-concave measure on $\mathbb{R}^n$. Let $R \in (0, \infty)$. Let $\varphi \in C^2(\mathbb{S}^{n-1})$ be even and strictly positive. Then there exists a sufficiently small $a > 0$ such that for every $\epsilon_1, \epsilon_2 \in (0, a)$ and for every $\lambda \in [0,1]$, one has*

$$\gamma(K_1^\lambda K_2^{1-\lambda}) \geq \gamma(K_1)^\lambda \gamma(K_2)^{1-\lambda},$$

where K_1 is the convex set with the support function $h_1 = R\varphi^{\epsilon_1}$ and K_2 is the convex set with the support function $h_2 = R\varphi^{\epsilon_2}$.

Theorem 1.10 can be used to obtain a local uniqueness result for Log-Minkowski problem (see Böröczky, Lutwak, Yang, Zhang [7], [8] and the references therein), and the corresponding investigation shall be carried out in a separate manuscript.

Remark 1.11. *Theorems 1.6 and 1.10 indicate an important difference between the Brunn-Minkowski conjecture for log-concave measures and the log-Brunn-Minkowski conjecture. While the second conjecture is stronger than the first one, their local behavior is surprisingly different. Indeed, one can see that the log-Brunn-Minkowski inequality necessarily fails for the simplest possible odd perturbation: the shift. Therefore, the inequality (7) is never correct when $K = RB_2^n$ and $L = RB_2^n + a$, for any $a \in \mathbb{R}^n$ and $R > 0$. In contrast, Theorem 1.6 tells us that the Brunn-Minkowski inequality for radially symmetric log-concave measures holds when K and L are obtained via perturbating RB_2^n, and the perturbation does not have to be even.*

Similarly to the previous case, we may use the representation formula for the volume to compute the second derivative of $\log(\gamma(K_s))$. In the case of Lebesgue measure we prove the following Theorem.

Theorem 1.12 (Infinitesimal form of Log-Brunn-Minkowski conjecture)**.** *Let $n \geq 2$ be an integer. If Conjecture 1.2 is true, then for every $h \in C_e^{2,+}(\mathbb{S}^{n-1})$, $\psi \in C^2(\mathbb{S}^{n-1})$, ψ even and strictly positive,*

$$\int_{\mathbb{S}^{n-1}} \psi^2 \frac{1 + \mathrm{tr}(Q^{-1}(h))h}{h^2} d\bar{V}_h - n\left(\int_{\mathbb{S}^{n-1}} \frac{\psi}{h} d\bar{V}_h\right)^2 \leq \int_{\mathbb{S}^{n-1}} \frac{1}{h} \langle Q^{-1}(h)\nabla\psi, \nabla\psi\rangle d\bar{V}_h. \tag{16}$$

Here $d\bar{V}_h$ stands for the normalized cone measure of the convex body K with support function h and $Q(h)$ is the curvature matrix of K (see definitions (9), (34) and (29)).

A corresponding infinitesimal Brunn-Minkowski inequality for Lebesgue measure was obtained by the first named author in [11] and reads as:
(17)
$$\int_{\mathbb{S}^{n-1}} \psi^2 \frac{\mathrm{tr}(Q^{-1}(h))}{h} d\bar{V}_h - (n-1)\left(\int_{\mathbb{S}^{n-1}} \frac{\psi}{h} d\bar{V}_h\right)^2 \leq \int_{\mathbb{S}^{n-1}} \frac{1}{h}\langle Q^{-1}(h)\nabla\psi, \nabla\psi\rangle d\bar{V}_h.$$
Note that by the Cauchy-Schwarz inequality,
$$\int_{\mathbb{S}^{n-1}} \frac{\psi^2}{h^2} d\bar{V}_h \geq \left(\int_{\mathbb{S}^{n-1}} \frac{\psi}{h} d\bar{V}_h\right)^2.$$
Hence, (16) is indeed a strengthening of (17).

The log-Brunn-Minkowski inequality has been proved in two special cases: when $n = 2$ (see Böröczky, Lutwak, Yang, Zhang [7]) and when K and L are unconditional (see Saroglou [36]). The latter condition is equivalent to require, in (16), that both h and φ are symmetric with respect to each coordinate hyperplane. Hence Theorem 1.12 implies the validity of (16) in the unconditional and planar cases.
In particular, letting $\varphi \equiv 1$ we arrive to the following corollary of Theorem 1.12.

Corollary 1.13 (A strengthening of Minkowski's second inequality.)**.** *Let K be a convex symmetric set in the plane, or a convex unconditional set in $\mathbb{R}^n$. Then,*

$$V_n(K)\left(V_{n-2}(K) + \int_{\partial K} \frac{1}{\langle y, \nu_K(y)\rangle} d\sigma(y)\right) \leq V_{n-1}(K)^2, \tag{18}$$

where V_{n-i} are the intrinsic volumes of K, $\nu_K(y)$ stands for the unit normal at $y \in \partial K$ and $d\sigma(y)$ is the surface area measure on ∂K.

Minkowski's second inequality, which states that for every convex set $K \subset \mathbb{R}^n$ one has
$$V_n(K)V_{n-2}(K) \leq \frac{n-1}{n} V_{n-1}(K)^2,$$
is deduced from (18) by using the Cauchy-Schwarz inequality. For a more general version of this inequality see, for example, Schneider [38, Chapter 4].

Additionally, the argument that we have described can be applied to obtain the following equivalent form of the B-conjecture. Let γ be a log-concave measure on $\mathbb{R}^n$, $n \geq 2$. Assume that γ is not supported on a lower-dimensional affine subspace of $\mathbb{R}^n$, and let f be its density.

Theorem 1.14 (Equivalent infinitesimal form of the B-conjecture)**.** *Conjecture 1.3 is true for γ if and only if for every 1-homogeneous, even, convex function H defined in $\mathbb{R}^n$ we have*

$$\begin{aligned} &n\int_{\mathbb{S}^{n-1}} \frac{f(\nabla H)}{\int_0^1 t^{n-1} f(t\nabla H)dt} d\bar{V}_{\gamma,K} - \left(\int_{\mathbb{S}^{n-1}} \frac{f(\nabla H)}{\int_0^1 t^{n-1} f(t\nabla H)dt} d\bar{V}_{\gamma,K}\right)^2 \leq \\ &-\int_{\mathbb{S}^{n-1}} \frac{\langle \nabla f(\nabla H), \nabla H\rangle}{\int_0^1 t^{n-1} f(t\nabla H)dt} d\bar{V}_{\gamma,K}. \end{aligned} \tag{19}$$

Here $d\bar{V}_{\gamma,h}$ is the normalized cone γ-measure of the convex body with support function H.

We remark that the result of Cordero-Erausquin, Fradelizi and Maurey implies that (19) is true when γ is the Gaussian measure, as well as for every unconditional log-concave measure γ whenever H is unconditional.

This paper is structured as follows. In Section 2 we intend to engage the reader in the method which we employ, presenting a new proof of the classical Brunn-Minkowski inequality for convex sets on the plane, which uses its infinitesimal version. Section 3 contains some preliminary material for the subsequent part of the paper. In Section 4 we establish the relations between dimensional Brunn-Minkowski inequality and log-Brunn-Minkowski inequality and their infinitesimal forms (i.e. we prove Lemmas 1.5 and 1.9). In Section 5 we prove Theorem 1.12. Theorem 1.14 is proved in Section 6. Theorems 1.6 and 1.10 are proved in Sections 7 and 8, respectively.

