Multiplicity of forced oscillations for the spherical pendulum acted on by a retarded periodic force☆

Alessandro Calamai

Dipartimento di Ingegneria Civile, Edile e Architettura, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy

Maria Patrizia Pera

Dipartimento di Matematica e Informatica 'U. Dini', Università di Firenze, Via Santa Marta 3, 50139 Firenze, Italy

Marco Spadini*

Dipartimento di Matematica e Informatica 'U. Dini', Università di Firenze, Via Santa Marta 3, 50139 Firenze, Italy

Abstract

We prove a multiplicity result for forced oscillations of a spherical pendulum (that is, a massive point moving on a sphere) subject to a periodic action, with or without friction, allowed to depend on the whole past of the motion. The approach is based on topological methods.

In particular, when the unperturbed forcing term is the gravity, we obtain two harmonic forced oscillations regardless of the presence of friction and of the form of the perturbing force field.

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1. Introduction

The pendulum equation has had a fundamental role in the development of classical mechanics and dynamical systems theory. Indeed, there has always

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*Corresponding author.
Email addresses: calamai@dipmat.univpm.it (Alessandro Calamai), mpatrizia.pera@unifi.it (Maria Patrizia Pera), marco.spadini@unifi.it (Marco Spadini)

been an interest in pendulum and pendulum-like equations in the mathematical literature. In particular, existence and multiplicity results for periodic solutions have always attracted attention. It is impossible to give here an exhaustive list of the many approaches that have been successfully pursued. As a very short list of papers representing different techniques we only mention [8, 10, 21, 26, 29], see also the survey papers [23, 24] and references therein. In spite of the fact that pendulum-like equations are still a field actively researched by mathematicians, the so-called *spherical pendulum* (i.e., a massive point constrained on a sphere) has been studied more extensively by the community of physicists and applied mathematicians.

In [17, Corollary 4.2] a simple argument, based on the topological structure of the set of harmonic solutions of a periodic perturbation of a differential equation on S^1, provided a multiplicity result for the forced pendulum. A similar argument, but in the considerably more complex framework of *retarded functional differential equations* (RFDEs), yielded in [16, Example 4.5] a multiplicity result for the delayed pendulum. On a parallel track, a set of somewhat more delicate topological arguments inspired by [12] gave in [18] a multiplicity result for the spherical pendulum (without delay). Indeed, these multiplicity results are, in a sense, "generic" as shown in [22].

The existence of periodic oscillations for the spherical pendulum has been proved in a series of papers, culminating in [12, 13], in the case when the perturbing force depends only on time and state and in the more recent papers [3, 6] when a, possibly infinite, delay is allowed. In the framework of delay differential equations, a preliminary study on first- and second-order RFDEs on possibly noncompact manifolds has been performed by some of the authors, mostly in collaboration with M. Furi and P. Benevieri. Namely, in [5] general properties of RFDEs with infinite delay on differentiable manifolds were studied. In [6, 16] we investigated the structure of the set of solutions of parameterized RFDEs, obtaining global continuation results for such equations. The existence results in [3, 6], as well as the already mentioned multiplicity result for the "retarded simple pendulum" in [16], are obtained as applications of these more general facts.

We point out that the problem of existence of forced oscillations for the spherical pendulum (where no delay is allowed) had been previously treated, in different contexts, also by other authors, see e.g. [1, 2]. In this direction we cite the recent paper [28], which deals with massive points moving on compact surfaces with boundary.

In the present paper, we focus on the physically meaningful case of forced oscillations of a spherical pendulum subject to a periodic forcing that depends possibly on the whole history of the pendulum's motion. We prove two multiplicity results, namely Theorems 3.1 and 3.2 below. In Theorem 3.2 we prove that, under mild assumptions, the gravitational spherical pendulum always admits at least two harmonic forced oscillations whatever the forcing term is and regardless of the presence of friction. The methods which we employ, like those of [16], are intrinsically topological. Indeed, in our setting, friction could well be absent and the stable equilibrium of the pendulum could be T-resonant (see Definition 3.1). Thus this result is not directly deducible from the implicit function theorem. Furthermore, Theorem 3.1 below actually allows us to obtain multiplicity results also when considering unperturbed force fields more general than the mere gravitational one (think about, e.g., systems of springs or electro-magnetic forces).

Our multiplicity results improve those of [18] in a natural sense, since differential equations with delay include ODEs as particular cases. On the other hand, the extension that we obtain here is only partial. In fact, in [18] the active force may depend also on the velocity, which is not the case in the present setting. Secondly, for technical reasons we assume that the retarded forcing term is locally Lipschitz, so we are not able to prove our results with the sole continuity assumption as it was done in [18].

Our results stem from the interplay between global and local aspects. A key notion for the "local" part of this approach is that of *ejecting set* or *point* (see Definition 2.1) which, broadly speaking, is analogous to the concept of bifurcation point. Although it is sometimes possible to prove directly that the property of being ejecting holds for some points, usually the most practical way is through a *condition of T-resonance* (see e.g. [9, Ch. 7]), or rather its contrary, i.e. that of "non-T-resonance". Roughly speaking a zero which is not T-resonant can be regarded as an ejecting point. The condition of T-resonance is linked to the physical notion of period for small oscillations about an equilibrium; a similar idea can be traced back to Poincaré (see [25] for an exposition).

In summary, we are going to provide conditions for the zeros of a certain vector field related to the equation governing the spherical pendulum to be ejecting for the set of T-periodic solutions, where $T > 0$ is the period of the forcing term (Lemma 3.8). This, when combined with a general topology lemma concerning ejecting sets (Theorem 2.1) and an *a priori* bound on branches of T-periodic solutions [18, Lemma 4.1], will yield our multiplicity results.

2. Preliminaries

2.1. Degree of a vector field

We now recall some basic notions about tangent vector fields on manifolds as well as the notion of degree of an admissible tangent vector field (see e.g. [15, 27]).

Let $M \subseteq \mathbb{R}^k$ be a smooth differentiable manifold. Let w be a tangent vector field on M, that is, a continuous map $w\colon M \to \mathbb{R}^k$ with the property that $w(p)$ belongs to the tangent space T_pM of M at p for any $p \in M$. Let W be an open subset of M in which we assume w *admissible* (for the degree); that is, the set $w^{-1}(0) \cap W$ is compact. Then, one can associate to the pair (w, W) an integer, $\deg(w, W)$, called the *degree (or characteristic) of the vector field w in W*, which, in a sense, counts (algebraically) the zeros of w in W.

