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Negative Orlicz-Sobolev norms and strongly nonlinear systems in fluid mechanics

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Abstract

We prove a version of the negative norm theorem in Orlicz-Sobolev spaces. A study of continuity properties of the Bogovskiĭ operator between Orlicz spaces is a crucial step, of independent interest, in our approach. Applications to the problem of pressure reconstruction for Non-Newtonian fluids governed by constitutive laws, which are not necessarily of power type, are presented. A key inequality for a numerical analysis of the underlying elliptic system is also derived.

1 Introduction

Assume that Ω is a domain, namely a connected open set, in $\mathbb{R}^n$, with $n \geq 2$, and let $1 \leq p \leq \infty$. The negative Sobolev norm of the distributional gradient of a function $u \in L^1(\Omega)$ can be defined as

$$\|\nabla u\|_{W^{-1,p}(\Omega,\mathbb{R}^n)} = \sup_{\boldsymbol{\varphi} \in C_0^\infty(\Omega,\mathbb{R}^n)} \frac{\int_\Omega u \operatorname{div} \boldsymbol{\varphi}\, dx}{\|\nabla \boldsymbol{\varphi}\|_{L^{p'}(\Omega,\mathbb{R}^{n\times n})}}\, dx\,. \tag{1.1}$$

Here, *div* stands for the divergence operator, and $p' = \frac{p}{p-1}$, the Hölder conjugate of p. Moreover, in (1.1), and in similar occurrences throughout the paper, we tacitly assume that

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the sumpremum is extended over all functions $\mathbf{v}$ which do not vanish identically. Observe that the notation $\|\nabla u\|_{W^{-1,p}(\Omega,\mathbb{R}^n)}$ is consistent with the fact that the quantity on the right-hand side of (1.1) agrees with the norm of ∇u, when regarded as an element of the dual of $W_0^{1,p'}(\Omega)$, where $W_0^{1,p'}(\Omega,\mathbb{R}^n)$ denotes the Sobolev space of $\mathbb{R}^n$-valued functions in Ω with zero traces.
Definition (1.1) goes back to Nečas [44], who showed that, if Ω is regular enough – a bounded Lipschitz domain, say – and $1 < p < \infty$, then the $L^p(\Omega)$ norm of a function is equivalent to the $W^{-1,p}(\Omega,\mathbb{R}^n)$ norm of its gradient. Namely, there exist positive constants $C_1 = C_1(\Omega,p)$ and $C_2 = C_2(n)$, such that

$$C_1\|u - u_\Omega\|_{L^p(\Omega)} \leq \|\nabla u\|_{W^{-1,p}(\Omega,\mathbb{R}^n)} \leq C_2\|u - u_\Omega\|_{L^p(\Omega)} \tag{1.2}$$

for every $u \in L^1(\Omega)$, where

$$u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx$$

the mean value of u over Ω, and $|\Omega|$ denotes the Lebesgue measure of Ω. This result is known as Nečas negative norm theorem.

In the present paper we are concerned with a version of this theorem when a negative norm is introduced with the Lebesgue space $L^p(\Omega)$ replaced with a general Orlicz space. Loosely speaking, Orlicz spaces extend Lebesgue spaces in that the role of the power t^p in their definition is played by a more general convex function (a precise definition is recalled in the next section). Clearly, a full analogue of (1.2) cannot hold for arbitrary Orlicz spaces, since (1.2) fails, for instance, in the borderline cases when either $p = 1$, or $p = \infty$. Our main result in this connection asserts that, however, an inequality in the spirit of (1.2) still holds if, on the leftmost side, an Orlicz norm appears which, in general, has to be slightly weaker than that on the rightmost side. A precise balance between the relevant norms for a conclusion of this kind to hold is the content of Theorem 3.1, Section 3. A key step in our approach is an analysis, of possible independent interest, of the divergence equation in Orlicz spaces, via boundedness properties of the (gradient of the) Bogovskiĭ operator in these spaces.

Our main motivation for a discussion of negative Orlicz norms are applications to a mathematical model for Non-Newtonian fluids. In the stationary case, the relevant model tells us that the velocity field $\mathbf{v} : \Omega \to \mathbb{R}^n$ and the pressure $\pi : \Omega \to \mathbb{R}$ of a fluid solve the following system of partial differential equations:

$$\begin{cases} -\operatorname{div}\mathbf{S} + \varrho\operatorname{div}\big(\mathbf{v}\otimes\mathbf{v}\big) + \nabla\pi = \varrho\operatorname{div}\mathbf{F} & \text{in } \Omega,\\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega,\\ \mathbf{v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

see, for instance, [5]. Here, ϱ is a positive constant, whose physical meaning is the density of the fluid, the operation $\otimes$ denotes tensor product, the function $\mathbf{F} : \Omega \to \mathbb{R}^{n\times n}$ describes the given volume forces, and the stress deviator $\mathbf{S} : \Omega \to \mathbb{R}^{n\times n}$ is related to $\mathbf{v}$ via a constitutive equation. Of course, the physically relevant dimensions are $n = 2$ and $n = 3$. In a common model for fluids with Non-Newtonian behavior, the dependence of $\mathbf{S}$ on $\mathbf{v}$ is through a nonlinear function of the symmetric part $\mathbf{D}(\mathbf{v})$ of its $\mathbb{R}^{n\times n}$-valued gradient, defined as $\mathbf{D}(\mathbf{v}) = \frac{1}{2}\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\big)$, where "$(\cdot)^T$" stands for transpose. Lebesgue and usual Sobolev spaces provide an appropriate functional framework for the study of existence, uniqueness and regularity of solutions to (1.3) when this nonlinear function is of power type. On the other hand, if nonlinearities of non-polynomial type are allowed, such as in the Eyring-Prandtl model [32], the more flexible Orlicz and Orlicz-Sobolev spaces have to be called into play. In particular, in Section 4 we show how our negative Orlicz norm theorem applies in the description of a suitable space for the pressure π in (1.3). A related result of use in a finite elements method for a numerical analysis of (1.3) is also established.

2 Orlicz and Orlicz-Sobolev spaces

In the present section we collect some definitions and fundamental results from the theory of Orlicz and Orlicz-Sobolev spaces. We refer to [45, 46] for a comprehensive treatment of this topic.

A function $A : [0,\infty) \to [0,\infty]$ is called a Young function if it is convex, left-continuous, and neither identically equal to 0, nor to ∞. Thus, with any such function, it is uniquely associated a (nontrivial) non-decreasing left-continuous function $a : [0,\infty) \to [0,\infty]$ such that

$$A(s) = \int_0^s a(r)\,dr \qquad \text{for } s \geq 0. \tag{2.1}$$

The Young conjugate $\widetilde{A}$ of A is the Young function defined by

$$\widetilde{A}(s) = \sup\{rs - A(r) : \, r \geq 0\} \qquad \text{for} \qquad s \geq 0\,.$$

Note the representation formula

$$\widetilde{A}(s) = \int_0^s a^{-1}(r)\,dr \qquad \text{for } s \geq 0,$$

where a^{-1} denotes the (generalized) left-continuous inverse of a. One has that

$$r \leq A^{-1}(r)\widetilde{A}^{-1}(r) \leq 2r \quad \text{for } r \geq 0. \tag{2.2}$$

Moreover,

$$\widetilde{\widetilde{A}} = A \tag{2.3}$$

for any Young function A. If A is any Young function and $\lambda \geq 1$, then

$$\lambda A(s) \leq A(\lambda s) \quad \text{for } s \geq 0. \tag{2.4}$$

As a consequence, if $\lambda \geq 1$, then

$$A^{-1}(\lambda s) \leq \lambda A^{-1}(s) \quad \text{for } s \geq 0, \tag{2.5}$$

where A^{-1} denotes the (generalized) right-continuous inverse of A.
A Young function A is said to satisfy the Δ_2-condition if there exists a positive constant C such that

$$A(2s) \leq CA(s) \quad \text{for} \quad s \geq 0. \tag{2.6}$$

We say that A satisfies the ∇_2-condition if there exists a constant $C > 2$ such that

$$A(2s) \geq CA(s) \tag{2.7}$$

for $s \geq 0$. If (2.6) [resp. (2.7)] just holds for $s \geq s_0$ for some $s_0 > 0$, then A is said to satisfy the Δ_2-condition [∇_2-condition] near infinity. We shall also write $A \in \Delta_2$ [$A \in \nabla_2$] to denote that A satisfies the Δ_2-condition [∇_2-condition].
One has that $A \in \Delta_2$ [near infinity] if and only if $\widetilde{A} \in \nabla_2$-condition [near infinity].
A Young function A is said to dominate another Young function B [near infinity] if there exists a positive constant C

$$B(s) \leq A(Cs) \qquad \text{for} \quad s \geq 0 \;\; [s \geq s_0 \quad \text{for some } s_0 > 0]\,. \tag{2.8}$$

The functions A and B are called equivalent [near infinity] if they dominate each other [near infinity].

Let Ω be a measurable subset of $\mathbb{R}^n$, and let A be a Young function. The Luxemburg norm, associated with A, is defined as

$$\|u\|_{L^A(\Omega)} = \inf\left\{\lambda : \int_\Omega A\Big(\frac{|u(x)|}{\lambda}\Big)\,dx \leq 1\right\}$$

for any measurable function $u : \Omega \to \mathbb{R}$. The collection of all functions u for which such norm is finite is called the Orlicz space $L^A(\Omega)$, and is a Banach function space. The subspace of $L^A(\Omega)$ of those functions u such that $\int_\Omega u(x)\,dx = 0$ will be denoted by $L^A_\perp(\Omega)$. A Hölder type inequality in Orlicz spaces takes the form

$$\|v\|_{L^{\widetilde{A}}(\Omega)} \leq \sup_{u \in L^A(\Omega)} \frac{\int_\Omega u(x)v(x)\,dx}{\|u\|_{L^A(\Omega)}} \leq 2\|v\|_{L^{\widetilde{A}}(\Omega)} \tag{2.9}$$

for every $v \in L^{\widetilde{A}}(\Omega)$. If A dominates B, then

$$L^A(\Omega) \to L^B(\Omega), \tag{2.10}$$

with embedding norm depending on the constant C appearing in (2.8). When $|\Omega| < \infty$, embedding (2.10) also holds if A dominates B just near infinity, but, in this case, the embedding constant also depends on B, s_0 and $|\Omega|$. The decreasing rearrangement $u^* : [0, \infty) \to [0, \infty]$

of measurable function $u : \Omega \to \mathbb{R}$ is the (unique) non-increasing, right-continuous function which is equimeasurable with u. Thus,

$$u^*(s) = \inf\{t \geq 0 : |\{x \in \ : |u(x)| > t\}| \leq s\} \qquad \text{for } s \geq 0.$$

The equimeasurability of u and u^* implies that

$$\|u\|_{L^A(\Omega)} = \|u^*\|_{L^A(0,|\Omega|)} \tag{2.11}$$

for every $u \in L^A(\Omega)$.

The Lebesgue spaces $L^p(\Omega)$, corresponding to the choice $A(t) = t^p$, if $p \in [1, \infty)$, and $A(t) = 0$ for $t \in [0, 1]$ and $A(t) = \infty$ for $t > 1$, if $p = \infty$, are a basic example of Orlicz spaces. Other customary instances of Orlicz spaces are provided by the Zygmund spaces $L^p \log^\alpha L(\Omega)$, and by the exponential spaces $\exp L^\beta(\Omega)$. If either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 0$, then $L^p \log^\alpha L(\Omega)$ is the Orlicz space associated with a Young function equivalent to $t^p(\log t)^\alpha$ near infinity. Given $\beta > 0$, $\exp L^\beta(\Omega)$ denotes the Orlicz space built upon a Young function equivalent to e^{t^β} near infinity.

The Orlicz space $L^A(\Omega, \mathbb{R}^n)$ of $\mathbb{R}^n$-valued measurable functions on Ω is defined as $L^A(\Omega, \mathbb{R}^n) = (L^A(\Omega))^n$, and is equipped with the norm given by $\|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} = \| |\mathbf{u}| \|_{L^A(\Omega)}$ for $\mathbf{u} \in L^A(\Omega, \mathbb{R}^n)$. The Orlicz space $L^A(\Omega, \mathbb{R}^{n\times n})$ of $\mathbb{R}^{n\times n}$ matrix-valued measurable functions on Ω is defined analogously.

Assume now that Ω is an open set. The Orlicz-Sobolev space $W^{1,A}(\Omega)$ is the set of all weakly differentiable functions in $L^A(\Omega)$ whose gradient also belongs to $L^A(\Omega)$. It is a Banach space endowed with the norm

$$\|u\|_{W^{1,A}(\Omega)} = \|u\|_{L^A(\Omega)} + \|\nabla u\|_{L^A(\Omega,\mathbb{R}^n)}.$$

We also define the subspace of $W^{1,A}(\Omega)$ of those functions which vanish on $\partial\Omega$ as

$$W^{1,A}_0(\Omega) = \{u \in W^{1,A}(\Omega) : \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \\ \text{is weakly differentiable in } \mathbb{R}^n\}.$$

In the case when $A(t) = t^p$ for some $p \geq 1$, and $\partial\Omega$ is regular enough, such definition of $W^{1,A}_0(\Omega)$ can be shown to reproduce the usual space $W^{1,p}_0(\Omega)$ defined as the closure in $W^{1,p}(\Omega)$ of the space $C^\infty_0(\Omega)$ of smooth compactly supported functions in Ω. In general, the set of smooth bounded functions is dense in $L^A(\Omega)$ only if A satisfies the Δ_2-condition (just near infinity when $|\Omega| < \infty$), and hence, for arbitrary A, our definition of $W^{1,A}_0(\Omega)$ yields a space which can be larger than the closure of $C^\infty_0(\Omega)$ in $W^{1,A}_0(\Omega)$ even for smooth domains. On the other hand, if Ω is a Lipschitz domain, then $W^{1,A}_0(\Omega) = W^{1,A}(\Omega) \cap W^{1,1}_0(\Omega)$, where $W^{1,1}_0(\Omega)$ is defined as usual. Recall that an open set Ω is called a Lipschitz domain if it is bounded and there exists a neighborhood $\mathcal{U}$ of each point of $\partial\Omega$ such that $\Omega \cap \mathcal{U}$ is the subgraph of a Lipschitz continuous function of $n-1$ variables. An open set Ω is said to have the cone property if there exists a finite cone Λ such that each point of Ω is the vertex of a finite cone contained in Ω and congruent to Λ. Clearly, any Lipschitz domain has the cone property, but the converse is not true in general.

A Poincaré type inequality in Orlicz-Sobolev spaces tells us that, if Ω is a Lipschitz domain, then there exists a constant C, depending on n and on the Lipschitz constant of Ω, such that

$$\|u - u_\Omega\|_{L^A(\Omega)} \leq C|\Omega|^{\frac{1}{n}} \|\nabla u\|_{L^A(\Omega,\mathbb{R}^n)} \tag{2.12}$$

for every $u \in W^{1,A}(\Omega)$. Inequality (2.12) is established in [18, Lemma 4.1] in the special case when Ω is a ball. Its proof makes use of a rearrangement type inequality for the norm $\|\nabla u\|_{L^A(\Omega)}$ which holds, in fact, for Sobolev functions u on any Lipschitz domain Ω [22, Lemma 4.1 and inequality (3.5)]. The same proof then applies to any Lipschitz domain, and one can verify that the constant in the resulting Poincaré inequality has the form claimed in (2.12).