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2. A PROOF OF THE BRUNN-MINKOWSKI INEQUALITY ON THE PLANE

In order to engage the reader with the method we shall employ in this manuscript, we outline a proof of the classical Brunn-Minkowski inequality for convex sets in the plane, i.e.

$$|\lambda K+(1-\lambda)L|^{\frac{1}{2}}\geq \lambda|K|^{\frac{1}{2}}+(1-\lambda)|L|^{\frac{1}{2}},\quad \forall\, K,L\in\mathcal{K}^2,\quad \forall\,\lambda\in[0,1]. \tag{20}$$

Proof. Assume that K and L are convex bodies on the plane which belong to the class $C^{2,+}$. We identify the unit circle $\mathbb{S}^1$ with the interval $[-\pi,\pi]$, so that every function on $\mathbb{S}^1$ is seen as a function on $[-\pi,\pi]$, which can be extended to $\mathbb{R}$ as a periodic function with period 2π. Note that if $n=2$, the matrix $Q(h)$ is 1×1 and its entry is $h+\ddot{h}$. Therefore, a function h defined in $[-\pi,\pi]$ is the support function of a $C^{2,+}$ convex body if and only if it admits a 2π-periodic, C^2 extension to $\mathbb{R}$ and

$$h(t)+\ddot{h}(t)>0\quad \forall\, t\in[-\pi,\pi].$$

Let ψ be of class C^2 and let h be the support function of a convex body L of class $C^{2,+}$, and assume that $h>0$ (i.e. the origin belongs to the interior of L). According to a well-known Santalo's formula (see Schneider [38]), the area of L may be expressed as

$$|K|=\frac{1}{2}\int_{-\pi}^{\pi}(h^2-\dot{h}^2)dt. \tag{21}$$

As the matter of fact, (11) implies (21) directly via integration by parts.

Let $a>0$ be sufficiently small so that $h_s:=h+s\psi$, $s\in[-a,a]$, is the support function of a convex body K_s. Consider the function

$$f(s):=|K_s|=\frac{1}{2}\int_{-\pi}^{\pi}\left[(h+s\psi)^2-(\dot{h}+s\dot{\psi})^2\right]dt. \tag{22}$$

By Lemma 1.5, (20) is equivalent to the fact that f is $\frac{1}{2}$–concave (for all h and ψ as above). The second derivative of $\sqrt{f}$ at $s=0$ is smaller or equal to zero if and only if

$$2f(0)f''(0)\leq f'(0)^2. \tag{23}$$

Combining (22) and (23) we arrive to

$$\left(\int(h^2-\dot{h}^2)dt\right)\left(\int(\psi^2-\dot{\psi}^2)dt\right)\leq\left(\int(h\psi-\dot{h}\dot{\psi})dt\right)^2. \tag{24}$$

To prove (24), we introduce the Fourier coefficients $a_k = \widehat{h}(k)$ and $b_k = \widehat{\psi}(k)$, $k \in \mathbb{N}$, of h and ψ, respectively. Then by Parseval's identity, (24) is equivalent to

$$\left(a_0^2 - \sum_{k\neq 0}(k^2-1)a_k^2\right)\left(b_0^2 - \sum_{k\neq 0}(k^2-1)b_k^2\right) \leq \left(a_0b_0 - \sum_{k\neq 0}(k^2-1)a_kb_k\right)^2. \tag{25}$$

Let

$$t = \sum_{k\neq 0}(k^2-1)a_kb_k, \quad A = \sqrt{\sum_{k\neq 0}(k^2-1)a_k^2}, \quad B = \sqrt{\sum_{k\neq 0}(k^2-1)b_k^2}.$$

By Cauchy's inequality, $|t| \leq AB$. Note that

$$a_0 = \int_{-\pi}^{\pi} h(t)dt > 0.$$

Note also that

$$a_0^2 - A^2 = \int_{-\pi}^{\pi}(h^2 - \dot{h}^2)dt = 2|K| > 0,$$

and hence $a_0 \geq A$. The goal is to prove that

$$(a_0^2 - A^2)(b_0^2 - B^2) \leq (a_0b_0 - t)^2 \tag{26}$$

for all $t \in [-AB, AB]$, provided that $a_0 \geq A > 0$, $B > 0$. The proof of (26) splits into three cases.

Case 1: $b_0 > B$. Then $a_0b_0 > AB$, and hence

$$\min\{(a_0b_0 - t)^2 \,:\, |t| \leq AB\} = (a_0b_0 - AB)^2.$$

Thus (26) amounts to the inequality

$$(a_0^2 - A^2)(b_0^2 - B^2) \leq (a_0b_0 - AB)^2,$$

which, in turn, is equivalent to the following true statement:

$$(a_0B - b_0A)^2 \geq 0.$$

Case 2: $|b_0| \leq |B|$. In this case the right hand side of (26) could be 0 but the left hand side is necessarily non positive.

Case 3: $b_0 < -B$. Then

$$\min\{(a_0b_0 - t)^2 \,:\, |t| \leq AB\} = (a_0b_0 + AB)^2,$$

and (26) follows from the inequality

$$(a_0^2 - A^2)(b_0^2 - B^2) \leq (a_0b_0 + AB)^2,$$

which, in turn, is true since

$$(a_0B + b_0A)^2 \geq 0.$$

This concludes the proof.

□

3. Preparatory material

We work in the $n-$dimensional Euclidean space $\mathbb{R}^n$ with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. We set $B_2^n := \{x \in \mathbb{R}^n : |x| \le 1\}$ and $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$, to denote the unit ball and the unit sphere, respectively.

We shall denote the Lebesgue measure (the *volume*) in $\mathbb{R}^n$ by $|\cdot|$. By σ we will denote the uniform measure on $\mathbb{S}^{n-1}$, i.e. the restriction to $\mathbb{S}^{n-1}$ of the $(n-1)$-dimensional Hausdorff measure.

We say that a set $A \subset \mathbb{R}^n$ is symmetric if for every $x \in A$ one has $-x \in A$.

The Minkowski addition of two subsets A and B of $\mathbb{R}^n$ is defined as

$$A + B = \{x + y : x \in A, y \in B\}.$$

The multiplication of a set A by a scalar $\lambda \ge 0$ is defined as the set

$$\lambda A = \{\lambda x : x \in A\}.$$

3.1. **Measures.** We will frequently consider measures on $\mathbb{R}^n$ different from the Lebesgue measure. A generic measure will be denoted by γ. All measures under consideration will be tacitly assumed to be Radon measures, and all sets will be assumed to be measurable. We will write that a measure γ has a density F if it is absolutely continuous with respect to the Lebesgue measure, and its Radon-Nikodym derivative with respect to the Lebesgue measure is F.

A measure γ on $\mathbb{R}^n$ is called symmetric if for every set $S \subset \mathbb{R}^n$, $\gamma(S) = \gamma(-S)$. If the measure has a density then it is symmetric whenever the density is an even function.

A measure γ on $\mathbb{R}^n$ is said to be rotation invariant if for every set $A \subset \mathbb{R}^n$, and for every rotation T, $\gamma(A) = \gamma(TA)$. If a rotation invariant measure γ has a density F, we may write F in the form:

$$F(x) = f(|x|),$$

for a suitable $f : [0,\infty) \to [0,\infty)$.

We recall that a function $f : \mathbb{R}^n \to [0,\infty)$ is log-concave if $-\log(f) : \mathbb{R}^n \to (\infty,\infty]$ is a convex function (with the convention $\log(0) = -\infty$).

