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A regularity criterion for a 2D tropical climate model with fractional dissipation

Luca Bisconti[1]

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Abstract
Tropical climate model derived by Frierson et al. (Commun Math Sci 2:591–626, 2004) and its modified versions have been investigated in a number of papers [see, e.g., Li and Titi (Discrete Contin Dyn Syst Series A 36(8):4495–4516, 2016), Wan (J Math Phys 57(2):021507, 2016), Ye (J Math Anal Appl 446:307–321, 2017) and more recently Dong et al. (Discrete Contin Dyn Syst Ser B 24(1):211–229, 2019)]. Here, we deal with the $2D$ tropical climate model with fractional dissipative terms in the equation of the barotropic mode u and in the equation of the first baroclinic mode v of the velocity, but without diffusion in the temperature equation, and we establish a regularity criterion for this system.

Keywords Regularity criterion · Tropical climate model · Tropical atmospheric dynamics · Navier–Stokes equations

Mathematics Subject Classification 35Q35 · 35Q30 · 35B65 · 76D03

1 Introduction

In this paper we consider the following $2D$ tropical climate model, i.e.

$$\begin{aligned} &\partial_t u + (u \cdot \nabla)u + \nu \Lambda^{2\alpha} u + \nabla p + \operatorname{div}(v \otimes v) = 0, \\ &\partial_t v + (u \cdot \nabla)v + (v \cdot \nabla)u + \eta \Lambda^{2\beta} v + \nabla\theta = 0, \\ &\partial_t \theta + (u \cdot \nabla)\theta + \operatorname{div} v = 0, \\ &\operatorname{div} u = 0 \\ &u(x,0) = u_0,\ \ v(x,0) = v_0,\ \ \theta(x,0) = \theta_0, \end{aligned} \tag{1}$$

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✉ Luca Bisconti
luca.bisconti@unifi.it

1 Dipartimento di Matematica e Informatica "U. Dini", Università degli Studi di Firenze, Via S. Marta 3, 50139 Florence, Italy

where $x \in \mathbb{R}^2$, $t \geq 0$, $u = (u_1(x,t), u_2(x,t))$ and $v = (v_1(x,t), v_2(x,t))$ denote the barotropic mode and the first baroclinic mode of the velocity, respectively, $p = p(x,t)$ denotes the pressure, and $\theta = \theta(x,t)$ the temperature. Here $\Lambda = (-\Delta)^{1/2}$, and $\nu \geq 0$, $\eta \geq 0$, $0 \leq \alpha \leq 2$ and $0 \leq \beta \leq 2$. Further conditions on the last two parameters will be introduced in a few lines.

When $\nu = \eta = 0$, the above system gives the original tropical climate model derived by Frierson et al. [5]. Instead, in the case of $\nu > 0$, $\eta > 0$, $\alpha = 1$ and $\beta = 1$, (1) reduces to the viscous version of the Frierson-Majda-Pauluis model that has been analyzed by Li and Titi [16]. Problem (1) with parameter ranges $0 \leq \alpha \leq 2$ and $0 \leq \beta \leq 2$ may have some pertinence in describing certain types of tropical atmospheric dynamics (see, e.g., [19–21] and the references therein).

We emphasize that in this paper the equation for θ contains no dissipation, and we also assume $\nu = 1$ and $\eta = 1$. Our goal here is mainly related to the problem of regularity in time for the solutions of system (1), in the $2D$ case.

Global well-posedness of solutions to a tropical climate model with dissipation in the equation of the first baroclinic mode of the velocity, under the hypotheses of small initial data, was studied by Wan [26] and Ma and Wan [17]. In fact, the issue of global regularity has been investigated in a number of articles and (partially) addressed in dependence of the values assumed by parameters α and β. In particular, in the $2D$ case, Ye [27] was able to prove global existence (adding $\Lambda^2\theta$ in $(1)_3$), in H^s-norm, $s \geq 2$, for $\beta = 1$ and $\alpha > 0$. Dong et al. [4] considered the $2D$ model assuming $\alpha + \beta = 2$ with $1 < \beta \leq 3/2$, and they proved that system (1) possesses a unique global solution when the initial data (u_0, v_0, θ_0) is sufficiently regular, i.e., $u_0, v_0 \in H^s(\mathbb{R}^2)$, $s > 2$, and $\theta_0 \in \dot{H}^{-1}(\mathbb{R}^2) \cap H^{s+1-\beta}(\mathbb{R}^2)$. In addition, in [4], the authors also studied the cases of $\alpha + \beta = 2$ with $3/2 < \beta < 2$, and $\alpha + \beta = 2$ with $\alpha = 2$ and $\beta = 0$. Let us also recall that Ma et al. [18] established the local well-posedness of strong solutions to the considered model.

We also mention an article of Zhu [30], in which the $3D$ system (1) is considered, with initial data in $H^3(\mathbb{R}^3)$. The author proved global existence of strong solutions $(u, v, \theta) \in L^\infty(0, T; H^3(\mathbb{R}^3))$, for any $T > 0$, removing $\Lambda^{2\beta} v$ in $(1)_2$, and assuming $\alpha \geq 5/2$.

In the present paper we consider problem (1) with $1/2 < \alpha < 1$, $0 < \beta < 1$, and $\beta + 2\alpha = 2$, which is a situation not addressed in the above mentioned works, and in any case it is new to the best of our knowledge. Taking initial data $u_0, v_0, \theta_0 \in H^s(\mathbb{R}^2)$, $s > 2$, and following an approach similar to the one given in [7] (see also [6] and [8]), in Theorem 3.1 we prove a continuation result for the solutions of (1). To be more precise, assuming that the solution $(u, v, \theta) \in C([0, T); H^s(\mathbb{R}^2))$, for some $T > 0$, it is possible to extend $(u, v, \theta)(t)$ beyond time T provided that the quantities $\|\nabla u(t)\|_{\dot{B}^0_{\infty,2}}$, $\|\nabla v(t)\|_{\dot{B}^0_{\infty,2}}$, and $\|\nabla \theta(t)\|_{\dot{B}^0_{\infty,2}}$ are bounded in $L^2(0, T)$, where $\dot{B}^0_{\infty,2} = \dot{B}^0_{\infty,2}(\mathbb{R}^2)$ is the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^2)$ with $s = 0$, $p = \infty$ and $q = 2$ (see below for details).

2 Preliminaries and basic facts

Let us recall some basic facts about the function spaces that will be used in the sequel. We also list the estimates needed to reach the claimed regularity result. Most of the controls we use in following have been established in [13]. We refer to this paper for a detailed overview (see also [14,24,25]), which is somehow aimed at our purposes, on the theory of Besov spaces $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n)$ (and homogeneous Besov spaces $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$), with $0 < p, q \le \infty$ and $s \in \mathbb{R}$.

Let us consider—only in this section—the case of $\mathbb{R}^n$ as domain of the considered vector and scalar fields and recall some inequalities and embeddings in Sobolev and Besov spaces. Here, $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$, with $0 < p, q \le \infty$ and $s \in \mathbb{R}$ indicate homogeneous Besov spaces and the non-homogeneous counterparts are $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n)$, $0 < p, q \le \infty$ and $s \in \mathbb{R}$ (see, e.g., [24,25]) with norms, respectively, $\|\cdot\|_{\dot{B}^s_{p,q}}$ and $\|\cdot\|_{B^s_{p,q}}$. In the sequel $BMO = BMO(\mathbb{R}^n)$ denotes the Bounded Mean oscillation space with norm $\|\cdot\|_{BMO}$ (see, e.g. [12–14]).

For $p \ge 1$, we indicate by $L^p = L^p(\mathbb{R}^n)$ the usual Lebesgue space, endowed with norm $\|\cdot\|_p = \|\cdot\|_{L^p}$. Also, for L^2, the norm is $\|\cdot\| = \|\cdot\|_2$. We denote by $W^{k,p} = W^{k,p}(\mathbb{R}^n)$ and $\|\cdot\|_{k,p} = \|\cdot\|_{W^{k,p}}$ a Sobolev space and its norm, respectively (see, e.g., [1]). When $p = 2$ we use the notation $H^k = W^{k,2}$ and $\|\cdot\|_{H^k} = \|\cdot\|_{W^{k,2}}$.