If w is (Fréchet) differentiable at $p \in M$ and $w(p) = 0$, then the differential $dw_p\colon T_pM \to \mathbb{R}^k$ maps T_pM into itself, so that the determinant $\det dw_p$ of dw_p is defined. If, in addition, p is a nondegenerate zero (i.e. $dw_p\colon T_pM \to \mathbb{R}^k$ is injective) then p is an isolated zero and $\det dw_p \neq 0$. In fact, if w is admissible for the degree in W, when the zeros of w are all nondegenerate, then the set $w^{-1}(0) \cap W$ is finite and

$$\deg(w, W) = \sum_{p \in w^{-1}(0) \cap W} \operatorname{sign} \det dw_p. \tag{2.1}$$

Observe that in the flat case, i.e. when $M = \mathbb{R}^k$, $\deg(w, W)$ is just the classical Brouwer degree with respect to zero, $\deg_B(w, V, 0)$, where V is any bounded open neighborhood of $w^{-1}(0) \cap W$ whose closure is contained in W. All the standard properties of the Brouwer degree for continuous maps on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, still hold in this more general context (see e.g. [15]).

The Excision Property allows the introduction of the notion of index of an isolated zero of a tangent vector field. Indeed, let $p \in M$ be an isolated zero of w. Clearly, $\deg(w, V)$ is well defined for each open $V \subseteq M$ such that $V \cap w^{-1}(0) = \{p\}$. By the Excision Property $\deg(w, V)$ is constant with respect to such V's. This common value of $\deg(w, V)$ is, by definition, the *index of w at p*, and is denoted by $\mathrm{i}\,(w, p)$. With this notation, if (w, W) is admissible and all the zeros of w in W are isolated, the Additivity Property yields that

$$\deg(w, W) = \sum_{p \in w^{-1}(0) \cap W} \mathrm{i}\,(w, p). \tag{2.2}$$

By formula (2.1) we have that, if p is a nondegenerate zero of w, then

$$\mathrm{i}\,(w, p) = \operatorname{sign}\det dw_p.$$

Notice that (2.1) and (2.2) differ in the fact that, in the latter, the zeros of w are not necessarily nondegenerate as they have to be in the former. In fact, in (2.2), w need not be differentiable at its zeros.

In the case when M is a compact boundaryless manifold, the celebrated Poincaré-Hopf Theorem states that $\deg(w, M)$ coincides with the Euler-Poincaré characteristic $\chi(M)$ of M and, therefore, is independent of w. In particular, if all the zeros of w are isolated, it follows that

$$\chi(M) = \sum_{p \in w^{-1}(0)} \mathrm{i}\,(w, p). \tag{2.3}$$

2.2. Ejecting sets

Let Y be a metric space and X a subset of $[0, +\infty) \times Y$. Given $\lambda \geq 0$, we denote by X_λ the slice $\{y \in Y : (\lambda, y) \in X\}$.

Definition 2.1 ([14])**.** *We say that $E \subseteq X_0$ is* ejecting *(for X) if it is relatively open in X_0 and there exists a connected subset of X which meets $\{0\} \times E$ and is not contained in $\{0\} \times X_0$.*

In [14, Theorem 3.3] the following result was essentially proved.

Theorem 2.1. *Let Y be a metric space and let X be a locally compact subset of $[0,+\infty)\times Y$. Assume that X_0 contains n pairwise disjoint ejecting subsets $E_1,\dots,E_n$. Suppose that $n-1$ of them are compact. Then, there are open neighborhoods $U_1,\dots,U_n$ in Y of $E_1,\dots,E_n$, respectively, with pairwise disjoint closure, and a positive number λ_* such that for $\lambda\in[0,\lambda_*)$*

$$X_\lambda\cap U_i\neq\emptyset,\quad i=1,\dots,n.$$

In particular, we have that the cardinality of X_λ is greater than or equal to n for any $\lambda\in[0,\lambda_*)$.

We point out that, although the assertion of Theorem 2.1 may seem quite intuitive, its set of assumptions is rather sharp in the sense that, as shown by examples in [14], none of the hypotheses can be dropped.

2.3. T-resonance

A handy notion for the local investigation of a stationary point is that of T-*resonance* ([9], see also [7, 14]) which we now briefly recall for a general smooth manifold $M\subseteq\mathbb{R}^k$. As we will see (Lemma 3.8 below), for a parametrized equation this notion or, rather, its negation is connected to that of ejecting set.

Consider on M the following differential equation:

$$x'(t)=g(x(t)) \tag{2.4}$$

where $g\colon M\to\mathbb{R}^k$ is a tangent vector field of class C^1. Given $T>0$, a point $p\in g^{-1}(0)$ is said to be T-*resonant* for g if the linearized equation (on T_pM)

$$z'(t)=g'(p)z(t)$$

admits T-periodic solutions other than the trivial one $z(t)\equiv 0$. If this is not true it is also customary to say informally that a point $p\in g^{-1}(0)$ is not T-resonant for the equation (2.4).

Observe that, if p is not T-resonant then $g'(p)$ is invertible, and so p is an isolated zero of g. One can check that p is not T-resonant for g if and only if $g'(p)$ has no purely imaginary eigenvalues of the form $\frac{2l\pi i}{T}$ with $l\in\mathbb{Z}$. Thus:

Remark 2.2. *Let $g\colon M\to\mathbb{R}^k$ be a tangent vector field and let p be a zero of g which is not T-resonant. Then, $\mathrm{i}\big(g,p\big)=\pm1$. Thus, for any sufficiently small neighborhood $U\subseteq M$ of p, we have $\deg(g,U)\neq 0$.*

2.4. Retarded functional differential equations

Here we collect some definitions and properties of RFDEs with infinite delay on possibly noncompact differentiable manifolds, which have been studied e.g. in [5]. As a general reference on RFDEs with finite delay in Euclidean spaces, see the monograph [20].

Given an arbitrary subset A of $\mathbb{R}^s$, we denote by $BU((-\infty,0],A)$ the set of bounded and uniformly continuous maps from $(-\infty,0]$ into A. Notice that $BU((-\infty,0],\mathbb{R}^s)$ is a Banach space, being closed in the space $BC((-\infty,0],\mathbb{R}^s)$ of the bounded and continuous functions from $(-\infty,0]$ into $\mathbb{R}^s$ (endowed with the standard supremum norm $\|\cdot\|$).

Let M be a boundaryless smooth manifold in $\mathbb{R}^k$. A continuous map

$$G\colon \mathbb{R}\times BU((-\infty,0],M)\to\mathbb{R}^k$$

is said to be a *retarded functional tangent vector field over* M if $G(t,\eta)\in T_{\eta(0)}M$ for all $(t,\eta)\in\mathbb{R}\times BU((-\infty,0],M)$. In the sequel, any map with this property will be briefly called a *functional field (over M).*

Let us consider a first order RFDE of the type

$$z'(t)=G(t,z_t), \tag{2.5}$$

where G is a functional field over M. Here, as usual and whenever it makes sense, given $t\in\mathbb{R}$, by $z_t\in BU((-\infty,0],M)$ we mean the function $\theta\mapsto z(t+\theta)$.