3.2. **Convex bodies.** A set in $\mathbb{R}^n$ is called convex if together with every two points it contains the interval connecting them. If a set in $\mathbb{R}^n$ is convex and compact with non-empty interior, we call it a convex body. As mentioned before, the family of convex bodies in $\mathbb{R}^n$ will be denoted by $\mathcal{K}^n$. For the theory of convex bodies we refer the reader to the books by Ball [1], Bonnesen, Fenchel [5], Koldobsky [20], Milman, Schechtman [30], Schneider [38] and others.

Note that for every $K, L \in \mathcal{K}^n$ and $\alpha, \beta \ge 0$, we have $\alpha K + \beta L \in K^n$.

For $K \in \mathcal{K}^n$, the *support function* of K, $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$, is defined as

$$h_K(u) = \sup_{x \in K} \langle x, u\rangle.$$

By the geometric viewpoint, $h_K(u)$ represents the (signed) distance from the origin of the supporting hyperplane to K with outer unit normal u. We shall use the notation $H_K(x)$ for the 1-homogenous extension of h_K, that is,

$$H_K(x) = \begin{cases} |x|\, h_K\left(\dfrac{x}{|x|}\right) & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The function H_K is convex in $\mathbb{R}^n$, for every $K \in \mathcal{K}^n$. Vice versa, for every continuous 1-homogeneous convex function H on $\mathbb{R}^n$, there exists a unique convex body K such that $H = H_K$.

Note that $K \in \mathcal{K}^n$ contains the origin (resp., in its interior) if and only if $h_K \geq 0$ (resp. $h_K > 0$) on $\mathbb{S}^{n-1}$. It is easy to see that for a convex set K and a scalar $\lambda \geq 0$, we have $h_{\lambda K}(u) = \lambda h_K(u)$ for every $u \in \mathbb{S}^{n-1}$. It is also well known that for convex bodies K and L, the support function of their Minkowski sum is the sum of their support functions. Hence:

$$h_{\alpha K+\beta L}(u) = \alpha h_K(u) + \beta h_L(u) \quad \forall\, K, L \in \mathcal{K}^n, \quad \forall\, \alpha, \beta \geq 0. \tag{27}$$

We recall that for $k = 0, \ldots, n$, the $k-$th intrinsic volume of a convex body K is defined as follows:

$$V_k = |K + \epsilon B_2^n|^{(n-k)}\big|_0 \,,$$

where the upper index $(n-k)$ stands for the $(n-k)-$th derivative. Note that $V_n(K)$ is the volume of K, while $V_{n-1}(K)$ is the $(n-1)$-dimensional Hausdorff measure of ∂K.

We say that a convex body K is $C^{2,+}$ if ∂K is of class C^2 and the Gauss curvature is strictly positive at every $x \in \partial K$. In particular, if K is $C^{2,+}$ then it admits outer unit normal $\nu_K(x)$ at every boundary point x. Recall that the Gauss map $\nu_K : \partial K \to \mathbb{S}^{n-1}$ is the map assigning the unit normal to each point of ∂K. For $K \in C^{2,+}$, the Gauss map is a diffeomorphism. Moreover, for every $x \in \partial K$ we have

$$h_K(\nu_K(x)) = \langle x, \nu_K(x) \rangle. \tag{28}$$

$C^{2,+}$ convex bodies can be characterized through their support function. We recall that an orthonormal frame on the sphere is a map which associates a collection of $n-1$ orthonormal vectors to every point of $\mathbb{S}^{n-1}$. Let $\psi \in C^2(\mathbb{S}^{n-1})$. We denote by $\psi_i(u)$ and $\psi_{ij}(u)$, $i, j \in \{1, \ldots, n-1\}$, the first and second covariant derivatives of ψ at $u \in \mathbb{S}^{n-1}$, with respect to a fixed local orthonormal frame on an open subset of $\mathbb{S}^{n-1}$. We define the matrix

$$Q(\psi; u) = (q_{ij})_{i,j=1,\ldots,n-1} = (\psi_{ij} + \psi\delta_{ij})_{i,j=1,\ldots,n-1} \,, \tag{29}$$

where the δ_{ij}'s are the usual Kronecker symbols. On an occasion, instead of $Q(\psi; u)$ we write $Q(\psi)$. Note that $Q(\psi; u)$ is symmetric by standard properties of covariant derivatives. The meaning of this matrix becomes particularly important when ψ is the support function of a convex body K. In this case we shall call it *curvature matrix* of K (see the following Remark 3.2). The proof of the following proposition can be deduced from Schneider [38, Section 2.5].

Proposition 3.1. *Let $K \in \mathcal{K}^n$ and let h be its support function. Then K is of class $C^{2,+}$ if and only if $h \in C^2(\mathbb{S}^{n-1})$ and*

$$Q(h; u) > 0 \quad \forall\, u \in \mathbb{S}^{n-1}.$$

In view of the previous results it is convenient to introduce the following set of functions

$$C^{2,+}(\mathbb{S}^{n-1}) = \{h \in C^2(\mathbb{S}^{n-1}) \,:\, Q(h; u) > 0\, \forall\, u \in \mathbb{S}^{n-1}\}.$$

Hence $C^{2,+}(\mathbb{S}^{n-1})$ is the set of support functions of convex bodies of class $C^{2,+}$.

Remark 3.2. *Let K be a $C^{2,+}$ convex body. Then $\nu_K : \partial K \to \mathbb{S}^{n-1}$ is a diffeomorphism. The matrix $Q(h; u)$ represents the inverse of the Weingarten map at $x = \nu_K^{-1}(u)$, and its eigenvalues are the principal radii of curvature of ∂K at x. Consequently we have*

$$\det(Q(h; u)) = \frac{1}{G(x)}$$

where G denotes the Gauss curvature.

Let K be a $C^{2,+}$ convex body, with support function h_K and its homogenous extension H_K. H_K is of class $C^1(\mathbb{R}^n \backslash \{0\})$. By ∇H_K we denote its gradient with respect to Cartesian coordinates. The following useful relation holds: for every $u \in \mathbb{S}^{n-1}$, $\nabla H_K(u)$ is the (unique) point on ∂K where the outer unit normal is u:

$$\nabla H_K(u) = \nu_K^{-1}(u) \quad \forall\, u \in \mathbb{S}^{n-1}.$$

Consequently,

$$\langle \nabla H_K(u), \nu_K(u) \rangle = H_K(u) \quad \forall\, u \in \mathbb{S}^{n-1}.$$

Remark 3.3. *Let $\psi \in C^1(\mathbb{S}^{n-1})$. The notation $\nabla_\sigma \psi$ stands for the* spherical gradient *of ψ, i.e. the vector $(\psi_1, \dots, \psi_{n-1})$, where ψ_i are the covariant derivatives of ψ with respect to the i-th element of a fixed orthonormal system on $\mathbb{S}^{n-1}$. Let Φ be the 1-homogeneous extension of ψ to $\mathbb{R}^n$. Then we have*

$$|\nabla \Phi(u)|^2 = \psi^2(u) + |\nabla_\sigma \psi(u)|^2 \tag{30}$$

for every $u \in \mathbb{S}^{n-1}$.

3.3. **A formula expressing a measure of a convex set in terms of its support function.** Let γ be a probability measure on $\mathbb{R}^n$; we assume without loss of generality that γ has a density F with respect to the Lebesgue measure, and that F is sufficiently regular (*e.g.* continuous).