In the sequel we will use the symbols C (or c) to denote generic constants, which may change from line–to–line, but are not dependent on the solution. Also, we denote a generic constant by $C(\cdot)$ (or by $c(\cdot)$), with the meaning that the constant depends mainly on the arguments between parentheses, or alternatively by using a subscript to make explicit the quantities the constant depends on.

2.1 Besov and Sobolev inequalities

Let us recall the hypotheses of [13, Theorem 2.1], under which we are going to introduce the following estimates. For any $p, \rho, \sigma \in [1, \infty], q \in [1, \infty)$ and $s > n/q$, there exists a constant C depending only on n, p and q, but not on ρ, σ such that for $f \in \dot{B}^{n/p}_{p,\rho} \cap B^s_{q,\sigma}$, we have (see [13, (2.2) p. 257, and pp. 260–261]) the following logarithmic control

$$\|f\|_{L^\infty} \le C\Big(1 + \|f\|_{\dot{B}^{n/p}_{p,\rho}}\big(\log^+(\|f\|_{B^s_{q,\sigma}})\big)^{1-1/\rho}\Big).$$

Now, let $p, q, \rho, \sigma, \nu \in [1, \infty]$ with $\nu \le \min(\rho, \sigma)$, $1/q = 1/p - s/n$, $1 \le r \le q$ and $s_1/n < 1/r - 1/q < s_2/n$. Then, for $f \in \dot{B}^{s_1}_{r,\sigma} \cap \dot{B}^{s_2}_{r,\sigma}$, we have ([13, p. 260])

$$\|f\|_{\dot{B}^0_{q,\nu}} \le C\|f\|_{\dot{B}^0_{q,\rho}}\left(1 + \left(1 + \log^+\left(\|f\|_{\dot{B}^{s_1}_{r,\sigma}} + \|f\|_{\dot{B}^{s_2}_{r,\sigma}}\right)\right)^{\frac{1}{\nu}-\frac{1}{\rho}}\right), \tag{2}$$

where $\nu \le \sigma, \rho$ and $\log^+(t) := \begin{cases} \log(t), & t > e, \\ 1, & e \ge t \ge 0. \end{cases}$ Then, by using the embedding $\dot{B}^s_{p,\rho} \subset \dot{B}^0_{q,\rho}$ with $s/n = 1/p - 1/q$, from (2) we obtain the following estimate ([13, (2.3) p. 257]), i.e.

$$\|f\|_{\dot{B}^0_{q,\nu}} \le C\left(1+\|f\|_{\dot{B}^s_{p,\rho}}\left(\log^+\left(\|f\|_{\dot{B}^{s_1}_{r,\sigma}}+\|f\|_{\dot{B}^{s_2}_{r,\sigma}}\right)\right)^{\frac{1}{\nu}-\frac{1}{\rho}}\right). \tag{3}$$

As a special case of (3), taking $p=q$, $\rho=\infty$, and $f\in W^{\alpha,r}$, $\alpha>n/r+1$, we have

$$\|\nabla f\|_{\dot{B}^0_{p,\nu}} \le C\left(1+\|\nabla f\|_{\dot{B}^0_{p,\infty}}\left(\log^+\left(\|f\|_{W^{\alpha,r}}\right)\right)^{\frac{1}{\nu}}\right).$$

Let us still assume $f\in W^{\alpha,r}$, $\alpha>n/r+1$, and recall the following control (see [13, Section 5, pp. 272–273] for details), i.e.

$$\|\nabla f\|_\infty \le C\|\nabla f\|_{\dot{B}^0_{\infty,1}}. \tag{4}$$

In particular, by using (4) along with (2), and taking $\nu=1$, $\rho=2$, $q=\infty$, we find the inequality

$$\begin{aligned}\|\nabla f\|_\infty &\le C\|\nabla f\|_{\dot{B}^0_{\infty,1}}\\ &\le C\|\nabla f\|_{\dot{B}^0_{\infty,2}}\left(1+\log^{\frac{1}{2}}\left(e+\|\nabla f\|_{W^{\alpha-1,r}}\right)\right),\end{aligned} \tag{5}$$

where we exploited the embedding $W^{\alpha-1,r}\subset B^s_{r,\sigma}$, if $\alpha-1>s$, or in alternative $W^{\alpha-1,r}\subset B^{\alpha-1}_{r,\max\{r,2\}}$, if $\alpha-1=s$ (see, e.g., [4]). When $\mathbb{R}^n=\mathbb{R}^2$, if we take $r=2$, so that $f\in W^{\alpha,2}=H^\alpha$, with $\alpha>2$, recalling that $H^{\alpha-1}\approx B^{\alpha-1}_{2,2}$, and setting $s=\alpha$, inequality (5) reduces to

$$\|\nabla f\|_\infty \le C\|\nabla f\|_{\dot{B}^0_{\infty,2}}\left(1+\log^{\frac{1}{2}}\left(e+\|\nabla f\|_{H^{s-1}}\right)\right). \tag{6}$$

2.2 Further inequalities

We will also use some elementary commutator type estimates as in the following lemma concerning the operator Λ^s, $s>0$ (see, e.g., [7,10,11]).

Lemma 2.1 *For $s>0$ and $1<r\le\infty$, and for smooth enough f and g*

$$\|\Lambda^s(fg)\|_r \le C(\|f\|_{p_1}\|\Lambda^s g\|_{q_1}+\|g\|_{p_2}\|\Lambda^s f\|_{q_2}), \tag{7}$$

where $q_1, q_2\in(1,\infty)$ and $1/r=1/p_1+1/q_1=1/p_2+1/q_2$, and C is a suitable positive constant.

Also, for the commutator $[\Lambda^s, f]g := \Lambda^s(fg)-f\Lambda^s g$, $s>0$, we have the following estimate

$$\|[\Lambda^s, f]g\|_r \le C(\|\nabla f\|_{p_1}\|\Lambda^{s-1}g\|_{q_1}+\|g\|_{p_2}\|\Lambda^s f\|_{q_2}), \tag{8}$$

where q_1, $q_2 \in (1, \infty)$ such that $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$, and C is a suitable positive constant.

In the sequel all the function spaces are taken on $\mathbb{R}^n = \mathbb{R}^2$, and so we always have $n = 2$ in the used Sobolev embeddings and interpolation inequalities. In particular $W^{k,p} = W^{k,p}(\mathbb{R}^2)$, $H^s = H^2(\mathbb{R}^2)$, $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^2)$, $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^2)$, and $BMO = BMO(\mathbb{R}^2)$.

As a further consequence of (3), in the case $\mathbb{R}^n = \mathbb{R}^2$, for $s = 0$, $p = \rho = \infty$, $\nu = 2$, and also assuming $f \in W^{\alpha,2}$, $\alpha > 2$, we have the following logarithmic control (see [7] and [13, Theorem 2.1, p. 257]), in which we set $s = \alpha$, i.e.

$$\|f\|_{\dot{B}^0_{\infty,2}} \le C\left(1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \log^{\frac{1}{2}}(e + \|f\|_{H^{s-1}})\right). \tag{9}$$

We also use the following interpolation inequalities (see, e.g, [3,9,15])

$$\|f\|_4 \le \|f\|^{\frac{1}{2}} \|\nabla f\|^{\frac{1}{2}}, \tag{10}$$

and (see [12, Lemma 1, p. 180])

$$\|f\|_4 \le \|f\|^{\frac{1}{2}} \|f\|^{\frac{1}{2}}_{BMO}. \tag{11}$$

3 Regularity result

Theorem 3.1 *Let $(u_0, v_0, \theta_0) \in H^s \times H^s \times H^s$, for any $s > 2$, with $\mathrm{div}\, u_0 = 0$. Assume the parameters α and β in (1) are such that*

$$\frac{1}{2} < \alpha < 1, \quad 0 < \beta < 1 \quad \textit{with} \quad \beta + 2\alpha = 2. \tag{12}$$

Let (u, v, θ) be a local solution to the system (1), defined on some time interval $[0, T)$, with $0 < T < \infty$, and having the following regularity

$$u, v, \theta \in C([0, \tilde{T}]; H^s) \;\; \textit{and} \;\; u \in L^2(0, \tilde{T}; H^{s+\alpha}), \;\; v \in L^2(0, \tilde{T}; H^{s+\beta}), \tag{13}$$

for any $0 < \tilde{T} < T$. Then $(u, v, \theta)(t)$ can be extended beyond time T, with the same regularity as in (13), provided that

$$\int_0^T \left(\|\nabla u(t)\|^2_{\dot{B}^0_{\infty,2}} + \|\nabla v(t)\|^2_{\dot{B}^0_{\infty,2}} + \|\nabla \theta(t)\|^2_{\dot{B}^0_{\infty,2}}\right) dt < \infty. \tag{14}$$

In this paper we will not deal with the two cases $(\alpha, \beta) = (1/2, 1)$ and $(\alpha, \beta) = (1, 0)$. In fact, to try to examine them, an alternative approach to the one used here seems necessary.

