Dyck Algebras, Interval Temporal Logic, and Posets of Intervals

DYCK ALGEBRAS, INTERVAL TEMPORAL LOGIC AND POSETS OF INTERVALS *

LUCA FERRARI†

Abstract. We investigate a natural Heyting algebra structure on the set of Dyck paths of the same length. We provide a geometrical description of the operations of pseudocomplement and relative pseudocomplement, as well as of regular elements. We also find a logic-theoretic interpretation of such Heyting algebras, which we call *Dyck algebras*, by showing that they are the algebraic counterpart of a certain fragment of a classical interval temporal logic (also known as Halpern-Shoham logic). Finally, we propose a generalization of our approach, suggesting a similar study of the Heyting algebra arising from the poset of intervals of a finite poset using Birkhoff duality. In order to illustrate this, we show how several combinatorial parameters of Dyck paths can be expressed in terms of the Heyting algebra structure of Dyck algebras together with a certain total order on the set of atoms of each Dyck algebra.

Key words. Dyck path, Heyting algebra, temporal logic, poset of intervals

AMS subject classifications. 03B44,06A07,06D20,68R05

1. Introduction. Among the plethora of different logics generalizing and extending the classical one, a family of logics which has proved very useful especially in computer science is that of *temporal logics*. A temporal logic is essentially a kind of logic which allows one to deal with statements whose truth values can vary in time. Applications in computer science concern, for example, formal verification, where temporal logics show their expressiveness in stating requirements of hardware or software systems. Starting from the generic idea stated above, one can conceive several different types of temporal logics, depending on the structure of time states and on how time states are managed. A particularly interesting class of temporal logics are the so-called *interval temporal logics*. An interval temporal logic is characterized by the fact that the truth of a statement depends on the *time interval* it is evaluated on (rather than the time instant). Such kinds of logic are useful, for instance, when it is important to work with properties which remain true (or false) for a certain amount of time. The relevance of these logics for computer science is even more evident: think, for instance, of processes, for which it is meaningful to reason in terms of time intervals rather than time instants. More generally, interval temporal logics have been successfully applied to temporal databases, specification, design and verification of hardware components and concurrent real-time processes; see, for instance [GMS] and the references therein.

To work with any interval temporal logic, it is important to understand which kinds of relations among intervals of time instants are relevant to the specific logic one wish to consider. The classification of all possible such relations has been pursued by Allen [A, AF], who has also defined an algebraic structure to deal with them. The modal logic of time intervals resulting by considering the whole set of Allen's relations is usually referred to as the *Halpern-Shoham logic* [HS]. Typically, one selects a subset of Allen's relations, thus defining the related fragment of the Halpern-Shoham logic. Most studied in this context are decidability questions, as witnessed by many works

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†Dipartimento di Matematica e Informatica "U. Dini", viale Morgagni 65, 50134 Firenze, Italy `luca.ferrari@unifi.it`.

appeared in recent years (an example related to a fragment which is relevant to our paper is [MM]).

In the present paper, we propose a combinatorial description of a specific interval temporal logic whose underlying time model is a finite linear order (an account of the behavior of interval temporal logic over strongly discrete linear orders, including finite linear orders, can be found in [BDMMSS]). Specifically, we consider what is sometimes called the *logic of sub-intervals*, that is, the interval temporal logic in which, from the truth of a statement on a certain interval of time instants I, the truth of that statement on all sub-intervals of I follows. We show that, given a linearly ordered set of time instants of cardinality $n-1$, the algebraic counterpart of the associated logic of sub-intervals is given by a certain Heyting algebra structure on the set of Dyck paths of semilength n, which is more precisely the canonical Heyting algebra structure associated with the distributive lattice structure on Dyck paths of semilength n induced by ordering them by *geometric inclusion* (i.e., a Dyck path P is declared to be less than or equal to a Dyck path Q whenever, in the usual two-dimensional drawing of Dyck paths, P lies weakly below Q, see [FP] and also the next section). We also give a fully geometric description of relative pseudocomplement and pseudocomplement in such Dyck algebras, thus supplementing similar results that have been illustrated in a more algebraic fashion by Mühle in [Muh]. In particular, we would like to point out that, in comparison with the work of Mühle, our approach has at least two main advantages. First of all, our geometric language is much simpler than that of [Muh], and this fact allows an easier visualization of the obtained results, as well as shorter and clearer proofs. A second benefit lies in the possibility of generalizing quite easily what have been obtained for Dyck paths to other classes of lattice paths; we will illustrate this fact by discussing the cases of Motzkin and Schröder paths, for which we will be able to describe (relative) pseudocomplement and regular elements in a very natural and effective way. Finally, we try to broaden the scope of our work, by proposing a possible generalization. The idea is to consider the poset of intervals (ordered by inclusion) of any poset $\mathcal{P}$ (rather than a totally ordered set) and to investigate properties of the Heyting algebra $\mathcal{H}$ obtained from $\mathcal{P}$ by classical (generalized) Birkhoff duality. More specifically, we ask what properties of $\mathcal{H}$ can be expressed in terms of the partial order $\mathcal{P}$. In the specific case of a finite totally ordered $\mathcal{P}$ (which is the case studied in the present paper), we illustrate the above project from a combinatorial point of view, namely, we express several statistics of combinatorial interest in terms of the Heyting algebra structure of Dyck paths together with the partial order structure on the atoms of such an algebra. We close our paper by proposing some further directions of future research.

2. Heyting algebras of Dyck paths. Given a Cartesian coordinate system, a *Dyck path* is a lattice path starting from the origin, ending on the x-axis, never falling below the x-axis and using only two kinds of steps, $u(p) = (1,1)$ and $d(own) = (1,-1)$. A Dyck path can be encoded by a word w on the alphabet $\{u, d\}$ such that in every prefix of w the number of u is greater than or equal to the number of d and the total number of u and d in w is the same (the resulting language is called *Dyck language* and its words *Dyck words*). The *length* of a Dyck path is the length of the associated Dyck word (which is necessarily an even number). A *peak* in a Dyck path is a pair of consecutive steps of the form ud; a *hill* is a peak at height 0 (i.e., lying on the x-axis). A *factor* of a Dyck path is any minimal subsequence of consecutive steps starting and ending on the x-axis; every Dyck path can be clearly decomposed in a unique way as the product (juxtaposition) of its factors. In particular, a hill is also called a *trivial* *factor*. A *pyramid* is a subsequence of consecutive steps of the form $u^k d^k$ ($k \geqslant 1$) starting and ending on the x-axis. In particular, a hill is a pyramid. A *return* is a point of the path, other than the starting one, lying on the x-axis. We will usually refer to a return by using its abscissa (which is necessarily an even number).

The set D_n of Dyck paths of semilength n can be endowed with a very natural poset structure. Given $P, Q \in D_n$, we say that $P \leqslant Q$ when, in the above described two-dimensional drawing of Dyck paths, P lies weakly below Q. Properties of the posets $\mathcal{D}_n = (D_n, \leqslant)$ have been investigated in [FM1, FM2, FM3, FP]. In particular, it is shown that $\mathcal{D}_n$ is a distributive lattice, and for this reason it will be called the *Dyck lattice of order* n. We point out that this last assertion is a consequence of the (easy to observe) fact that $\mathcal{D}_n$ is isomorphic to the dual of the Young lattice of integer partitions whose Ferrers diagrams fit into the staircase diagram $(n-1, n-2, \ldots, 2, 1)$ [S]. The language of Dyck paths, however, gives a geometric flavor to the subject which allows one to express several properties in a more fascinating way, as well as to suggest possible analogies with other families of lattice paths.