Theorem 3.4. *Let K be a $C^{2,+}$ convex body, with support function h and its homogenous extension H. Assume that the origin is in the interior of K. Then*

$$\gamma(K) = \int_{\mathbb{S}^{n-1}} h(y) \det Q(h; y) \int_0^1 t^{n-1} F\left(t \nabla H(y)\right) dt dy. \tag{31}$$

Proof. Firstly, we consider a polar coordinate system associated with the body. Let

$$X : \partial K \times [0,1] \to \mathbb{R}^n$$

be the map defined by

$$X(x,t) = tx.$$

Note that, by convexity of K, X establishes a bijection between $\partial K \times [0,1]$ and K. A simple computation shows that the Jacobian of this map, J_X, is given by

$$J_X(x,t) = t^{n-1} \langle x, \nu_K(x) \rangle d\sigma(x) = t^{n-1} h_K(\nu_K(x)) d\sigma(x),$$

where $d\sigma(x)$ is the area element of ∂K at x (see Nazarov [33] or the second named author [26]). Hence, by the area formula,

$$\gamma(K) = \int_K F(x) dx = \int_{\partial K} h_K(\nu_k(x)) \int_0^1 t^{n-1} F(tx)\, dt d\sigma(x).$$

Next, we make the change of variables $y = \nu_K(x)$. In view of Remark 3.2, its Jacobian is equal to $\det Q(h; \cdot)$. The proof is complete. □

Remark 3.5. *Though we will use the previous representation formula only for $C^{2,+}$ convex bodies, it is easy to see that it can be extended, by an approximation argument, to arbitrary convex bodies. In the general case the integration term* $\det Q(h; y) dy$ *must be replaced by $dS_{n-1}(K, y)$, where $S_{n-1}(K, \cdot)$ is the* area measure *of K (see Schneider* [38, Chapter 4]*).*

Corollary 3.6. *Let γ be a rotation invariant probability measure measure on $\mathbb{R}^n$ with density $F(y) = f(|y|)$. Let K be a convex body of class $C^{2,+}$ and assume that the origin is in the interior of K. Then*

$$\gamma(K) = \int_{\mathbb{S}^{n-1}} h \det Q(h;y) \int_0^1 t^{n-1} f(t|\nabla H(y)|) dt dy,$$

where h is the support function of K and H is its 1-homogeneous extension.

Remark 3.7. *We note that the above implies a well known formula for Lebesgue measure (corresponding to the case $f \equiv 1$) of a convex body:*

$$|K| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(u) \det Q(h(u)) du,$$

which we already encountered in Section 2, for $n = 2$.

For $K \in C^{2,+}$ the *cone-volume measure* V_K of K is a Borel measure on the unit sphere $\mathbb{S}^{n-1}$ defined for a Borel set $A \subset \mathbb{S}^{n-1}$ via

$$V_K(A) = \frac{1}{n} \int_{y \in \nu_K^{-1}(A)} \langle y, \nu_K(y) \rangle d\sigma(y), \tag{32}$$

where σ stands for the $(n-1)-$dimensional Hausdorff measure (restricted to ∂K). We refer, for instance, to Schneider [38, Section 9.1], Henk, Linke [19], Böröczky, Lutwak, Yang, Zhang [7], Naor [31], for a more detailed presentation of this notion, and for its definition for general convex bodies. As justified by Remark 3.7, the cone-volume measure of a smooth convex set K has a density with respect to the Haar measure on the sphere, and this density is expressible in terms of the support function of K as follows:

$$dV_K(u) = \frac{1}{n} h_K(u) \det Q(h_K(u)) du. \tag{33}$$

We shall use the notation $d\bar{V}_K$ for a cone volume measure normalized to be a probability measure on the sphere, that is

$$d\bar{V}_K(u) = \frac{1}{|K|} \frac{1}{n} h_K(u) \det Q(h_K(u)) du. \tag{34}$$

We will frequently identify a convex body K with its support function h and we shall sometimes use the notation V_h instead of V_K.

Additionally, given a measure γ on $\mathbb{R}^n$ with density $f(x)$, and a $C^{2,+}$ convex body K with support function h_K, we shall use the notion of cone γ-measure, defined on the sphere by the following relation:

$$dV_{\gamma,K}(u) = h_K(u) \det Q(h_K(u)) \int_0^1 t^{n-1} f(t \nabla H_K(u)) dt du. \tag{35}$$

Here H_K stands for 1-homogenous extension of h_K to $\mathbb{R}^n$. The expression (35) is justified by Theorem 3.4.

We shall use the notation $d\bar{V}_{\gamma,K}$ for a cone γ-measure normalized to be a probability measure on the sphere, that is

$$d\bar{V}_{\gamma,K}(u) = \frac{1}{\gamma(K)} h_K(u) \det Q(h_K(u)) \int_0^1 t^{n-1} f(t \nabla H(u)) dt du. \tag{36}$$

3.4. **The co-factor matrix and related notions.** In what follows we will need a lemma due to Cheng and Yau (see [10]), which will be particularly useful for applying the divergence theorem on $\mathbb{S}^{n-1}$. To state the lemma we need some preparation. In particular we will use some notions related to matrices.

Let $M = (m_{ij})$ be an $N \times N$ symmetric matrix, $N \in \mathbb{N}$. We define $C[M]$, the *cofactor matrix* of M, as follows

$$C[M] = (c_{ij}[M])_{i,j=1,\dots,N} \quad \text{where} \quad c_{ij}[M] = \frac{\partial \det}{\partial m_{ij}}(M) \quad i,j = 1,\dots,N.$$

$C[M]$ is an $N \times N$ symmetric matrix. If M is invertible then

$$C[M] = \det(M)\, M^{-1}. \tag{37}$$

Taking the trace on both sides and using symmetry of the matrices M and $C[M]$, we get

$$\sum_{i,j=1}^{N} c_{ij}[M] m_{ij} = N \det(M). \tag{38}$$

We shall also consider the second derivatives of the determinant of a matrix with respect to its entries:

$$c_{ij,kl}[M] = \frac{\partial^2 \det}{\partial m_{ij} \partial m_{kl}}(M).$$

By homogeneity we have that, for every $i, j = 1, \dots, N$

$$\sum_{k,l=1}^{N} c_{ij,kl}[M]\, m_{kl} = (N-1) c_{ij}[M]. \tag{39}$$

Throughout the paper we shall use the Einstein summation convention for repeated indices.

3.5. **The Cheng-Yau lemma and an extension.** Let $h \in C^{2,+}(\mathbb{S}^{n-1})$, and assume additionally that $h \in C^3(\mathbb{S}^{n-1})$. Consider the cofactor matrix $y \to C[Q(h;y)]$. This is a matrix of functions on $\mathbb{S}^{n-1}$. The lemma of Cheng and Yau asserts that each column of this matrix is divergence-free.

Lemma 3.8 (Cheng-Yau.)**.** *Let $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$. Then, for every index $j \in \{1,\dots,n-1\}$ and for every $y \in \mathbb{S}^{n-1}$,*

$$\sum_{i=1}^{n-1} \left(c_{ij}[Q(h;y)]\right)_i = 0,$$

where the sub-script i denotes the derivative with respect to the i-th element of an orthonormal frame on $\mathbb{S}^{n-1}$.

For simplicity of notation we shall often write $C(h)$, $c_{ij}(h)$ and $c_{ij,kl}(h)$ in place of $C[Q(h)]$, $c_{ij}[Q(h)]$ and $c_{ij,kl}[Q(h)]$ respectively.