The Orlicz-Sobolev space $W^{1,A}(\Omega,\mathbb{R}^n)$ of $\mathbb{R}^n$-valued functions is defined as $W^{1,A}(\Omega,\mathbb{R}^n) = \big(W^{1,A}(\Omega)\big)^n$, and equipped with the norm $\|\mathbf{u}\|_{W^{1,A}(\Omega,\mathbb{R}^n)} = \|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} + \|\nabla \mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})}$. The space $W^{1,A}_0(\Omega,\mathbb{R}^n)$ is defined accordingly.

3 The negative norm theorem and the Bogovskiĭ operator

Let A be a Young function, and let Ω be a bounded domain in $\mathbb{R}^n$. We define the negative Orlicz-Sobolev norm associated with A of the distributional gradient of a function $u \in L^1(\Omega)$ as

$$\|\nabla u\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} = \sup_{\boldsymbol{\varphi}\in C^\infty_0(\Omega,\mathbb{R}^n)} \frac{\int_\Omega u \operatorname{div} \boldsymbol{\varphi}\, dx}{\|\nabla \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}. \tag{3.1}$$

The alternative notation $W^{-1}L^A(\Omega,\mathbb{R}^n)$ will also occasionally be employed to denote the negative Orlicz-Sobolev norm $W^{-1,A}(\Omega,\mathbb{R}^n)$ associated with the Orlicz space $L^A(\Omega)$.

Our Orlicz-Sobolev space version of the negative norm theorem involves pairs of Young functions A and B which obey the following balance conditions:

$$t\int_0^t \frac{B(s)}{s^2}\, ds \leq A(ct) \qquad \text{for } t \geq 0, \tag{3.2}$$

and

$$t\int_0^t \frac{\widetilde{A}(s)}{s^2}\, ds \leq \widetilde{B}(ct) \qquad \text{for } t \geq 0, \tag{3.3}$$

for some positive constant c.
Let us mention that assumptions (3.2) and (3.3) also come into play in a version of the Korn inequality for the symmetric gradient in Orlicz spaces [21].

Theorem 3.1. *Let A and B be Young functions fulfilling* (3.2) *and* (3.3)*. Assume that Ω is a bounded domain with the cone property in $\mathbb{R}^n$, $n \geq 2$. Then there exist constants $C_1 = C_1(\Omega, c)$ and $C_2 = C_2(n)$ such that*

$$C_1\|u - u_\Omega\|_{L^B(\Omega)} \leq \|\nabla u\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} \leq C_2\|u - u_\Omega\|_{L^A(\Omega)} \tag{3.4}$$

for every $u \in L^1(\Omega)$. Here, c denotes the constant appearing in (3.2) *and* (3.3)*.*

Remark 3.2. Inequality (3.4) continues to hold even if conditions (3.2) and (3.3) are just fulfilled for $t \geq t_0$ for some $t_0 > 0$, but with constants C_1 and C_2 depending also on A, B, t_0 and $|\Omega|$. Indeed, the Young functions A and B can be replaced, if necessary, with Young functions equivalent near infinity, and fulfilling (3.2) and (3.3) for every $t > 0$. Owing (2.10), such replacement leaves the quantities $\|\cdot\|_{L^A(\Omega)}$, $\|\cdot\|_{L^B(\Omega)}$ and $\|\nabla\cdot\|_{W^{-1,A}(\Omega,\mathbb{R}^n)}$ unchanged, up to multiplicative constants depending on A, B, t_0 and $|\Omega|$.

If either (3.2) or (3.3) holds, then A dominates B globally [21, Proposition 3.5]. In a sense, assumptions (3.2) and (3.3) provide us with a quantitative information on if, and how much, the norm $\|\cdot\|_{L^B(\Omega)}$ has to be weaker than $\|\cdot\|_{L^A(\Omega)}$ for a version of the negative-norm Theorem to be restored in Orlicz-Sobolev spaces. The situations when (3.2), or (3.3), holds with $B = A$ can be precisely characterized. Membership of A to Δ_2 is a necessary and sufficient condition for (3.3) to hold with $B = A$ [38, Theorem 1.2.1]. Therefore, under this condition, assumption (3.3) can be dropped in Theorem 3.1. On the other hand, $A \in \nabla_2$ if and only if $\widetilde{A} \in \Delta_2$, and hence membership of A to ∇_2 is a necessary and sufficient condition for (3.2) to hold with $B = A$. Thus, under this condition, assumption (3.2) can be dropped in Theorem 3.1. In particular, if $A \in \Delta_2 \cap \nabla_2$, then both conditions (3.2) and (3.3) are fulfilled with $B = A$. Hence, we have the following corollary which can also be derived from the results of [29].

Corollary 3.3. *Assume that Ω is a bounded domain with the cone property in $\mathbb{R}^n$, $n \geq 2$. Let A be a Young function in $\Delta_2 \cap \nabla_2$. Then there exist a constants $C = C(\Omega, A)$ and $C_2 = C_2(n)$ such that*

$$C_1\|u - u_\Omega\|_{L^A(\Omega)} \leq \|\nabla u\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} \leq C_2\|u - u_\Omega\|_{L^A(\Omega)} \tag{3.5}$$

for every $u \in L^1(\Omega)$.

A typical situation where condition (3.2) does not hold with with $B = A$ is when A grows linearly, or "almost linearly", near infinity. In this case, $A \notin \nabla_2$. In fact, as recalled in Section 1, the standard negative-norm Theorem expressed by (1.2) breaks down in the borderline cases when $p = 1$. On the other hand, condition (3.3) fails, with $B = A$, if, for example, A has a very fast – faster than any power – growth. In this case, $A \notin \Delta_2$. Loosely speaking, the norm $\|\cdot\|_{L^A(\Omega)}$ is now "close" to $\|\cdot\|_{L^\infty(\Omega)}$, and, as a matter of fact, equation (1.2) in not true with $p = \infty$.

These are not, however, the only situations when (3.2), or (3.3), fail with $B = A$. For instance, functions A which neither satisfy the Δ_2 condition, nor the ∇_2 condition, for which hence neither (3.3) nor (3.2) can hold with $B = A$, are those where $A(t)$ "oscillates" between two different powers t^p and t^q, with $1 < p < q < \infty$. Functions of this kind are usually called of (p, q)-growth in the literature. Partial differential equations, and associated variational problems, whose nonlinearity is governed by this growth, have been extensively studied. In the framework of Non-Newtonian fluids, they have been analyzed in [8].

All the circumstances described above can be handled via Theorem (3.1). A few examples involving customary families of Young functions are presented hereafter.

Example 3.4. Assume that $A(t)$ is a Young function equivalent to $t^p \log^\alpha(1+t)$ near infinity, where either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 1$. Hence, if $|\Omega| < \infty$, then

$$L^A(\Omega) = L^p \log^\alpha L(\Omega).$$

Assume that Ω is a bounded domain with the cone property in $\mathbb{R}^n$. If $p > 1$, then $A \in \Delta_2 \cap \nabla_2$, and hence Corollary 3.3 tells us that

$$C_1 \|u - u_\Omega\|_{L^p \log^\alpha L(\Omega)} \leq \|\nabla u\|_{W^{-1}L^p \log^\alpha L(\Omega,\mathbb{R}^n)} \leq C_2 \|u - u_\Omega\|_{L^p \log^\alpha L(\Omega)} \tag{3.6}$$

for every $u \in L^1(\Omega)$. However, if $p = 1$, then $A \in \Delta_2$, but $A \notin \nabla_2$. An application of Theorem 3.1 now yields

$$C_1 \|u - u_\Omega\|_{L \log^{\alpha-1} L(\Omega)} \leq \|\nabla u\|_{W^{-1}L \log^\alpha L(\Omega,\mathbb{R}^n)} \leq C_2 \|u - u_\Omega\|_{L \log^\alpha L(\Omega)} \tag{3.7}$$

for every $u \in L^1(\Omega)$. In particular,

$$C_1 \|u - u_\Omega\|_{L^1(\Omega)} \leq \|\nabla u\|_{W^{-1}L \log L(\Omega,\mathbb{R}^n)} \leq C_2 \|u - u_\Omega\|_{L \log L(\Omega)} \tag{3.8}$$

for every $u \in L^1(\Omega)$.

Example 3.5. Let $\beta > 0$, and let $A(t)$ be a Young function equivalent to $\exp(t^\beta)$ near infinity. Then

$$L^A(\Omega) = \exp L^\beta(\Omega)$$

if $|\Omega| < \infty$. One has that $A \in \nabla_2$, but $A \notin \Delta_2$. Theorem 3.1 ensures that, if Ω is a bounded domain with the cone property in $\mathbb{R}^n$, then

$$C_1 \|u - u_\Omega\|_{\exp L^{\frac{\beta}{\beta+1}}(\Omega)} \leq \|\nabla u\|_{W^{-1}\exp L^\beta(\Omega,\mathbb{R}^n)} \leq C_2 \|u - u_\Omega\|_{\exp L^\beta(\Omega)} \tag{3.9}$$

for every $u \in L^1(\Omega)$. Moreover,

$$C_1 \|u - u_\Omega\|_{\exp L(\Omega)} \leq \|\nabla u\|_{W^{-1}L^\infty(\Omega,\mathbb{R}^n)} \leq C_2 \|u - u_\Omega\|_{L^\infty(\Omega)} \tag{3.10}$$

for every $u \in L^1(\Omega)$.

Our proof of Theorem 3.1 relies upon an analysis of the divergence equation

$$\begin{cases} \operatorname{div} \mathbf{u} = f & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

in Orlicz spaces. This is the objective of the next result. In what follows, we set

$$C^\infty_{0,\perp}(\Omega) = \{u \in C^\infty_0(\Omega) : u_\Omega = 0\}$$

and

$$L^A_\perp(\Omega) = \{u \in L^A(\Omega) : u_\Omega = 0\}.$$

Theorem 3.6. *Assume that Ω is a bounded domain with the cone property in $\mathbb{R}^n$, $n \geq 2$. Let A and B be Young functions fulfilling* (3.2) *and* (3.3)*. Then there exists a bounded linear operator*

$$\mathcal{B}_\Omega : L^A_\perp(\Omega) \to W^{1,B}_0(\Omega, \mathbb{R}^n) \tag{3.12}$$

such that

$$\mathcal{B}_\Omega : C^\infty_{0,\perp}(\Omega) \to C^\infty_0(\Omega, \mathbb{R}^n) \tag{3.13}$$

and

$$\operatorname{div}(\mathcal{B}_\Omega f) = f \quad \text{in } \Omega \tag{3.14}$$

for every $f \in L^A_\perp(\Omega)$. In particular, there exists a constant $C = C(\Omega, c)$ such that

$$\|\nabla(\mathcal{B}_\Omega f)\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C\|f\|_{L^A(\Omega)} \tag{3.15}$$

and

$$\int_\Omega B(|\nabla(\mathcal{B}_\Omega f)|)\, dx \leq \int_\Omega A(C|f|)\, dx \tag{3.16}$$

for every $f \in L^A_\perp(\Omega)$. Here, c denotes the constant appearing in (3.2) *and* (3.3)*.*

Although it will not be used for our main purposes, we state in Theorem 3.7 below a result parallel to Theorem 3.6, dealing with a version of problem (3.11) in the case when the right-hand side of the equation is in divergence form. Namely,

$$\begin{cases} \operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{g} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.17}$$

where $\mathbf{g} : \Omega \to \mathbb{R}^n$ is a prescribed function satisfying the compatibility condition (in a weak sense) that its normal component on $\partial\Omega$ vanishes. A precise formulation of this condition requires the introduction of the following function spaces. Given a Young function A, denote by $H^A(\Omega)$ the Banach space of those vector-valued functions $\mathbf{u} : \Omega \to \mathbb{R}^n$ such that the norm

$$\|\mathbf{u}\|_{H^A(\Omega)} = \|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} + \|\operatorname{div} \mathbf{u}\|_{L^A(\Omega)} \tag{3.18}$$

is finite. We also denote by $H^A_0(\Omega)$ its subspace of those functions $\mathbf{u} \in H^A(\Omega)$ whose normal component on $\partial\Omega$ vanishes, in the sense that

$$\int_\Omega \varphi \operatorname{div} \mathbf{u}\, dx = -\int_\Omega \mathbf{u} \cdot \nabla\varphi\, dx \tag{3.19}$$

for every $\varphi \in C^\infty_0(\Omega)$.

Theorem 3.7. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let A and B be Young functions fulfilling* (3.2) *and* (3.3). *Then there exists a bounded linear operator*

$$\mathcal{E}_\Omega : H_0^A(\Omega) \to W_0^{1,B}(\Omega, \mathbb{R}^n) \tag{3.20}$$

such that

$$\operatorname{div}(\mathcal{E}_\Omega \mathbf{g}) = \operatorname{div} \mathbf{g} \quad \text{in } \Omega \tag{3.21}$$

for every $\mathbf{g} \in H_0^A(\Omega)$. In particular, there exists a constant $C = C(\Omega, c)$ such that

$$\|\nabla(\mathcal{E}_\Omega \mathbf{g})\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C \|\operatorname{div} \mathbf{g}\|_{L^A(\Omega)} \tag{3.22}$$

and

$$\|\mathcal{E}_\Omega \mathbf{g}\|_{L^B(\Omega,\mathbb{R}^n)} \leq C\|\mathbf{g}\|_{L^A(\Omega)} \tag{3.23}$$

for every $\mathbf{g} \in H_0^A(\Omega)$. Here, c denotes the constant appearing in (3.2) *and* (3.3).

The proofs of Theorems 3.6 and 3.7 in turn make use of a rearrangement estimate, which extends those of [4, Theorem 16.12] and [3], for a class of singular integral operators of the form

$$Tf(x) = \lim_{\varepsilon \to 0^+} \int_{\{y:|y-x|>\varepsilon\}} K(x, x-y) f(y)\, dy \quad \text{for } x \in \mathbb{R}^n, \tag{3.24}$$

for an integrable function $f : \mathbb{R}^n \to \mathbb{R}$. Here, the kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ fulfils the following properties: (i)

$$K(x, x - \lambda y) = \lambda^{-n} K(x, x-y) \quad \text{for } x, y \in \mathbb{R}^n; \tag{3.25}$$

(ii)

$$\int_{\mathbb{S}^{n-1}} K(x, x-y)\, d\mathcal{H}^{n-1}(y) = 0 \quad \text{for } x \in \mathbb{R}^n; \tag{3.26}$$

(iii) For every $\sigma \in [1, \infty)$, there exists a constant C_1 such that

$$\left(\int_{\mathbb{S}^{n-1}} |K(x, x-y)|^\sigma \, d\mathcal{H}^{n-1}(y) \right)^{\frac{1}{\sigma}} \leq C_1 (1 + |x|)^n \quad \text{for } x \in \mathbb{R}^n, \tag{3.27}$$

where $\mathbb{S}^{n-1}$ denotes the unit sphere, centered at 0, in $\mathbb{R}^n$, and $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure;
(iv) There exists a constant C_2 such that

$$|K(x,y)| \leq C_2 \frac{(1+|x|)^n}{|x-y|^n} \quad \text{for } x, y \in \mathbb{R}^n,\ x \neq y, \tag{3.28}$$

and, if $2|x - z| < |x - y|$, then

$$|K(x,y) - K(z,y)| \leq C_2 (1+|y|)^n \frac{|x-z|}{|x-y|^{n+1}}, \tag{3.29}$$

$$|K(y,x) - K(y,z)| \leq C_2 (1+|y|)^n \frac{|x-z|}{|x-y|^{n+1}}. \tag{3.30}$$

Theorem 3.8. *Let Ω be a bounded open set in $\mathbb{R}^n$, and let $K(x,y)$ be a kernel satisfying* (3.25)–(3.30). *If $f \in L^1(\mathbb{R}^n)$ and $f = 0$ in $\mathbb{R}^n \setminus \Omega$, then the singular integral operator T given by* (3.24) *is well defined for a.e. $x \in \mathbb{R}^n$, and there exists a constant $C = C(C_1, C_2, n, \mathrm{diam}(\Omega))$ such that*

$$(3.31) \qquad (Tf)^*(s) \le C\left(\frac{1}{s}\int_0^s f^*(r)\,dr + \int_s^{|\Omega|} f^*(r)\,\frac{dr}{r}\right) \quad \text{for } s \in (0, |\Omega|).$$

As a consequence of Theorem 3.8, the boundedness of singular integral operators given by (3.24) between Orlicz spaces associated with Young functions A and B fulfilling (3.2) and (3.3) can be established.