3.1 Proof of Theorem 3.1

The proof consists in proving suitable a priori estimates for the considered solution (u, v, θ) showing explicitly that it can extended after time $T > 0$. Thus, the procedure is divided in a number of steps in which we establish the needed bounds in L^2, H^1, H^2 and H^s, $s > 2$. These steps parallels the formal estimates in the global existence results given in [4] and [27], although in our case they are carried out with different techniques borrowed from [7,22] and [13, Theorem 5.1] (see also [28,29], and [2]).

L^2-estimates

Taking the L^2-inner product of $(1)_1$, $(1)_2$ and $(1)_3$ with u, v and θ, respectively, and adding them up, therefore we get, for any $t > 0$

$$\begin{aligned}&\|u(t)\|^2 + \|v(t)\|^2 + \|\theta(t)\|^2 + 2\int_0^t \left(\|\Lambda^\alpha u(s)\|^2 + \|\Lambda^\beta v(s)\|^2\right) ds \\ &\quad = \|u_0\|^2 + \|v_0\|^2 + \|\theta_0\|^2,\end{aligned} \tag{15}$$

where the following identities have been applied

$$\begin{aligned}&\int_{\mathbb{R}^2} \operatorname{div}(v \otimes v) \cdot u \, dx + \int_{\mathbb{R}^2} (v \cdot \nabla) u \cdot v \, dx = 0,\\ &\int_{\mathbb{R}^2} \nabla\theta \cdot v \, dx + \int_{\mathbb{R}^2} \operatorname{div} v \cdot \theta \, dx = 0,\\ &\int_{\mathbb{R}^2} (u \cdot \nabla) u \cdot u \, dx = 0, \quad \int_{\mathbb{R}^2} (u \cdot \nabla) v \cdot v \, dx = 0 \text{ and } \int_{\mathbb{R}^2} (u \cdot \nabla)\theta \cdot \theta \, dx = 0.\end{aligned}$$

Thanks to (15), for any $t > 0$, it follows that $u \in L^\infty(0, t; L^2) \cap L^2(0, t; H^\alpha)$, $v \in L^\infty(0, t; L^2) \cap L^2(0, t; H^\beta)$, and $\theta \in L^\infty(0, t; L^2)$.

In what follows we use the notation $\|(u, v, \theta)(t)\|^2 := \|u(t)\|^2 + \|v(t)\|^2 + \|\theta(t)\|^2$, with $t \geq 0$.

Next, we consider higher order estimates.

H^1-estimates

Multiplying $(1)_1$ by $-\Delta u$, integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2 + \|\nabla\Lambda^\alpha u(t)\|^2 = \int_{\mathbb{R}^2} (v \cdot \nabla) v \cdot \Delta u \, dx + \int_{\mathbb{R}^2} v \operatorname{div} v \cdot \Delta u \, dx \tag{16}$$

where, due the divergence-free condition $\operatorname{div} u = 0$, we used the following relation

$$\int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot \Delta u \, dx = 0.$$

Multiplying $(1)_2$, $(1)_3$ by $-\Delta v$ and $-\Delta\theta$, respectively, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2+\|\nabla\Lambda^\beta v(t)\|^2 &= \int_{\mathbb{R}^2}(u\cdot\nabla)v\cdot\Delta v\,dx+\int_{\mathbb{R}^2}(v\cdot\nabla)u\cdot\Delta v\,dx\\ &\quad+\int_{\mathbb{R}^2}\nabla\theta\cdot\Delta v\,dx,\end{aligned}\tag{17}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|^2=\int_{\mathbb{R}^2}(u\cdot\nabla)\theta\cdot\Delta\theta\,dx+\int_{\mathbb{R}^2}\mathrm{div}v\cdot\Delta\theta\,dx.\tag{18}$$

Summing (16), (17) and (18), and observing that

$$\int_{\mathbb{R}^2}\nabla\theta\cdot\Delta v\,dx+\int_{\mathbb{R}^2}\mathrm{div}v\cdot\Delta\theta\,dx=0,$$

we reach the following relation

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\nabla u(t)\|^2+\|\nabla v(t)\|^2+\|\nabla\theta(t)\|^2\big)+\|\nabla\Lambda^\alpha u(t)\|^2+\|\nabla\Lambda^\beta v(t)\|^2\\ &\quad=\int_{\mathbb{R}^2}(v\cdot\nabla)v\cdot\Delta u\,dx+\int_{\mathbb{R}^2}v\,\mathrm{div}v\cdot\Delta u\,dx+\int_{\mathbb{R}^2}(u\cdot\nabla)v\cdot\Delta v\,dx\\ &\qquad+\int_{\mathbb{R}^2}(v\cdot\nabla)u\cdot\Delta v\,dx+\int_{\mathbb{R}^2}(u\cdot\nabla)\theta\cdot\Delta\theta\,dx.\end{aligned}\tag{19}$$

Exploiting the fact that u is divergence free, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^2}(u\cdot\nabla)v\cdot\Delta v\,dx\right| &\le \int_{\mathbb{R}^2}|\nabla u||\nabla v|^2\,dx\\ &\le c\|\nabla v\|_4^2\|\nabla u\|\\ &\le c\|\nabla v\|_{BMO}\|\nabla v\|\|\nabla u\|\\ &\le c\|\nabla v\|_{BMO}\big(\|\nabla v\|^2+\|\nabla u\|^2\big)\\ &\le c\|\nabla v\|_{\dot{B}^0_{\infty,2}}\big(\|\nabla v\|^2+\|\nabla u\|^2\big)\\ &\le c\|\nabla v\|_{\dot{B}^0_{\infty,\infty}}\big(1+\log^{\frac{1}{2}}(e+\|\nabla v\|_{H^{s-1}})\big)\big(\|\nabla v\|^2+\|\nabla u\|^2\big),\end{aligned}\tag{20}$$

where we used (11) along with the embedding $\dot{B}^0_{\infty,2} \subset BMO$ (see, e.g., [8,13]), and control (9). Similarly, for the last integral term in the right-hand side of (3.1), we find

$$\begin{aligned}\left|\int_{\mathbb{R}^2}(u\cdot\nabla)\theta\cdot\Delta\theta\,dx\right| &\le \int_{\mathbb{R}^2}|\nabla u||\nabla\theta|^2\,dx\\ &\le c\|\nabla\theta\|_4^2\|\nabla u\|\\ &\le c\|\nabla\theta\|_{BMO}\big(\|\nabla\theta\|^2+\|\nabla u\|^2\big)\\ &\le c\|\nabla\theta\|_{\dot{B}^0_{\infty,\infty}}\big(1+\log^{\frac12}(e+\|\nabla\theta\|_{H^{s-1}})\big)\big(\|\nabla u\|^2+\|\nabla\theta\|^2\big),\end{aligned} \tag{21}$$

where we applied again relation (11).

Remark 3.2 Controls (20) and (21) can be reached somehow more directly by using Hölder's inequality $\|f\|_4^2 \le \|f\|\|f\|_\infty$ along with (6), but so obtaining estimates involving the stronger norm $\|\cdot\|_{\dot{B}^0_{\infty,2}}$ –which is however used in the sequel– in place of $\|\cdot\|_{\dot{B}^0_{\infty,\infty}}$.