Recall that a *join-irreducible element* of a poset $\mathcal{P}$ is an element a such that, if $a = x \vee y$, then $a = x$ or $a = y$. In particular, if $\mathcal{P}$ has minimum 0, an *atom* is an element covering 0 (hence an atom is join-irreducible). Moreover, a subset I of $\mathcal{P}$ is a *down-set* whenever, for every x, y in $\mathcal{P}$, if $y \in I$ and $x \leqslant y$, then $x \in I$. The well-known *Birkhoff representation theorem* (see, for instance, [DP]) states that every finite distributive lattice is isomorphic to the lattice of down-sets of the poset of its join-irreducibles. As a consequence, every element of a finite distributive lattice is the join of the join-irreducibles below it. Concerning Dyck lattices, a join-irreducible is a path all of whose factors are hills except for a single pyramid having at least 4 steps (see [FM1]). In particular, an atom is a join-irreducible in which the unique nontrivial pyramid has *exactly* 4 steps.

Since Dyck lattices are finite distributive lattices, they also have a canonical Heyting algebra structure. Recall that a *Heyting algebra* is a lattice $\mathcal{H}$ with minimum 0 and maximum 1 such that the relative pseudocomplement of x with respect to y exists for all $x, y \in \mathcal{H}$. By definition, the *relative pseudocomplement of* x *with respect to* y is the element $x \rightsquigarrow y$ defined as follows:

$$x \rightsquigarrow y = \bigvee \{z \in \mathcal{H} \mid x \wedge z \leqslant y\}.$$

The Heyting algebra of Dyck paths of semilength n will be denoted $\mathfrak{D}_n$, and we will call it the *Dyck algebra of order* n.

In a Heyting algebra $\mathcal{H}$, two important notions are those of pseudocomplement and of regular element. The *pseudocomplement* of x is defined as $\sim x = x \rightsquigarrow 0$. It can be shown that $x \leqslant \sim\sim x$. The converse, however, does not hold in general. An element x of $\mathcal{H}$ is said to be *regular* whenever $x = \sim\sim x$. The subposet of regular elements of a Heyting algebra forms a Boolean algebra.

The main aim of the present section is to give a combinatorial description of relative pseudocomplement and pseudocomplement in Dyck algebras, as well as to characterize the Boolean algebra of the regular elements. We point out that similar results have been obtained in [Muh]. Our statements, however, have a more geometric flavor, which would hopefully result in a more natural way of capturing the above mentioned notions.

For any pair of Dyck paths (P, Q) of semilength n, we define the *crossing set* $C(P, Q) \subseteq [2n] \cup \{0\} = \{0, 1, 2, \ldots, 2n\}$ of (P, Q) by declaring $x \in C(P, Q)$ whenever exactly one of the following conditions holds:

1. $x \in \{0, 2n\}$;
2. P and Q have a common point having abscissa x; moreover, P has an up step starting at that point and Q has a down step starting at that point;
3. P and Q have a common point having abscissa x; moreover, P has a down step arriving at that point and Q has an up step arriving at that point.

Roughly speaking, an element of the crossing set of (P, Q) is either the abscissa of the starting/ending point of the two paths or the abscissa of a point in which the two paths cross in a specific way. More precisely, suppose that $C(P, Q) = \{x_0, x_1, x_2, \ldots, x_k\}$, where the x_i's are listed in increasing order (so that $x_0 = 0$ and $x_k = 2n$). If i is even, then P lies weakly below Q between x_i and x_{i+1} ("weakly" meaning that P and Q may coincide in some point other than those of abscissas x_i and x_{i+1}); if i is odd, then P lies strictly above Q between x_i and x_{i+1}. Notice that k is necessarily an odd number (or, which is the same, the cardinality of $C(P, Q)$ is even): indeed, both at the beginning and at the end P lies weakly below Q (since both paths necessarily start with an up step and end with a down step). Finally, observe that clearly $C(P, Q) \neq C(Q, P)$ in general. In Figure 1, the elements of $C(P, Q)$ are the abscissas of the points represented by black bullets.

Proposition 1. *Let $P, Q \in D_n$ and let $C(P, Q) = \{x_0, x_1, x_2, \ldots, x_k\}$ be the crossing set of (P, Q). Then $P \rightsquigarrow Q \in D_n$ is the Dyck path constructed as follows:*

1. *if i is even, then the portion of $P \rightsquigarrow Q$ between x_i and x_{i+1} is the unique subpath of the form $u^\alpha d^\beta$ whose starting and ending points are the same as P and Q, for suitable nonnegative integers α and β;*
2. *if i is odd, then $P \rightsquigarrow Q$ coincides with Q between x_i and x_{i+1}.*

Proof. We observe that, if $i \neq 0$ is even, then necessarily P has an up step starting at abscissa x_i and Q has a down step starting at abscissa x_i, whereas, if $i \neq k$ is odd, then P has a down step ending at abscissa x_i and Q has an up step ending at abscissa x_i. Thus, between x_i and x_{i+1}, if i is even then P lies weakly below Q, otherwise (i.e. if i is odd) P lies strictly above Q (this last statement is true also in the cases $i = 0, k$). As a consequence, if i is even, $P \rightsquigarrow Q$ can run as high as possible between x_i and x_{i+1}; this is achieved by putting as many up steps as possible immediately after x_i, followed by the correct number of down steps, which means that the portion of $P \rightsquigarrow Q$ between x_i and x_{i+1} is of the form $u^\alpha d^\beta$, as required. On the other hand, if i is odd, then $P \rightsquigarrow Q$ must coincide with Q between x_i and x_{i+1}, in order to have $(P \rightsquigarrow Q) \wedge P \leqslant Q$, and this is clearly the maximum subpath between x_i and x_{i+1} which satisfies such a condition. ∎

The result of the above proposition can be restated less formally, but maybe more expressively, as follows: $P \rightsquigarrow Q$ is obtained from Q by replacing those portions of path in which P lies weakly below Q with the highest possible Dyck factors.

In Figure 1 we give an example of how to compute $P \rightsquigarrow Q$ starting from P and Q, as described in the above proposition.

As a consequence, we have the following result, which gives us a recipe to compute pseudocomplements in Dyck algebras (see Figure 2). In the statement of the corollary, we will use the expression *"sequence of k consecutive hills"*, which should be clear in the case $k > 0$. By convention, with the expression *"sequence of 0 consecutive hills"* we will mean a point of the path lying on the x-axis (other than the starting and the ending ones) and neither preceded nor followed by a hill (in other words, a return between two nontrivial factors).