As a corollary of the previous result we have the following integration by parts formula. If $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$ and $\psi, \phi \in C^2(\mathbb{S}^{n-1})$, then

$$\int_{\mathbb{S}^{n-1}} \phi\, c_{ij}(h)(\psi_{ij} + \psi\, \delta_{ij}) dy = \int_{\mathbb{S}^{n-1}} \psi\, c_{ij}(h)(\phi_{ij} + \phi\, \delta_{ij}) dy. \tag{40}$$

The Lemma of Cheng and Yau admits the following extension (see the paper by the first-named author, Hug and Saorin-Gomez [14]).

Lemma 3.9. *Let $\psi \in C^2(\mathbb{S}^{n-1})$ and $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$. Then, for every $k \in \{1, \dots, n-1\}$ and for every $y \in \mathbb{S}^{n-1}$*

$$\sum_{i=1}^{n-1} \left(c_{ij,kl}[Q(h;y)](\psi_{ij} + \psi\delta_{ij})\right)_l = 0.$$

Correspondingly we have, for every $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$, $\psi, \varphi, \phi \in C^2(\mathbb{S}^{n-1})$ and $i, j \in \{1, \dots, n-1\}$

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} \psi\, c_{ij,kl}(h)(\varphi_{ij} + \varphi\delta_{ij})((\phi)_{kl} + \phi\,\delta_{kl})dy \\ &= \int_{\mathbb{S}^{n-1}} \phi\, c_{ij,kl}(h)(\varphi_{ij} + \varphi\delta_{ij})((\psi)_{kl} + \psi\,\delta_{kl})dy. \end{aligned} \tag{41}$$

4. Proof of Lemmas 1.5 and 1.9

Proof of Lemma 1.5. Assume first that γ satisfies (5) for all pairs of symmetric convex sets K and L. Consider a system $\mathbf{K}(h, \psi, I)$. Then the equality $h_{K_s} = h + s\psi$, $s \in I$, and the linearity of support function with respect to Minkowski addition, imply that for every $s, t \in I$ and for every $\lambda \in [0, 1]$

$$K_{\lambda s+(1-\lambda)t} = \lambda K_s + (1-\lambda)K_t.$$

By (5),

$$\gamma(K_{\lambda s+(1-\lambda)t})^{\frac{1}{n}} = \gamma(\lambda K_s + (1-\lambda)K_t)^{\frac{1}{n}} \geq \lambda\gamma(K_s)^{\frac{1}{n}} + (1-\lambda)\gamma(K_t)^{\frac{1}{n}},$$

which means that the function $\gamma(K_s)^{\frac{1}{n}}$ is concave on I.

Conversely, suppose that for every system $\mathbf{K}(h, \psi, I)$ the function $\gamma(K_s)^{\frac{1}{n}}$ verifies (12). We firstly observe that this implies concavity of $\gamma(K_s)^{\frac{1}{n}}$ on the entire interval I. Indeed, given s_0 in the interior of I, consider $\tilde{h} = h + s_0\psi$, and define a new system $\tilde{\mathbf{K}}(\tilde{h}, \psi, J)$, where J is a new interval such that $\tilde{h} + s\psi = h + (s + s_0)\psi \in C^{2,+}$ for every $s \in J$. Then the second derivative of $\gamma(K_s)^{\frac{1}{n}}$ at $s = s_0$ is negative, so is the second derivative of $\gamma(\tilde{K}_s)^{\frac{1}{n}}$ at $s = 0$.

Next, note that for every pair of $C^{2,+}$ convex bodies K and L there exists a system $\mathbf{K}(h, \psi, I)$ to which they both belong. Indeed, pick $h = h_K$ and $\psi = h_L - h_K$, then for every $s \in [0, 1]$,

$$K_s = (1-s)K + sL.$$

It is important to observe here that for every $s \in [0, 1]$,

$$h_s = h_K + s(h_L - h_K) = (1-s)h_K + sh_L,$$

and hence h_s is a support function. Thus the system $\mathbf{K}(h_K, h_L - h_K, [0, 1])$ is well-defined. Since $\gamma(K_s)^{\frac{1}{n}}$ is concave on $[0, 1]$, we get

$$\gamma((1-s)K + sL)^{\frac{1}{n}} = \gamma(K_s)^{\frac{1}{n}} \geq (1-s)\gamma(K)^{\frac{1}{n}} + s\gamma(L)^{\frac{1}{n}},$$

which finishes the proof of (5) for convex bodies of class $C^{2,+}$. The general case is achieved by a standard approximation argument (see, for example, Schneider [38]).

□

Proof of Lemma 1.9. Let $h \in C^{2,+}(\mathbb{S}^{n-1})$ and $\varphi \in C^2(\mathbb{S}^{n-1})$ be strictly positive even functions on $\mathbb{S}^{n-1}$; by Remark 1.7 there exists $a > 0$ such that $h_s := h\varphi^s$ is the support function of a convex body Q_s for all $s \in [-a, a]$. Note that for $s, t \in [-a, a]$ we get

$$h_{\lambda s+(1-\lambda)t} = h_s^{\lambda} h_t^{1-\lambda},$$

and thus

$$Q_{\lambda s+(1-\lambda)t} = Q_s^{\lambda} Q_t^{1-\lambda}.$$

Inequality (14) implies

$$\gamma(Q_{\lambda s+(1-\lambda)t}) = \gamma(Q_s^{\lambda} Q_t^{1-\lambda}) \geq \gamma(Q_s)^{\lambda} \gamma(Q_t)^{1-\lambda},$$

which means that $\gamma(Q_s)$ is log-concave in $[-a, a]$.

□

5. The infinitesimal form of log-Brunn-Minkowski inequality, and proof of Theorem 1.12.

Let $h \in C^{2,+}(\mathbb{S}^{n-1})$ and $\psi \in C^2(\mathbb{S}^{n-1})$. As before, denote by $\mathbf{K}(h, \psi, I) = \{K_s\}$ the collection of sets with support functions $h_s = h + s\psi$. Consider the function $f(s) = |K_s|$. Then, by Remark 3.7,

$$f(s) = |K_s| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} (h + s\psi)(u) \det Q((h + s\psi)(u)) du. \tag{42}$$

It was shown by the first named author [11] that

$$f'(0) = \int_{\mathbb{S}^{n-1}} \psi \det Q(h) du, \tag{43}$$

and

$$f''(0) = \int_{\mathbb{S}^{n-1}} \psi^2 \mathrm{tr}(Q^{-1}(h)) \det Q(h) du - \int_{\mathbb{S}^{n-1}} \langle Q^{-1}(h) \nabla \psi, \nabla \psi \rangle \det Q(h) du. \tag{44}$$

Theorem 1.12 will follow from the next Lemma.

Lemma 5.1. *Let $n \geq 2$. Let γ be a measure on $\mathbb{R}^n$. Fix $h \in C^{2,+}(\mathbb{S}^{n-1})$, $\varphi \in C^2(\mathbb{S}^{n-1})$, $\varphi > 0$ and set $\psi = h \log \varphi$. Let $\boldsymbol{K}(h, \psi, I)$, with $I = [-a, a]$ and $a > 0$, be a one-parameter family as in Definition 1.4, i.e. the collection of sets with support functions $h_s = h + s\psi$, for $s \in [-a, a]$. Consider the function $f(s) = \gamma(K_s)$. Introduce the additional notation for the operator $F(h, \psi) := f'(0)$. Set*

$$A(h, \psi) := \left. \frac{dF\left(h, \frac{h+s\psi}{h}\psi\right)}{ds} \right|_{s=0}. \tag{45}$$

Consider the one-parameter family $\boldsymbol{Q}(h, \varphi, [-a, a])$, as in Definition 1.8, i.e. the collection of sets with support functions $h_s = h\varphi^s$, $s \in [-a, a]$. Let $g(s) = \gamma(Q_s)$. Then

- *$g(0) = f(0)$;*
- *$g'(0) = f'(0)$;*
- *$g''(0) = f''(0) + A(h, \psi)$.*

The proof of the Lemma immediately follows from the fact that

$$h\varphi^s = h + sh \log \varphi + o(s), \quad \text{as } s \to 0,$$

with the selection $\psi = h \log \varphi$.