For the remaining terms in the right-hand side of (3.1), recalling that

$$\int_{\mathbb{R}^2}(v\cdot\nabla)\nabla v\cdot\nabla u\,dx+\int_{\mathbb{R}^2}(v\cdot\nabla)\nabla u\cdot\nabla v\,dx=-\int_{\mathbb{R}^2}\mathrm{div}v\nabla v\cdot\nabla u\,dx,$$

and therefore by making use of the integration by parts, rearranging the terms and exploiting (7)–(8), we infer

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}v\,\mathrm{div}v\cdot\Delta u\,dx+\int_{\mathbb{R}^2}(v\cdot\nabla)v\cdot\Delta u\,dx+\int_{\mathbb{R}^2}(v\cdot\nabla)u\cdot\Delta v\,dx\right|\\ &\le\left|\int_{\mathbb{R}^2}\nabla\big(\mathrm{div}v\,v\big)\cdot\nabla u\,dx\right|+\left|\int_{\mathbb{R}^2}[\nabla,v\cdot\nabla]v\cdot\nabla u\,dx\right|\\ &\quad+\left|\int_{\mathbb{R}^2}[\nabla,v\cdot\nabla]u\cdot\nabla v\,dx\right|+\left|\int_{\mathbb{R}^2}\mathrm{div}v\nabla v\cdot\nabla u\,dx\right|\\ &\le\|\Lambda^{1-\alpha}\big(\mathrm{div}\,v\,v\big)\|\|\Lambda^{1+\alpha}u\|+\|[\nabla,v\cdot\nabla]v\|\|\nabla u\|\\ &\quad+\|[\nabla,v\cdot\nabla]u\|\|\nabla v\|+\|\mathrm{div}v\|_\infty\|\nabla v\|\|\nabla u\|\\ &\le c(\|\nabla u\|_\infty+\|\nabla v\|_\infty)(\|\nabla v\|^2+\|\nabla u\|^2)+c\|\mathrm{div}\,v\|_{p_1}\|\Lambda^{1-\alpha}v\|_{q_1}\|\Lambda^{1+\alpha}u\|\\ &\quad+c\|v\|_{p_2}\|\Lambda^{1-\alpha}\mathrm{div}\,v\|_{q_2}\|\Lambda^{1+\alpha}u\|\\ &\le c(\|\nabla u\|_\infty+\|\nabla v\|_\infty)(\|\nabla v\|^2+\|\nabla u\|^2)+c\|\nabla v\|_{p_1}\|\Lambda^{1-\alpha}v\|_{q_1}\|\Lambda^{1+\alpha}u\|\\ &\quad+c\|v\|_{p_2}\|\Lambda^{1+(1-\alpha)}v\|_{q_2}\|\Lambda^{1+\alpha}u\|=:K_1+K_2+K_3,\end{aligned}$$

where $[\nabla, f\cdot\nabla]g := \nabla((f\cdot\nabla)g)-(f\cdot\nabla)\nabla g$, and $1/p_i+1/q_i=1/2$, $i=1,2$.

For K_1, using (6), it follows that

$$\begin{aligned}K_1\le c(\|\nabla u\|^2_{\dot{B}^0_{\infty,2}}+\|\nabla v\|^2_{\dot{B}^0_{\infty,2}})&\big(1+\log(e+\|\nabla u\|_{H^{s-1}}+\|\nabla v\|_{H^{s-1}})\big)\\ &\big(\|\nabla u\|^2+\|\nabla v\|^2\big).\end{aligned}$$

Consider K_2. Taking $p_1 = \infty$ and $q_1 = 2$, have that

$$
\begin{aligned}
K_2 &\le c_\varepsilon \|\nabla v\|_\infty^2 \|\Lambda^{1-\alpha} v\|^2 + \varepsilon \|\Lambda^{1+\alpha} u\|^2 \\
&\le c_\varepsilon \|\nabla v\|_\infty^2 \big(\|v\|^{1-\sigma} \|\nabla v\|^\sigma\big)^2 + \varepsilon \|\Lambda^{1+\alpha} u\|^2 \\
&\le c_\varepsilon \|\nabla v\|_\infty^2 \big(\|v\|^2 + \|\nabla v\|^2\big) + \varepsilon \|\Lambda^{1+\alpha} u\|^2 \\
&\le c_\varepsilon \|\nabla v\|_\infty^2 \big(1 + \|\nabla v\|^2\big) + \varepsilon \|\Lambda^{1+\alpha} u\|^2 \\
&\le c_\varepsilon \|\nabla v\|_{\dot{B}^0_{\infty,2}}^2 \big(1 + \log(e + \|\nabla v\|_{H^{s-1}})\big)\big(1 + \|\nabla v\|^2\big) + \varepsilon \|\Lambda^{1+\alpha} u\|^2,
\end{aligned}
$$

where we used Young's inequality, Gagliardo-Nirenberg's inequality (see [23]) with $\sigma = 1 - \alpha$, and we also employed relation (6).

Let us take into account K_3. In this case, setting $p_2 = q_2 = 4$, we get

$$
\begin{aligned}
K_3 &= \|v\|_4 \|\Lambda^{1+(1-\alpha)} v\|_4 \|\Lambda^{1+\alpha} u\| \\
&\le c_\varepsilon \|v\|_4^2 \|\Lambda^{1+(1-\alpha)} v\|_4^2 + \varepsilon \|\Lambda^{1+\alpha} u\|^2 \\
&\le c_\varepsilon \|v\|_4^2 \big(\|\nabla v\|_\infty^{1-\sigma} \|\Lambda^{1+\beta} v\|^\sigma\big)^2 + \varepsilon \|\Lambda^{1+\alpha} u\|^2 \\
&= c_\varepsilon \big(\|v\|_4^2 \|\nabla v\|_\infty^{2(1-\sigma)}\big) \|\Lambda^{1+\beta} v\|^{2\sigma} + \varepsilon \|\Lambda^{1+\alpha} u\|^2 \\
&\le c_{\varepsilon,\delta} \|v\|_4^{\frac{2}{1-\sigma}} \|\nabla v\|_\infty^2 + \delta \|\Lambda^{1+\beta} v\|^2 + \varepsilon \|\Lambda^{1+\alpha} u\|^2 \\
&\le c_{\varepsilon,\delta} \big(\|v\|^{\frac{1}{2}} \|\nabla v\|^{\frac{1}{2}}\big)^{\frac{2}{1-\sigma}} \|\nabla v\|_\infty^2 + \delta \|\Lambda^{1+\beta} v\|^2 + \varepsilon \|\Lambda^{1+\alpha} u\|^2,
\end{aligned}
\tag{22}
$$

where we used Young's inequality, Gagliardo-Nirenberg's inequality with parameter $\sigma = (2\alpha - 1)/2(1 - \beta)$ and control (10). Let us recall that in our case we have $1/2 < \alpha < 1$ and $0 < \beta < 1$. Observe that

$$
\frac{1-\alpha}{\beta} \le \sigma < 1 \iff \beta + 2\alpha \ge 2 \ \text{ and } \ \alpha + \beta < \frac{3}{2}, \tag{23}
$$

and the last two conditions above are always satisfied under the assumptions in (12). Then, relation (22) reduces to

$$
\begin{aligned}
K_3 &\le c_{\varepsilon,\delta} \|\nabla v\|^{\frac{1}{1-\sigma}} \|\nabla v\|_\infty^2 + \delta \|\Lambda^{1+\beta} v\|^2 + \varepsilon \|\Lambda^{1+\alpha} u\|^2 \\
&= c \|\nabla v\|_\infty^2 \|\nabla v\|^{\frac{1}{1-\sigma}} + \delta \|\Lambda^{1+\beta} v\|^2 + \varepsilon \|\Lambda^{1+\alpha} u\|^2.
\end{aligned}
$$

Notice that

$$
2 \ge \frac{1}{1-\sigma} = \frac{2(1-\beta)}{3 - 2\alpha - 2\beta} \iff \beta + 2\alpha \le 2,
$$

which is guaranteed by the hypotheses in (12). Thus, we reach

$$
\begin{aligned}
K_3 &\le c \|\nabla v\|_\infty^2 \big(1 + \|\nabla v\|^2\big) + \delta \|\Lambda^{1+\beta} v\|^2 + \varepsilon \|\Lambda^{1+\alpha} u\|^2 \\
&\le c \|\nabla v\|_{\dot{B}^0_{\infty,2}}^2 \big(1 + \log(e + \|\nabla v\|_{H^{s-1}})\big)\big(1 + \|\nabla v\|^2\big) + \delta \|\Lambda^{1+\beta} v\|^2 + \varepsilon \|\Lambda^{1+\alpha} u\|^2,
\end{aligned}
$$

(24)

where in the last step we used (6).