A *solution* of (2.5) is a function $z\colon J\to M$, defined on an open real interval J with $\inf J=-\infty$, bounded and uniformly continuous on any closed half-line $(-\infty,b]\subset J$, and which verifies eventually the equality $z'(t)=G(t,z_t)$. That is, $z\colon J\to M$ is a solution of (2.5) if $z_t\in BU((-\infty,0],M)$ for all $t\in J$ and there exists $\tau\in J$ such that z is C^1 on the interval $(\tau,\sup J)$ and $z'(t)=G(t,z_t)$ for all $t\in(\tau,\sup J)$.

It can be proved (see e.g. [5]) that if a functional field G is locally Lipschitz in the second variable, then two maximal solutions of equation (2.5) coinciding in the past must coincide also in the future.

3. Multiplicity results

In this section we obtain the main results of the paper, Theorems 3.1 and 3.2 below. We work on the compact boundaryless manifold $\boldsymbol{S}=\{q\in\mathbb{R}^3:|q|=r\}$, where $|\cdot|$ is the Euclidean norm, that is the homothetic sphere $\boldsymbol{S} = rS^2$. A crucial observation, following from the Poincaré-Hopf theorem, will be that for any tangent vector field v on $\boldsymbol{S}$, $\deg(v, \boldsymbol{S}) = \chi(\boldsymbol{S}) = \chi(S^2) = 2$.

We consider the following family of parametrized equations, depending on $\lambda \geq 0$:

$$mx''(t) = -m(|x'(t)|^2/r^2)x(t) - \eta x'(t) + h(x(t)) + \lambda F(t, x_t). \tag{3.1}$$

where:

- $m > 0$;
- $h : \boldsymbol{S} \to \mathbb{R}^3$ is a C^1 tangent vector field on $\boldsymbol{S}$;
- $\eta \geq 0$ is given;
- $F : \mathbb{R} \times BU((-\infty, 0], \boldsymbol{S}) \to \mathbb{R}^3$ is a functional field over $\boldsymbol{S}$ which is T-periodic in the first variable and locally Lipschitz in the second one, i.e., given $(\tau, \varphi) \in \mathbb{R} \times BU((-\infty, 0], \boldsymbol{S})$, there exist an open neighborhood U of (τ, φ) and $L \geq 0$ such that

$$|F(t, \varphi_1) - F(t, \varphi_2)| \leq L \|\varphi_1 - \varphi_2\|,$$

for all $(t, \varphi_1), (t, \varphi_2) \in U$.

Equation (3.1) represents the motion equation of a particle of mass m constrained to $\boldsymbol{S}$ and acted on by the sum of three forces: a tangent vector field h depending only on the position, a possible friction and a T-periodic forcing term λF which depends on the whole past history of the process. The term $R(q, v) = -m(|v|^2/r^2)q$ in equation (3.1) is the reactive force of the constraint. A physically relevant example is obtained when h is the tangential component of the gravitational force. That is,

$$h(q) = h_g(q) = \frac{mg}{r^2}\big(q_3q_1, q_3q_2, -(r^2 - q_3^2)\big).$$

In order to clarify what we mean by a solution of (3.1), we introduce in a natural way a first order RFDE on the tangent bundle

$$T\boldsymbol{S} = \big\{(q, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \ q \in \boldsymbol{S},\ \langle q, v\rangle = 0\big\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^3$. Let, for $(q, v) \in T\boldsymbol{S}$ and $(t, (\varphi, \psi)) \in \mathbb{R} \times BU((-\infty, 0], T\boldsymbol{S})$:

- $\kappa(q,v) = \Big(0, \dfrac{v}{m}\Big)$

- $\hat{h}(q,v) = \left(v, -\dfrac{|v|^2}{r^2} q + \dfrac{h(q)}{m}\right)$

- $\hat{F}(t,(\varphi,\psi)) = \left(0, \dfrac{1}{m} F(t,\varphi)\right)$.

One can prove that $\hat{h}$ and κ are tangent vector fields on $T\boldsymbol{S}$, and $\hat{F}$ is a functional field over $T\boldsymbol{S}$. Setting $\xi = (q,v)$, the following is a RFDE on $T\boldsymbol{S}$ in the sense discussed in Section 2.4:

$$\xi'(t) = \hat{h}(\xi(t)) - \eta\kappa(\xi(t)) + \lambda \hat{F}(t,\xi_t). \tag{3.2}$$

That is, (3.2) is of the form (2.5) with, for any $\lambda \in [0,\infty)$,

$$G\big(t,(\varphi,\psi)\big) = \left(\psi(0), -\frac{|\psi(0)|^2}{r^2}\varphi(0) - \frac{\eta}{m}\psi(0) + \frac{1}{m}h\big(\varphi(0)\big) + \frac{\lambda}{m}F(t,\varphi)\right). \tag{3.3}$$

We regard a solution of (3.1) as a map $x\colon J \to \boldsymbol{S}$, defined on an open real interval J with $\inf J = -\infty$, such that the pair $(x,x')\colon J \to T\boldsymbol{S}$ is a solution of (3.2).

We now introduce the notion of T-resonance for equation (3.1) (see also the appendix for a more general discussion).

Definition 3.1. *We say that a point $q \in h^{-1}(0)$ is T-resonant for* (3.1) *if $(q,0)$ is T-resonant for $\hat{h} - \eta\kappa$, that is, for equation* (3.2) *with $\lambda = 0$.*

Physically, q is a T-resonant zero of h if T is the period for small oscillations of the pendulum about the equilibrium q.

According to the above definition, $q \in h^{-1}(0)$ is not T-resonant for (3.1) if and only if equation (3.2) for $\lambda = 0$ linearized about $\xi_0 = (q,0)$, namely the equation on $T_{\xi_0}T\boldsymbol{S}$

$$\zeta'(t) = \hat{h}'(\xi_0)\zeta(t) - \eta\kappa'(\xi_0)\zeta(t), \tag{3.4}$$

has only the trivial solution. Straightforward computations (see e.g. [22]) show that this is the case if and only if

$$\det\left(\frac{1}{m}h'(q) - \frac{2\pi i\ell\eta}{mT}I + \left(\frac{2\ell\pi}{T}\right)^2 I\right) \neq 0, \qquad \forall \ell \in \mathbb{Z}, \tag{3.5}$$

where $I\colon T_q\boldsymbol{S} \to T_q\boldsymbol{S}$ denotes the identity and i is the imaginary unit.