5.1. **Proof of Theorem 1.12.** Suppose the Conjecture 1.2 holds. By Lemma 1.9 we get that for every one-parameter family $Q_s \in \mathbf{Q}(h, \varphi, I)$, with h and φ even,

$$\frac{d^2}{ds^2}\log(\gamma(Q_s))\bigg|_{s=0} \leq 0.$$

When γ is the Lebesgue measure on $\mathbb{R}^n$, then, by (43),

$$F(h, \psi) = \int_{\mathbb{S}^{n-1}} \psi \det Q(h) du,$$

and hence, by (45),

$$A(h, \psi) = \frac{d}{ds}\left(\int_{\mathbb{S}^{n-1}} \frac{h + s\psi}{h}\psi \det Q(h) du\right)\bigg|_{s=0} = \int_{\mathbb{S}^{n-1}} \frac{\psi^2}{h} \det Q(h) du. \tag{46}$$

Theorem 1.12 then follows from (42), (43), (44), (46) and (34). □

6. The proof of Theorem 1.14 about the infinitesimal form of B-conjecture.

Proof. Let $K \in \mathcal{K}^n$ with support function h; as usual we denote its homogenous extension by H. Let γ be a measure on $\mathbb{R}^n$ with density f. Consider the function $B : [0, \infty) \to \mathbb{R}^+$ defined as follows:

$$B(s) = \gamma(\mathrm{e}^s K) = \int_{\mathbb{S}^{n-1}} h_s \det Q(h_s) \int_0^1 t^{n-1} f\left(t\nabla H_s\right) dt du, \tag{47}$$

where $h_s = \mathrm{e}^s \cdot h$, and $H_s = \mathrm{e}^s \cdot H$. Thus,

$$B(s) = \int_{\mathbb{S}^{n-1}} \mathrm{e}^{sn} h \det Q(h) \int_0^1 t^{n-1} f\left(t\mathrm{e}^s \nabla H\right) dt du,$$

and

$$B'(s) = \int_{\mathbb{S}^{n-1}} n\mathrm{e}^{sn} h \det Q(h) \int_0^1 t^{n-1} f\left(t\mathrm{e}^s \nabla H\right) dt du$$
$$+ \int_{\mathbb{S}^{n-1}} \mathrm{e}^{sn} h \det Q(h) \int_0^1 t^n \mathrm{e}^s \langle \nabla H, \nabla f(t\mathrm{e}^s \nabla H)\rangle dt du.$$

Integrating by parts in t, we get

$$\int_0^1 t^n \mathrm{e}^s \langle \nabla H, \nabla f(t\mathrm{e}^s \nabla H)\rangle dt = f(\mathrm{e}^s \nabla H) - \int_0^1 n t^{n-1} f\left(t\mathrm{e}^s \nabla H\right) dt.$$

Hence,

$$B'(s) = \int_{\mathbb{S}^{n-1}} \mathrm{e}^{sn} h \det Q(h) f(\mathrm{e}^s \nabla H) du,$$

and,

$$B''(s) = \int_{\mathbb{S}^{n-1}} \mathrm{e}^{sn} h \det Q(h) \left[n f(\mathrm{e}^s \nabla H) + \mathrm{e}^s \langle \nabla H, \nabla f(\mathrm{e}^s \nabla H)\rangle\right] du.$$

Using (35), we obtain

$$B'(0) = \int_{\mathbb{S}^{n-1}} \frac{f(\nabla H)}{\int_0^1 t^{n-1} f(t\nabla H) dt} dV_{\gamma,K}, \tag{48}$$

and

$$B''(0) = n\int_{\mathbb{S}^{n-1}} \frac{f(\nabla H)}{\int_0^1 t^{n-1} f(t\nabla H) dt} dV_{\gamma,K} + \int_{\mathbb{S}^{n-1}} \frac{\langle \nabla f(\nabla H), \nabla H\rangle}{\int_0^1 t^{n-1} f(t\nabla H) dt} dV_{\gamma,K}. \tag{49}$$

The B-conjecture is equivalent to logarithmic concavity of $B(s)$, and hence Theorem 1.14 follows from (47), (48) and (49). □

7. Proof of Theorem 1.6, about the stability for dimensional Brunn-Minkowski inequality.

7.1. **First and second variation of the measure.** As usual, γ is a radially symmetric log-concave measure on $\mathbb{R}^n$, with density F with respect to Lebesgue measure; in particular, we write F in the form:

$$F(x) = f(|x|).$$

We will assume that f is smooth, more precisely $f \in C^2([0,\infty))$. Let us fix $h \in C^{2,+}(\mathbb{S}^{n-1})$ and let K be a convex body with support function h. Let $\psi \in C^2(\mathbb{S}^{n-1})$ and consider the one-parameter system of convex bodies $\mathbf{K}(h,\psi,[-a,a])$ for a suitably small $a > 0$. In particular for every $s \in [-a,a]$ there exists a convex body K_s such that $h_{K_s} = h_s$. Hence we may consider the function

$$g\,:\,[-a,a] \to \mathbb{R}, \quad g(s) = \gamma(K_s).$$

The aim of this subsection is to derive formulas for the first and second derivative of $g(s)$ at $s = 0$. We start from the expression:

$$g(s) = \int_{\mathbb{S}^{n-1}} h_s(u)\,\det(Q(h_s;u)) \int_0^1 t^{n-1} f(t\sqrt{h_s^2(u) + |\nabla_\sigma h_s(u)|^2})dtdu,$$

where we used Theorem 3.4, the rotation invariance of γ, and Remark 3.3. To simplify notations we set

$$\begin{aligned}
&Q_s = Q(h_s;u)\,, \quad Q = Q_0; \quad D_s = \left[h_s^2(u) + |\nabla_\sigma h_s(u)|^2\right]^{1/2}, \quad D = D_0;\\
&A_s = \int_0^1 t^{n-1} f(tD_s)dt\,, \quad A = A_0; \quad B_s = \int_0^1 t^n f'(tD_s)dt\,, \quad B = B_0;\\
&C_s = \int_0^1 t^{n+1} f''(tD_s)dt\,, \quad C = C_0.
\end{aligned}$$

Then

$$\begin{aligned}
g'(s) &= \int_{\mathbb{S}^{n-1}} \psi \det(Q_s) A_s du + \int_{\mathbb{S}^{n-1}} h_s c_{ij}(h_s)(\psi_{ij} + \psi\delta_{ij}) A_s du\\
&+ \int_{\mathbb{S}^{n-1}} h_s \det(Q_s) B_s \frac{h_s\psi + \langle \nabla_\sigma h_s, \nabla_\sigma \psi\rangle}{D_s} du.
\end{aligned} \tag{50}$$