From relation (3.1), along with the estimates (20)–to–(24), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(e+\|\nabla u(t)\|^2+\|\nabla v(t)\|^2+\|\nabla\theta(t)\|^2\big)+(1-2\varepsilon)\|\Lambda^{1+\alpha}u(t)\|^2+(1-\delta)\|\Lambda^{1+\beta}v(t)\|^2\\
&\quad\le C\big(\|\nabla u(t)\|^2_{\dot B^0_{\infty,2}}+\|\nabla v(t)\|^2_{\dot B^0_{\infty,2}}+\|\nabla\theta(t)\|^2_{\dot B^0_{\infty,\infty}}\big)\big(1+\log\big(e+\|\nabla u(t)\|_{H^{s-1}}\\
&\qquad+\|\nabla v(t)\|_{H^{s-1}}+\|\nabla\theta(t)\|_{H^{s-1}}\big)\big)\big(e+\|\nabla u\|^2+\|\nabla v\|^2+\|\nabla\theta\|^2\big)\\
&\quad\le C I_{u,v,\theta}(t)\big(1+\log\big(e+\|\nabla u(t)\|_{H^{s-1}}+\|\nabla v(t)\|_{H^{s-1}}+\|\nabla\theta(t)\|_{H^{s-1}}\big)\big)\\
&\qquad\times\big(e+\|\nabla u\|^2+\|\nabla v\|^2+\|\nabla\theta\|^2\big),
\end{aligned}
\tag{25}
$$

where

$$
I_{u,v,\theta}(t):=\|\nabla u(t)\|^2_{\dot B^0_{\infty,2}}+\|\nabla v(t)\|^2_{\dot B^0_{\infty,2}}+\|\nabla\theta(t)\|^2_{\dot B^0_{\infty,2}}.
\tag{26}
$$

Setting

$$
\begin{aligned}
\|(\nabla u,\nabla v,\nabla\theta)(t)\|^2&:=\|\nabla u(t)\|^2+\|\nabla v(t)\|^2+\|\nabla\theta(t)\|^2,\quad\text{and}\\
\|\!|(\nabla u,\nabla v,\nabla\theta)(t)|\!\|_{H^{s-1}}&:=\|\nabla u(t)\|_{H^{s-1}}+\|\nabla v(t)\|_{H^{s-1}}+\|\nabla\theta(t)\|_{H^{s-1}},
\end{aligned}
$$

relation (25) can be rewritten as

$$
\begin{aligned}
&\frac{d}{dt}\big(e+\|(\nabla u,\nabla v,\nabla\theta)(t)\|^2\big)+2(1-2\varepsilon)\|\Lambda^{1+\alpha}u(t)\|^2+2(1-\delta)\|\Lambda^{1+\beta}v(t)\|^2\\
&\quad\le C I_{u,v,\theta}(t)\,\big(1+\log\big(e+\|\!|(\nabla u,\nabla v,\nabla\theta)(t)|\!\|_{H^{s-1}}\big)\big)\,\big(e+\|\nabla u,\nabla v,\nabla\theta(t)\|^2\big).
\end{aligned}
\tag{27}
$$

For any $T_*\le t<T$, $T_*\ge 0$, we set

$$
y(t):=\sup_{T_*\le s\le t}\|\!|(\nabla u,\nabla v,\nabla\theta)(s)|\!\|_{H^{s-1}},
\tag{28}
$$

and applying Gronwall's inequality to (27), for any $T_*\le t<T$, it follows that

$$
\begin{aligned}
&\big(e+\|(\nabla u,\nabla v,\nabla\theta)(t)\|^2\big)+2\int_{T_*}^t\big((1-2\varepsilon)\|\Lambda^{1+\alpha}u(s)\|^2+(1-\delta)\|\Lambda^{1+\beta}v(s)\|^2\big)ds\\
&\quad\le\big(e+\|(\nabla u,\nabla v,\nabla\theta)(T_*)\|^2\big)\exp\left\{C\int_{T_*}^t I_{u,v,\theta}(s)\log\big(e^2+e\|\!|(\nabla u,\nabla v,\nabla\theta)(s)|\!\|_{H^{s-1}}\big)ds\right\}\\
&\quad\le C_*\big(e+y(t)\big)^{c\epsilon},
\end{aligned}
$$

where $\epsilon>0$ is small constant, depending on T_*, such that

$$
\int_{T_*}^t I_{u,v,\theta}(s)ds<\epsilon<<1,
\tag{29}
$$

where we used (14), and the constant C_* depends on $\|(\nabla u, \nabla v, \nabla\theta)(T_*)\|$.

H^2-estimates

Applying the operator Δ to $(1)_1$, $(1)_2$ and $(1)_3$, and multiplying them in L^2 by Δu, Δv and $\Delta\theta$, respectively, and adding them up, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\Delta u(t)\|^2+\|\Delta v(t)\|^2+\|\Delta\theta(t)\|^2\big)+\|\Lambda^{2+\alpha}u\|^2+\|\Lambda^{2+\frac{1}{2}}v\|^2\\
&\quad=-\int_{\mathbb{R}^2}\Delta\big((u\cdot\nabla)u\big)\cdot\Delta u\,dx-\int_{\mathbb{R}^2}\Delta\big((u\cdot\nabla)v\big)\cdot\Delta v\,dx-\int_{\mathbb{R}^2}\Delta\big((u\cdot\nabla)\theta\big)\cdot\Delta\theta\,dx\\
&\qquad-\int_{\mathbb{R}^2}\Delta\big((v\cdot\nabla)v\big)\cdot\Delta u\,dx-\int_{\mathbb{R}^2}\Delta\big(\mathrm{div}v\,v\big)\cdot\Delta u\,dx-\int_{\mathbb{R}^2}\Delta\big((v\cdot\nabla)u\big)\cdot\Delta v\,dx\\
&\quad=:I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}\tag{30}$$

Let us first consider the three worst terms, i.e. $I_4+I_5+I_6$. Thus, we have

$$\begin{aligned}
|I_4+I_5+I_6| &= \bigg|-\int_{\mathbb{R}^2}\Delta\big(\mathrm{div}v\,v\big)\cdot\Delta u\,dx-\int_{\mathbb{R}^2}[\Delta,v\cdot\nabla]v\cdot\Delta u\,dx\\
&\qquad-\int_{\mathbb{R}^2}[\Delta,v\cdot\nabla]u\cdot\Delta v\,dx+\int_{\mathbb{R}^2}\mathrm{div}v\Delta v\cdot\Delta u\,dx\bigg|\\
&\le\bigg|\int_{\mathbb{R}^2}\Delta\big(\mathrm{div}v\,v\big)\cdot\Delta u\,dx\bigg|+\bigg|\int_{\mathbb{R}^2}[\Delta,v\cdot\nabla]v\cdot\Delta u\,dx\bigg|\\
&\qquad+\bigg|\int_{\mathbb{R}^2}[\Delta,v\cdot\nabla]u\cdot\Delta v\,dx\bigg|+\bigg|\int_{\mathbb{R}^2}\mathrm{div}v\Delta v\cdot\Delta u\,dx\bigg|\\
&\le\|\Lambda^{2-\alpha}\big(\mathrm{div}v\,v\big)\|\|\Lambda^{2+\alpha}u\|+\|[\Delta,v\cdot\nabla]v\|\|\Delta u\|\\
&\qquad+\|[\Delta,v\cdot\nabla]u\|\|\Delta v\|+\|\mathrm{div}v\|_\infty\|\Delta v\|\|\Delta u\|\\
&=:II_1+II_2+II_3+II_4,
\end{aligned}\tag{31}$$

where $[\Delta, f\cdot\nabla]g:=\Delta((f\cdot\nabla)g)-(f\cdot\nabla)\Delta g$.