Corollary 2. *Let $P \in D_n$. Then $\sim P = P \rightsquigarrow 0$ is obtained from P by*

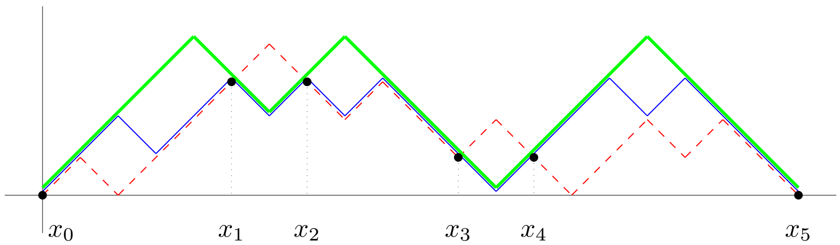

FIG. 1. *P is red and dashed, Q is blue and solid and $P \rightsquigarrow Q$ is green and thick.*

1. *replacing each sequence of $k \geqslant 0$ consecutive hills starting at abscissa x and ending at abscissa x' with a pyramid of suitable height starting at* $\max(0, x-2)$ *and ending at* $\min(x'+2, 2n)$*, and*
2. *completing the path by suitably adding a (finite) set of hills.*

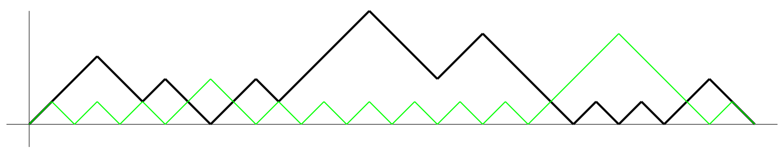

FIG. 2. *A Dyck path (black) and its pseudocomplement (green).*

To conclude this section, we will give a characterization of regular elements of Dyck algebras. Similarly to the previous results, our description will be in terms of the geometric shape of the path.

PROPOSITION 3. *A Dyck path is regular if and only if its factors are all pyramids.*

Proof. For any Dyck path P, it follows from the previous corollary that all factors of $\sim P$ are pyramids. Therefore, if P is regular, then $P = \sim\sim P$, and all factors of P are pyramids.

For the converse, observe that the pseudocomplement operation exchanges returns and non-returns of a Dyck path (that is, $(x, 0)$ is a return of P if and only if $(x, 0)$ is not a return of $\sim P$). Now, if P is a concatenation of pyramids, then P is uniquely determined by its returns, and the above observation implies that $\sim\sim P = P$, i.e. P is regular. □

Recall that, given a poset $\mathcal{P}$, a *closure operator* is a map $^{-} : \mathcal{P} \to \mathcal{P}$ such that, for all x, y in $\mathcal{P}$, (i) $x \leqslant \overline{x}$, (ii) $x \leqslant y \Rightarrow \overline{x} \leqslant \overline{y}$ and (iii) $\overline{\overline{x}} = \overline{x}$. A general fact of the theory of Heyting algebras is that performing twice the pseudocomplement operation gives a closure operator. Thus, in the specific case of Dyck algebras, given a path P, its closure $\overline{P} = \sim\sim P$ is obtained by turning each of its factors into the unique pyramid greater than it and having the same number of steps.

We close by noticing that the Boolean algebra structure of regular elements of $\mathfrak{D}_n$ can be naturally described in terms of *compositions*. Indeed, the map which associates a concatenation of pyramids in $\mathfrak{D}_n$ with the integer composition (of n) whose parts are the heights of the pyramids (read from left to right) is clearly a bijection. The partial order induced by $\mathfrak{D}_n$ on the subset of its regular elements can be translated along such a bijection into the so called *refinement order* on compositions of n, whose covering relation is defined as follows: a composition λ is covered by a composition η when η is obtained from λ by summing two consecutive parts (see Figure 3). These Boolean algebras on compositions have occasionally surfaced in the literature, see for instance [AS, BLvW, EJ].

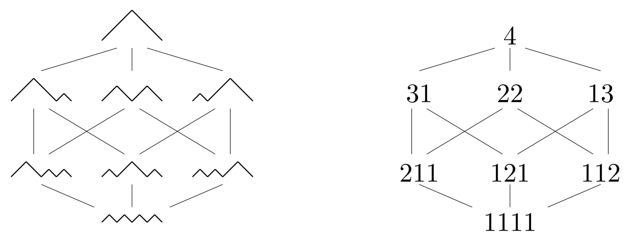

FIG. 3. *The Boolean algebra of regular elements of $\mathfrak{D}_4$ and its isomorphic representation in terms of compositions of 4.*

2.1. Motzkin and Schröder algebras. The same approach we have developed for Dyck algebras can be tried also for algebras defined by other classes of paths. Here we give a brief description of the cases of Motzkin and Schröder paths, in order to illustrate how essentially the same arguments allow to find similar results.

A *Motzkin path* is defined like a Dyck path, with the difference that the allowed steps are $u(p) = (1,1)$, $d(own) = (1,-1)$ and $h(orizontal) = (1,0)$. The set M_n of Motzkin paths of length n can be endowed with a poset structure analogous to that of Dyck paths, which turns out to be a distributive lattice as well. So we can define the *Motzkin algebra of order* n to be the Heyting algebra $\mathfrak{M}_n$ of Motzkin paths of length n canonically associated with the above mentioned distributive lattice structure.

Our next goal is to find a way to compute relative pseudocomplements in Motzkin algebras. As we will see, the strategy is the same as in Dyck algebras, and is based on the correct definition of the crossing set of a pair of Motzkin paths. To properly state it, we need a few preliminary notations, which will be given below. Apart from the technical differences with the Dyck case, the role of the crossing sets of the pair (P,Q) is the same: providing the (unique) common factorization of P and Q into factors of maximal length such that, in each piece of the factorization, either P lies weakly below Q or P lies strictly above Q.

In the set $\{u,d,h\}$ of possible steps of a Motzkin path we introduce a total (strict) order $\lhd$ defined as follows: $d \lhd h \lhd u$. Moreover, given two Motzkin paths P and Q of the same length, we say that a pair of steps $\{s_P, s_Q\}$, where s_P belongs to P and s_Q belongs to Q, constitutes a *fake crossing* whenever s_P and s_Q intersect in a point having noninteger coordinates. It is clear that, in a fake crossing, either $s_P = u$ and $s_Q = d$ or $s_P = d$ and $s_Q = u$. To distinguish the two cases, we will call the former a Q-fake crossing and the latter a P-fake crossing.

Given a pair (P,Q) of Motzkin paths of the same length, the *crossing set* of (P,Q) is the set $C(P,Q) \subseteq [n] \cup \{0\}$ defined by declaring $x \in C(P,Q)$ whenever exactly one of the following conditions holds:

1. $x \in \{0,n\}$;
2. P and Q have a common point having abscissa x; moreover, denoting with s_P and s_Q the steps of P and Q (respectively) starting at that point, we have $s_Q \lhd s_P$;

3. P and Q have a common point having abscissa x; moreover, denoting with s_P and s_Q the steps of P and Q (respectively) arriving at that point, we have $s_P \lhd s_Q$;
4. there is a Q-fake crossing between abscissas x and $x+1$;
5. there is a P-fake crossing between abscissas $x-1$ and x.

We can now state the promised characterization of relative pseudocomplement in $\mathfrak{M}_n$. It is an almost *verbatim* transcription of Proposition 1, and also the proof follows essentially the same lines, so we will just sketch it.