Theorem 3.9. *Let Ω, K and T be as in Theorem 3.8. Assume that A and B are Young functions satisfying* (3.2) *and* (3.3). *Then there exists a constant $C = C(C_1, C_2, n, \mathrm{diam}(\Omega), c)$ such that*

$$(3.32) \qquad \|Tf\|_{L^B(\Omega)} \le C\|f\|_{L^A(\Omega)},$$

and

$$(3.33) \qquad \int_\Omega B(|Tf|)\,dx \le \int_\Omega A(C|f|)\,dx$$

for every $f \in L^A(\Omega)$. Here, c denotes the constant appearing in (3.2) *and* (3.3).

Proof. By [19, Lemma 1], we have that, if A and B are Young functions satisfying (3.2), then there exists a constant $C = C(c)$ such that

$$(3.34) \qquad \left\|\frac{1}{s}\int_0^s \varphi(r)\,dr\right\|_{L^B(0,\infty)} \le C\|\varphi\|_{L^A(0,\infty)}$$

for every $\varphi \in L^A(0,\infty)$. Moreover, if A and B fulfill (3.3), then there exists a constant $C = C(c)$ such that

$$(3.35) \qquad \left\|\int_s^\infty \varphi(r)\,\frac{dr}{r}\right\|_{L^B(0,\infty)} \le C\|\varphi\|_{L^A(0,\infty)}$$

for every $\varphi \in L^A(0,\infty)$. Combining (3.31), (3.34) and (3.35), and making use of property (2.11) yield inequality (3.32).
As far as (3.33) is concerned, observe that, inequalities (3.2) and (3.3) continue to hold, with the same constant c, if A and B are replaced with kA and kB, where k is any positive constant. Thus, inequality (3.32) continues to hold, with the same constant C, after this replacement, whatever K is, namely

$$(3.36) \qquad \|Tf\|_{L^{kB}(\Omega)} \le C\|f\|_{L^{kA}(\Omega)}$$

for every $f \in L^A(\Omega)$. Now, given any such f, choose $k = \frac{1}{\int_\Omega A(|f|)\,dx}$. The very definition of Luxemburg norm tells us that $\|f\|_{L^{kA}(\Omega)} \le 1$. Hence, by (3.36), $\|Tf\|_{L^{kB}(\Omega)} \le C$. The definition of Luxemburg norm again implies that $\int_\Omega k\, B\big(\frac{|Tf|}{C}\big)\,dx \le 1$, namely (3.33). $\square$

Proof of Theorem 3.8. Let $R > 0$ be such that $\Omega \subset B_R(0)$, the ball centered at 0, with radius R. Fix a smooth function $\eta : [0,\infty) \to [0,\infty)$ such that $\eta = 1$ in $[0, 3R]$ and $\eta = 0$ in $[4R,\infty)$. Define

$$\widehat{K}(x,y) = \eta(|x|)K(x,y) \quad \text{for } x, y \in \mathbb{R}^n.$$

By properties (3.25)–(3.30) of $K(x,y)$, one has that:

$$\widehat{K}(x, x-\lambda y) = \lambda^{-n}\widehat{K}(x, x-y) \quad \text{for } x, y \in \mathbb{R}^n; \tag{3.37}$$

$$\int_{\mathbb{S}^{n-1}} \widehat{K}(x, x-y)\, d\mathcal{H}^{n-1}(y) = 0 \quad \text{for } x \in \mathbb{R}^n; \tag{3.38}$$

for every $\sigma \in [1,\infty)$, there exists a constant $\widehat{C}_1 = \widehat{C}_1(C_1, \sigma, R, n)$ such that

$$\left(\int_{\mathbb{S}^{n-1}} |\widehat{K}(x, x-y)|^\sigma\, d\mathcal{H}^{n-1}(y)\right)^{\frac{1}{\sigma}} \le \widehat{C}_1 \quad \text{for } x \in \mathbb{R}^n, \tag{3.39}$$

where C_1 is the constant appearing in (3.27); there exists a constant $\widehat{C}_2 = \widehat{C}_2(C_2, R, n)$ such that

$$|\widehat{K}(x,y)| \le \frac{\widehat{C}_2}{|x-y|^n} \text{for } x, y \in \mathbb{R}^n,\ x \ne y, \tag{3.40}$$

and, if $x \in \mathbb{R}^n$, $y \in \Omega$ and $2|x-z| < |x-y|$, then

$$|\widehat{K}(x,y) - \widehat{K}(z,y)| \le \widehat{C}_2 \frac{|x-z|}{|x-y|^{n+1}}, \tag{3.41}$$

$$|\widehat{K}(y,x) - \widehat{K}(y,z)| \le \widehat{C}_2 \frac{|x-z|}{|x-y|^{n+1}}, \tag{3.42}$$

where C_2 is the constant appearing in (3.28)–(3.30).
Define

$$\widehat{T}_\varepsilon f(x) = \int_{\{y:|y-x|>\varepsilon\}} \widehat{K}_\varepsilon(x,y) f(y)\,dy,$$
$$\widehat{T}_S f(x) = \sup_{\varepsilon>0} |\widehat{T}_\varepsilon(f)(x)|.$$

Inequality (3.31) will follow if we prove that

$$(\widehat{T}_S f)^*(s) \le C\left(\frac{1}{s}\int_0^s f^*(r)\,dr + \int_s^{|\Omega|} f^*(r)\,\frac{dr}{r}\right) \quad \text{for } s \in (0,\infty) \tag{3.43}$$

for some constant $C = C(C_1, C_2, n, R)$, and for every $f \in L^1(\mathbb{R}^n)$ such that $f = 0$ in $\mathbb{R}^n \setminus B_R(0)$. A proof of inequality (3.43) can be accomplished along the same lines as that of Theorem 1 of [3], which in turn relies upon similar techniques as in [23]. For completeness, we sketch such proof hereafter.

The key step in the derivation of (3.43) consists in showing that, for every $\gamma \in (0,1)$, there exists a constant $C = C(C_1, C_2, \gamma, n, R)$ such that

$$(\widehat{T}_S f)^*(s) \le C(Mf)^*(\gamma s) + (\widehat{T}_S f)^*(2s) \quad \text{for } s \in (0,\infty) \tag{3.44}$$

for every $f \in L^1(\mathbb{R}^n)$ such that $f = 0$ in $\mathbb{R}^n \setminus B_R(0)$. Fix $s > 0$, and define

$$E = \{x \in \mathbb{R}^n : \widehat{T}_S f(x) > (\widehat{T}_S f)^*(2s)\}.$$

Then, there exists an open set $U \supset E$ such that $|U| \le 3s$. By Whitney's covering theorem, there exist a family of disjoint cubes $\{Q_k\}$ such that $U = \cup_{k=1}^\infty Q_k$, $\sum_{k=1}^\infty |Q_k| = |U| \le 3s$, and

$$\mathrm{diam}(Q_k) \le \mathrm{dist}(Q_k, \mathbb{R}^n \setminus U) \le 4\mathrm{diam}(Q_k) \quad \text{for } k \in \mathbb{N}.$$

The operator $\widehat{T}_S$ is of weak type (1,1), namely, there exists a constant C' such that

$$|\{x \in \mathbb{R}^n : \widehat{T}_S f(x) > \lambda\}| \le \frac{C'}{\lambda}\|f\|_{L^1(\mathbb{R}^n)} \tag{3.45}$$

for $f \in L^1(\mathbb{R}^n)$, as proved in [27, Proof of Lemma 6.3].

We shall now show that there exists a constant $\overline{C}$ such that

$$|\{x \in Q_k : \widehat{T}_S f(x) > \overline{C} Mf(x) + (\widehat{T}_S f)^*(2s)\}| \le \frac{1-\gamma}{3}|Q_k| \quad \text{for } k \in \mathbb{N}. \tag{3.46}$$

Fix any $k \in \mathbb{N}$, choose $x_k \in \mathbb{R}^n \setminus U$ such that $\mathrm{dist}(x_k, Q_k) \le 4\mathrm{diam}(Q_k)$, and denote by Q the cube, centered at x_k, with $\mathrm{diam}(Q) = 20\mathrm{diam}(Q_k)$. Define

$$g = f\chi_Q, \quad h = f\chi_{\mathbb{R}^n \setminus Q},$$

so that $f = g + h$. If we prove that there exist constants $\overline{C}_1$ and $\overline{C}_2$ such that

$$\widehat{T}_S h(x) \le \overline{C}_1 Mf(x) + (\widehat{T}_S f)^*(2s) \quad \text{for } x \in Q_k, \tag{3.47}$$

and

$$|\{x \in Q_k : \widehat{T}_S g(x) > \overline{C}_2 Mf(x)\}| \le \frac{1-\gamma}{3}|Q_k|, \tag{3.48}$$

then (3.46) follows with $\overline{C} = \overline{C}_1 + \overline{C}_2$. Consider (3.48) first. Let $\overline{C}_2$ be such that $\frac{C'|Q|}{\overline{C}_2} \le \frac{1-\gamma}{3}|Q_k|$. Let $\lambda = \frac{\overline{C}_2}{|Q|}\int_Q |g|dx$. Since $\overline{C}_2 Mf(x) \ge \lambda$ for $x \in Q_k$, an application of (3.45) with this choice of λ tells us that

$$|\{x \in Q_k : \widehat{T}_S g(x) > \overline{C}_2 Mf(x)\}| \le |\{\widehat{T}_S g(x) > \lambda\}|$$

$$\le \frac{C'}{\lambda}\int_Q |g|dx \le \frac{C'|Q|}{\overline{C}_2} \le \frac{1-\gamma}{3}|Q_k|,$$

namely (3.48). In order to establish (3.47), it suffices to prove that, for every $\varepsilon > 0$,

$$|\widehat{T}_\varepsilon h(x)| \le \overline{C}_1 Mf(x) + \widehat{T}_S f(x_k) \quad \text{for } x \in Q_k. \tag{3.49}$$

Indeed, since $x_k \notin U$, we have that $\widehat{T}_S f(x_k) \le (\widehat{T}_S f)^*(2s)$, and hence (3.49) implies (3.47). We may thus focus on (3.49). Fix $\varepsilon > 0$, and set $r = \max\{\varepsilon, \operatorname{dist}(x_k, \mathbb{R}^n \setminus Q)\}$. Observe that $r > 10\operatorname{diam}(Q_k)$. Given any $x \in Q_k$, define $V = B_\varepsilon(x) \triangle B_\varepsilon(x_k)$. One has that

$$\begin{aligned} |\widehat{T}_\varepsilon h(x)| &= \left|\int_{\{y:|y-x|>\varepsilon\}} \widehat{K}(x,y)h(y)\,dy\right| \\ &\le \left|\int_{\{y:|y-x_k|>\varepsilon\}} \widehat{K}(x,y)h(y)\,dy\right| + \int_V |\widehat{K}(x,y)h(y)|\,dy. \end{aligned} \tag{3.50}$$

Observe that, if $y \in \operatorname{supp} h$, then $|x-y| > \frac{r}{2}$ and hence $\frac{1}{|x-y|^n} < \frac{2^n}{r^n}$. Thus, owing to (3.40),

$$|\widehat{K}(x,y)| \le \frac{\widehat{C}_2}{r^n}.$$

Moreover, $V \subset B_{3r}(x)$. Therefore, there exists a constant $\widehat{C}$ such that

$$\int_V |\widehat{K}(x,y)h(y)|\,dy \le \frac{\widehat{C}}{|B_{3r}(x)|} \int_{B_{3r}(x)} |h(y)|\,dy \le \widehat{C}Mh(x) \le \widehat{C}Mf(x). \tag{3.51}$$

On the other hand,

$$\begin{aligned} \left|\int_{\{y:|y-x_k|>\varepsilon\}} \widehat{K}(x,y)h(y)\,dy\right| &\le \left|\int_{\{y:|y-x_k|>r\}} \widehat{K}(x,y)h(y)\,dy\right| \\ &\le \left|\int_{\{y:|y-x_k|>r\}} \widehat{K}(x_k,y)f(y)\,dy\right| + \int_{\{y:|y-x_k|>r\}} |\widehat{K}(x_k,y) - \widehat{K}(x,y)|\,|f(y)|\,dy, \\ &\le \widehat{T}_S(x_k) + \int_{\{y:|y-x_k|>r\}} |\widehat{K}(x_k,y) - \widehat{K}(x,y)|\,|f(y)|\,dy, \end{aligned} \tag{3.52}$$

where the first inequality holds since $h(y) = 0$ in $\{y : |y-x_k| \le r\}$ if $r = \operatorname{dist}(x_k, \mathbb{R}^n \setminus Q)$, and trivially holds (with equality) if $r = \varepsilon$. Since $2|x-x_k| \le |x-y|$ in the last integral in (3.52), and f vanishes in $\mathbb{R}^n \setminus B_R(0)$, by (3.41)

$$|\widehat{K}(x_k,y) - \widehat{K}(x,y)| \le \widehat{C}_2 \frac{|x_k - x|}{|x-y|^{n+1}} \le \widehat{C}_2 \frac{\operatorname{diam}(Q_k)}{|x-y|^{n+1}}.$$

Hence,

$$
\begin{aligned}
(3.53)\qquad & \int_{\{y:|y-x_k|>r\}} |\widehat{K}(x_k,y)-\widehat{K}(x,y)|\,|f(y)|\,dy \\
&\qquad \le \int_{\{y:|y-x|>\mathrm{diam}(Q_k)\}} |f(y)|\frac{\mathrm{diam}(Q_k)}{|x-y|^{n+1}}\,dy \le \widetilde{C}Mf(x)
\end{aligned}
$$

for some constant $\widetilde{C}$. Note that, in the first inequality, we have made use of the inclusion $\{y : |y-x_k| > r\} \subset \{y : |y-x| > \mathrm{diam}(Q_k)\}$, which holds since $|x-x_k| < 5\,\mathrm{diam}(Q_k)$, and $10\,\mathrm{diam}(Q_k) < r$.
Combining inequalities (3.50)– (3.53) yields (3.49). Inequality (3.46) is fully established. Via summation in $k \in Q_k$, we obtain from (3.46) that

$$
(3.54)\qquad |\{x \in \mathbb{R}^n : \widehat{T}_S f(x) > \widehat{C}Mf(x) + (\widehat{T}_S f)^*(2s)\}| \le (1-\gamma)s\,.
$$

Coupling (3.54) with the inequality

$$
(3.55)\qquad |\{x \in \mathbb{R}^n : Mf(x) > (Mf)^*(\gamma s)\}| \le \gamma s
$$

tells us that

$$
\begin{aligned}
|\{x \in \mathbb{R}^n : \widehat{T}_S f(x) > \widehat{C}(Mf)^*(\gamma s) + (\widehat{T}_S f)^*(2s)\}| &\le |\{x \in \mathbb{R}^n : \widehat{T}_S f(x) > \widehat{C}Mf(x) + (\widehat{T}_S f)^*(2s)\}| \\
&\quad + |\{x \in \mathbb{R}^n : Mf(x) > (Mf)^*(\gamma s)\}| \le s\,,
\end{aligned}
$$

whence (3.44) follows, by the very definition of decreasing rearrangement.
Starting from inequality (3.44) leads to (3.31), via the same iteration argument as in the proof of [3, Theorem 1]. □

Lemma 3.10. *Let Ω be a bounded domain with the cone property in $\mathbb{R}^n$, with $n \ge 2$. Then there exist $N \in \mathbb{N}$ and a finite family $\{\Omega_i\}_{i=0,\dots N}$ of domains which are starshaped with respect to balls, such that $\Omega = \cup_{i=0}^N \Omega_i$. Moreover, given $f \in L^A_\perp(\Omega)$, there exist $f_i \in L^A_\perp(\Omega)$, $i = 0, \dots N$, such that $f_i = 0$ in $\Omega \setminus \Omega_i$,*

$$
f = \sum_{i=0}^N f_i
$$

and

$$
(3.56)\qquad \|f_i\|_{L^A(\Omega)} \le C\|f\|_{L^A(\Omega)} \quad \textit{for } i = 0, \dots, N,
$$

for some constant $C = C(\Omega)$.