Then, we have

$$\begin{aligned}
II_2+II_3+II_4&\le c(\|\nabla v\|_\infty\|\Delta v\|\|\Delta u\|+\|\nabla u\|_\infty\|\Delta v\|\|\Delta u\|+\|\nabla u\|_\infty\|\Delta v\|^2)\\
&\le c(\|\nabla u\|_\infty+\|\nabla v\|_\infty)(\|\Delta v\|^2+\|\Delta u\|^2).
\end{aligned}$$

Consider II_1. By using (7) we have that

$$\begin{aligned}
II_1&\le c\|\nabla v\|_{p_1}\|\Lambda^{1+(1-\alpha)}v\|_{q_1}\|\Lambda^{2+\alpha}u\|+c\|v\|_{p_2}\|\Lambda^{2+(1-\alpha)}v\|_{q_2}\|\Lambda^{2+\alpha}u\|\\
&=:II_{11}+II_{12},
\end{aligned}$$

with $1/p_i+1/q_i=1/2$, $i=1,2$.

Let us take into account II_{11}. Then, setting $p_1 = \infty$ and $q_1 = 2$, we have that

$$\begin{aligned}
II_{11} &= \|\nabla v\|_\infty \|\Lambda^{1+(1-\alpha)} v\| \|\Lambda^{2+\alpha} u\| \\
&\leq c_\varepsilon \|\nabla v\|_\infty^2 \|\Lambda^{2-\alpha} v\|^2 + \varepsilon \|\Lambda^{2+\alpha} u\|^2 \\
&\leq c_\varepsilon \|\nabla v\|_\infty^2 \big(\|v\|^{1-\sigma} \|\Delta v\|^\sigma\big)^2 + \varepsilon \|\Lambda^{2+\alpha} u\|^2 \\
&\leq c \|\nabla v\|_\infty^2 (\|v\|^2 + \|\Delta v\|^2) + \varepsilon \|\Lambda^{2+\alpha} u\|^2 \\
&\leq c \|\nabla v\|_\infty^2 (1 + \|\Delta v\|^2) + \varepsilon \|\Lambda^{2+\alpha} u\|^2,
\end{aligned}$$

where we used Young's inequality, and Gagliardo-Nirenberg's inequality

$$\|\Lambda^{2-\alpha} v\|^2 \leq (\|v\|^{1-\sigma} \|\Delta v\|^\sigma)^2, \quad \text{with } \sigma = (2-\alpha)/2.$$

For the term II_{12}, setting $p_2 = \infty$ and $q_2 = 2$, we reach

$$\begin{aligned}
II_{12} &= \|v\|_\infty \|\Lambda^{2+(1-\alpha)} v\| \|\Lambda^{2+\alpha} u\| \\
&\leq c_\varepsilon \|v\|_\infty^2 \|\Lambda^{2+(1-\alpha)} v\|^2 + \varepsilon \|\Lambda^{2+\alpha} u\|^2 \\
&\leq c_\varepsilon \|v\|_\infty^2 \big(\|\Delta v\|^{1-\sigma} \|\Lambda^{2+\beta} v\|^\sigma\big)^2 + \varepsilon \|\Lambda^{2+\alpha} u\|^2 \\
&\leq c_{\varepsilon,\delta} \|v\|_\infty^{\frac{2}{1-\sigma}} \|\Delta v\|^2 + \delta \|\Lambda^{2+\beta} v\|^2 + \varepsilon \|\Lambda^{2+\alpha} u\|^2 \\
&\leq c_{\varepsilon,\delta} \big(\|v\|^{\frac{1}{2}} \|\nabla v\|_\infty^{\frac{1}{2}}\big)^{\frac{2}{1-\sigma}} \|\Delta v\|^2 + \delta \|\Lambda^{2+\beta} v\|^2 + \varepsilon \|\Lambda^{2+\alpha} u\|^2 \\
&\leq c \|\nabla v\|_\infty^{\frac{1}{1-\sigma}} \|\Delta v\|^2 + \delta \|\Lambda^{2+\beta} v\|^2 + \varepsilon \|\Lambda^{2+\alpha} u\|^2,
\end{aligned} \tag{32}$$

where we applied Young and Gagliardo-Nirenberg's inequalities. In particular, parameter $\sigma = (1-\alpha)/\beta$ is such that $\sigma < 1$ if and only if $\alpha + \beta > 1$. Moreover, observe that

$$2 \geq \frac{1}{1-\sigma} = \frac{\beta}{\beta + \alpha - 1} \iff \beta + 2\alpha \geq 2,$$

and also this last condition is satisfied assuming (12). Hence, using Young's inequality, from (32) we get

$$II_{12} \leq c(1 + \|\nabla v\|_\infty^2) \|\Delta v\|^2 + \delta \|\Lambda^{2+\beta} v\|^2 + \varepsilon \|\Lambda^{2+\alpha} u\|^2.$$

Let us take into account the integral term I_1, to get

$$\begin{aligned}
|I_1| &= \left| \int_{\mathbb{R}^2} [\Delta, u \cdot \nabla] u \cdot \Delta u \, dx \right| \\
&\leq \|[\Delta, u \cdot \nabla] u\| \|\Delta u\| \\
&\leq c \|\nabla u\|_\infty \|\Delta u\|^2,
\end{aligned}$$

where we used (8). We also have that

$$
\begin{aligned}
|I_2| &= \left| \int_{\mathbb{R}^2} [\Delta, u \cdot \nabla] v \cdot \Delta v \, dx \right| \\
&\leq \| [\Delta, u \cdot \nabla] v \| \| \Delta v \| \\
&\leq c \big(\| \nabla u \|_\infty \| \Delta v \|^2 + \| \nabla v \|_\infty \| \Delta u \| \| \Delta v \| \big) \\
&\leq c \big(\| \nabla u \|_\infty + \| \nabla v \|_\infty) (\| \Delta u \|^2 + \| \Delta v \|^2).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
|I_3| &= \left| \int_{\mathbb{R}^2} [\Delta, u \cdot \nabla] \theta \cdot \Delta \theta \, dx \right| \\
&\leq \| [\Delta, u \cdot \nabla] \theta \| \| \Delta \theta \| \\
&\leq c \big(\| \nabla u \|_\infty \| \Delta \theta \|^2 + \| \nabla \theta \|_\infty \| \Delta u \| \| \Delta \theta \| \big) \\
&\leq c \big(\| \nabla u \|_\infty + \| \nabla \theta \|_\infty) (\| \Delta u \|^2 + \| \Delta \theta \|^2).
\end{aligned} \tag{33}
$$

Combining (30) with (31)–to–(33), and setting $\|(\Delta u, \Delta v, \Delta\theta)(t)\|^2 := \|\Delta u(t)\|^2 + \|\Delta v(t)\|^2 + \|\Delta\theta(t)\|^2$, we finally get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(e + \|(\Delta u, \Delta v, \Delta\theta)(t)\|^2\big) + (1-2\varepsilon)\|\Lambda^{2+\alpha}u(t)\|^2 + (1-\delta)\|\Lambda^{2+\beta}v(t)\|^2 \\
&\quad \leq C\big(1 + \|\nabla u(t)\|_\infty^2 + \|\nabla v(t)\|_\infty^2 + \|\nabla\theta(t)\|_\infty^2\big)\big(e + \|(\Delta u, \Delta v, \Delta\theta)(t)\|^2\big) \\
&\quad \leq C(1 + I_{u,v,\theta}(t))\,\big(1 + \log\big(e + \|(\nabla u, \nabla v, \nabla\theta)(t)\|_{H^{s-1}}\big)\big)\,\big(e + \|(\Delta u, \Delta v, \Delta\theta)(t)\|^2\big),
\end{aligned} \tag{34}
$$

where $I_{u,v,\theta}(t)$ is defined as in (26) and, in particular, in the last inequality we used relation (6). We conclude as in (27)-to-(29) by an application of Gronwall's lemma and exploiting hypothesis (14), to get the control

$$
\begin{aligned}
&\big(e + \|(\Delta u, \Delta v, \Delta\theta)(t)\|^2\big) + 2\int_{T_*}^t \big((1-2\varepsilon)\|\Lambda^{2+\alpha}u(s)\|^2 + (1-\delta)\|\Lambda^{2+\beta}v(s)\|^2\big)ds \\
&\quad \leq \big(e + \|(\Delta u, \Delta v, \Delta\theta)(T_*)\|^2\big)\exp\left\{C\int_{T_*}^t \big(1 + I_{u,v,\theta}(s)\big)\log\right. \\
&\qquad \left.\big(e^2 + e\|(\Delta u, \Delta v, \Delta\theta)(s)\|_{H^{s-1}}\big)\big)ds\right\} \\
&\quad \leq C_* e^{c(t-T_*)}\big(e + y(t)\big)^{c\epsilon},
\end{aligned}
$$

where $y(t)$ is defined as in (28), $\epsilon > 0$ is the small quantity introduced in (29), and we still use C_* that, in this case, also depends on $\|(\Delta u, \Delta v, \Delta\theta)(T_*)\|^2$.