Given $\lambda \geq 0$, by a *T-periodic solution*, or *forced oscillation*, of equation (3.1) we mean a solution which is globally defined on $\mathbb{R}$ and is T-periodic.

We are interested in a multiplicity result for the T-periodic solutions of (3.1) when $\lambda > 0$ is small. Namely, our main result is the following:

Theorem 3.1. *Let h, F and η be as above. Assume that $q_1, \ldots, q_{n-1} \in h^{-1}(0)$ are non T-resonant for* (3.1)*. Assume also that*

$$\sum_{i=1}^{n-1} \mathrm{i}(h, q_i) \neq \chi(\boldsymbol{S}) = 2$$

Then, for $\lambda > 0$ sufficiently small, equation (3.1) *admits at least n, T-periodic solutions whose images are pairwise not coincident.*

In the gravitational case $h = h_g$ there are two zeros of h_g, the "north" $(0, 0, r)$ and the "south" $(0, 0, -r)$ poles. As it follows from (3.5) (see also the appendix), the north pole is necessarily not T-resonant. Therefore we have the following important consequence of Theorem 3.1:

Theorem 3.2. *When $h = h_g$, for $\lambda > 0$ sufficiently small, equation* (3.1) *admits at least two T-periodic solutions whose images are not coincident.*

We wish to emphasize the fact that in Theorem 3.2 no assumption is made on the T-resonance properties of the south pole. Indeed, this result holds even in absence of friction ($\eta = 0$) and when the period for small oscillations about the south pole is T. Because of this peculiarity, one has that Theorem 3.2 is not a mere consequence of the implicit function theorem.

To prove Theorem 3.1 we need some notions and results taken mostly from [14, 16, 18, 19]. In what follows, we will mainly work with equation (3.2). First we recall a result, Theorem 3.4 below, which concerns the existence of a "global bifurcating branch" for (3.2). We need some preliminary notions.

We will denote by $C_T(T\boldsymbol{S})$ the set of the T-periodic continuous maps from $\mathbb{R}$ into $T\boldsymbol{S}$. This will be regarded as a metric subspace of the Banach space $C_T(\mathbb{R}^6)$ of the T-periodic continuous maps from $\mathbb{R}$ into $\mathbb{R}^6$ with the usual supremum norm. Observe in particular that, $T\boldsymbol{S}$ being complete, so is the metric space $C_T(T\boldsymbol{S})$.

A pair $(\lambda, \xi) \in [0, +\infty) \times C_T(T\boldsymbol{S})$, where ξ is a solution of (3.2), is called a *T-periodic pair* (for (3.2)). Those T-periodic pairs that are of the particular form $(0, \overline{\zeta})$, $\overline{\zeta}$ being the map constantly equal to ζ, are said to be *trivial*. Observe that any T-periodic pair $(0, \overline{\zeta})$ is trivial if and only if $\zeta = (q, 0)$ with $h(q) = 0$.

The following immediate consequence of Lemma 3.1 of [4] expresses a crucial property of the set of T-periodic pairs.

Lemma 3.3. *The set of T-periodic pairs for* (3.2) *is closed and locally compact.*

For the sake of simplicity, we will identify $T\boldsymbol{S}$ with its image in $[0, +\infty) \times C_T(T\boldsymbol{S})$ under the embedding which associates to any $\zeta \in T\boldsymbol{S}$ the pair $(0, \overline{\zeta})$. In particular, given $q \in \boldsymbol{S}$, according to our convention $(q, 0)$ can be seen as an element of $[0, \infty) \times C_T(T\boldsymbol{S})$. Moreover, with a slight abuse of notation, if Ξ is a subset of $[0, +\infty) \times C_T(T\boldsymbol{S})$, by $\Xi \cap \boldsymbol{S}$ we mean the subset of $\boldsymbol{S}$ given by all $q \in \boldsymbol{S}$ such that the pair $(0, \overline{(q, 0)})$ belongs to Ξ. Observe that if $\Omega \subseteq [0, +\infty) \times C_T(T\boldsymbol{S})$ is open, then $\Omega \cap \boldsymbol{S}$ is open in $\boldsymbol{S}$.

We need the following consequence of [19, Lemma 3.2] and [16, Theorem 4.1].

Theorem 3.4. *Let h, κ, $\hat{h}$, F and $\hat{F}$ be as above. Let Ω be an open subset of* $[0, +\infty) \times C_T(T\boldsymbol{S})$*, and assume that* $\deg(h, \Omega \cap \boldsymbol{S})$ *is defined and nonzero. Then Ω contains a connected set Γ of nontrivial T-periodic pairs for* (3.2) *whose closure in Ω is not compact and meets the set of trivial T-periodic pairs.*

Proof. By assumption, h is admissible on $\Omega \cap \boldsymbol{S}$. Now, taking into account that $\kappa|_{\boldsymbol{S}} = 0$, we get that the vector field

$$\frac{h}{m} - \eta\kappa|_{\boldsymbol{S}} = \frac{h}{m}$$

is admissible on $\Omega \cap \boldsymbol{S}$ as well. Consequently, by [19, Lemma 3.2], it follows that $\hat{h} - \eta\kappa$ is admissible on $\Omega \cap T\boldsymbol{S}$ and

$$\begin{aligned}\deg(\hat{h} - \eta\kappa, \Omega \cap T\boldsymbol{S}) &= \deg\Big(-\left(\hat{h} - \eta\kappa\right)\Big|_{\boldsymbol{S}}, \Omega \cap \boldsymbol{S}\Big) \\ &= \deg\left(-\Big(\frac{h}{m} - \eta\kappa|_{\boldsymbol{S}}\Big), \Omega \cap \boldsymbol{S}\right) = \deg\left(-\frac{h}{m}, \Omega \cap \boldsymbol{S}\right).\end{aligned}$$

Since

$$\deg(-h/m, \Omega \cap \boldsymbol{S}) = (-1)^{\dim \boldsymbol{S}} \deg(h, \Omega \cap \boldsymbol{S}) = \deg(h, \Omega \cap \boldsymbol{S}) \neq 0,$$

we have that $\deg(\hat{h} - \eta\kappa, \Omega \cap T\boldsymbol{S}) \neq 0$ and the assertion follows directly from [16, Theorem 4.1]. □

Remark 3.5. *Let Ω and Γ be as in Theorem 3.4. Assume that Υ is the connected component in $[0,\infty) \times C_T(T\boldsymbol{S})$ of the set of T-periodic pairs for* (3.2) *that contains Γ. The Theorem of Ascoli-Arzelà implies that any bounded set of T-periodic pairs is relatively compact. Then, the closed set Υ cannot be both bounded and contained in Ω. In particular, if $\Upsilon \cap \mathrm{Fr}\,(\Omega) = \emptyset$ then Υ cannot be bounded (compare [16, Remark 4.2]). Here and in the sequel the symbol* $\mathrm{Fr}\,(\cdot)$ *denotes the boundary.*

The following crucial result, that will play a key role in our argument, is a generalization of [18, Lemma 3.3], see also Lemma 3.1 in [14].