Proposition 4. *Let $P, Q \in M_n$ and let $C(P,Q) = \{x_0, x_1, x_2, \dots, x_k\}$ be the crossing set of (P,Q). Then $P \rightsquigarrow Q \in M_n$ is the Motzkin path constructed as follows:*

- *denote with $s_P^{(1)}$ and $s_Q^{(1)}$ the first step of P and Q, respectively; if $s_Q^{(1)} \lhd s_P^{(1)}$, then*
  1. *if i is odd, then the portion of $P \rightsquigarrow Q$ between x_i and x_{i+1} is the unique subpath of the form $u^\alpha h^\omega d^\beta$ whose starting and ending points are the same as P and Q, for suitable nonnegative integers α and β and $\omega \in \{0,1\}$;*
  2. *if i is even, then $P \rightsquigarrow Q$ coincides with Q between x_i and x_{i+1}.*
- *otherwise, just swap the two previous cases.*

Proof. The crossing set $C(P,Q) = \{x_0, x_1, \dots, x_k\}$ determines a partition of the linear order $[0,n]$ into intervals in such a way that, in each interval, either P lies strictly above Q (except possibly for the first and the last steps, either of which may constitute a fake crossing) or P lies weakly below Q. In particular, notice that, in the latter case (P lies weakly below Q), there cannot be fake crossings neither at the beginning nor at the end of the interval. As a consequence, if in the interval delimited by x_i and x_{i+1} P lies weakly below Q, the related portion of path of $P \rightsquigarrow Q$ must run as high as possible, and so it must be of the form $u^\alpha h^\omega d^\beta$; moreover, necessarily $\omega \leqslant 1$, otherwise we would not get the highest factor in the considered interval (since any pair of consecutive horizontal steps could be replaced with a peak). On the other hand, if in the interval delimited by x_i and x_{i+1} we are not in the previous situation, then necessarily the related portion of path of $P \rightsquigarrow Q$ must coincide with Q, and it is not difficult to realize that this is also true if there are fake crossings. ∎

Starting from the above characterization of the relative pseudocomplement operation, we can derive descriptions of the pseudocomplement and of the regular elements that are strikingly similar the the analogous ones for Dyck algebras. We will then limit ourselves to provide the statements of such results.

Proposition 5. *Let $P \in M_n$. Then $\sim P = P \rightsquigarrow 0$ is obtained from P by*
1. *replacing each sequence of $k \geqslant 0$ consecutive horizontal steps on the x-axis starting at abscissa x and ending at abscissa x' with a factor of the form $u^\alpha h^\omega d^\beta$ of suitable height, with $\omega \leqslant 1$, starting at $\max(0, x-1)$ and ending at $\min(x'+1, n)$, and*
2. *completing the path by suitably adding a (finite) set of horizontal steps on the x-axis.*

Proposition 6. *A Motzkin path is regular if and only if it is the concatenation of horizontal steps on the x-axis and factors of the form $u^\alpha h^\omega d^\alpha$, with $\omega \leqslant 1$.*

Analogously to what happens for Dyck algebras, also the Boolean algebras of regular elements of the Motzkin algebras are isomorphic to the same algebras of compositions. More specifically, the isomorphism maps a regular Motzkin path into the composition obtained by reading the path from left to right and replacing each h step on the x-axis with 1 and each factor $u^{\alpha}h^{\omega}d^{\alpha}$ with $2\alpha+\omega$.

The case of Schröder paths is even easier than that of Motzkin paths, since it shows even more similarities with the case of Dyck paths. Recall that a Schröder path is defined as Dyck and a Motzkin path, but now the allowed steps are $u(p)=(1,1)$, $d(own)=(1,-1)$ and double horizontal $h^2=(2,0)$. As for the previous cases, we can consider the *Schröder algebra* $\mathfrak{S}_n$ canonically obtained from the usual distributive lattice structure on the set S_n of Schröder paths of semilength n. As for Motzkin paths, define the partial order $\lhd$ on the set of steps of Schröder paths by setting $d \lhd h^2 \lhd u$. Given $P,Q \in S_n$, the *crossing set* $C(P,Q) \subseteq [2n] \cup \{0\}$ is defined by declaring $x \in C(P,Q)$ whenever exactly one of the following conditions holds:

1. $x \in \{0, 2n\}$;
2. P and Q have a common point having abscissa x; moreover, denoting with s_P and s_Q the steps of P and Q (respectively) starting at that point, we have $s_Q \lhd s_P$;
3. P and Q have a common point having abscissa x; moreover, denoting with s_P and s_Q the steps of P and Q (respectively) arriving at that point, we have $s_P \lhd s_Q$;

Proposition 7. *Let $P,Q \in S_n$ and let $C(P,Q)=\{x_0,x_1,x_2,\dots,x_k\}$ be the crossing set of (P,Q). Then $P \rightsquigarrow Q \in S_n$ is the Schröder path constructed as follows:*

- *denote with $s_P^{(1)}$ and $s_Q^{(1)}$ the first step of P and Q, respectively; if $s_Q^{(1)} \lhd s_P^{(1)}$, then*
  1. *if i is odd, then the portion of $P \rightsquigarrow Q$ between x_i and x_{i+1} is the unique subpath of the form $u^{\alpha}d^{\beta}$ whose starting and ending points are the same as P and Q, for suitable nonnegative integers α and β;*
  2. *if i is even, then $P \rightsquigarrow Q$ coincides with Q between x_i and x_{i+1}.*
- *otherwise, just swap the two previous cases.*

Proposition 8. *Let $P \in S_n$. Then $\sim P = P \rightsquigarrow 0$ is obtained from P by*

1. *replacing each sequence of $k \geqslant 0$ consecutive double horizontal steps on the x-axis with a pyramid of suitable height, and*
2. *completing the path by suitably adding a (finite) set of double horizontal steps on the x-axis.*

Proposition 9. *A Schröder path is regular if and only if it is the concatenation of double horizontal steps on the x-axis and pyramids.*

Again, there is a description of the Boolean algebra of regular elements of Schröder algebras in terms of the usual algebras of compositions, where the isomorphism is the function which maps a regular Schröder paths into the composition obtained by reading the path from left to right and replacing each double horizontal step with 1 and each pyramid $u^{\alpha}d^{\alpha}$ with α.

3. The logic of sub-intervals. The aim of this section is to give a logic-theoretic interpretation of Dyck algebras. More specifically, it turns out that Dyck algebras provide the natural algebraic counterpart of a special sort of intuitionistic logics, which are more precisely a certain class of interval temporal logics.

Let $\mathcal{T}_n=\{t_1,t_2,\dots,t_n\}$ be a finite linearly ordered set, with $t_1<t_2<\cdots<t_n$. The elements of $\mathcal{T}_n$ will be sometimes called *time states*. Denote by $Int(\mathcal{T}_n)$ the set of all *intervals* of $\mathcal{T}_n$, i.e. $I \in Int(\mathcal{T}_n)$ when there exist $t_i,t_j \in \mathcal{T}_n$ such that $I =$

$[t_i, t_j] = \{t \mid t_i \leqslant t \leqslant t_j\}$. In the following we will consider $Int(\mathcal{T}_n)$ partially ordered by inclusion.

Next we define a set of propositions in a recursive fashion, as usual. We point out that the logic we are going to describe is related to the Halpern-Shoham logic [HS], which is one of the logics of time intervals. In particular, the propositional logic of interest to us appears to be intimately related to the fragment of the Halpern-Shoham logic in which a single modal operator is considered, namely the so-called operator "during". A paper dealing with this fragment is [MPS], where the authors show that it is decidable over finite linear orders. We also remark that, on the other hand, in [MM] a strictly related fragment is shown to be undecidable over discrete structures. Another interesting reference is [SS], where a connection between the logic of (strict) sub-intervals and the logic of Minkowski spacetime is explored.

The set of propositions ITL_n is defined as follows, by means of the usual connectives:

- $\bot, \top \in ITL_n$; for all $1 \leqslant i \leqslant n$, $\varepsilon_i \in ITL_n$ (the ε_i's are the *propositional variables*);
- if $\varphi, \psi \in ITL_n$, then $\varphi \vee \psi, \varphi \wedge \psi, \varphi \to \psi, \neg\varphi \in ITL_n$.