Proof, sketched. Any bounded open set with the cone property can be decomposed into a finite union of Lipschitz domains [1, Lemma 4.22]. On the other hand, any Lipschitz domain can be decomposed into a finite union of open sets which are starshaped with respect to balls [36, Lemma 3.4, Chapter 3]. This proves the existence of the domains $\{\Omega_i\}_{i=0,\dots N}$ as in the statement. The same argument as in the proof of [36, Lemma 3.2, Chapter 3] then enables one to construct the desired family of functions f_i on Ω, $i = 1, \dots, N$, according to the following iteration scheme. We set $G_i = \cup_{j=i+1}^N \Omega_j$, $g_0 = f$, and, for $i = 1, \dots, N-1$,

$$g_i(x) = \begin{cases} \big(1 - \chi_{\Omega_i\cap G_i}(x)\big) g_{i-1}(x) - \frac{\chi_{\Omega_i\cap G_i}(x)}{|\Omega_i\cap G_i|}\int_{G_i\setminus\Omega_i} g_{i-1}(y)dy & \text{if } x \in G_i, \\ 0 & \text{otherwise,} \end{cases} \tag{3.57}$$

and

$$f_i(x) = \begin{cases} g_{i-1}(x) - \frac{\chi_{\Omega_i\cap G_i}(x)}{|\Omega_i\cap G_i|}\int_{\Omega_i} g_{i-1}(y)dy & \text{if } x \in \Omega_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.58}$$

Observe that, since Ω is connected, we can always relabel the sets $\Omega_i \cap G_i$ in such a way that $|\Omega_i \cap G_i| > 0$ for $i = 1, \dots, N-1$. Finally, we define

$$f_N = g_{N-1}. \tag{3.59}$$

The family $\{f_i\}$ satisfies the required properties. The only nontrivial one is (3.56). To verify the latter, fix i, and observe that, by (3.58), the second inequality in (2.9), inequality (2.2), and inequality (2.5)

$$\begin{aligned} \|f_i\|_{L^A(\Omega)} &\le \|g_{i-1}\|_{L^A(\Omega)}\Big(1 + \frac{2}{|\Omega_i\cap G_i|}\|1\|_{L^A(\Omega_i\cap G_i)}\|1\|_{L^{\widetilde{A}}(\Omega_i)}\Big) \\ &= \|g_{i-1}\|_{L^A(\Omega)}\Big(1 + \frac{2}{|\Omega_i\cap G_i| A^{-1}(1/|\Omega_i\cap G_i|)}\frac{1}{\widetilde{A}^{-1}(1/|\Omega_i|)}\Big) \\ &\le \|g_{i-1}\|_{L^A(\Omega)}\Big(1 + 4\frac{\widetilde{A}^{-1}(1/|\Omega_i\cap G_i|)}{\widetilde{A}^{-1}(1/|\Omega_i|)}\Big) \\ &\le \|g_{i-1}\|_{L^A(\Omega)}\Big(1 + 4\frac{|\Omega_i|}{|\Omega_i\cap G_i|}\Big). \end{aligned} \tag{3.60}$$

On the other hand, by (3.57) and a chain similar to (3.60), one has that

$$\begin{aligned} \|g_{i-1}\|_{L^A(\Omega)} &\le \|g_{i-2}\|_{L^A(\Omega)}\Big(1 + \frac{2}{|\Omega_{i-1}\cap G_{i-1}|}\|1\|_{L^A(\Omega_{i-1}\cap G_{i-1})}\|1\|_{L^{\widetilde{A}}(G_{i-1})}\Big) \\ &\le \|g_{i-2}\|_{L^A(\Omega)}\Big(1 + 4\frac{\widetilde{A}^{-1}(|\Omega_{i-1}\cap G_{i-1}|)}{\widetilde{A}^{-1}(|G_{i-1}|)}\Big) \\ &\le \|g_{i-2}\|_{L^A(\Omega)}\Big(1 + 4\frac{\widetilde{A}^{-1}(|\Omega_{i-1}\cap G_{i-1}|)}{\widetilde{A}^{-1}(|G_{i-1}|)}\Big) \\ &\le \|g_{i-2}\|_{L^A(\Omega)}\Big(1 + 4\max\Big\{1, \frac{|G_{i-1}|}{|\Omega_{i-1}\cap G_{i-1}|}\Big\}\Big). \end{aligned} \tag{3.61}$$

From (3.60), and an iteration of (3.61), one infers that

$$\|f_i\|_{L^A(\Omega)} \leq \left(1 + 4\frac{|\Omega_i|}{|\Omega_i \cap G_i|}\right) \prod_{j=1}^{i-1} \left(1 + 4 \max\left\{1, \frac{|G_j|}{|\Omega_j \cap G_j|}\right\}\right) \|f\|_{L^A(\Omega)}, \tag{3.62}$$

and (3.56) follows. □

Proof of Theorem 3.6 By Lemma 3.10, it suffices to prove the statement in the case when Ω is a domain starshaped with respect to a ball B, which, without loss of generality, can be assumed to be centered at the origin and with radius 1. In this case, we shall show that the (gradient of the) Bogovskii operator $\mathcal{B}_\Omega$, defined at a function $f \in L^A_\perp(\Omega)$ as

$$\mathcal{B}_\Omega f(x) = \int_\Omega f(y)\Big(\frac{x-y}{|x-y|^n} \int_{|x-y|}^\infty \omega\Big(y + \zeta\frac{x-y}{|x-y|}\Big)\zeta^{n-1}\, d\zeta\Big)\, dy \quad \text{for } x \in \Omega, \tag{3.63}$$

where ω is any (nonnegative) function in $C^\infty_0(B)$ with $\int_B \omega\, dx = 1$, agrees with a singular integral operator, whose kernel fulfills (3.25)–(3.30), plus two operators enjoying stronger boundedness properties. To be more precise, we set $\mathbf{u} = \mathcal{B}_\Omega f$ and claim that $\mathbf{u} \in W^{1,1}_0(\Omega, \mathbb{R}^n)$, and

$$\frac{\partial u_i}{\partial x_j} = H_{ij} f \quad \text{for a.e. } x \in \Omega, \tag{3.64}$$

where H_{ij} is the linear operator defined at f as

$$\begin{aligned}(H_{ij} f)(x) = &\int_\Omega K_{ij}(x,y) f(y)\, dy + \int_\Omega G_{ij}(x,y) f(y)\, dy \\ &+ f(x) \int_\Omega \frac{(x-y)_i (x-y)_j}{|x-y|^2} \omega(y)\, dy \quad \text{for } x \in \Omega,\end{aligned} \tag{3.65}$$

for $i, j = 1, \dots n$. Here, K_{ij} is the kernel of a singular integral operator satisfying the same assumptions as the kernel K in Theorem 3.8, and the kernels G_{ij} satisfy

$$|G_{ij}(x,y)| \leq \frac{c}{|x-y|^{n-1}} \quad \text{for } x, y \in \mathbb{R}^n,\ x \neq y. \tag{3.66}$$

To verify this assertion, recall that, if $f \in C^\infty_{0,\perp}(\Omega)$, then $\mathbf{u} \in C^\infty_0(\Omega, \mathbb{R}^n)$, and moreover equations (3.64) and (3.14) hold for every $x \in \Omega$ [36, Proof of Lemma III.3.1]. Consider next the general case when $f \in L^A_\perp(\Omega)$. Owing to (3.2), $L^A_\perp(\Omega) \to L\mathrm{Log}L_\perp(\Omega)$, since $B(t)$ grows at least linearly near infinity, and hence $A(t)$ dominates the function $t\log(1+t)$ near infinity. Since the space $C^\infty_{0,\perp}(\Omega)$ is dense in $L\mathrm{log}L_\perp(\Omega)$, there exists a sequence of functions $\{f_k\} \subset C^\infty_{0,\perp}(\Omega)$ such that $f_k \to f$ in $L\mathrm{log}L(\Omega)$. One has that

$$\mathcal{B}_\Omega : L\mathrm{log}L(\Omega) \to L^1(\Omega, \mathbb{R}^n)$$

(in fact, $\mathcal{B}_\Omega$ is also bounded into $L\mathrm{log}L(\Omega, \mathbb{R}^n)$). Furthermore,

$$H_{ij} : L\mathrm{log}L(\Omega) \to L^1(\Omega),$$

as a consequence of (3.66) and of a special case of Theorem 3.9, with $L^A(\Omega) = L\log L(\Omega)$ and $L^B(\Omega) = L^1(\Omega)$. Thus, $\mathcal{B}_\Omega f_k \to \mathcal{B}_\Omega f$ in $L^1(\Omega, \mathbb{R}^n)$ and $H_{ij} f_k \to H_{ij} f$ in $L^1(\Omega)$. This implies that $\mathbf{u} \in W_0^{1,1}(\Omega, \mathbb{R}^n)$, and (3.64) and (3.14) hold.
By Theorem 3.9, the singular integral operator defined by the first addend on the right-hand side of (3.65) is bounded from $L^A(\Omega)$ into $L^B(\Omega)$. By inequality (3.66), the operator defined by the second addend on the right-hand side of (3.65) has (at least) the same boundedness properties as a Riesz potential operator with kernel $\frac{1}{|x-y|^{n-1}}$. Such an operator is bounded in $L^1(\Omega)$ and in $L^\infty(\Omega)$, with norms depending only on $|\Omega|$ and on n. An interpolation theorem by Calderon [7, Theorem 2.12, Chap. 3] then ensures that it is also bounded from $L^A(\Omega)$ into $L^A(\Omega)$, and hence, a fortiori, from $L^A(\Omega)$ into $L^B(\Omega)$, with norm depending on n and $|\Omega|$. Finally, the operator given by the last addend on the right-hand side of (3.65) is pointwise bounded (in absolute value) by $|f(x)|$. Thus, it is bounded from $L^A(\Omega)$ into $L^A(\Omega)$, and hence from $L^A(\Omega)$ into $L^B(\Omega)$. Equations (3.12) and (3.15) are thus established.
Inequality (3.16) can be derived from (3.15) via a scaling argument analogous to that which leads to (3.33) from (3.32) – see the Proof of Theorem 3.9. □

Proof of Theorem 3.7, sketched. By an analogous argument as in the proofs of [36, Lemmas 3.4 and 3.5, and Theorem 3.3], it suffices to show that, if Ω and G are bounded Lipschitz domains, such that the domain $\Omega_0 = \Omega \cap G$ is star-shaped with respect to a ball $B \subset\subset \Omega_0$, and f has the form

$$f = \zeta \,\mathrm{div}\, \mathbf{g} + \theta \int_\Omega \phi \,\mathrm{div}\, \mathbf{g}\, dy,$$

for some functions $\zeta \in C_0^\infty(G)$, $\theta \in C_0^\infty(\Omega_0)$ and $\varphi \in C^\infty(\overline{\Omega})$, and fulfills

$$\int_{\Omega_0} f(x)\, dx = 0,$$

then there exists a function $\mathbf{w} \in W_0^{1,B}(\Omega, \mathbb{R}^n)$ such that

$$\mathrm{div}\, \mathbf{w} = f \qquad \text{in } \Omega_0, \tag{3.67}$$

$$\|\nabla \mathbf{w}\|_{L^B(\Omega_0, \mathbb{R}^{n\times n})} \le C \|\,\mathrm{div}\, \mathbf{g}\|_{L^A(\Omega)}, \tag{3.68}$$

and

$$\|\mathbf{w}\|_{L^B(\Omega_0, \mathbb{R}^n)} \le C \|\mathbf{g}\|_{L^A(\Omega, \mathbb{R}^n)}, \tag{3.69}$$

for some constant $C = C(\varphi, \theta, \zeta, c, B, G, \Omega)$, where c is the constant appearing in (3.2) and (3.3).
Since $f \in L^A_\perp(\Omega_0)$, an inspection of the proof of Theorem 3.6 then reveals that the function $\mathbf{w}$, given by

$$\mathbf{w}(x) = \int_{\Omega_0} f(y) \mathbf{N}(x, y) dy \quad \text{for } x \in \Omega_0, \tag{3.70}$$

where

$$\mathbf{N}(x,y) = \frac{x-y}{|x-y|^n}\int_{|x-y|}^{\infty}\omega\Big(y+\zeta\frac{x-y}{|x-y|}\Big)\zeta^{n-1}\,d\zeta \quad \text{for } x,y\in\Omega, \tag{3.71}$$

and ω is any (nonnegative) function in $C_0^\infty(B)$ with $\int_B \omega\,dx = 1$, satisfies (3.67), and

$$\|\nabla \mathbf{w}\|_{L^B(\Omega_0,\mathbb{R}^{n\times n})} \le C\|f\|_{L^A(\Omega_0)} \tag{3.72}$$

for some constant C. Since

$$\|f\|_{L^A(\Omega_0)} \le C\|\operatorname{div}\,\mathbf{g}\|_{L^A(\Omega)} \tag{3.73}$$

for some constant C, inequality (3.68) follows. It remains to prove (3.69). To this purpose, assume, for the time being, that div $\mathbf{g}\in C_0^\infty(\Omega)$. Then, by [36, Equation 3.35],

$$\begin{aligned} w_i(x) = &-\int_{\Omega_0} N_i(x,y)\mathbf{g}(y)\cdot\nabla\zeta(y)\,dy - \int_{\Omega_0} K_{ij}(x,x-y)\zeta(y)g_j(y)\,dy\\ &-\int_{\Omega_0}\sum_{j=1}^n G_{ij}(x,y)\zeta(y)g_j(y)\,dy - \zeta(x)\sum_{j=1}^n g_j(x)\int_{\Omega_0}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\omega(y)\,dy\\ &-\Big(\int_\Omega \mathbf{g}\cdot\nabla\varphi\,dy\Big)\int_{\Omega_0} N_i(x,y)\theta(y)\,dy \quad \text{for a.e. } x\in\Omega_0, \end{aligned} \tag{3.74}$$