Lemma 3.6. *Let h, κ, $\hat{h}$, F and $\hat{F}$ be as above. Assume that $(q,0)$ is an isolated zero of $\hat{h}$. Then, for any sufficiently small neighborhood V of $\overline{(q,0)}$ in $C_T(T\boldsymbol{S})$ there exists a real number $\delta_V > 0$ such that $[0,\delta_V] \times \mathrm{Fr}\,(V)$ does not contain any T-periodic pair of* (3.2)*.*

In order to give the proof of this lemma we need to recall some notions. A multivalued map $\phi\colon \mathcal{X} \multimap \mathcal{Y}$ between two metric spaces is said to be *upper semicontinuous* if it has compact (possibly empty) values and for any open subset V of $\mathcal{Y}$ the upper inverse image of V, i.e. the set $\phi^{-1}(V) = \{x \in \mathcal{X} : \phi(x) \subseteq V\}$, is an open subset of $\mathcal{X}$.

The following remark will be used in the proof of Lemma 3.6.

Remark 3.7. *Given a compact subset K of $\mathcal{X} \times \mathcal{Y}$, the multivalued map that associates to $x \in \mathcal{X}$ the slice K_x (whose graph is K) is upper semicontinuous. To see this, let V be any open subset of $\mathcal{Y}$ and assume, by contradiction, that the set $U = \{x \in \mathcal{X} : K_x \subseteq V\}$ is not open. Then, there exists a sequence $\{x_n\}$ in $\mathcal{X} \backslash U$ which converges to some $x_0 \in U$. For any $n \in \mathbb{N}$, choose $y_n \in K_{x_n} \cap (\mathcal{Y} \backslash V)$. Because of the compactness of K, we may assume $(x_n, y_n) \to (x_0, y_0) \in K$. Thus, y_0 belongs to K_{x_0} which is a subset of V, contradicting the fact that y_0 also belongs to the closed set $\mathcal{Y} \backslash V$.*

Proof of Lemma 3.6. Let $X \subseteq [0,+\infty) \times C_T(T\boldsymbol{S})$ denote the set of the T-periodic pairs of (3.2) and let X_0 be the slice of X at $\lambda = 0$. Since $(q,0)$

is isolated, there exists an open neighborhood V of $\overline{(q,0)}$ in $C_T(T\boldsymbol{S})$ such that the closure $\mathrm{cl}\,(V \cap X_0) = \{\overline{(q,0)}\}$. By Lemma 3.3, X is locally compact. Hence, there exists an open neighborhood W of $\overline{(q,0)}$ in $C_T(T\boldsymbol{S})$ and a number $\mu > 0$ such that $([0,\mu] \times \mathrm{cl}\,(W)) \cap X$ is compact. By restricting V, if necessary, we may assume that $\mathrm{cl}\,(V) \subseteq W$. By Remark 3.7, the multimap $\Psi\colon [0,\mu] \multimap \mathrm{cl}\,(W)$ given by $\Psi(\lambda) = X_\lambda \cap \mathrm{cl}\,(W)$ is upper semicontinuous. Thus, since $\Psi(0) = \{\overline{(q,0)}\} \subseteq V$, there exists $\delta_V > 0$ such that $\Psi([0,\delta_V]) \subseteq V$. Whence the assertion. $\square$

Lemma 3.8 below shows, roughly speaking, that the condition of T-resonance is strictly related to the notion of ejecting set. In other words, if $q \in h^{-1}(0)$ is not T-resonant for (3.1), then $\{\overline{(q,0)}\}$ can be regarded as an ejecting set for the set X of the T-periodic pairs of (3.2).

Lemma 3.8. *Let $(q,0)$ be a zero of $\hat{h} - \eta\kappa$ which is not T-resonant. Then, $\{\overline{(q,0)}\}$ is an ejecting set for the set of T-periodic pairs of* (3.2).

Proof. Since $(q,0)$ is a zero of $\hat{h} - \eta\kappa$ which is not T-resonant, then it is an isolated zero of $\hat{h} - \eta\kappa$ and, thus, of $\hat{h}$. Hence, the set $\{\overline{(q,0)}\}$ is relatively open in the slice X_0 of the set X of T-periodic pairs of (3.2). Now, Remark 2.2 applied to $\hat{h} - \eta\kappa$ ensures the existence of a small neighborhood U of $(q,0)$ in $T\boldsymbol{S}$ such that

$$\deg(\hat{h} - \eta\kappa, U) \neq 0. \tag{3.6}$$

Set

$$\Omega = \{(\lambda,\xi) \in [0,+\infty) \times C_T(T\boldsymbol{S}) : \xi(t) \in U \text{ for all } t \in \mathbb{R}\}.$$

Clearly, Ω is an open set and, because of the previous identifications, $\Omega \cap T\boldsymbol{S} = U$. Hence,

$$\deg(\hat{h} - \eta\kappa, U) = \deg(\hat{h} - \eta\kappa, \Omega \cap T\boldsymbol{S}). \tag{3.7}$$

As in the proof of Theorem 3.4, we have that $\hat{h} - \eta\kappa$ is admissible on $\Omega \cap T\boldsymbol{S}$ and $\deg(\hat{h} - \eta\kappa, \Omega \cap T\boldsymbol{S}) = \deg(h, \Omega \cap \boldsymbol{S})$. Thus, using (3.6) and (3.7), we get $\deg(h, \Omega \cap \boldsymbol{S}) \neq 0$. Theorem 3.4 applies yielding the existence in Ω of a connected set Γ_q of nontrivial T-periodic pairs whose closure in Ω is not compact and meets the set of trivial T-periodic pairs for (3.2). By Lemma 3.6, Γ_q cannot be contained in X_0. This completes the proof. $\square$

Our main result will be deduced from the following fact concerning equation (3.2):

Theorem 3.9. *Let h, κ, $\hat{h}$, F and $\hat{F}$ be as above. Assume that $(q_1, 0), \dots, (q_{n-1}, 0)$ are zeros of $\hat{h} - \eta\kappa$ which are not T-resonant. Assume in addition that*

$$\sum_{i=1}^{n-1} \mathrm{i}(h, q_i) \neq \chi(\boldsymbol{S}) = 2.$$

Then, for $\lambda > 0$ sufficiently small, equation (3.2) *admits at least n, T-periodic solutions whose projections on the base space $\boldsymbol{S}$ are pairwise not coincident. Thus, in particular, their images are pairwise not coincident.*

Proof. Observe first that the points $(q_1, 0), \dots, (q_{n-1}, 0)$, being not T-resonant zeros of $\hat{h} - \eta\kappa$, are isolated zeros of $\hat{h} - \eta\kappa$ and, thus, of $\hat{h}$. As previously, denote by X the set of T-periodic pairs for (3.2). Lemma 3.6 implies, in particular, that $\overline{(q_i, 0)}$, $i = 1, \dots, n-1$, are isolated points of X_0.