We give an *interval-based semantics*, for which each proposition φ can be true or false depending on how it is evaluated on a specific interval $I \in Int(\mathcal{T}_n)$. More formally, if we denote by $\mathbf{2}^A$ the set of all maps from a set A to the set $\mathbf{2} = \{0, 1\}$, we define a map v as follows:

$$\begin{aligned} v : ITL_n &\longrightarrow \mathbf{2}^{Int(\mathcal{T}_n)} \\ : \varphi &\longmapsto v_\varphi : Int(\mathcal{T}_n) \longrightarrow \{0, 1\} \end{aligned}$$

where $v_\varphi(I) = 0$ (resp., 1) if φ is false (resp., true) when evaluated on the interval I. In the following we will usually write $\varphi(I)$ in place of $v_\varphi(I)$. In particular, we say that φ is *valid* when $\varphi(I) = 1$ for all $I \in Int(\mathcal{T}_n)$.

Thus we have a general *evaluation map* v, which associates with every proposition φ a specific *valuation* v_φ which says on which intervals φ is true. The behavior of valuations with respect to connectives is defined as usual. More precisely:

- $(\varphi \vee \psi)(I) = 1$ whenever $\varphi(I) = 1$ or $\psi(I) = 1$;
- $(\varphi \wedge \psi)(I) = 1$ whenever $\varphi(I) = 1$ and $\psi(I) = 1$;
- $(\neg\varphi)(I) = 1$ whenever $\varphi(I) = 0$;
- $(\varphi \to \psi)(I) = 1$ whenever holds: if $\varphi(I) = 1$ then $\psi(I) = 1$.

Moreover, concerning propositional variables, we define $\varepsilon_i(I)$ to be true if and only if $I = [t_i, t_i] = \{t_i\}$. We have therefore all that we need to evaluate any proposition $\varphi \in ITL_n$.

Notice that, at this point, the partial order structure of $Int(\mathcal{T}_n)$ does not play any role. We now introduce two new connectives whose semantics instead depend on such partial order. These connectives are denoted by $\Box$ and $\Diamond$, and their semantics is defined as follows:

- $(\Box\varphi)(I) = 1$ when, for all intervals $J \subseteq I$, $\varphi(J) = 1$;
- $(\Diamond\varphi)(I) = 1$ when there exists an interval $J \subseteq I$ such that $\varphi(J) = 1$.

Notice that $\Box$ is "idempotent", in the sense that, for all intervals I, $(\Box\Box\varphi)(I) = (\Box\varphi)(I)$.

We are now ready to describe the subset of ITL_n which will be relevant to us. Define $\Theta_n = \{\varphi \in ITL_n \mid \varphi \to \Box\varphi \text{ is valid}\}$. Intuitively, this means that, if φ is true in I, then φ is true in all sub-intervals of I.

We remark here that, from a purely logic-theoretic point of view, the construction of the set Θ_n can be suitably described in the framework of modal companions of an superintuitionistic logic, see for instance [CZ]. However, the main goal of this section is to provide a combinatorial description of the logic of Θ_n, which we believe to be new.

As a subset of ITL_n, it is not clear a priori if Θ_n is interesting from a semantic point of view. We will now clarify this point, by showing that Θ_n is closed with respect to some, but not all, of the classical connectives.

Proposition 10. *If $\varphi, \psi \in \Theta_n$, then $\varphi \wedge \psi, \varphi \vee \psi \in \Theta_n$.*

Proof. Given $I \in Int(\mathcal{T}_n)$, suppose that $(\varphi \wedge \psi)(I) = 1$, that is $\varphi(I) = \psi(I) = 1$. Since $\varphi, \psi \in \Theta_n$, we have that, for all intervals $J \subseteq I$, it is $\varphi(J) = \psi(J) = 1$, which means that $(\Box(\varphi \wedge \psi))(I) = 1$, i.e. $\varphi \wedge \psi \in \Theta_n$.

Similarly, if we suppose that $(\varphi \vee \psi)(I) = 1$, we then have that $\varphi(I) = 1$ or $\psi(I) = 1$. Assume, for instance, that $\varphi(I) = 1$. Then, for all intervals $J \subseteq I$, it is $\varphi(J) = 1$, which implies $(\varphi \vee \psi)(J) = 1$. We can thus conclude that $(\Box(\varphi \vee \psi))(I) = 1$, i.e. $\varphi \vee \psi \in \Theta_n$. ∎

Proposition 11. *Θ_n is not closed with respect to $\neg$, that is there exists a proposition $\varphi \in \Theta_n$ such that $\neg\varphi \notin \Theta_n$.*

Proof. Consider the proposition $\varphi = \varepsilon_1 \vee \varepsilon_2$, and take the interval $I = \{t_1, t_2\}$. We have clearly $\varphi(I) = 0$, and so $(\neg\varphi)(I) = 1$. Now let $J = \{t_1\} \subseteq I$: we then get $\varphi(J) = 1$. Therefore we have found an interval $J \subseteq I$ such that $(\neg\varphi)(J) = 0$, which implies that $(\Box(\neg\varphi))(I) = 0$. We can thus conclude that $((\neg\varphi) \to \Box(\neg\varphi))(I) = 0$, as desired. Notice that this argument clearly works for any proposition of the type $\varepsilon_i \vee \varepsilon_{i+1}$. ∎

Proposition 12. *Θ_n is not closed with respect to $\to$, that is there exist propositions $\varphi, \psi \in \Theta_n$ such that $\varphi \to \psi \notin \Theta_n$.*

Proof. This proposition can be seen as a corollary of the previous one, since it is not difficult to prove that, for any interval I, $(\neg\varphi)(I) = (\varphi \to \bot)(I)$. However we will explicitly provide an example not of that form.

Given $\varphi = \varepsilon_1 \vee \varepsilon_2$ and $\psi = \varepsilon_2$, we clearly have that $\varphi, \psi \in \Theta_n$. Now, given $I = \{t_2, t_3\}$, we have $\varphi(I) = 0$, hence $(\varphi \to \psi)(I) = 1$. Set $J = \{t_2\} \subseteq I$, we get $\varphi(J) = 1$ and $\psi(J) = 0$, that is $(\varphi \to \psi)(J) = 0$. What we have proved so far is that there is an interval I such that $(\varphi \to \psi)(I) = 1$ having a sub-interval J for which $(\varphi \to \psi)(J) = 0$. The very last statement (the one concerning J) means that $(\Box(\varphi \to \psi))(J) = 0$. Therefore we can conclude that $((\varphi \to \psi) \to (\Box(\varphi \to \psi)))(I) = 0$, and so $\varphi \to \psi \notin \Theta_n$. ∎

The facts that we have recorded so far tell us that the connectives $\vee$ and $\wedge$ have a nice behavior inside Θ_n; the same cannot be said for the connectives $\neg$ and $\to$. We now define two new connectives $\sim$ and $\rightsquigarrow$ which can afford better notions of negation and implication inside Θ_n.

Given an interval I of $\mathcal{T}_n$, we define the semantics of $\sim$ and $\rightsquigarrow$ as follows:

- $(\sim\varphi)(I) = 1$ whenever $\forall J \subseteq I$, $\varphi(J) = 0$;
- $(\varphi \rightsquigarrow \psi)(I) = 1$ whenever $\forall J \subseteq I$, if $\varphi(J) = 1$, then $\psi(J) = 1$.

Thus, roughly speaking, we say that $\sim\varphi$ is true on I whenever φ is false on all sub-intervals of I, and that $\varphi \rightsquigarrow \psi$ is true on I whenever ψ is true on all sub-intervals of I on which φ is true. We will call $\sim$ and $\rightsquigarrow$ *pseudonegation* and *pseudoimplication*, respectively.