where the kernels K_{ij} and G_{ij} satisfy the same assumptions as the kernels in (3.65), and $|\mathbf{N}(x,y)|\le C|x-y|^{1-n}$ for some constant C. Note that condition (3.19) has been used in writing the last addend on the right-hand side of equation (3.74).
We now drop the assumption that div $\mathbf{g}\in C_0^\infty(\Omega)$. Condition (3.2) entails that $L^A(\Omega)\to L\log L(\Omega)$, and hence $H_0^A(\Omega)\to H_0^{L\log L}(\Omega)$, where the latter space denotes $H_0^A(\Omega)$ with $A(t)$ equivalent to $t\log(1+t)$ near infinity. Thus, $\mathbf{g}\in H_0^{L\log L}(\Omega)$. One can show that any such function can be approximated by a sequence of functions $\{\mathbf{g}_k\}\subset C_0^\infty(\Omega,\mathbb{R}^n)$ in such a way that $g_k\to g$ in $H^{L\log L}(\Omega,\mathbb{R}^n)$. This follows from an analogous argument as in the proof of [48, Theorem 1.3]. In particular, the fact that $C_0^\infty(\Omega)$ is dense in $L\log L(\Omega)$, since the function $t\log(1+t)\in\Delta_2$ near infinity, plays a role here. The first and third addend on the right-hand side of (3.74) are integral operators applied to $\mathbf{g}$ whose kernel is bounded by a multiple of $|x-y|^{1-n}$. The fourth addend is just bounded by a constant multiple of $|\mathbf{g}|$. The last addend is a constant multiple of an integral of $\mathbf{g}$ against a the bounded vector-valued function $\nabla\varphi$. Hence, all these operators are bounded from $L^A(\Omega,\mathbb{R}^n)$ into $L^A(\Omega_0)$. The second addend is a singular integral operator enjoying the same properties as the singular integral operator K in Theorems 3.8 and 3.9. Thus, since all operators appearing on the right-hand side of (3.74) are bounded from $L\log L(\Omega,\mathbb{R}^n)$ into $L^1(\Omega_0)$, the right-hand side of (3.74), evaluated with $\mathbf{g}$ replaced by $\mathbf{g}_k$, converges in $L^1(\Omega_0)$ to the right-hand side of (3.74). On the other hand, equation (3.70) and the properties of the Bogovskiĭ operator tell us that the left-hand side of (3.74), with w_i corresponding to $\mathbf{g}_k$, converges in $L^1(\Omega_0)$ to the

left-hand side of (3.74). Altogether, we conclude that (3.74) actually holds even if $\mathbf{g}$ is just in $H_0^A(\Omega)$.
The properties of the operators on the right-hand side of (3.74) mentioned above ensure that they are bounded from $L^A(\Omega,\mathbb{R}^n)$ into $L^B(\Omega_0)$. Inequality (3.69) thus follows from (3.74). □

We need a last preliminary result in preparation for the proof of Theorem 3.1.

Proposition 3.11. *Let Ω be an open subset in $\mathbb{R}^n$ such that $|\Omega|<\infty$, and let A be a Young function. Assume that $u\in L^A(\Omega)$. Then:*

$$\sup_{v\in L^{\widetilde{A}}(\Omega)}\frac{\int_\Omega uv\,dx}{\|v\|_{L^{\widetilde{A}}(\Omega)}}=\sup_{\varphi\in C_0^\infty(\Omega)}\frac{\int_\Omega u\varphi\,dx}{\|\varphi\|_{L^{\widetilde{A}}(\Omega)}}, \tag{3.75}$$

and

$$\sup_{v\in L_\perp^{\widetilde{A}}(\Omega)}\frac{\int_\Omega uv\,dx}{\|v\|_{L^{\widetilde{A}}(\Omega)}}=\sup_{\varphi\in C_{0,\perp}^\infty(\Omega)}\frac{\int_\Omega u\varphi\,dx}{\|\varphi\|_{L^{\widetilde{A}}(\Omega)}}. \tag{3.76}$$

Note that equation (3.75) is well known under the assumption that $A\in\nabla_2$ near infinity, namely $\widetilde{A}\in\Delta_2$ near infinity, since $C_0^\infty(\Omega)$ is dense in $L^{\widetilde{A}}(\Omega)$ in this case. Equation (3.76) also easily follows from this property when $A\in\nabla_2$ near infinity. The novelty of Proposition 3.11 is in the arbitrariness of A.

Proof of Proposition 3.11. Consider first (3.75). It clearly suffices to show that

$$\sup_{v\in L^{\widetilde{A}}(\Omega)}\frac{\int_\Omega uv\,dx}{\|v\|_{L^{\widetilde{A}}(\Omega)}}=\sup_{v\in L^\infty(\Omega)}\frac{\int_\Omega uv\,dx}{\|v\|_{L^{\widetilde{A}}(\Omega)}}, \tag{3.77}$$

and

$$\sup_{v\in L^\infty(\Omega)}\frac{\int_\Omega uv\,dx}{\|v\|_{L^{\widetilde{A}}(\Omega)}}=\sup_{\varphi\in C_0^\infty(\Omega)}\frac{\int_\Omega u\varphi\,dx}{\|\varphi\|_{L^{\widetilde{A}}(\Omega)}}. \tag{3.78}$$

Given any $v\in L^{\widetilde{A}}(\Omega)$, define, for $k\in\mathbb{N}$, the function $v_k:\Omega\to\mathbb{R}$ as

$$v_k=\operatorname{sign}(v)\,\min\{|v|,k\}. \tag{3.79}$$

Clearly, $v_k\in L^\infty(\Omega)$, and $0\le|v_k|\nearrow|v|$ a.e. in Ω as $k\to\infty$. Hence,

$$\int_\Omega|uv_k|\,dx\nearrow\int_\Omega|uv|\,dx\quad\text{as } k\to\infty,$$

by the monotone convergence theorem for integrals, and, by the Fatou property of the Luxemburg norm,

$$\|v_k\|_{L^{\widetilde{A}}(\Omega)}\nearrow\|v\|_{L^{\widetilde{A}}(\Omega)}\quad\text{as } k\to\infty.$$

Thus, since

$$\sup_{v\in L^{\widetilde{A}}(\Omega)} \frac{\int_\Omega uv\,dx}{\|v\|_{L^{\widetilde{A}}(\Omega)}} = \sup_{v\in L^{\widetilde{A}}(\Omega)} \frac{\int_\Omega |uv|\,dx}{\|v\|_{L^{\widetilde{A}}(\Omega)}},$$

equation (3.77) follows.
As far as (3.78) is concerned, consider an increasing sequence of compact sets E_k such that $\mathrm{dist}(E_k, \mathbb{R}^n \setminus \Omega) \geq \frac{2}{k}$, $E_k \subset E_{k+1} \subset \Omega$ for $k \in \mathbb{N}$, and $\cup_k E_k = \Omega$. Moreover, let $\{\varrho_k\}$ be a family of (nonnegative) smooth mollifiers in $\mathbb{R}^n$, such that $\mathrm{supp}\varrho_k \subset B_{\frac{1}{k}}(0)$ and $\int_{\mathbb{R}^n} \varrho_k\,dx = 1$ for $k \in \mathbb{N}$. Given $v \in L^\infty(\Omega)$, define $w_k : \mathbb{R}^n \to \mathbb{R}$ as

$$w_k = \begin{cases} v & \text{in } E_k, \\ 0 & \text{elsewhere,} \end{cases}$$

and $\varphi_k : \mathbb{R}^n \to \mathbb{R}$ as

$$\varphi_k(x) = \int_{\mathbb{R}^n} w_k(y)\varrho_k(x-y)\,dy \quad \text{for } x \in \mathbb{R}^n. \tag{3.80}$$

Classical properties of mollifiers ensure that

$$\varphi_k \in C_0^\infty(\Omega), \quad \varphi_k \to v \text{ a.e. in } \Omega \text{ as } k \to \infty, \quad \|\varphi_k\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega)} \text{ for } k \in \mathbb{N}.$$

Thus, if $u \in L^A(\Omega)$, then

$$\int_\Omega u\varphi_k\,dx \to \int_\Omega uv\,dx \quad \text{as } k \to \infty, \tag{3.81}$$

by the dominated convergence theorem for integrals. Moreover,

$$\|\varphi_k\|_{L^{\widetilde{A}}(\Omega)} \to \|v\|_{L^{\widetilde{A}}(\Omega)} \quad \text{as } k \to \infty. \tag{3.82}$$

Indeed, by dominated convergence and the definition of Luxemburg norm,

$$\int_\Omega \widetilde{A}\left(\frac{|\varphi_k|}{\|v\|_{L^{\widetilde{A}}(\Omega)}}\right) dx \to \int_\Omega \widetilde{A}\left(\frac{|v|}{\|v\|_{L^{\widetilde{A}}(\Omega)}}\right) dx \leq 1 \quad \text{as } k \to \infty.$$

In particular, for every $\varepsilon > 0$, there exists k_ε such that

$$\int_\Omega \widetilde{A}\left(\frac{|\varphi_k|}{\|v\|_{L^{\widetilde{A}}(\Omega)}}\right) dx < 1 + \varepsilon \quad \text{if } k > k_\varepsilon.$$

Hence, by the arbitrariness of ε and the definition of Luxemburg norm,

$$\liminf_{k\to\infty} \|\varphi_k\|_{L^{\widetilde{A}}(\Omega)} \geq \|v\|_{L^{\widetilde{A}}(\Omega)}. \tag{3.83}$$

We also have that

$$\limsup_{k\to\infty} \|\varphi_k\|_{L^{\widetilde{A}}(\Omega)} \leq \|v\|_{L^{\widetilde{A}}(\Omega)}. \tag{3.84}$$

Indeed, assume that (3.84) fails. Then, there exists $\sigma > 0$ and a subsequence of $\{\varphi_k\}$, still denoted by $\{\varphi_k\}$, such that

$$1 < \int_\Omega \widetilde{A}\bigg(\frac{|\varphi_k|}{\|v\|_{L^{\widetilde{A}}(\Omega)} + \sigma}\bigg)\, dx \to \int_\Omega \widetilde{A}\bigg(\frac{|v|}{\|v\|_{L^{\widetilde{A}}(\Omega)} + \sigma}\bigg)\, dx \le 1\,,$$

a contradiction. Equation (3.82) follows from (3.83) and (3.84). Coupling (3.81) with (3.82) yields (3.78). The proof of (3.75) is complete.
The proof of (3.76) follows along the same lines, and, in particular, via the equations

$$\sup_{v \in L^{\widetilde{A}}_\perp(\Omega)} \frac{\int_\Omega uv\, dx}{\|v\|_{L^{\widetilde{A}}(\Omega)}} = \sup_{v \in L^\infty_\perp(\Omega)} \frac{\int_\Omega uv\, dx}{\|v\|_{L^{\widetilde{A}}(\Omega)}}, \tag{3.85}$$

and

$$\sup_{v \in L^\infty_\perp(\Omega)} \frac{\int_\Omega uv\ \, dx}{\|v\|_{L^{\widetilde{A}}(\Omega)}} = \sup_{\varphi \in C^\infty_{0,\perp}(\Omega)} \frac{\int_\Omega u\varphi\, dx}{\|\varphi\|_{L^{\widetilde{A}}(\Omega)}}. \tag{3.86}$$

On defining, for any $v \in L^{\widetilde{A}}_\perp(\Omega)$, the sequence of functions $\{\overline{v}_k\} \subset L^\infty_\perp(\Omega)$ as

$$\overline{v}_k = v_k - (v_k)_\Omega$$

for $k \in \mathbb{N}$, where v_k is given by (3.79), one can prove equation (3.85) via a slight variant of the argument employed for (3.77). Here, one has to use the fact that $(v_k)_\Omega \to 0$ as $k \to \infty$. Similarly, equation (3.86) can be established similarly to (3.78) on replacing, for any given $v \in L^\infty_\perp(\Omega)$, the sequence $\{\varphi_k\}$ defined by (3.80) with the sequence $\{\overline{\varphi}_k\} \subset C^\infty_{0,\perp}(\Omega)$ defined as

$$\overline{\varphi}_k = \varphi_k - (\varphi_k)_\Omega\, \psi \quad \text{for } k \in \mathbb{N},$$

where ψ is any function in $C^\infty_0(\Omega)$ such that $\int_\Omega \psi\, dx = 1$. Note that now, for every $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that $\|\overline{\varphi}_k\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega)} + \varepsilon$, provided that $k > k_\varepsilon$. $\square$

Proof of Theorem 3.1. Let $u \in L^1(\Omega)$.Then

$$\begin{aligned} \|u - u_\Omega\|_{L^B(\Omega)} &= \sup_{v \in L^{\widetilde{A}}(\Omega)} \frac{\int_\Omega (u - u_\Omega)\, v\, dx}{\|v\|_{L^{\widetilde{B}}(\Omega)}} = \sup_{v \in L^{\widetilde{A}}(\Omega)} \frac{\int_\Omega (u - u_\Omega)\, (v - v_\Omega)\, dx}{\|v\|_{L^{\widetilde{B}}(\Omega)}} \\ &= \sup_{v \in L^{\widetilde{A}}(\Omega)} \frac{\int_\Omega u\, (v - v_\Omega)\, dx}{\|v\|_{L^{\widetilde{B}}(\Omega)}} \le 3 \sup_{v \in L^{\widetilde{A}}(\Omega)} \frac{\int_\Omega u\, (v - v_\Omega)\, dx}{\|v - v_\Omega\|_{L^{\widetilde{B}}(\Omega)}} \\ &= 3 \sup_{v \in L^{\widetilde{A}}_\perp(\Omega)} \frac{\int_\Omega u\, v\, dx}{\|v\|_{L^{\widetilde{B}}(\Omega)}} = 3 \sup_{\varphi \in C^\infty_{0,\perp}(\Omega)} \frac{\int_\Omega u\varphi\, dx}{\|\varphi\|_{L^{\widetilde{A}}(\Omega)}}. \end{aligned} \tag{3.87}$$

Note that the inequality in (3.87) holds since, by the first inequality in (2.2),

$$\|v - v_\Omega\|_{L^{\widetilde{B}}(\Omega)} \le \|v\|_{L^{\widetilde{B}}(\Omega)} + \|v_\Omega\|_{L^{\widetilde{B}}(\Omega)} \le \|v\|_{L^{\widetilde{B}}(\Omega)} + |v_\Omega| \|1\|_{L^{\widetilde{B}}(\Omega)}$$

$$
\begin{aligned}
&\le \|v\|_{L^{\widetilde{B}}(\Omega)} + \tfrac{2}{|\Omega|}\|v\|_{L^{\widetilde{B}}(\Omega)}\|1\|_{L^{B}(\Omega)}\|1\|_{L^{\widetilde{B}}(\Omega)} \\
&= \|v\|_{L^{\widetilde{B}}(\Omega)} + \tfrac{2}{|\Omega|}\|v\|_{L^{\widetilde{B}}(\Omega)}\tfrac{1}{B^{-1}(|\Omega|)}\tfrac{1}{\widetilde{B}^{-1}(|\Omega|)} \le 3\|v\|_{L^{\widetilde{B}}(\Omega)},
\end{aligned}
$$

and the last equality in (3.87) relies upon (3.76). By Theorem 3.6, applied with A and B replaced with $\widetilde{B}$ and $\widetilde{A}$, respectively, there exists a constant $C = C(\Omega, c)$ such that

$$
\begin{aligned}
\sup_{\varphi\in C^\infty_{0,\perp}(\Omega)} \frac{\int_\Omega u\varphi\,dx}{\|\varphi\|_{L^{\widetilde{A}}(\Omega)}} &= \sup_{\varphi\in L^{\widetilde{B}}_\perp(\Omega)} \frac{\int_\Omega u\ \mathrm{div}\,(\mathcal{B}_\Omega\varphi)\,dx}{\|\varphi\|_{L^{\widetilde{B}}(\Omega)}} \le C \sup_{\varphi\in L^{\widetilde{B}}_\perp(\Omega)} \frac{\int_\Omega u\ \mathrm{div}\,(\mathcal{B}_\Omega\varphi)\,dx}{\|\nabla\mathcal{B}_\Omega\varphi\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}} \\
&\le C \sup_{\boldsymbol{\varphi}\in C^\infty_0(\Omega,\mathbb{R}^n)} \frac{\int_\Omega u\ \mathrm{div}\ \boldsymbol{\varphi}\,dx}{\|\nabla\boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}\,dx = C\,\|u\|_{W^{-1,A}(\Omega)}.
\end{aligned}
\tag{3.88}
$$