By Lemma 3.8, the sets $\{\overline{(q_i, 0)}\}$, $i = 1, \dots, n-1$, are ejecting for X. Our aim is to apply Theorem 2.1. To this end, we need to prove the existence in the slice X_0 of a further (not necessarily compact) ejecting set.

Let $W_1, \dots, W_{n-1}$ be pairwise disjoint open neighborhoods of $\overline{(q_1, 0)}, \dots, \overline{(q_{n-1}, 0)}$ in $C_T(T\boldsymbol{S})$, respectively, with the property that

$$\mathrm{cl}\,(W_i) \cap X_0 = \left\{\overline{(q_i, 0)}\right\} \quad \text{for } i = 1, \dots, n-1. \tag{3.8}$$

Set

$$\Omega = [0, \infty) \times \left(C_T(T\boldsymbol{S}) \setminus \bigcup_{i=1}^{n-1} \mathrm{cl}\,(W_i) \right).$$

By the Poincaré-Hopf Theorem and the additivity property of the degree, we get

$$\deg(h, \Omega \cap \boldsymbol{S}) = \chi(\boldsymbol{S}) - \deg\left(h, \bigcup_{i=1}^{n-1} (W_i \cap S)\right) = 2 - \sum_{i=1}^{n-1} \mathrm{i}(h, q_i) \neq 0.$$

Thus, Theorem 3.4 yields the existence of a connected set $\Gamma \subseteq \Omega$ of nontrivial T-periodic pairs for (3.2) whose closure in Ω is not compact and meets the set of trivial T-periodic pairs. Let Υ be the connected component of X containing Γ.

We claim that Υ_0, which is obviously relatively open in X_0, is an ejecting set. To see this, it is sufficient to show that Υ is not contained in $\{0\} \times X_0$. Assume by contradiction that this is not the case, that is, assume that $\Upsilon = \{0\} \times \Upsilon_0$. Now, observe that any connected set of solutions in $C_T(T\boldsymbol{S})$ of the equation (3.2) for $\lambda = 0$ is bounded. Indeed, Lemma 4.1 in [18] shows that any connected set of solutions in $C^1_T(\boldsymbol{S})$ of the second-order equation (3.1) for $\lambda = 0$ is unbounded. Here $C^1_T(\boldsymbol{S})$ denotes the subset of the Banach space $C^1_T(\mathbb{R}^3)$ of the T-periodic C^1 maps from $\mathbb{R}$ into $\mathbb{R}^3$ with the induced topology; as well known, the Banach space $C^1_T(\mathbb{R}^3)$ is isometric to a subset of $C_T(\mathbb{R}^6)$. Since the two equations are equivalent, it follows that any connected set of solutions in $C_T(T\boldsymbol{S})$ of (3.2) for $\lambda = 0$ is bounded as well and, in particular, so is Υ. By the contradictory hypothesis and (3.8) we have

$$\Upsilon \cap \operatorname{Fr}(\Omega) = \{0\} \times (\Upsilon_0 \cap \operatorname{Fr}(\Omega_0)) \subseteq \{0\} \times \left(X_0 \cap \left(\bigcup_{i=1}^{n-1} \operatorname{Fr}(W_i) \right) \right) = \emptyset.$$

Remark 3.5 yields a contradiction and the claim is proved.

So far we have proved that the subsets of X_0

$$E_1 := \{\overline{(q_1, 0)}\}, \ \ldots, \ E_{n-1} := \{\overline{(q_{n-1}, 0)}\}, \ E_n := \Upsilon_0$$

are indeed n ejecting sets, the first $n-1$ of which are compact.

By Theorem 2.1, there exists $\lambda_* > 0$ such that for $\lambda \in [0, \lambda_*)$ there are in $C_T(T\boldsymbol{S})$ open neighborhoods $U_1, \ldots, U_n$ of $E_1, \ldots, E_n$ with pairwise disjoint closure and such that each $\{\lambda\} \times U_i$ contains at least one T-periodic pair, say (λ, ξ_i^λ) for any $\lambda \in [0, \lambda_*)$. Put $\xi_i^\lambda = (x_i^\lambda, y_i^\lambda)$ for $i = 1, \ldots, n$. By Lemmas 3.6 and 3.8, reducing λ_* if necessary, we can assume that, for $i = 1, \ldots, n-1$, U_i is a ball in $C_T(T\boldsymbol{S})$ centered at $\overline{(q_i, 0)}$ having radius strictly smaller than

$$\mathfrak{r} := \frac{1}{2} \min_{1 \le j < k \le n-1} \left| q_j - q_k \right|.$$

So that, for $j, k = 1, \ldots, n-1$, the images of ξ_j^λ and ξ_k^λ are disjoint if $j \neq k$. Indeed, since these images are, respectively, confined to the balls $B\big((q_j, 0), \mathfrak{r}\big)$ and $B\big((q_k, 0), \mathfrak{r}\big)$, whose projections onto the base space $\boldsymbol{S}$ are disjoint, we have that of x_j^λ and x_k^λ have disjoint images if $j \neq k$. So far, concerning ξ_n^λ, we can only say this: since ξ_n^λ is contained in $U_n \subseteq C_T(T\boldsymbol{S}) \setminus (U_1 \cup \ldots \cup U_{n-1})$, its image must contain, for each $i = 1, \ldots, n-1$, at least one point that lies outside the ball $B\big((q_i, 0), \mathfrak{r}\big)$. Hence its image cannot coincide with any of those of $\xi_1^\lambda, \ldots, \xi_{n-1}^\lambda$.