H^s-estimates, $s > 2$

We now derive and close the H^s-controls for u, v and θ. Applying the operator Λ^s to $(1)_1$, $(1)_2$ and $(1)_3$, multiplying the resulting equations in L^2, by $\Lambda^s u$, $\Lambda^s v$ and $\Lambda^s \theta$, respectively, and adding them up, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\Lambda^s u(t)\|^2 + \|\Lambda^s v(t)\|^2 + \|\Lambda^s \theta(t)\|^2\big) + \|\Lambda^{s+\alpha} u(t)\|^2 + \|\Lambda^{s+\beta} v(t)\|^2 \\
&\quad = -\int_{\mathbb{R}^2} \Lambda^s\big((u\cdot\nabla)u\big)\cdot\Lambda^s u\,dx - \int_{\mathbb{R}^2} \Lambda^s\big((u\cdot\nabla)v\big)\cdot\Lambda^s v\,dx - \int_{\mathbb{R}^2} \Lambda^s\big((u\cdot\nabla)\theta\big)\cdot\Lambda^s \theta\,dx \\
&\qquad - \int_{\mathbb{R}^2} \Lambda^s\big((v\cdot\nabla)v\big)\cdot\Lambda^s u\,dx - \int_{\mathbb{R}^2} \Lambda^s\big(\mathrm{div} v\, v\big)\cdot\Lambda^s u\,dx - \int_{\mathbb{R}^2} \Lambda^s\big((v\cdot\nabla)u\big)\cdot\Lambda^s v\,dx \\
&\quad =: \hat{I}_1 + \hat{I}_2 + \hat{I}_3 + \hat{I}_4 + \hat{I}_5 + \hat{I}_6.
\end{aligned} \tag{35}$$

Let us first consider the integral term $\hat{I}_1$. By exploiting relation (8), we get

$$\begin{aligned}
|\hat{I}_1| &= \left|\int_{\mathbb{R}^2} [\Lambda^s, u\cdot\nabla]u\cdot\Lambda^s u\,dx\right| \\
&\le \|[\Lambda^s, u\cdot\nabla]u\|\|\Lambda^s u\| \\
&\le C\|\nabla u\|_\infty \|\Lambda^s u\|^2.
\end{aligned} \tag{36}$$

Consider $\hat{I}_2$ to find

$$\begin{aligned}
|\hat{I}_2| &= \left|\int_{\mathbb{R}^2} \Lambda^s\big((u\cdot\nabla)v\big)\cdot\Lambda^s v\,dx\right| \\
&\le \|[\Lambda^s, u\cdot\nabla]v\|\|\Lambda^s v\| \\
&\le C\big(\|\nabla u\|_\infty\|\Lambda^{s-1}\nabla v\| + \|\Lambda^s u\|\|\nabla v\|_\infty\big)\|\Lambda^s v\| \\
&\le C\big(\|\nabla u\|_\infty + \|\nabla v\|_\infty\big)\big(\|\Lambda^s u\|^2 + \|\Lambda^s v\|^2\big).
\end{aligned}$$

We now estimate $\hat{I}_3$. Observe that

$$\begin{aligned}
|\hat{I}_3| &= \left|\int_{\mathbb{R}^2} [\Lambda^s, u\cdot\nabla]\theta\cdot\Lambda^s \theta\,dx\right| \\
&\le \|[\Lambda^s, u\cdot\nabla]\theta\|\|\Lambda^s \theta\| \\
&\le \big(\|\nabla u\|_\infty\|\Lambda^{s-1}\nabla \theta\| + \|\nabla \theta\|_\infty\|\Lambda^s u\|\big)\|\Lambda^s \theta\| \\
&\le C\big(\|\nabla u\|_\infty + \|\nabla \theta\|_\infty\big)\big(\|\Lambda^s u\|^2 + \|\Lambda^s \theta\|^2\big).
\end{aligned}$$

Let us take into account $\hat{I}_4$, $\hat{I}_5$ and $\hat{I}_6$. We have that

$$\begin{aligned}
|\hat{I}_4+\hat{I}_5+\hat{I}_6| &= \Big| -\int_{\mathbb{R}^2} \Lambda^s\big(\operatorname{div} v\, v\big)\cdot\Lambda^s u\,dx - \int_{\mathbb{R}^2}[\Lambda^s, v\cdot\nabla]v\cdot\Lambda u\,dx \\
&\quad - \int_{\mathbb{R}^2}[\Lambda^s, v\cdot\nabla]u\cdot\Lambda^s v\,dx + \int_{\mathbb{R}^2}\operatorname{div} v\Lambda^s v\cdot\Lambda^s u\,dx\Big| \\
&\le \Big|\int_{\mathbb{R}^2}\Lambda^s\big(\operatorname{div} v v\big)\cdot\Lambda^s u\,dx\Big| + \Big|\int_{\mathbb{R}^2}[\Lambda^s, v\cdot\nabla]v\cdot\Lambda^s u\,dx\Big| \\
&\quad + \Big|\int_{\mathbb{R}^2}[\Lambda^s, v\cdot\nabla]u\cdot\Lambda^s v\,dx\Big| + \Big|\int_{\mathbb{R}^2}\operatorname{div} v\Lambda^s v\cdot\Lambda^s u\,dx\Big| \\
&\le \|\Lambda^{s-\alpha}\big(\operatorname{div} v\, v\big)\|\|\Lambda^{s+\alpha}u\| + \|[\Lambda^s, v\cdot\nabla]v\|\|\Lambda^s u\| \\
&\quad + \|[\Lambda^s, v\cdot\nabla]u\|\|\Lambda^s v\| + \|\operatorname{div} v\|_\infty\|\Lambda^s v\|\|\Lambda^s u\| \\
&=: \widehat{II}_1 + \widehat{II}_2 + \widehat{II}_3 + \widehat{II}_4,
\end{aligned}$$

where $[\Lambda^s, f\cdot\nabla]g := \Lambda^s((f\cdot\nabla)g) - (f\cdot\nabla)\Lambda^s g$.

In particular, using (7) we have that

$$\begin{aligned}
\widehat{II}_1 &\le C\big(\|\nabla v\|_{p_1}\|\Lambda^{s-\alpha}v\|_{q_1} + \|v\|_{p_2}\|\Lambda^{s+(1-\alpha)}v\|_{q_2}\big)\|\Lambda^{s+\alpha}u\| \\
&=: \widehat{II}_{11} + \widehat{II}_{12},
\end{aligned}$$

with $1/p_i + 1/q_i = 1/2$, $i = 1, 2$.