The first inequality in (3.4) follows from (3.87) and (3.88). The second inequality is trivial, since

$$
\begin{aligned}
\|u\|_{W^{-1,A}(\Omega)} &= \sup_{\boldsymbol{\varphi}\in C^\infty_0(\Omega,\mathbb{R}^n)} \int_\Omega \frac{u\ \mathrm{div}\ \boldsymbol{\varphi}}{\|\nabla\boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}\,dx = \sup_{\boldsymbol{\varphi}\in C^\infty_0(\Omega,\mathbb{R}^n)} \frac{\int_\Omega \big(u-u_\Omega\big)\ \mathrm{div}\ \boldsymbol{\varphi}\,dx}{\|\nabla\boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}} \\
&\le C \sup_{\boldsymbol{\varphi}\in C^\infty_0(\Omega,\mathbb{R}^n)} \frac{\int_\Omega \big(u-u_\Omega\big)\ \mathrm{div}\ \boldsymbol{\varphi}\,dx}{\|\ \mathrm{div}\ \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega)}}\,dx \le C \sup_{\varphi\in C^\infty_\perp(\Omega)} \frac{\int_\Omega \big(u-u_\Omega\big)\,\varphi\,dx}{\|\varphi\|_{L^{\widetilde{A}}(\Omega)}}\,dx \\
&\le 2C\|u-u_\Omega\|_{L^A(\Omega)},
\end{aligned}
$$

for some constant $C = C(n)$. □

4 Nonlinear systems in fluid mechanics

In many customary mathematical models, the stationary flow of a homogeneous incompressible fluid in a bounded domain $\Omega \subset \mathbb{R}^n$ is described by a system with the structure (1.3). With a slight abuse of notation with respect to (1.3), we also denote by $\mathbf{S} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ the function, acting on the symmetric gradient $\mathbf{D}(\mathbf{v})$ of the velocity field $\mathbf{v}$, which yields the stress deviator of the fluid. Thus, we shall consider systems of the form

$$
\begin{cases}
-\,\mathrm{div}\ \mathbf{S}(\mathbf{D}(\mathbf{v})) + \varrho\,\mathrm{div}\ \big(\mathbf{v}\otimes\mathbf{v}\big) + \nabla\pi = \varrho\,\mathrm{div}\ \mathbf{F} & \text{in } \Omega,\\
\mathrm{div}\ \mathbf{v} = 0 & \text{in } \Omega,\\
\mathbf{v} = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{4.1}
$$

In the simplest case of a Newtonian fluid, the function $\mathbf{S}$ is linear, and div $\mathbf{S}(\mathbf{D}(\mathbf{v})) = \Delta\mathbf{v}$, the Laplacian of $\mathbf{v}$. However only fluids with an easy molecular structure, such as water, oil, and several gases are governed by this low. More complex liquids are not, and are called Non-Newtonian fluids – see e.g. [2, 5]. The most common nonlinear model among rheologists is the power law model, corresponding to the choice

$$
\mathbf{S}(\boldsymbol{\xi}) = \nu_0\big(\kappa_0 + |\boldsymbol{\xi}|\big)^{p-2}\boldsymbol{\xi} \quad \text{for } \xi \in \mathbb{R}^{n\times n}.
\tag{4.2}
$$

Here, $\nu_0 \in (0,\infty)$ and $\kappa_0 \in [0,\infty)$ are constants, and $p \in (1,\infty)$ is an exponent which need to be specified via physical experiments. An extensive list of specific p-values for different fluids can be found in [5].
A more general constitutive equation for Non-Newtonian fluids, which allows for non-polynomial type nonlinearities, takes the form

$$\mathbf{S}(\boldsymbol{\xi}) = \frac{\Phi'(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}\boldsymbol{\xi} \quad \text{for } \boldsymbol{\xi} \in \mathbb{R}^{n\times n}\,, \tag{4.3}$$

where Φ is a Young function.
In various instances of interest in applications, the term

$$\operatorname{div}\big(\mathbf{v}\otimes\mathbf{v}\big) = (\nabla\mathbf{v})\mathbf{v} \tag{4.4}$$

is negligible in (4.1), compared with the other terms appearing in the first equation. This is the case, for example, if the modulus of the velocity $\mathbf{v}$ is small. Another situation where the role of the term (4.4) is immaterial is that of plastic or pseudo-plastic fluids. Indeed, (4.4) accounts for the inner rotation in the fluid flow, and for such fluids the impact of this term is very limited. Dropping the term (4.4) reduces (4.1) to the simplified system

$$\begin{cases} -\operatorname{div}\,\mathbf{S}(\mathbf{D}(\mathbf{v})) + \nabla\pi = \varrho\operatorname{div}\,\mathbf{F} & \text{in } \Omega,\\ \operatorname{div}\,\mathbf{v} = 0 & \text{in } \Omega,\\ \mathbf{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

A standard approach to (4.1) or (4.5) consists in two steps. Firstly, a velocity field $\mathbf{v}$ is exhibited such that

$$\int_\Omega \mathbf{H} : \nabla\boldsymbol{\phi}\,dx = 0 \tag{4.6}$$

for every $\boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(\Omega,\mathbb{R}^n)$, where either

$$\mathbf{H} = \mathbf{S}(\mathbf{D}(\mathbf{v})) + \rho\,\mathbf{F}\,, \tag{4.7}$$

or

$$\mathbf{H} = \mathbf{S}(\mathbf{D}(\mathbf{v})) + \rho\,\mathbf{F}\, - \rho\,\mathbf{v}\otimes\mathbf{v}\,, \tag{4.8}$$

according to weather the convective term $\mathbf{v}\otimes\mathbf{v}$ is included in the model or not. Here, " : " stands for scalar product between matrices, and $C^\infty_{0,\mathrm{div}}(\Omega,\mathbb{R}^n)$ denotes the space of compactly supported, infinitely differentiable $\mathbb{R}^n$-valued functions whose divergence vanishes in Ω. The function $\mathbf{v}$ belongs to a proper Sobolev type space depending on the constitutive equation underlying the definition of the function $\mathbf{S}$. Secondly, the pressure π is reconstructed.

A discussion of the first issue falls beyond the scopes of the present paper, and will not be addressed here. Let us just mention that the standard power type model (4.2) has been investigated in the classical contributions [39, 40, 41, 42], and in the recent papers

[34, 35, 31, 30, 13]. Stationary flows of fluids whose constitutive equation satisfies (4.3) with a Young-function $\Phi \in \Delta_2 \cap \nabla_2$ are studied in [14, 10, 26, 11]. An unconventional constitutive equation, where $\mathbf{S}$ has the form (4.3) with $\Phi'(t) \approx \log(1+t)$ near infinity, and hence $\Phi \notin \nabla_2$ near infinity, was introduced by Eyring in [32], where it is assumed that

$$\mathbf{S}(\boldsymbol{\xi}) = \nu_0 \frac{\operatorname{arsinh}(\lambda_0|\boldsymbol{\xi}|)}{\lambda_0|\boldsymbol{\xi}|}\boldsymbol{\xi} \quad \text{for } \boldsymbol{\xi} \in \mathbb{R}^{n\times n}, \tag{4.9}$$

for some physical constants $\nu_0, \lambda > 0$. Similar results are due to Prandtl (see e.g. [17] for an overview on this kind of models). An analysis of the simplified system (4.5) for the Eyring-Prandtl model is the object of [33], whereas the complete system (4.1), in the case $n = 2$, is considered in [12]. System (4.5), with $\mathbf{S}$ given by (4.9), is also included, as a special case, in the papers [15, 16], where constitutive equations with much more general growths, possibly "oscillating" between two different powers, are also considered. Parabolic versions are treated in [37].

In the remaining part of this paper, we focus, instead, on the second question, namely the reconstruction of the pressure π in a correct Orlicz space. In case of fluids governed by a general constitutive low of the form (4.3), the function $\mathbf{H}$ belongs to some Orlicz space $L^A(\Omega, \mathbb{R}^{n\times n})$. If $A \in \Delta_2 \cap \nabla_2$, then $\pi \in L^A(\Omega)$ as well. However, in general, one can only expect that π belongs to some larger Orlicz space $L^B(\Omega)$. The balance between the Young functions A and B is determined by conditions (3.2) and (3.3), as stated in the following result.

Theorem 4.1. *Let A and B be Young functions fulfilling* (3.2) *and* (3.3)*. Let Ω be a bounded domain with the cone property in $\mathbb{R}^n$, $n \geq 2$. Assume that $\mathbf{H} \in L^A(\Omega, \mathbb{R}^{n\times n})$ and satisfies*

$$\int_\Omega \mathbf{H} : \nabla\boldsymbol{\phi}\, dx = 0$$

for every $\boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^n)$. Then there exists a unique function $\pi \in L^B_\perp(\Omega)$ such that

$$\int_\Omega \mathbf{H} : \nabla\boldsymbol{\phi}\, dx = \int_\Omega \pi \ \mathrm{div}\ \boldsymbol{\varphi}\, dx \tag{4.10}$$

for every $\boldsymbol{\varphi} \in C^\infty_0(\Omega, \mathbb{R}^n)$. Moreover, there exists a constant $C = C(\Omega, c)$ such that

$$\|\pi\|_{L^B(\Omega)} \leq C\|\mathbf{H} - \mathbf{H}_\Omega\|_{L^A(\Omega,\mathbb{R}^{n\times n})}, \tag{4.11}$$

and

$$\int_\Omega B(|\pi|)\, dx \leq \int_\Omega A(C|\mathbf{H} - \mathbf{H}_\Omega|)\, dx. \tag{4.12}$$

Here, c denotes the constant appearing in (3.2) *and* (3.3)*.*

In particular, Theorem 4.1 reproduces, within a unified framework, various results appearing in the literature. For instance, when the constitutive relation (4.2) is in force, the function $A(t)$ is just a power t^q, where the exponent $q > 1$, and depends on p, on $\mathbf{F}$, and on whether the system (4.1) or (4.5) is taken into account. In any case, $L^A(\Omega, \mathbb{R}^{n\times n})$ agrees with the Lebesgue space $L^q(\Omega, \mathbb{R}^{n\times n})$, and Theorem 4.1 recovers the fact that π belongs to the same Lebesgue space $L^q(\Omega)$.

As far as the simplified system (4.5) for the Eyring-Prandtl model (4.9) is concerned, under appropriate assumptions on $\mathbf{F}$ one has that $\mathbf{H} \in \exp L(\Omega, \mathbb{R}^{n\times n})$. Hence, via Theorem 4.1, we infer the existence of a pressure $\pi \in \exp L^{\frac{1}{2}}(\Omega)$. More generally, if $\mathbf{H} \in \exp L^{\beta}(\Omega, \mathbb{R}^{n\times n})$ for some $\beta > 0$, one has that $\pi \in \exp L^{\beta/(\beta+1)}(\Omega)$. The complete system (4.1) for the Eyring-Prandtl model, in the 2-dimensional case, admits a weak solution $\mathbf{v}$ such that $\mathbf{v} \otimes \mathbf{v} \in L\log L^2(\Omega, \mathbb{R}^{n\times n})$ and hence $\mathbf{H} \in L\log L^2(\Omega, \mathbb{R}^{n\times n})$ [12]. Again, one cannot expect that the pressure π belongs to the same space. In fact, Theorem 4.1 yields the existence of a pressure $\pi \in L\log L(\Omega)$, thus reproducing a result from [12]. In general, if $\mathbf{H} \in L\log L^{\alpha}(\Omega, \mathbb{R}^{n\times n})$ for some $\alpha \geq 1$, then we obtain that $\pi \in L\log L^{\alpha-1}(\Omega)$.

Proof of Theorem 4.1. By De Rahms Theorem, in the version of [47], there exists a distribution Ξ such that

$$\int_\Omega \mathbf{H} : \nabla \boldsymbol{\phi}\, dx = \Xi(\operatorname{div} \boldsymbol{\varphi}) \tag{4.13}$$

for every $\boldsymbol{\varphi} \in C_0^\infty(\Omega, \mathbb{R}^n)$. Replacing $\boldsymbol{\varphi}$ with $\mathcal{B}_\Omega\big(\varphi - \varphi_\Omega\big)$ in (4.13), where $\varphi \in C_0^\infty(\Omega)$, yields

$$\int_\Omega \mathbf{H} : \nabla \mathcal{B}_\Omega\big(\varphi - \varphi_\Omega\big)\, dx = \Xi\big(\varphi - \varphi_\Omega\big)$$

for every $\varphi \in C_0^\infty(\Omega)$. We claim that the linear functional $C_0^\infty(\Omega) \ni \varphi \mapsto \Xi\big(\varphi - \varphi_\Omega\big)$ is bounded on $C_0^\infty(\Omega)$ equipped with the $L^\infty(\Omega)$ norm. Indeed, by (3.2), one has that $L^A(\Omega, \mathbb{R}^{n\times n}) \to L\log L(\Omega, \mathbb{R}^{n\times n})$. Moreover, by a special case of Theorem 3.6, $\nabla \mathcal{B}_\Omega : L^\infty_\perp(\Omega) \to \exp L(\Omega, \mathbb{R}^{n\times n})$. Thus, since $L\log L(\Omega, \mathbb{R}^{n\times n})$ and $\exp L(\Omega, \mathbb{R}^{n\times n})$ are Orlicz spaces built upon Young functions which are conjugate of each other,

$$\begin{aligned}\left| \int_\Omega \mathbf{H} : \nabla \mathcal{B}_\Omega\big(\varphi - \varphi_\Omega\big)\, dx \right| &\leq C \|\mathbf{H}\|_{L\log L(\Omega, \mathbb{R}^{n\times n})} \|\nabla \mathcal{B}_\Omega\big(\varphi - \varphi)_\Omega\big)\|_{\exp L(\Omega, \mathbb{R}^{n\times n})} \\ &\leq C' \|\mathbf{H}\|_{L^A(\Omega, \mathbb{R}^{n\times n})} \|\varphi - \varphi_\Omega\|_{L^\infty(\Omega)} \\ &\leq C'' \|\mathbf{H}\|_{L^A(\Omega, \mathbb{R}^{n\times n})} \|\varphi\|_{L^\infty(\Omega)}\,,\end{aligned} \tag{4.14}$$

for every $\varphi \in C_0^\infty(\Omega)$, where $C = C(|\Omega|, n)$ and $C' = C'(\Omega, c)$. Hence, the relevant functional can be continued to a bounded linear functional on $\varphi \in C_0^0(\Omega)$, with the same norm.