In order to conclude the proof let us show that, reducing $\lambda_* > 0$ if necessary, we have that for $\lambda \in [0, \lambda_*)$ the projection x_n^λ of ξ_n^λ on $\boldsymbol{S}$ cannot coincide with that of any of the other solutions $\xi_1^\lambda, \dots, \xi_{n-1}^\lambda$. Assume the contrary. Then, there exists a sequence $\{\lambda_s\}_{s\in\mathbb{N}}$, with $\lambda_s \searrow 0$, such that the image of $x_n^{\lambda_s}$ coincides with at least one of the images of $x_1^{\lambda_s}, \dots, x_{n-1}^{\lambda_s}$ (not necessarily the same for all λ_s's). Since the images of these solutions are disjoint, selecting a subsequence and reordering the solutions, we can assume that $x_n^{\lambda_s}([0,T]) = x_1^{\lambda_s}([0,T])$ for all s. Letting $s \to \infty$, we have that $x_n^{\lambda_s}(t)$ converges uniformly on $[0,T]$ to the constant function $t \mapsto q_1$.

We claim that $(x_n^{\lambda_s})'(t)$ converges to zero uniformly on $[0,T]$. To see this, let d_s be the diameter of the orbit of $x_n^{\lambda_s}$, that is, $d_s := \max_{t_1,t_2\in[0,T]} \left|x_n^{\lambda_s}(t_1) - x_n^{\lambda_s}(t_2)\right|$. Thus, clearly, $d_s \to 0$ as $s \to \infty$. Lemma 4.1 in [6] implies that

$$\left(\max_{t\in[0,T]} \left|(x_n^{\lambda_s})'(t)\right|\right)^2 \leq \frac{d_s}{1 - d_s/r} \max_{t\in[0,T]} \left|(x_n^{\lambda_s})''_\pi(t)\right|, \tag{3.9}$$

where, given $t \in [0,T]$, $(x_n^{\lambda_s})''_\pi(t)$ denotes the projection onto the tangent space of $\boldsymbol{S}$ at $x_n^{\lambda_s}(t)$ of the acceleration of $(x_n^{\lambda_s})''(t)$. In other words, $(x_n^{\lambda_s})''_\pi(t)$ is the tangential component of $(x_n^{\lambda_s})''(t)$.

Now, observe that if ξ is a T-periodic solution of (3.2) with $\xi(t) = (x(t), y(t))$, then x is a T-periodic solution of (3.1) with $x'(t) = y(t)$. Hence, so is $x_n^{\lambda_s}$. Thus,

$$\max_{t\in[0,T]} \left|(x_n^{\lambda_s})''_\pi(t)\right| \leq \frac{1}{m}\Bigg(\eta \max_{t\in[0,T]} \left|(x_n^{\lambda_s})'(t)\right| + \\ + \max_{t\in[0,T]} \left|h\big(x_n^{\lambda_s}(t)\big)\right| + \lambda_s \max_{t\in[0,T]} \left|F\big(t, (x_n^{\lambda_s})_t\big)\right|\Bigg).$$

Since $x_n^{\lambda_s}(t) \to q_1$ as $s \to \infty$ and $h(q_1) = 0$, by Lemma 2.2 in [6] we can assume that that there exists a positive constant C that bounds from above the sum of the last two terms in the parenthesis. So that we have

$$\max_{t\in[0,T]} \left|(x_n^{\lambda_s})''_\pi(t)\right| \leq \frac{1}{m}\Big(\eta \max_{t\in[0,T]} \left|(x_n^{\lambda_s})'(t)\right| + C\Big). \tag{3.10}$$

By (3.9) and (3.10) we get, for s large enough,

$$\begin{aligned}\left(\max_{t\in[0,T]} \left|(x_n^{\lambda_s})'(t)\right|\right)^2 &\leq \frac{d_s}{m(1 - d_s/r)}\Big(\eta \max_{t\in[0,T]} \left|(x_n^{\lambda_s})'(t)\right| + C\Big) \\ &\leq 2\frac{d_s}{m}\Big(\eta \max_{t\in[0,T]} \left|(x_n^{\lambda_s})'(t)\right| + C\Big),\end{aligned}$$

whence,

$$\max_{t\in[0,T]} \left|(x_n^{\lambda_s})'(t)\right| \le \frac{d_s\eta}{m} + \sqrt{\left(\frac{d_s\eta}{m}\right)^2 + 2\frac{d_sC}{m}}.$$

This implies that, as $s \to \infty$, $\left|(x_n^{\lambda_s})'(t)\right| \to 0$ uniformly on $[0,T]$, as claimed.

Thus, $\xi_n^{\lambda_s} \to \overline{(q_1, 0)}$ so that, eventually, $\xi_n^{\lambda_s} \in U_1$. This is impossible since $\xi_n^{\lambda_s} \in U_n$ and $U_1 \cap U_n = \emptyset$. □

Finally, we are in a position to prove our main multiplicity result.

Proof of Theorem 3.1. As observed previously, if ξ is a T-periodic solution of (3.2) with $\xi(t) = (x(t), y(t))$, then x is a T-periodic solution of (3.1) with $x'(t) = y(t)$. Thus, solutions of (3.2) with different images (in $T\boldsymbol{S}$) yield solutions of (3.1) that have different images (in $\boldsymbol{S}$) as well.

By Definition 3.1, the assumption on the points q_i means that all the points $(q_i, 0)$ are not T-resonant for $\hat{h} - \eta\kappa$. The assertion now follows from Theorem 3.9. □

Appendix A.

In this appendix we are merely concerned with ODEs. Thus, for simplicity, we will write all the equations without the explicit dependence on t.

Let $M \subseteq \mathbb{R}^k$ be a smooth manifold. Consider the following second order ordinary differential equation on M

$$x'' = R(x, x') + f(x, x') \tag{A.1}$$

where $f\colon TM \to \mathbb{R}^k$ is tangent to M, that is, $f(q, v) \in T_qM$ for all $(q, v) \in TM$, and $R\colon TM \to \mathbb{R}^k$ is the reactive force of the constraint M. Namely, R is the unique function as above with the property that $R(q, v) \in TM^\perp$ for all $(q, v) \in TM$, R is quadratic in v and $\hat{f}(q, v) := (v, R(q, v) + f(q, v))$ is tangent to TM (see e.g. [11, 14]). Indeed, (A.1) is equivalent to the following first order ODE on TM:

$$\xi' = \hat{f}(\xi) \tag{A.2}$$

where $\xi = (q, v)$.