Passing to the second derivative (for $s = 0$) we get

$$
\begin{aligned}
g''(0) &= 2\int_{\mathbb{S}^{n-1}} \psi c_{ij}(h)(\psi_{ij}+\psi\delta_{ij})A du \\
&+ 2\int_{\mathbb{S}^{n-1}} \psi \det(Q) B \frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D} du \\
&+ 2\int_{\mathbb{S}^{n-1}} h c_{ij}(h)(\psi_{ij}+\psi\delta_{ij}) B \frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D} du \\
&+ \int_{\mathbb{S}^{n-1}} A h c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(\psi_{kl}+\psi\delta_{kl}) du \\
&+ \int_{\mathbb{S}^{n-1}} h \det(Q) C \left[\frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D}\right]^2 du \\
&+ \int_{\mathbb{S}^{n-1}} h \det(Q) B \left[D(h^2 + |\nabla_\sigma \psi|^2) - \frac{[h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle]^2}{D}\right] \frac{1}{D^2}\, du. \qquad (51)
\end{aligned}
$$

We now focus on the fourth summand of the last expression. Applying formulas (41) and (39) we get

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-1}} A h c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(\psi_{kl}+\psi\delta_{kl}) du \\
&= \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})((Ah)_{kl}+Ah\delta_{kl}) du \\
&= \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(A(h_{kl}+h\delta_{kl}) + 2A_k h_l + hA_{kl}) du \\
&= \int_{\mathbb{S}^{n-1}} A\psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(h_{kl}+h\delta_{kl}) du \\
&+ \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(2A_k h_l + hA_{kl}) du \\
&= (n-2)\int_{\mathbb{S}^{n-1}} A\psi c_{ij}(h)(\psi_{ij}+\psi\delta_{ij}) du \\
&+ \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(2A_k h_l + hA_{kl}) du.
\end{aligned}
$$

Hence

$$
\begin{aligned}
g''(0) &= n\int_{\mathbb{S}^{n-1}} \psi c_{ij}(h)(\psi_{ij}+\psi\delta_{ij})A du + 2\int_{\mathbb{S}^{n-1}} \psi \det(Q) B \frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D} du \\
&+ 2\int_{\mathbb{S}^{n-1}} h c_{ij}(h)(\psi_{ij}+\psi\delta_{ij}) B \frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D} du \\
&+ \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(2A_k h_l + hA_{kl}) du \\
&+ \int_{\mathbb{S}^{n-1}} h \det(Q) C \left[\frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D}\right]^2 du \\
&+ \int_{\mathbb{S}^{n-1}} h \det(Q) B \left[D(\psi^2 + |\nabla_\sigma \psi|^2) - \frac{[h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle]^2}{D}\right] \frac{1}{D^2}\, du. \qquad (52)
\end{aligned}
$$

7.2. **The case of Euclidean balls.** Let $h \equiv R$, $R > 0$. This choice considerably simplifies the situation as:

$$Q = RI_{n-1}; \quad \nabla_\sigma \equiv R; \quad D \equiv R; \quad c_{ij}(h) \equiv R^{n-2}\delta_{ij};$$
$$A = \int_0^1 t^{n-1} f(Rt)dt, \quad B = \int_0^1 t^n f'(Rt)dt, \quad C = \int_0^1 t^{n+1} f''(Rt)dt.$$

Here I_{n-1} denotes the $(n-1)\times(n-1)$ identity matrix. In particular A does not depend on the point u on $\mathbb{S}^{n-1}$, so that

$$A_i \equiv A_{ij} \equiv 0 \quad \text{on } \mathbb{S}^{n-1}.$$

Hence $g(0) = |\mathbb{S}^{n-1}|R^n A$, and

$$\begin{aligned} g'(0) &= R^{n-1}A\int_{\mathbb{S}^{n-1}} \psi du + R^{n-1}A\int_{\mathbb{S}^{n-1}} (\Delta_\sigma\psi + (n-1)\psi)du + R^n B\int_{\mathbb{S}^{n-1}} \psi du \\ &= R^{n-1}(nA + RB)\int_{\mathbb{S}^{n-1}} \psi du. \qquad (53) \end{aligned}$$

Here we used the fact that, by the divergence theorem on $\mathbb{S}^{n-1}$,

$$\int_{\mathbb{S}^{n-1}} \Delta_\sigma \psi du = 0.$$

As for the second derivative, we have

$$\begin{aligned} g''(0) &= nR^{n-2}A\int_{\mathbb{S}^{n-1}} \psi(\Delta_\sigma\psi + (n-1)\psi)du + 2R^{n-1}B\int_{\mathbb{S}^{n-1}} \psi^2 du \\ &+ 2R^{n-1}B\int_{\mathbb{S}^{n-1}} \psi(\Delta_\sigma\psi + (n-1)\psi))du + R^n C\int_{\mathbb{S}^{n-1}} \psi^2 du \\ &+ R^{n-1}B\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du. \end{aligned}$$

By the divergence theorem,

$$\int_{\mathbb{S}^{n-1}} \psi\Delta_\sigma\psi du = -\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du,$$

and thus

(54)
$$g''(0) = R^{n-2}(An(n-1) + 2nRB + R^2C)\int_{\mathbb{S}^{n-1}} \psi^2 du - R^{n-2}(nA + RB)\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du.$$

Integrating by parts in t, we get

$$f(R) = nA + RB,$$

and

$$f'(R) = (n+1)B + RC.$$

Thus we obtain

$$g'(0) = R^{n-1}f(R)\int_{\mathbb{S}^{n-1}} \psi du. \qquad (55)$$

and

$$\begin{aligned} g''(0) &= R^{n-2}\left[(n-1)f(R) + Rf'(R)\right]\int_{\mathbb{S}^{n-1}} \psi^2 du - R^{n-2}f(R)\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du \\ &= R^{n-2}f(R)\left((n-1)\int_{\mathbb{S}^{n-1}} \psi^2 du - \int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du\right) + R^{n-1}f'(R)\int_{\mathbb{S}^{n-1}} \psi^2 du. \qquad (56) \end{aligned}$$

The validity of (12) for $h \equiv R$ is equivalent to the validity of the following inequality for every $\psi \in C^2(\mathbb{S}^{n-1})$:

$$\begin{aligned}&\frac{Af(R)}{|\mathbb{S}^{n-1}|}\left((n-1)\int_{\mathbb{S}^{n-1}}\psi^2du - \int_{\mathbb{S}^{n-1}}|\nabla_\sigma\psi|^2du\right) + \frac{ARf'(R)}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\psi^2du \leq \\ &\frac{n-1}{n}f(R)^2\left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\psi du\right)^2.\end{aligned} \tag{57}$$

Let us denote the quadratic operators appearing in the left-hand side and in the right-hand side of the last inequality by $B_1(\psi)$ and $B_2(\psi)$, correspondingly. That is,

$$B_1(\psi) = \frac{Af(R)}{|\mathbb{S}^{n-1}|}\left((n-1)\int_{\mathbb{S}^{n-1}}\psi^2du - \int_{\mathbb{S}^{n-1}}|\nabla_\sigma\psi|^2du\right) + \frac{ARf'(R)}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\psi^2du,$$

and

$$B_2(\psi) = \frac{n-1}{n}f(R)^2\left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\psi du\right)^2.$$

The next step is to decompose ψ as the sum of a constant function and a function which is orthogonal to constant functions. Let us write

$$\psi = \psi_0 + \psi_1$$

where

$$\psi_0 = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\psi du \quad \text{and} \quad \int_{\mathbb{S}^{n-1}}\psi_1 du = 0.$$

Note that

$$\int_{\mathbb{S}^{n-1}}\psi^2d\sigma = \int_{\mathbb{S}^{n-1}}\psi_0^2d\sigma + \int_{\mathbb{S}^{n-1}}\psi_1^2d\sigma.$$