Let us start with $\widehat{II}_{12}$. Taking $p_2 = \infty$ and $q_2 = 2$, and applying Young and Gagliardo-Nirenberg's inequalities, we infer

$$\begin{aligned}
\widehat{II}_{12} &\le c_\varepsilon\|v\|_\infty^2\|\Lambda^{s+(1-\alpha)}v\|^2 + \varepsilon\|\Lambda^{s+\alpha}u\|^2 \\
&\le c_\varepsilon\|v\|_\infty^2\big(\|\Delta v\|^{(1-\vartheta)}\|\Lambda^{s+\beta}v\|^{\vartheta}\big)^2 + \varepsilon\|\Lambda^{s+\alpha}u\|^2 \\
&\le c_{\varepsilon,\delta}\|v\|_\infty^{\frac{2}{1-\vartheta}}\|\Delta v\|^2 + \delta\|\Lambda^{s+\beta}v\|^2 + \varepsilon\|\Lambda^{s+\alpha}u\|^2 \\
&\le c_{\varepsilon,\delta}\big(\|v\|^{\frac{1}{2}}\|\Delta v\|^{\frac{1}{2}}\big)^{\frac{2}{1-\vartheta}}\|\Delta v\|^2 + \delta\|\Lambda^{s+\beta}v\|^2 + \varepsilon\|\Lambda^{s+\alpha}u\|^2 \\
&\le c_{\varepsilon,\delta}\|\Delta v\|^{\frac{1}{1-\vartheta}}\|\Delta v\|^2 + \delta\|\Lambda^{s+\beta}v\|^2 + \varepsilon\|\Lambda^{s+\alpha}u\|^2 \\
&\le c\|\Delta v\|^{2+\frac{1}{1-\vartheta}} + \delta\|\Lambda^{s+\beta}v\|^2 + \varepsilon\|\Lambda^{s+\alpha}u\|^2,
\end{aligned}$$

where $\vartheta = (s-1-\alpha)/(s-2+\beta)$. Here, as usual, $s > 2$, $1/2 < \alpha < 1$, and $0 < \beta < 1$, and it holds true

$$\vartheta = \frac{s-1-\alpha}{s-2+\beta} < 1 \iff \alpha+\beta > 1, \tag{37}$$

which is verified under the assumptions in (12). Moreover, we have that

$$\frac{1}{1-\vartheta} = \frac{s-2+\beta}{\beta+\alpha-1}.$$

For the term $\widehat{II}_{11}$, taking $p_1 = \infty$ and $q_1 = 2$, we have the following control

$$\begin{aligned}\widehat{II}_{11} &= \|\nabla v\|_\infty \|\Lambda^{s-\alpha} v\|_2 \|\Lambda^{s+\alpha} u\| \\ &\le c_\varepsilon \|\nabla v\|_\infty^2 \|\Lambda^{s-\alpha} v\|^2 + \varepsilon \|\Lambda^{s+\alpha} u\|^2 \\ &\le c_\varepsilon \|\nabla v\|_\infty^2 \big(\|v\|^{1-\sigma} \|\Lambda^s v\|^\sigma\big)^2 + \varepsilon \|\Lambda^{s+\alpha} u\|^2 \\ &\le c_\varepsilon \|\nabla v\|_\infty^2 (1 + \|\Lambda^s v\|^2) + \varepsilon \|\Lambda^{s+\alpha} u\|^2,\end{aligned}$$

where we used Gagliardo-Nirenberg's inequality with $\sigma = (s-\alpha)/\alpha$.

For the remaining terms $\widehat{II}_2$, $\widehat{II}_3$, and $\widehat{II}_4$, we get

$$|\widehat{II}_2 + \widehat{II}_3 + \widehat{II}_4| \le c(\|\nabla u\|_\infty + \|\nabla v\|_\infty)(\|\Lambda^s u\|^2 + \|\Lambda^s v\|^2). \tag{38}$$

Setting $\|(\Lambda^s u, \Lambda^s v, \Lambda^s \theta)(t)\|^2 := \|\Lambda^s u(t)\|^2 + \|\Lambda^s v(t)\|^2 + \|\Lambda^s \theta(t)\|^2$, recalling that $|||(\nabla u, \nabla v, \nabla\theta)(t)|||_{H^{s-1}} = \|\nabla u(t)\|_{H^{s-1}} + \|\nabla v(t)\|_{H^{s-1}} + \|\nabla\theta(t)\|_{H^{s-1}}$, and using (35) along with the above estimates (36)–to–(38), for $\hat{I}_i$, $i = 1, \ldots, 8$, we obtain the following differential inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(e + \|(\Lambda^s u, \Lambda^s v, \Lambda^s\theta)(t)\|^2\big) + (1-2\varepsilon)\|\Lambda^{s+\alpha} u(t)\|^2 + (1-\delta)\|\Lambda^{s+\beta} v(t)\|^2 \\ &\quad\le C\big(\|\nabla u(t)\|_\infty^2 + \|\nabla v(t)\|_\infty^2 + \|\nabla\theta(t)\|_\infty^2\big)\big(e + \|(\Lambda^s u, \Lambda^s v, \Lambda^s\theta)(t)\|^2\big) \\ &\qquad + C\|\Delta v(t)\|^{2+\frac{1}{1-\vartheta}} \\ &\quad\le C I_{u,v,\theta}(t)\,\big(1 + \log\big(e + |||(\nabla u, \nabla v, \nabla\theta)(t)|||_{H^{s-1}}\big)\big) \\ &\qquad \times \big(e + \|(\Lambda^s u, \Lambda^s v, \Lambda^s\theta)(t)\|^2\big) + C\|\Delta v(t)\|^{2+\frac{1}{1-\vartheta}},\end{aligned} \tag{39}$$

where in the last step we used again (6), with $I_{u,v,\theta}$ defined as in (26), and parameter ϑ is given in (37).

Then, up to use explicitly Gagliardo-Nirenberg's inequality to control lower-order terms in $|||(\nabla u, \nabla v, \nabla\theta)(t)|||_{H^{s-1}}$ with $\|(\Lambda^s u, \Lambda^s v, \Lambda^s\theta)(t)\|^2$ and $\|(u, v, \theta)(t)\|^2$, and up to introduce a constant C_0 larger than "e" in the above logarithm (which is needed to reabsorb the terms coming from the interpolations and the subsequent manipulations), setting

$$Y(t) := \|(\Lambda^s u, \Lambda^s v(t), \Lambda^s\theta)(t)\|^2,$$

from the differential inequality (39) we infer

$$\frac{d}{dt}\big(e + Y(t)\big) \le C I_{u,v,\theta}(t)\,\big(1 + \log\big(e + Y(t)\big)\big)\,\big(e + Y(t)\big) + C\|\Delta v(t)\|^{2+\frac{1}{1-\vartheta}}.$$

As a direct consequence of Gronwall's inequality, we get

$$e + Y(t) \leq \left(\big(e + Y(T_*)\big) + C \int_{T_*}^{t} \|\Delta v(s)\|^{2+\frac{1}{1-\vartheta}} \, ds \right) \times \exp\left\{ C \int_{T_*}^{t} I_{u,v,\theta}(s) \log\left(e^2 + eY(s)\right) ds \right\}.$$

Hence, with a slight abuse of notation we still use the quantity $y(t) = \sup_{T_* \leq s \leq t} Y(s)$, $T_* \leq t < T$, introduced in (28), and recalling that $\epsilon > 0$ is the small parameter in (29), we have that

$$\begin{aligned} e + y(t) &\leq \left[\big(e + Y(T_*)\big) + C_*(t - T_*) e^{c\gamma(t-T_*)} \big(e + y(t)\big)^{c\gamma\epsilon} \right] \big(e + y(t)\big)^{c\epsilon} \\ &\leq c\left[\big(e + Y(T_*)\big) + (t - T_*) e^{c\gamma(t-T_*)} \right] \big(e + y(t)\big)^{c(\gamma+1)\epsilon}, \end{aligned}$$

where $\gamma = 1 + 1/2(1-\vartheta)$, and we employed relation (34) and its direct consequence. Here, the constant C_* depends on $\|u(T_*)\|_{H^2}$, $\|v(T_*)\|_{H^2}$, and $\|\theta(T_*)\|_{H^2}$. Then, taking $\epsilon < 1/c(\gamma+1)$, we finally obtain the bound

$$\sup_{0 \leq t \leq T} \left(\|\Lambda^s u(t)\|^2 + \|\Lambda^s v(t)\|^2 + \|\Lambda^s \theta(t)\|^2 \right) < +\infty,$$

and so the solution $(u, v, \theta)(t)$ remains uniformly bounded in H^s-norm on $[0, T]$. In particular, by using a standard argument about the continuation of local solutions, it follows that $(u, v, \theta)(t)$ can be extended beyond $T > 0$, and the proof is completed.

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