Observe that the semantics of pseudonegation and pseudoimplication can be described in terms of classical negation and implication and the connectives $\Box$ and $\Diamond$. In fact, for any interval I, $(\sim\varphi)(I) = (\neg\Diamond\varphi)(I) = (\Box\neg\varphi)(I)$ and $(\varphi \rightsquigarrow \psi)(I) = (\Box(\varphi \to \psi))(I)$. Moreover, as an immediate consequence of the definitions, we have $(\sim\varphi)(I) = (\varphi \rightsquigarrow \bot)(I)$.

It is an easy task (and so we leave it to the reader) to prove that, if $\varphi, \psi \in \Theta_n$, then $\sim\varphi, \varphi \rightsquigarrow \psi \in \Theta_n$. We now show that pseudonegation has the typical behavior of an intuitionistic negation.

Proposition 13. *Given $\varphi \in \Theta_n$ and $I \in Int(\mathcal{T}_n)$, if $\varphi(I) = 1$, then $(\sim\sim\varphi)(I) = 1$. The converse, however, does not hold in general.*

Proof. We observe that $(\sim\sim\varphi)(I) = 1$ if and only if, for all intervals $J \subseteq I$, there exists an interval $K \subseteq J$ such that $\varphi(K) = 1$. Since $\varphi \in \Theta_n$, if we suppose that $\varphi(I) = 1$, then we have that, for all intervals $J \subseteq I$, $\varphi(J) = 1$, hence the thesis follows.

To show that the converse does not hold in general, consider the proposition $\varphi = \varepsilon_1 \vee \varepsilon_2$ and the interval $I = \{t_1, t_2\}$. We immediately see that $\varphi(I) = 0$. Moreover, the fact that $(\sim\sim\varphi)(I) = 1$ is equivalent to the fact that, for all intervals $J \subseteq \{t_1, t_2\}$, there exists an interval $K \subseteq J$ such that $(\varepsilon_1 \vee \varepsilon_2)(K) = 1$. It is now easy to realize that the last statement is true. □

Proposition 14. *Given $\varphi \in \Theta_n$ and $I \in Int(\mathcal{T}_n)$, $(\sim\varphi)(I) = 1$ if and only if $(\sim\sim\sim\varphi)(I) = 1$.*

Proof. $\Rightarrow$) This is a special case of the previous proposition.

$\Leftarrow$) Suppose that $(\sim\sim\sim\varphi)(I) = 1$, then we have that, for all intervals $J \subseteq I$, $(\sim\sim\varphi)(J) = 0$. Thanks to the previous proposition, this implies that, for all intervals $J \subseteq I$, $\varphi(J) = 0$, that is $(\sim\varphi)(I) = 1$, as required. □

We are now ready to show that pseudonegation and pseudoimplication are the "right connectives" in order to describe the Heyting algebra structure of Dyck paths. Given $\varphi, \psi \in \Theta_n$, we say that φ and ψ are *equivalent* when $v(\varphi) = v(\psi)$. In this case we write $\varphi \models\mid \psi$. It is now left to the reader to show that $\models\mid$ is an equivalence relation on Θ_n which preserves $\vee, \wedge, \rightsquigarrow, \sim$; this means that, denoting with $\star$ any of the above mentioned binary connectives, if $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \Theta_n$ are such that $\varphi_1 \models\mid \varphi_2$ and $\psi_1 \models\mid \psi_2$, then $\varphi_1 \star \psi_1 \models\mid \varphi_2 \star \psi_2$ (and a similar fact holds for the unary connective $\sim$). Thus we can endow $\Theta_n / \models\mid$ with the distributive lattice structure in which $\vee$ and $\wedge$ are well-defined on equivalence classes thanks to the above considerations. Denote with $[\Theta_n]$ the resulting distributive lattice. Thus, for instance, given $\varphi, \psi \in \Theta_n$, denoting with $[\varphi], [\psi] \in \Theta_n / \models\mid$ the associated equivalence classes, in $[\Theta_n]$ we have that $[\varphi] \vee [\psi] = [\varphi \vee \psi]$, $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$ and $[\varphi] \rightsquigarrow [\psi] = [\varphi \rightsquigarrow \psi]$. Our next goal is to show that the canonical Heyting algebra structure on $[\Theta_n]$ is given precisely by the pseudoimplication operation $\rightsquigarrow$.

Proposition 15. *For every $\varphi, \psi \in \Theta_n$, we have:*

$$[\varphi] \rightsquigarrow [\psi] = \bigvee\{[\alpha] \in \Theta_n/\ \dashv\vdash\ \mid [\varphi] \wedge [\alpha] \leqslant [\psi]\}.$$

In other words, $\rightsquigarrow$ is the relative pseudocomplement operation in the canonical Heyting algebra structure of $[\Theta_n]$.

Proof. We start by observing that the partial order relation $\leqslant$ associated with the lattice structure of $[\Theta_n]$ can be described as follows: $[\varphi] \leqslant [\psi]$ whenever $\varphi(I) \leqslant \psi(I)$, for all intervals I (which means that, if $\varphi(I) = 1$, then $\psi(I) = 1$; this is the usual partial order derived from an algebra of propositions). The reader is invited to see that $\leqslant$ is well defined since, if the above condition is satisfied, then the same inequalities hold when φ and ψ are replaced by φ' and ψ', for any $\varphi' \in [\varphi]$, $\psi' \in [\psi]$.

Now suppose that $S = \{[\alpha] \in \Theta_n/\ \dashv\vdash\ \mid [\varphi] \wedge [\alpha] \leqslant [\psi]\} = \{[\alpha_1], [\alpha_2], \ldots, , [\alpha_r]\}$. Thus we wish to show that $[\varphi \rightsquigarrow \psi] = [\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_r]$. The first step will be to prove that $[\varphi \rightsquigarrow \psi] \in S$. Indeed, recall that the propositions α_i are characterized by the fact that $[\varphi \wedge \alpha_i] \leqslant [\psi]$. Now, given $I \in Int(\mathcal{T}_n)$, suppose that $(\varphi \wedge (\varphi \rightsquigarrow \psi))(I) = 1$. This implies that $\varphi(I) = 1$. Then, in order to have $(\varphi \rightsquigarrow \psi)(I) = 1$, necessarily $\psi(I) = 1$. This is enough to conclude that $[\varphi \wedge (\varphi \rightsquigarrow \psi)] \leqslant [\psi]$, and so that $[\varphi \rightsquigarrow \psi] \in S$, as desired.

To conclude the proof, we will now show that $[\varphi \rightsquigarrow \psi]$ is an upper bound of S, i.e. $[\varphi \rightsquigarrow \psi] \geqslant [\alpha_i]$, for all $i \leqslant r$. To this aim, suppose that $\alpha_i(I) = 1$, for some interval I; it will be enough to show that $(\varphi \rightsquigarrow \psi)(I) = 1$. Given $J \subseteq I$ such that $\varphi(J) = 1$, then we also have $\alpha_i(J) = 1$ (since $\alpha_i \in \Theta_n$), and so $(\varphi \wedge \alpha_i)(J) = 1$, hence $\psi(J) = 1$. We have thus shown that $(\varphi \rightsquigarrow \psi)(I) = 1$, as desired. □

As usual, to avoid heavy notations, the whole Heyting algebra structure on the set $[\Theta_n]$ will simply be denoted $[\Theta_n]$. The next lemma is crucial in the proof of our main theorem.