Therefore,

$$B_1(\psi) = B_1(\psi_0) + B_1(\psi_1),$$

as well as

$$B_2(\psi) = B_2(\psi_0) + B_2(\psi_1).$$

Since γ is radially symmetric, one has $f' \leq 0$. Moreover, by the standard Poincaré inequality on the unit sphere, one has

$$(n-1)\int_{\mathbb{S}^{n-1}}\psi^2du - \int_{\mathbb{S}^{n-1}}|\nabla_\sigma\psi|^2du \leq 0, \tag{58}$$

for every ψ such that

$$\int_{\mathbb{S}^{n-1}}\psi du = 0. \tag{59}$$

Thus

$$B_1(\psi_1) \leq 0 = B_2(\psi_1).$$

To prove (57) it remains to show that

$$B_1(\psi_0) \leq B_2(\psi_0). \tag{60}$$

This condition is equivalent to

$$ng_\psi(0)g_\psi''(0) \leq (n-1)[g_\psi'(0)]^2 \tag{61}$$

in the special case in which ψ is a constant function. The inequality (61) is nothing but the dimensional Brunn-Minkowski inequality for spherically invariant measures when K and L are Euclidean balls. As was shown in [27] (see also the third named author [28]), this statement follows from Log-Brunn-Minkowski conjecture in the case of log-concave spherically invariant measures when K and L are Euclidean balls. Indeed, spherically invariant case is a very partial case of the unconditional case, and the Log-Brunn-Minkowski for the unconditional sets and measures was independently established by Cordero, Fradelizi, Maurey [15], and Saroglou [36].

For the reader's convenience we present the self-sufficient short proof of this fact in the Appendix. □

8. Proof of the Theorem 1.10

Proof. When $h \equiv R > 0$, the additional term introduced in Lemma 5.1 can be written as follows:

$$A(h,\psi) = f(R)\int_{\mathbb{S}^{n-1}} \psi^2 du.$$

By Lemmas 1.9 and 5.1, and by the computations carried out in the previous section, the claim of the theorem is equivalent to the following inequality:

$$\begin{aligned} &A\left[nf(R)+Rf'(R)\right]\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\psi^2du - Af(R)\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}|\nabla_\sigma\psi|^2du \le \\ &f(R)^2\left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\psi d\sigma\right)^2, \end{aligned} \tag{62}$$

for every $\psi \in C^2(\mathbb{S}^{n-1})$.

We follow the argument of the previous section and split the proof into two cases.

Case 1. Consider an even $\psi \in C^2(\mathbb{S}^{n-1})$ such that $\int \psi = 0$. Then the inequality (62) amounts to

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\psi^2du \le \frac{f(R)}{nf(R)+Rf'(R)}\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}|\nabla_\sigma\psi|^2du. \tag{63}$$

Indeed, under these conditions ψ is orthogonal to the first and the second eigenfunctions of the Laplace operator on $\mathbb{S}^{n-1}$. The third eigenvalue of this operator is $2n$. Hence

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\psi^2du \le \frac{1}{2n}\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}|\nabla_\sigma\psi|^2du. \tag{64}$$

Since f is decreasing, we have $f'(R) \le 0$, and hence

$$\frac{f(R)}{nf(R)+Rf'(R)} \ge \frac{1}{n} > \frac{1}{2n}. \tag{65}$$

The inequalities (64) and (65) imply (63).

Case 2. Let ψ be a constant function.The inequality (62) holds for constant functions because, once again, the Log-Brunn-Minkowski inequality holds in the case of spherically invariant measures and Euclidean balls (see the Appendix).

To summarize, we here established (62) separately for constant functions and centered functions. A polarization argument analogous to the one presented in the proof of Theorem 1.6 finishes the proof. □

Appendix

We provide a direct proof of the fact that for a spherically invariant log-concave measure γ on $\mathbb{R}^n$ and for $a, b > 0$ we have

$$\gamma\left((aB_2^n)^\lambda(bB_2^n)^{1-\lambda}\right) \ge \left(\gamma(aB_2^n)\right)^\lambda\left(\gamma(bB_2^n)\right)^{1-\lambda}. \tag{66}$$

To show it, we first state a well-known result proved by Borell [6] and rediscovered by Uhrin [39] (see also Ball [2]).

Proposition 8.1. *Let* $f, g, h : [0, +\infty) \to [0, +\infty)$ *be such that*

$$h(x^{1-\lambda}y^{\lambda}) \geq f(x)^{1-\lambda}g(y)^{\lambda}$$

for every $x, y \in [0, +\infty)$ *and every* $\lambda \in [0, 1]$. *Then,*

$$\int_0^{+\infty} h \geq \left(\int_0^{+\infty} f\right)^{1-\lambda} \left(\int_0^{+\infty} g\right)^{\lambda}.$$

To prove the Proposition, apply the Prékopa-Leindler inequality (see Prékopa [34], Leindler [24], or Gardner [16]) to $\overline{f}(x) = f(\mathrm{e}^x)\,\mathrm{e}^x$, $\overline{g}(x) = g(\mathrm{e}^x)\,\mathrm{e}^x$ and $\overline{h}(x) = h(\mathrm{e}^x)\,\mathrm{e}^x$, and perform the change of variables $t = \mathrm{e}^x$.

Corollary 8.2. *Let* ϕ *be a log-concave non-increasing function defined in* $[0, \infty)$. *Set*

$$F(R) = \int_0^R t^{n-1}\phi(t)dt, \quad \forall\, R > 0.$$

Then $F(\mathrm{e}^x)$ *is log-concave.*

Proof. Apply Proposition 8.1 to $f(x) = 1_{[0,a]}(x)x^{n-1}\phi(x)$, $g(x) = 1_{[0,b]}(x)x^{n-1}\phi(x)$ and $h(x) = 1_{[0,a^{1-\lambda}b^{\lambda}]}(x)x^{n-1}\phi(x)$. Indeed, if $x \notin [0, a]$ or $y \notin [0, b]$, we have

$$h(x^{1-\lambda}y^{\lambda}) \geq 0 = f(x)^{1-\lambda}g(y)^{\lambda}.$$

As the density ϕ is log-concave and non-increasing, we have

$$\phi(x^{1-\lambda}y^{\lambda}) \geq \phi((1-\lambda)x + \lambda y) \geq \phi(x)^{1-\lambda}\phi(y)^{\lambda}.$$

To obtain the first inequality above, we used the arithmetic mean - geometric mean inequality.

Hence in the case when $x \in [0, a]$ and $y \in [0, b]$ we have

$$\begin{aligned} h(x^{1-\lambda}y^{\lambda}) &= (x^{1-\lambda}y^{\lambda})^{n-1}\phi(x^{1-\lambda}y^{\lambda}) \\ &\geq (x^{1-\lambda}y^{\lambda})^{n-1}\phi(x)^{1-\lambda}\phi(y)^{\lambda} \\ &= f(x)^{1-\lambda}g(y)^{\lambda}. \end{aligned}$$

It follows that

$$\int_0^{+\infty} h \geq \left(\int_0^{+\infty} f\right)^{1-\lambda} \left(\int_0^{+\infty} g\right)^{\lambda},$$

which entails $F(a^{1-\lambda}b^{\lambda}) \geq F(a)^{1-\lambda}F(b)^{\lambda}$. □

To conclude, we observe that for a spherically invariant log-concave measure γ with density $\phi(x)$,

$$\gamma(RB_2^n) = |\mathbb{S}^{n-1}| \int_0^R t^{n-1}\phi(t)dt,$$

and, therefore, (66) follows from the Corollary 8.2. □

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