Now, as a consequence of Riesz's representation Theorem, there exists a Radon measure Ξ such that

$$\Xi\big(\varphi - \varphi_\Omega\big) = \int_\Omega \varphi\, d\mu$$

for every $\varphi \in C_0^0(\Omega)$. Fix any open set $E \subset \Omega$. By Theorem 3.6 again, there exists a constant C such that

$$\mu(E) = \sup_{\varphi \in C_0^0(E),\ \|\varphi\|_\infty = 1} \Xi\big(\varphi - \varphi_\Omega\big) = \sup_{\varphi \in C_0^0(E),\ \|\varphi\|_\infty = 1} \int_\Omega \mathbf{H} : \nabla \mathcal{B}_\Omega\big(\varphi - \varphi_\Omega\big)\, dx \tag{4.15}$$

$$\leq \sup_{\varphi \in C_0^0(E),\, \|\varphi\|_\infty = 1} \|\mathbf{H}\|_{L\log L(E,\mathbb{R}^{n\times n})} \|\nabla \mathcal{B}_\Omega\big(\varphi - \varphi_\Omega\big)\|_{\exp L(\Omega,\mathbb{R}^{n\times n})}$$
$$\leq C \sup_{\varphi \in C_0^0(E),\, \|\varphi\|_\infty = 1} \|\mathbf{H}\|_{L\log L(E,\mathbb{R}^{n\times n})} \|\varphi - (\varphi)_\Omega\|_{L^\infty(\Omega)} \leq C\, \|\mathbf{H}\|_{L\log L(E,\mathbb{R}^{n\times n})}.$$

One can verify that the norm $\|\cdot\|_{L\log L(E)}$ is absolutely continuous, in the sense that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{H}\|_{L\log L(E,\mathbb{R}^{n\times n})} < \varepsilon$ if $|E| < \delta$, and since any Lebesgue measurable set can be approximated from outside by open sets, inequality (4.15) implies that the measure μ is absolutely continuous with respect to the Lebesgue measure. Hence, μ has a density with respect to the Lebesgue measure. So Ξ can be represented by a function $\pi \in L^1(\Omega)$ fulfilling (4.10) holds. The function π is uniquely determined if we assume that $\pi_\Omega = 0$. By this assumption, Theorem 3.1, and equation (4.10) we have that

$$\|\pi\|_{L^B(\Omega)} \leq C\|\nabla\pi\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} = C \sup_{\boldsymbol{\varphi} \in C_0^\infty(\Omega,\mathbb{R}^n)} \frac{\int_\Omega \pi \operatorname{div} \boldsymbol{\varphi}\, dx}{\|\nabla \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}$$
$$= C \sup_{\boldsymbol{\varphi} \in C_0^\infty(\Omega,\mathbb{R}^n)} \frac{\int_\Omega \mathbf{H} : \nabla \boldsymbol{\varphi}\, dx}{\|\nabla \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}} = C \sup_{\boldsymbol{\varphi} \in C_0^\infty(\Omega,\mathbb{R}^n)} \frac{\int_\Omega (\mathbf{H} - \mathbf{H}_\Omega) : \nabla \boldsymbol{\varphi}\, dx}{\|\nabla \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}$$
$$\leq 2C\|\mathbf{H} - \mathbf{H}_\Omega\|_{L^A(\Omega,\mathbb{R}^{n\times n})},$$

where $C = C(\Omega, c)$. This proves inequality (4.11). Inequality (4.12) follows from (4.11), on replacing A and B with kA and kB, respectively, with $k = \frac{1}{\int_\Omega A(|\mathbf{H} - \mathbf{H}_\Omega|)\, dx}$, via an argument analogous to that of the proof of (3.33). □

Let us turn to a further consequence of Theorem 4.1, which is related to a numerical analysis of problem (4.1), or of its simplified version (4.5). We shall adopt the scheme of the finite element method for the p-Stokes system exploited in [6] in the special case when $\mathbf{S}$ is given by (4.2). In what follows, we assume that Ω is a polyhedron. The goal is to compute an approximate solution to problem (4.5) via discretization. To this purpose, one needs a triangulation $\mathscr{T}_h$ of Ω into simplices of diameter bounded by $h > 0$. Recall that a simplex in $\mathbb{R}^n$ is the convex hull of $n+1$ points which do not lie on the same hyperplane. We also need that such a triangulation is regular enough for the Lipschitz constant of the functions, which locally represent the boundaries of the relevant simplices, to be uniformly bounded in h. We denote by $\mathscr{P}^{0,h}(\Omega)$ the space of those functions in Ω whose restriction to each simplex of $\mathscr{T}_h$ is constant, and by $\mathscr{P}^{0,h}_\perp(\Omega)$ its subspace of those functions from $\mathscr{P}^{0,h}(\Omega)$ whose mean value over Ω is zero. Also, for $\ell \geq 1$, we denote by $\mathscr{P}^{\ell,h}(\Omega)$ the space of first-order weakly differentiable functions in Ω whose restriction to each simplex of $\mathscr{T}_h$ is a polynomial of degree ℓ, and by $\mathscr{P}^{\ell,h}_0(\Omega)$ its subspace of those functions from $\mathscr{P}^{\ell,h}(\Omega)$ which vanish on $\partial\Omega$. Finally $\mathscr{P}^{\ell,h}_\perp(\Omega)$ stands for the space of all functions from $\mathscr{P}^{\ell,h}(\Omega)$ whose mean value over Ω is zero. Clearly, given any Young function A, one has that

$$\mathscr{P}^{\ell,h}(\Omega) \subset L^A(\Omega), \quad \mathscr{P}^{\ell,h}_\perp(\Omega) \subset L^A_\perp(\Omega) \quad \text{if } \ell \geq 0,$$

and

$$\mathscr{P}^{\ell,h}(\Omega) \subset W^{1,A}(\Omega), \quad \mathscr{P}^{\ell,h}_0(\Omega) \subset W^{1,A}_0(\Omega) \quad \text{if } \ell \geq 1.$$

The spaces $\mathscr{P}^{\ell,h}(\Omega,\mathbb{R}^n)$, $\mathscr{P}_0^{\ell,h}(\Omega,\mathbb{R}^n)$ and $\mathscr{P}_\perp^{\ell,h}(\Omega,\mathbb{R}^n)$ of $\mathbb{R}^n$-valued functions are defined accordingly.

An important tool for the numerical analysis of problem (4.1) is a projection operator Π^h which, for given $m \in \mathbb{N}\cup\{0\}$ and $k \in \mathbb{N}$, is such that $\Pi^h : W^{1,1}(\Omega,\mathbb{R}^n) \to \mathscr{P}^{k,h}(\Omega,\mathbb{R}^n)$, and enjoys the following properties:
(i) Π^h preserves zero boundary values, i.e.

$$\Pi^h : W_0^{1,1}(\Omega,\mathbb{R}^n) \to \mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n); \tag{4.16}$$

(ii) Π^h is divergence preserving, namely,

(4.17)
$$\int_\Omega p_h \operatorname{div} \mathbf{u}\, dx = \int_\Omega p_h \operatorname{div} \Pi^h\mathbf{u}\, dx \quad \text{for every}\, \mathbf{u} \in W^{1,1}(\Omega,\mathbb{R}^n)\, \text{and for every}\, p_h \in \mathscr{P}^{m,h}(\Omega);$$

(iii) Π^h is continuous in the $W^{1,1}$-sense, i.e. there exists a constant $C = C(\Omega)$ such that

$$\fint_{\mathcal{S}} |\Pi^h\mathbf{u}|\, dx + \fint_{\mathcal{S}} h_{\mathcal{S}}|\nabla\Pi^h\mathbf{u}|\, dx \le C \fint_{M_{\mathcal{S}}} |\mathbf{u}|\, dx + C \fint_{M_{\mathcal{S}}} h_{\mathcal{S}}|\nabla\mathbf{u}|\, dx \tag{4.18}$$

for every $\mathcal{S} \subset \mathscr{T}_h$, and every $\mathbf{u} \in W^{1,1}(\Omega,\mathbb{R}^n)$. Here $h_{\mathcal{S}}$ denotes the diameter of $\mathcal{S}$, and $M_{\mathcal{S}}$ the union of $\mathcal{S}$ and all its direct neighbors in the triangulation $\mathscr{T}_h$. The existence of such an operator for suitable couples of m and k is standard – see e.g. [6, appendix]. For instance, the choice $k = 2$ and $m = 0$ is admissible, whereas $k = 1$ and $m = 0$ is not.

Now, the discrete version of problem (4.5) amounts to finding a couple $(\mathbf{v}_h,\pi_h) \in \mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n) \times \mathscr{P}_\perp^{m,h}(\Omega)$ such that

$$\int_\Omega \mathbf{S}(\boldsymbol{\varepsilon}(\mathbf{v}_h)) : \nabla\boldsymbol{\phi}_h\, dx = \int_\Omega \pi_h \operatorname{div} \boldsymbol{\phi}_h\, dx - \rho\int_\Omega \mathbf{F} : \nabla\boldsymbol{\phi}_h\, dx,$$

and

$$\int_\Omega p_h \operatorname{div} \mathbf{v}_h\, dx = 0,$$

for every $(\boldsymbol{\varphi}_h, p_h) \in \mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n) \times \mathscr{P}_\perp^{m,h}(\Omega)$.

This can again be accomplished in two steps. First, on setting

$$\mathscr{P}_{0,\mathrm{div}}^{k,h}(\Omega,\mathbb{R}^n) = \left\{\mathbf{u}_h \in \mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n) : \int_\Omega p_h \operatorname{div} \mathbf{u}_h\, dx = 0 \text{ for every } p_h \in \mathscr{P}^{m,h}(\Omega)\right\},$$

and

$$\mathbf{H}_h = \mathbf{S}(\boldsymbol{\varepsilon}(\mathbf{v}_h)) + \rho\mathbf{F}\,,$$

one has to find $\mathbf{v}_h \in \mathscr{P}_{0,\mathrm{div}}^{k,h}(\Omega,\mathbb{R}^n)$ such that

$$\int_\Omega \mathbf{H}_h : \nabla\boldsymbol{\phi}_h\, dx = 0 \quad \text{for every } \boldsymbol{\varphi}_h \in \mathscr{P}_{0,\mathrm{div}}^{k,h}(\Omega,\mathbb{R}^n). \tag{4.19}$$

The existence of a unique function $\mathbf{v}_h$ satisfying (4.19) follows easily from the theory of monotone operators. If $\mathbf{S}$ has variational structure one may equivalently solve the corresponding strictly convex minimizing problem on the finite dimensional function space $\mathscr{P}^{k,h}_{0,\mathrm{div}}(\Omega, \mathbb{R}^n)$. Next, the pressure has be to reconstructed. Precisely, one has to find $\pi_h \in \mathscr{P}^{m,h}_{\perp}(\Omega)$ such that

$$\int_\Omega \mathbf{H}_h : \nabla \boldsymbol{\phi}_h \, dx = \int_\Omega \pi_h \operatorname{div} \boldsymbol{\phi}_h \, dx \quad \text{for every } \boldsymbol{\varphi}_h \in \mathscr{P}^{k,h}_0(\Omega, \mathbb{R}^n).$$

This is the discrete analogone of the problem from Theorem 4.1. As far as existence is concerned, we only need to solve an algebraic linear system. The following questions then arise:
(i) Is the pressure π_h unique?
(ii) Does the pressure π_h depend continuously on the data of the problem, namely on $\mathbf{F}$?
(iii) Does the family of discretized pressure functions $\{\pi_h\}$ converge when $h \to 0$?

The answer to all this questions follows from the so-called inf-sup condition. Such a condition, in the standard case when $\mathbf{S}$ has a power type growth as in (4.2), reads as

$$\text{(4.20)} \qquad \inf_{p_h \in P^{m,h}_{\perp}(\Omega)} \sup_{\boldsymbol{\varphi}_h \in \mathscr{P}^{k,h}_0(\Omega, \mathbb{R}^n)} \frac{\int_\Omega p_h \operatorname{div} \boldsymbol{\phi}_h \, dx}{\|p_h\|_{L^{p'}(\Omega)} \|\nabla \boldsymbol{\varphi}_h\|_{L^p(\Omega, \mathbb{R}^{n\times n})}} \geq C,$$

for some positive constant C independent of h. In fact, the uniqueness of π_h does not even require C to be independent of h.

Our next result provides us with an Orlicz space version of (4.20).

Theorem 4.2. *Let A and B be Young functions fulfilling* (3.2) *and* (3.3)*. Assume that Ω is a polyhedron in $\mathbb{R}^n$, $n \geq 2$, and that $m \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$ are such that there exists an operator $\Pi^h : W^{1,1}(\Omega, \mathbb{R}^n) \to \mathscr{P}^{k,h}(\Omega, \mathbb{R}^n)$ satisfying (4.16)-(4.18). Then,*

$$\text{(4.21)} \qquad \inf_{p_h \in \mathscr{P}^{m,h}_{\perp}(\Omega)} \sup_{\boldsymbol{\varphi}_h \in \mathscr{P}^{k,h}_0(\Omega, \mathbb{R}^n)} \frac{\int_\Omega p_h \operatorname{div} \boldsymbol{\phi}_h \, dx}{\|p_h\|_{L^{B}(\Omega)} \|\nabla \boldsymbol{\varphi}_h\|_{L^{\tilde{A}}(\Omega, \mathbb{R}^{n\times n})}} \geq C,$$

for some positive constant $C = C(\Omega, c)$, where c is the constant appearing in (3.2) *and* (3.3)

Proof. As a first step, we show that the operator Π^h is continuous in every Orlicz space, in the sense that there exists a constant $C = C(\Omega)$ such that, for every Young function A,

$$\text{(4.22)} \qquad \|\nabla \Pi^h \mathbf{u}\|_{L^A(\Omega, \mathbb{R}^{n\times n})} \leq C \|\nabla \mathbf{u}\|_{L^A(\Omega, \mathbb{R}^{n\times n})} \quad \text{for every } \mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n).$$