Assume now that f is C^1. Let $q_0 \in M$ be such that $f(q_0, 0) = 0$ and let $\xi_0 = (q_0, 0)$. Since R is quadratic in the second variable, we get that equation (A.2) linearized about ξ_0 is the equation on $T_{\xi_0}TM$

$$\zeta' = \hat{f}'(\xi_0)\zeta \tag{A.3}$$

which, in turn, is equivalent to a second order equation on $T_{q_0}M$, namely

$$z'' = \partial_1 f(q_0, 0)z + \partial_2 f(q_0, 0)z'. \tag{A.4}$$

The above argument applied to $M = \boldsymbol{S}$ and equation (3.1) for $\lambda = 0$ yields the linear equation (3.4). Thus, a point $q_0 \in h^{-1}(0)$ is not T-resonant for (3.1) if and only if the second order equation on the tangent plane $T_{q_0}\boldsymbol{S}$

$$mz'' = h'(q_0)z - \eta z' \tag{A.5}$$

has the constant $z(t) \equiv 0$ as its unique T-periodic solution. Observe that, in particular, when all the eigenvalues of $h'(q_0)$ are positive the unique periodic solution is $z(t) \equiv 0$. Thus, in this case, q_0 is not T-resonant. Consequently, if $h = h_g$ we have that the north pole of $\boldsymbol{S}$ cannot be T-resonant.

References

[1] V. Benci, *Periodic solutions of Lagrangian systems on a compact manifold.* J. Differential Equations, **63** (1986), 135–161.

[2] V. Benci and M. Degiovanni, *Periodic solutions of dissipative dynamical systems.* Variational methods (Paris, 1988), 395–411, Progr. Nonlinear Differential Equations Appl., 4, Birkhäuser Boston, Boston, MA, 1990.

[3] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *On the existence of forced oscillations for the spherical pendulum acted on by a retarded periodic force*, J. Dyn. Diff. Equat. **23** (2011), 541–549.

[4] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *Global continuation of periodic solutions for retarded functional differential equations on manifolds*, Boundary Value Problems (2013), Art. ID 21, 19 pp.

[5] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *On general properties of retarded functional differential equations on manifolds*, Discrete Contin. Dyn. Syst. **33** (2013), no. 1, 27–46.

[6] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *Global continuation of forced oscillations of retarded motion equations on manifolds*, J. Fixed Point Theory Appl. **16** (2014), no. 1-2, 273–300.

[7] L. Bisconti and M. Spadini, *About the notion of non-T-resonance and applications to topological multiplicity results for ODEs on differentiable manifolds*, Math. Methods Appl. Sci. 38 (2015), no. 18, 4760–4773. DOI: 10.1002/mma.3390

[8] A. Boscaggin, R. Ortega and F. Zanolin, *Subharmonic solutions of the forced pendulum equation: a symplectic approach*, Arch. Math. (Basel) 102 (2014), no. 5, 459–468.

[9] J. Cronin, *Differential Equations, Introduction and Qualitative Theory 2nd Ed.*, Pure and Appl. Math 180. Marcel Dekker Inc.: New York,1994.

[10] A. Fonda and R. Toader, *Periodic solutions of pendulum-like Hamiltonian systems in the plane*, Adv. Nonlinear Stud. 12 (2012), 395–408.

[11] M. Furi, *Second order differential equations on manifolds and forced oscillations*, in: Proc. Conference on 'Topological methods in Differential Equations and Inclusions', Université de Montréal, in: NATO-ASI Series C, Kluwer Acad. Publ., Dordrecht, NL, 1994.

[12] M. Furi and M.P. Pera, *The Forced Spherical Pendulum does have Forced Oscillations*, Delay Differential Equations and Dynamical Systems, Busenberg S. and Martelli M. Editors, Lecture Notes in Math. 1475, Springer Verlag, Berlin, 1991, 176–183, Proceedings of the Conference on Fixed-Point Theory and Applications.

[13] M. Furi and M.P. Pera, *On the Existence of Forced Oscillations for the Spherical Pendulum*, Boll. Un. Mat. Ital. Ser. VII **4-B** (1990), 381–390.

[14] M. Furi, M.P. Pera and M. Spadini, *Forced oscillations on manifolds and multiplicity results for periodically perturbed autonomous systems*, J. Computational and App. Mathematics 113 (2000), 241–254.

[15] M. Furi, M.P. Pera, M. Spadini, *The fixed point index of the Poincaré operator on differentiable manifolds*, Handbook of topological fixed point theory, Brown R. F., Furi M., Górniewicz L., Jiang B. (Eds.), Springer, 2005.

[16] M. Furi, M.P. Pera and M. Spadini, *Periodic solutions of retarded functional perturbations of autonomous differential equations on manifolds*, Commun. Appl. Anal. 15 (2011), 381–394.

[17] M. Furi and M. Spadini, *On the set of harmonic solutions of periodically perturbed autonomous differential equations on manifolds*, Nonlinear Analysis TMA Vol. 29 (1997) no. 8., 963–970.

[18] M. Furi and M. Spadini, *Multiplicity of forced oscillations for the spherical pendulum,* Topological Methods in Nonlinear Analysis **11** (1998), 147–157.

[19] M. Furi and M. Spadini, *Branches of forced oscillations for periodically perturbed autonomous second order ODE's on manifolds*, J. of Differential Equations 154 (1999), 96–106.

[20] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, New York, 1993.

[21] G. Katriel, *Many periodic solutions for pendulum-type equations*, Nonlinear Anal. 34 (1998), no. 5, 687–699.

[22] M. Lewicka and M. Spadini, *On the genericity of the multiplicity results for forced oscillations on compact manifolds*, NoDEA Nonlinear Differential Equations Appl. 6 (1999), no. 4, 357–369.

[23] J. Mawhin, *Seventy-five years of global analysis around the forced pendulum equation*, In: R.P. Agarwal, F. Neuman and J. Vosmanský (eds.): Proceedings of Equadiff 9, Conference on Differential Equations and Their Applications, Brno, August 25-29, 1997, Survey papers. Masaryk University, Brno, 1998. pp. 115–145.

[24] J. Mawhin, *Global results for the forced pendulum equations*, Handbook on Differential Equations. Ordinary Differential Equations, A. Cañada, P. Drábek, A. Fonda eds., vol. 1, Elsevier, Amsterdam, 2004, 533–589.

[25] J. Mawhin, *The implicit function theorem and its substitutes in Poincaré's qualitative theory of differential equations*, Studies in History and Philosophy of Science. Part B. Studies in History and Philosophy of Modern Physics 2014; 47:124–130.

[26] J. Mawhin and M. Willem, *Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations*, J. Differential Equations 52 (1984), no. 2, 264–287.

[27] J.W. Milnor, *Topology from the differentiable viewpoint*, Univ. press of Virginia, Charlottesville, 1965.

[28] I. Polekhin, *Forced oscillations of a massive point on a compact surface with a boundary*, Nonlinear Anal. 128 (2015), 100–105.

[29] A.J. Ureña, *Using closed orbits to bifurcate many periodic solutions for pendulum-type equations*, J. Math. Anal. Appl. 302 (2005), no. 2, 318–341.