Lemma 16. *For any $\varphi \in ITL_n$, set $\overline{\varphi} = \sim\sim \varphi$. Given an interval I of $[n]$, set $\varepsilon_I = \overline{\bigvee_{i \in I} \varepsilon_i}$. Then, for any $\varphi \in \Theta_n$, there exists an antichain of intervals $I_1, I_2, \ldots, I_r$ of $[n]$ such that*

$$\varphi\ \dashv\vdash\ \varepsilon_{I_1} \vee \varepsilon_{I_2} \vee \cdots \vee \varepsilon_{I_r}.$$

Moreover, when the intervals are listed in increasing order of their minima, the above one is the unique proposition of that form equivalent to φ.

Proof. Fix $\varphi \in \Theta_n$. Denote with $\mathcal{I} \subseteq Int(\mathcal{T}_n)$ the set of all maximal intervals such that $\varphi(I) = 1$ (where "maximal" is intended with respect to the inclusion order). By construction, any two elements of $\mathcal{I}$ are incomparable; in particular, no two intervals in $\mathcal{I}$ can have either of the two endpoints in common. Totally order the elements of $\mathcal{I} = \{I_1, I_2, \ldots I_r\}$ with respect to their smallest elements (notice that we would obtain the same total order if we do the same with respect to the greatest elements). Moreover, identify each element $t_i \in \mathcal{T}_n$ with its index $i \in [n]$. In this way, we have that $\mathcal{I} \subseteq Int([n])$ and, for each $\alpha \leqslant r$, $I_\alpha \in Int([n])$. Our aim is now to prove

$$\varphi\ \dashv\vdash \bigvee_{1 \leqslant \alpha \leqslant r} \varepsilon_{I_\alpha}. \tag{1}$$

Before starting to prove this equivalence, it is convenient to observe the following two facts:

- for all intervals $J \subseteq I$, $\varepsilon_I(J) = 1$;
- for all intervals $J \nsubseteq I$, $\varepsilon_I(J) = 0$.

Indeed, given an interval $J \subseteq I$, we have $\varepsilon_I(J) = \sim\sim(\bigvee_{i\in I} \varepsilon_i)(J) = 1$ if and only if, for all intervals $K \subseteq J$, there exists an interval $M \subseteq K$ such that

$$\bigvee_{i\in I} \varepsilon_i(M) = 1.$$

The last statement is in fact true: for a given $K \subseteq J$, it is enough to choose an element $\tau \in K$ in order to have $\bigvee_{i\in I} \varepsilon_i(\{\tau\}) \geqslant \varepsilon_\tau(\{\tau\}) = 1$.

On the other hand, given an interval $J \nsubseteq I$, we have $\varepsilon_I(J) = \sim\sim(\bigvee_{i\in I} \varepsilon_i)(J) = 0$ if and only if there exists an interval $K \subseteq J$ such that, for all intervals $M \subseteq K$,

$$(\bigvee_{i\in I} \varepsilon_i)(M) = 0.$$

Once again, it is not difficult to see that the last equality is true: choosing, for instance, $K = J\backslash I$, one immediately realizes that, for every $i \in I$, $\varepsilon_i(M) = 0$ (since $i \notin M$, and so $M \neq \{i\}$).

We are now ready to proceed with the announced proof of (1). Given an interval I, since the only possible truth values are 0 and 1, it will be enough to prove what follows:

(i) if $\varphi(I) = 1$, then $\bigvee_\alpha \varepsilon_{I_\alpha}(I) = 1$;
(ii) if $\varphi(I) = 0$, then $\bigvee_\alpha \varepsilon_{I_\alpha}(I) = 0$.

Let us prove the two above statements separately.

(i) Suppose that $\varphi(I) = 1$. Then there exists s such that $I_s \in \mathcal{I}$ and $I \subseteq I_s$. Therefore $\bigvee_\alpha \varepsilon_{I_\alpha}(I) \geqslant \varepsilon_{I_s}(I) = 1$.
(ii) Suppose that $\varphi(I) = 0$. This means that $I \nsubseteq I_s$, for all $s \leqslant r$. Therefore $\varepsilon_{I_s}(I) = 0$, for all s, hence $\bigvee_\alpha \varepsilon_{I_\alpha}(I) = 0$. □

For any given $\varphi \in \Theta_n$, the above lemma provides a canonical form for φ, which will be called its *closed disjunctive form* (briefly, *CDF*).

The next theorem is the main result of the present paper.

THEOREM 17. *The Heyting algebra $\mathfrak{D}_n$ of Dyck paths of semilength n is isomorphic to the Heyting algebra $[\Theta_{n-1}]$.*

Proof. By the previous lemma, we can (and in fact will) identify each equivalence class of $[\Theta_n]$ with the unique proposition in CDF contained in the class. Moreover, we recall that, in $\mathfrak{D}_n$, the atoms are those paths all of whose factors are hills except for a single pyramid having exactly 4 steps. If P is an atom of $\mathfrak{D}_n$, we denote with x_P the abscissa of the unique nontrivial peak of P, and we call $x_P/2$ the *order* of the atom P.

Define the function $f : [\Theta_{n-1}] \to \mathfrak{D}_n$ as follows: given pairwise incomparable intervals $I_1, I_2, \ldots, I_r \subseteq [n-1]$, set $f(\varepsilon_{I_1} \vee \varepsilon_{I_2} \vee \cdots \vee \varepsilon_{I_r})$ equal to the Dyck path P of semilength n whose decomposition into join-irreducibles $P = P_1 \vee P_2 \vee \cdots \vee P_r$ has cardinality r and is such that, for every $j \leqslant r$, the interval of atoms below P_j is made by the atoms of order i, for all $i \in I_j$. We claim that f is a Heyting algebra isomorphism.

We start by showing that f is onto. Indeed, given any Dyck path P in $\mathfrak{D}_n$, its decomposition into join-irreducibles uniquely determines an antichain of intervals of $[n-1]$, which is given by the intervals $I_1, \ldots I_r$ of the orders of the atoms lying below each join-irreducible. By construction, the proposition (in CDF) $\varepsilon_{I_1} \vee \cdots \vee \varepsilon_{I_r}$ is mapped by f onto P.

Next we prove that f is order-preserving. To this aim, we first give an alternative description of the partial order of the Heyting algebra $[\Theta_{n-1}]$, based on the CDF representatives of equivalence classes. Given φ, ψ in $[\Theta_{n-1}]$, suppose that $\varphi = \varepsilon_{I_1} \vee \cdots \vee \varepsilon_{I_r}$ and $\psi = \varepsilon_{J_1} \vee \cdots \vee \varepsilon_{J_s}$, for suitable antichains of intervals in $[n-1]$. Recall that $\varphi \leqslant \psi$ if and only if, for all $I \subseteq [n-1]$, $\varphi(I) \leqslant \psi(I)$. Our assumptions on φ and ψ implies that $\varphi(I) = 1$ if and only if $I \subseteq I_h$, for some $h \leqslant r$ (and analogously for ψ). Thus we get that $\varphi \leqslant \psi$ if and only if, for every $h \leqslant r$, there exists $k \leqslant s$ such that $I_h \subseteq J_k$. Now suppose that $\varphi \leqslant \psi$. If $r = s = 1$, then $f(\varphi) = P$ and $f(\psi) = Q$ are join-irreducibles in $\mathfrak{D}_n$, i.e. they consist of a series of hills and a unique pyramid having at least 4 steps. Saying that $\varphi \leqslant \psi$ means in this case that $I_1 \subseteq J_1$, hence the interval of atoms dominated by P is contained in the interval of atoms dominated by Q, that is $P \leqslant Q$. In the general case, set $f(\varphi) = P = P_1 \vee \cdots \vee P_r$ and $f(\psi) = Q = Q_1 \vee \cdots \vee Q_s$; if $\varphi \leqslant \psi$, then, for every $h \leqslant r$, there exists $k \leqslant s$ such that $I_h \subseteq J_k$, which implies that $P_h \leqslant Q_k$. From here it follows that $P \leqslant Q$.