Inequality (4.22) has been established in [28, Thm. 4.5] and in [6, Thm. 3.2] under the additional assumption that A fulfils a global Δ_2-condition. A variant of those proofs shows that, in fact, this assumption can be dropped. We outline the argument hereafter. Since $|\nabla \Pi^h \mathbf{u}|$ belongs to a finite dimensional function space, it follows from (4.18) that

$$\fint_{\mathcal{S}} A(h_{\mathcal{S}} |\nabla \Pi^h \mathbf{u}|) \, dx \leq \fint_{\mathcal{S}} A\bigg(C \fint_{\mathcal{S}} h_{\mathcal{S}} |\nabla \Pi^h \mathbf{u}| \, dy \bigg) \, dx$$

$$\leq \fint_{\mathcal{S}} A\Big(C' \fint_{M_{\mathcal{S}}} |\mathbf{u}|\, dy + C' \fint_{M_{\mathcal{S}}} h_{\mathcal{S}} |\nabla \mathbf{u}|\, dy\Big)\, dx$$

for some constants $C = C(\Omega)$ and $C' = C'(\Omega)$, and for every $\mathbf{u} \in W^{1,1}(\Omega, \mathbb{R}^n)$. Hence, owing to Jensen's inequality and the convexity of A,

$$\fint_{\mathcal{S}} A(h_{\mathcal{S}} |\nabla \Pi^h \mathbf{u}|)\, dx \leq \fint_{M_{\mathcal{S}}} A(C'|\mathbf{u}|)\, dy + \fint_{M_{\mathcal{S}}} A(C' h_{\mathcal{S}} |\nabla \mathbf{u}|)\, dy \tag{4.23}$$

for every $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$. Now, given any $\mathbf{q} \in \mathbb{R}^n$, we deduce from the convexity of A, (4.16) and (4.23)

$$\begin{aligned}\fint_{\mathcal{S}} A(h_{\mathcal{S}} |\nabla \mathbf{u} - \nabla \Pi^h \mathbf{u}|)\, dx &\leq \frac{1}{2} \fint_{\mathcal{S}} A(2h_{\mathcal{S}} |\nabla (\mathbf{u} - \mathbf{q})|)\, dx + \frac{1}{2} \fint_{\mathcal{S}} A(2h_{\mathcal{S}} |\nabla \Pi^h (\mathbf{u} - \mathbf{q})|)\, dx \\ &\leq \fint_{M_{\mathcal{S}}} A(Ch_{\mathcal{S}} |\nabla (\mathbf{u} - \mathbf{q})|)\, dx + \fint_{M_{\mathcal{S}}} A(C|\mathbf{u} - \mathbf{q}|)\, dx, \end{aligned} \tag{4.24}$$

for some constant $C = C(\Omega)$ and for every $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$. Finally we choose $\mathbf{q} = \mathbf{u}_{M_{\mathcal{S}}}$. By inequality (2.12) and a scaling argument as in the proof of Theorem 3.6 one can show that there exists a constant $C = C(\Omega)$ such that

$$\fint_{M_{\mathcal{S}}} A(|\mathbf{u} - \mathbf{q}|)\, dx \leq \fint_{M_{\mathcal{S}}} A(Ch_{\mathcal{S}} |\nabla \mathbf{u}|)\, dx \tag{4.25}$$

for every $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$. Inequalities (4.24) and (4.25) imply that

$$\fint_{\mathcal{S}} A(h_{\mathcal{S}} |\nabla \mathbf{u} - \nabla \Pi^h \mathbf{u}|)\, dx \leq \fint_{M_{\mathcal{S}}} A(Ch_{\mathcal{S}} |\nabla \mathbf{u}|)\, dx \tag{4.26}$$

for every $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$. The convexity of A and inequality (4.26) yield

$$\begin{aligned}\fint_{\mathcal{S}} A(h_{\mathcal{S}} |\nabla \Pi^h \mathbf{u}|)\, dx &\leq \frac{1}{2} \fint_{\mathcal{S}} A(2h_{\mathcal{S}} |\nabla \mathbf{u}|)\, dx + \frac{1}{2} \fint_{\mathcal{S}} A(2h_{\mathcal{S}} |\nabla \mathbf{u} - \nabla \Pi^h \mathbf{u}|)\, dx \\ &\leq \fint_{M_{\mathcal{S}}} A(Ch_{\mathcal{S}} |\nabla \mathbf{u}|)\, dx \end{aligned}$$

for some constant $C = C(\Omega)$, and for every $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$. This implies (4.18), on replacing $\mathbf{u}$ with $h_{\mathcal{S}}^{-1} \mathbf{u}$ and summing up over all simplexes $\mathcal{S}$. Note, in this connection, that the number of neighbours of each $\mathcal{S}$ only depends on n.

We are now in a position to prove inequality (4.21). By Theorem 3.1, there exists a constant $C = C(\Omega, c)$ such that

$$\|p_h\|_{L^B(\Omega)} \leq C \|\nabla p_h\|_{W^{-1,A}(\Omega, \mathbb{R}^n)} = C \sup_{\boldsymbol{\varphi} \in C_0^\infty(\Omega, \mathbb{R}^n)} \frac{\int_\Omega p_h \operatorname{div} \boldsymbol{\phi}\, dx}{\|\nabla \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega)}} \tag{4.27}$$

for every $p_h \in \mathscr{P}_\perp^{0,h}(\Omega)$. Owing to (4.27) and to properties (4.17) and (4.18) of the operator Π^h,

$$\begin{aligned}\|p_h\|_{L^B(\Omega)} &\le C \sup_{\boldsymbol{\varphi}\in C_0^\infty(\Omega,\mathbb{R}^n)} \frac{\int_\Omega p_h \operatorname{div} \Pi^h \boldsymbol{\phi}\, dx}{\|\nabla \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}} \le C' \sup_{\boldsymbol{\varphi}\in C_0^\infty(\Omega,\mathbb{R}^n)} \frac{\int_\Omega p_h \operatorname{div} \Pi^h \boldsymbol{\phi}\, dx}{\|\nabla \Pi^h \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}\\ &\le C' \sup_{\boldsymbol{\varphi}_h\in \mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n)} \frac{\int_\Omega p_h \operatorname{div} \boldsymbol{\phi}_h\, dx}{\|\nabla \boldsymbol{\varphi}_h\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}},\end{aligned}$$

for some constants $C = C(\Omega, c)$ and $C' = C'(\Omega, c)$, and for every $p_h \in \mathscr{P}_\perp^{m,h}(\Omega)$. Hence, (4.21) follows. □

Remark 4.3. Since the functions p_h belong to finite dimensional spaces it would be possible to replace $\|p_h\|_{L^B(\Omega)}$ in Theorem 4.2 by $\|p_h\|_{L^A(\Omega)}$. However, the constant then depends on the dimension of $\mathscr{P}_\perp^{m,h}(\Omega)$ and hence on h. In that form, the result would be of no use in the study of the convergence of the finite element method approximation.

Corollary 4.4. *Let A, B and Ω be as in Theorem 4.2. Assume that the function $\mathbf{H}_h \in L^A(\Omega, \mathbb{R}^{n\times n})$ fulfils*

$$\int_\Omega \mathbf{H}_h : \nabla \boldsymbol{\phi}_h\, dx = 0 \tag{4.28}$$

for every $\boldsymbol{\varphi}_h \in \mathscr{P}_{0,\mathrm{div}}^{k,h}(\Omega, \mathbb{R}^n)$. Then:

i) There exists a unique function $\pi_h \in \mathscr{P}_\perp^{m,h}(\Omega)$ such that

$$\int_\Omega \mathbf{H}_h : \nabla \boldsymbol{\phi}_h\, dx = \int_\Omega \pi_h \operatorname{div} \boldsymbol{\phi}_h\, dx \tag{4.29}$$

for every $\boldsymbol{\varphi}_h \in \mathscr{P}_0^{k,h}(\Omega, \mathbb{R}^n)$.

ii) There exists a constant C, independent of h, such that

$$\|\pi_h\|_{L^B(\Omega)} \le C \|\mathbf{H}_h\|_{L^A(\Omega,\mathbb{R}^{n\times n})}. \tag{4.30}$$

iii) There exists a constant C, independent of h, such that, if $\mathbf{H} \in L^A(\Omega, \mathbb{R}^{n\times n})$ and $\pi \in L_\perp^A(\Omega)$ satisfy

$$\int_\Omega \mathbf{H} : \nabla \boldsymbol{\phi}\, dx = \int_\Omega \pi \operatorname{div} \boldsymbol{\phi}\, dx \tag{4.31}$$

for every $\boldsymbol{\varphi} \in C_0^\infty(\Omega, \mathbb{R}^n)$,then

$$\|\pi_h - \pi\|_{L^B(\Omega)} \le C\Big(\|\mathbf{H}_h - \mathbf{H}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} + \inf_{\mu_h\in \mathscr{P}_\perp^{0,h}(\Omega)} \|\mu_h - \pi\|_{L^A(\Omega)}\Big). \tag{4.32}$$

Proof. (i) Consider a basis $\{p_h^j\}$, $i=1,\dots,N_h$ of $\mathscr{P}_\perp^{m,h}(\Omega)$ and a basis $\{\boldsymbol{\varphi}_h^j\}$, $j=1,\dots,M_h$ of $\mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n)$. Note that $N_h\le M_h$. Then the problem

$$\int_\Omega \mathbf{H}_h : \nabla\boldsymbol{\phi}_h\,dx = \int_\Omega \pi_h \operatorname{div}\boldsymbol{\phi}_h\,dx$$

for every $\boldsymbol{\varphi}_h\in\mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n)$ is equivalent to the algebraic linear system $\mathcal{A}\mathbf{z}=\mathbf{b}$, where

$$\mathcal{A}_{ij}=\int_\Omega p_h^j \operatorname{div}\boldsymbol{\phi}_h^i\,dx,\quad b_i=\int_\Omega \mathbf{H}_h:\nabla\boldsymbol{\phi}_h^i\,dx.$$

Such system has a solution, provided that $\mathbf{b}$ belongs to the image of the matrix $\mathcal{A}$, or, equivalently, if it is orthogonal to $\operatorname{Ker}(\mathcal{A}^T)$. Now, observe that a vector $\mathbf{y}$ belongs to $\operatorname{Ker}(\mathcal{A}^T)$ if and only if

$$0=\sum_i \mathcal{A}_{ij}y_i=\int_\Omega p_h^j \operatorname{div}\Big(\sum_i y_i\boldsymbol{\phi}_h^i\Big)\,dx\quad\text{for every } j,$$

whence, $\sum_i y_i\boldsymbol{\phi}_h^i\in\mathscr{P}_{0,\mathrm{div}}^{k,h}(\Omega,\mathbb{R}^n)$. Thus, owing to (4.28),

$$\langle\mathbf{b},\mathbf{y}\rangle=\int_\Omega \mathbf{H}_h:\nabla\Big(\sum_i y_i\boldsymbol{\phi}_h^i\Big)\,dx=0$$

for every $\mathbf{y}\in\operatorname{Ker}(\mathcal{A}^T)$. Hence, $\mathbf{b}\in\big(\operatorname{Ker}\mathcal{A}^T\big)^\perp$.
Next, we show that $\mathcal{A}$ is injective, whence the uniqueness of π_h follows. To verify the injectivity of $\mathcal{A}$, assume that $\mathbf{z}$ is such that $\mathcal{A}\mathbf{z}=0$. Hence, on setting $p_h=\sum_j z_j\,p_h^j$, we have that $\int_\Omega p_h \operatorname{div}\boldsymbol{\phi}_h^i\,dx=0$ for every i. Let us apply Theorem 4.2 with any pair of Young functions A and B fulfilling (3.2) and (3.3), for instance $A(t)=B(t)=t^2$ for $t\ge 0$ (this means that we are in fact applying (4.20) with $p=2$). We then obtain that

$$\|p_h\|_{L^2(\Omega)}\le C\sup_{\boldsymbol{\varphi}_h\in\mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n)}\frac{\int_\Omega p_h \operatorname{div}\boldsymbol{\phi}_h\,dx}{\|\nabla\boldsymbol{\varphi}_h\|_{L^2(\Omega,\mathbb{R}^{n\times n})}}=0,$$

whence $\mathbf{z}=0$.
(ii) By Theorem 4.2 and (2.9)

$$\begin{aligned}\|\pi_h\|_{L^B(\Omega)} &\le C\sup_{\boldsymbol{\varphi}_h\in\mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n)}\frac{\int_\Omega \pi_h \operatorname{div}\boldsymbol{\phi}_h\,dx}{\|\nabla\boldsymbol{\varphi}_h\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}=C\sup_{\boldsymbol{\varphi}_h\in\mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n)}\frac{\int_\Omega \mathbf{H}_h:\nabla\boldsymbol{\phi}_h\,dx}{\|\nabla\boldsymbol{\varphi}_h\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}\\ &\le C\sup_{\boldsymbol{\varphi}\in W_0^{1,\widetilde{A}}(\Omega,\mathbb{R}^n)}\frac{\int_\Omega \mathbf{H}_h:\nabla\boldsymbol{\phi}\,dx}{\|\nabla\boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}\le C\sup_{\boldsymbol{\Psi}\in L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}\frac{\int_\Omega \mathbf{H}_h:\boldsymbol{\Psi}\,dx}{\|\boldsymbol{\Psi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}\\ &\le C'\|\mathbf{H}_h\|_{L^A(\Omega,\mathbb{R}^{n\times n})}\end{aligned}$$

for some constants $C = C(\Omega, c)$ and $C' = C'(\Omega, c)$. This proves inequality (4.30).
(iii) The triangle inequality and the embedding $L^A(\Omega) \to L^B(\Omega)$ ensure that, for every $\mu_h \in \mathscr{P}_\perp^{m,h}(\Omega)$,

$$\begin{aligned}\|\pi_h - \pi\|_{L^B(\Omega)} &\leq \|\pi_h - \mu_h\|_{L^B(\Omega)} + \|\mu_h - \pi\|_{L^B(\Omega)} \\ &\leq \|\pi_h - \mu_h\|_{L^B(\Omega)} + C\|\mu_h - \pi\|_{L^A(\Omega)},\end{aligned}$$

for some constant $C = C(c)$. By Theorem 4.2,

$$\|\pi_h - \mu_h\|_{L^B(\Omega)} \leq C \sup_{\boldsymbol{\varphi}_h \in \mathscr{P}_0^{k,h}(\Omega,\mathbb{R}^n)} \frac{\int_\Omega (\pi_h - \mu_h) \operatorname{div} \boldsymbol{\phi}_h \, dx}{\|\nabla \boldsymbol{\varphi}_h\|_{L^{\tilde{A}}(\Omega,\mathbb{R}^{n\times n})}}$$

for some constant $C(\Omega, c)$, and for every $\mu_h \in \mathscr{P}_\perp^{m,h}(\Omega)$. Moreover, by (4.14), (4.31), and an approximation argument for functions in $\mathscr{P}_0^{k,h}(\Omega, \mathbb{R}^n)$ via functions in $C_0^\infty(\Omega, \mathbb{R}^n)$, we have that

$$\begin{aligned}\int_\Omega (\pi_h - \mu_h) \operatorname{div} \boldsymbol{\phi}_h \, dx &= \int_\Omega (\pi_h - \pi) \operatorname{div} \boldsymbol{\phi}_h \, dx + \int_\Omega (\pi - \mu_h) \operatorname{div} \boldsymbol{\phi}_h \, dx \\ &= \int_\Omega (\mathbf{H}_h - \mathbf{H}) : \nabla \boldsymbol{\phi}_h \, dx + \int_\Omega (\pi - \mu_h) \operatorname{div} \boldsymbol{\phi}_h \, dx,\end{aligned}$$

for every $\mu_h \in \mathscr{P}_\perp^{m,h}(\Omega)$ and $\boldsymbol{\varphi}_h \in \mathscr{P}_0^{k,h}(\Omega, \mathbb{R}^n)$. Hence, via Hölder's inequality in Orlicz spaces,

$$\int_\Omega (\pi_h - \mu_h) \operatorname{div} \boldsymbol{\phi}_h \, dx \leq C\Big(\|\mathbf{H}_h - \mathbf{H}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} + \|\mu_h - \pi\|_{L^A(\Omega)}\Big)\|\nabla \boldsymbol{\phi}_h\|_{L^{\tilde{A}}(\Omega,\mathbb{R}^{n\times n})}$$

for some constant $C = C(n)$. Altogether, we conclude that

$$\|\pi_h - \pi\|_{L^B(\Omega)} \leq C\Big(\|\mathbf{H}_h - \mathbf{H}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} + \inf_{\mu_h \in \mathscr{P}_\perp^{m,h}(\Omega)} \|\mu_h - \pi\|_{L^A(\Omega)}\Big)$$

for some constant C independent of h, namely (4.32). □

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