All the above arguments can be reversed, thus showing that f is also order-reflecting, i.e. that $f(\varphi) \leqslant f(\psi)$ implies that $\varphi \leqslant \psi$.

Therefore we have shown that f is onto, order-preserving and order-reflecting. It is known that this is enough to conclude that f is an order isomorphism. As a consequence, f is also a lattice isomorphism. Finally, thanks to Proposition 15, if we consider the canonical Heyting algebra structure induced by the finite distributive lattice structure, we have that f is a Heyting algebra isomorphism between $\mathfrak{D}_n$ and $[\Theta_{n-1}]$, as desired. ∎

4. Posets of intervals. The results of the previous sections suggest that every element of a Dyck algebra can be described by means of the underlying Heyting algebra structure together with a natural linear order structure on the set of the atoms of the algebra. Below we will try to clarify this statement.

Given a Dyck path P, denote with $\overline{P}$ its Heyting algebra closure, that is $\overline{P} = \sim\sim P$. The set of atoms of a Dyck algebra can be given a total order structure (which has nothing to do with the partial order of the algebra) by declaring an atom P strictly less than another atom Q whenever $x_P < x_Q$ (we refer to the notation introduced in the proof of theorem 17 for the order of an atom). In this case we will write $P \ll Q$, to avoid confusion with the partial order on Dyck paths. The (finite) set of atoms of $\mathfrak{D}_n$ will then be denoted $\{\pi_1, \pi_2, \ldots, \pi_{n-1}\}$, where π_i is the atom of order i. As we have already noticed, a join-irreducible path is uniquely determined by the set of atoms lying below it. Such a set of atoms is obviously an interval with respect to $\ll$. More specifically, if P is a join-irreducible and $\pi_i, \pi_{i+1}, \ldots, \pi_{i+j}$ are the atoms below P, then $P = \overline{\pi_i \vee \pi_{i+1} \vee \cdots \vee \pi_{i+j}}$. Summing up, every Dyck path can be expressed (via Birkhoff representation theorem) as the join of the closure of the join of $\ll$-intervals of atoms.

A further step towards abstraction consists of identifying an interval of atoms of $\mathfrak{D}_n$ with the interval of the orders of such atoms (which is a subset of $[n-1]$). Thus a Dyck path of semilength n can be identified with a family of incomparable intervals (i.e., an antichain of intervals) of $[n-1]$. This observation leads to a possible generalization of the approach we have developed so far for Dyck algebras, which we attempt to sketch in the remainder of this section.

Let $\mathcal{P}$ be a poset and denote with $Int(\mathcal{P})$ the poset of bounded intervals of $\mathcal{P}$ ordered by inclusion. The generic element of $Int(\mathcal{P})$ is then $[x, y] = \{z \in \mathcal{P} \mid x \leqslant z \leqslant y\}$. We are interested in the set $\mathcal{O}(Int(\mathcal{P}))$ of all down-sets of $Int(\mathcal{P})$. When ordered by inclusion, $\mathcal{O}(Int(\mathcal{P}))$ is a complete distributive lattice. This kind of lattices is often relevant from a theoretical point of view. For instance, we recall here that, when $\mathcal{P}$ is locally finite (i.e. every interval of $\mathcal{P}$ is finite), $\mathcal{O}(Int(\mathcal{P}))$ is isomorphic to the lattice of two-sided ideals of the incidence algebra of $\mathcal{P}$. This is a crucial fact in showing that two locally finite posets are order-isomorphic if and only if their incidence algebras are isomorphic (see, for instance, [DRS]).

Lemma 18. *The lattice $\mathcal{O}(Int(\mathcal{P}))$ is atomic (i.e. every element of $\mathcal{O}(Int(\mathcal{P}))$ contains at least one atom), and the set of its atoms is in bijection with P.*

Proof. For any $x \in \mathcal{P}$, the interval $[x, x]$ is a minimal element of $Int(\mathcal{P})$ (and every minimal element is of this form). Therefore the set $\mathcal{A} = \{\{\varnothing, [x, x]\} \subseteq Int(\mathcal{P}) \mid x \in \mathcal{P}\}$ is the set of atoms of $\mathcal{O}(Int(\mathcal{P}))$. Since every nonempty down-set of $Int(\mathcal{P})$ contains at least one interval I, if $x \in I$, then obviously $\{\varnothing, [x, x]\}$ is contained in the given down-set, which is enough to conclude. ∎

The above lemma asserts that there is a natural partial order on the set of atoms of $\mathcal{O}(Int(\mathcal{P}))$ (inherited from the partial order of $\mathcal{P}$), which has of course nothing to do with the inclusion order on $\mathcal{O}(Int(\mathcal{P}))$. It would be very interesting to deduce properties of the complete distributive lattice $\mathcal{O}(Int(\mathcal{P}))$ from properties of $\mathcal{P}$. Since lattices of down-sets are completely distributive, they are also Heyting algebras (in the same canonical way as finite lattices are), thus the same project can be developed for the Heyting algebra structure of $\mathcal{O}(Int(\mathcal{P}))$. To the best of our knowledge, it seems that this approach to the study of posets of intervals has never been considered before. To justify it, we now briefly mention some remarkable examples.

Examples.
1. If $\mathcal{P}$ is a discrete poset (i.e., an antichain), then clearly $Int(\mathcal{P}) \simeq \mathcal{P}$, hence any element of $\mathcal{O}(Int(\mathcal{P}))$ can be seen as a subset of $\mathcal{P}$. This means that $\mathcal{O}(Int(\mathcal{P}))$ is a complete and atomic Boolean algebra.
2. If $\mathcal{P}$ is totally ordered, then, in the finite case, $\mathcal{O}(Int(\mathcal{P}))$ is isomorphic to a Dyck lattice of suitable order, see also [FM1]. In case $\mathcal{P}$ is infinite, we obtain a natural infinite analog of Dyck lattices which still deserves to be studied.
3. If $\mathcal{P}$ is a finite Boolean algebra, then $Int(\mathcal{P})$ is the sup-semilattice of the nonempty faces of a cube of suitable dimension (see [BO1, BO2]). However, the distributive lattice $\mathcal{O}(Int(\mathcal{P}))$ has never been studied; a better understanding of its structure, as well as of its logic-theoretic properties as a Heyting algebra, is surely desirable. Also, we are not aware of what happens for infinite Boolean algebras.

5. Combinatorial properties of Dyck paths in terms of atoms of Dyck lattices. In this final section we will give a glimpse of the potential applications of the general approach outlined in the previous section in the particular case of Dyck algebras. More specifically, we will focus on combinatorics, and we will express several combinatorial properties of Dyck paths in terms of the Heyting algebra structure of Dyck algebras and the natural linear order $\ll$ on their atoms.

We recall once again that every path of the Dyck algebra $\mathfrak{D}_n$ can be identified with an antichain of intervals of the totally ordered set $[n-1]$ (namely, the family of pairwise incomparable intervals each of which represents the indices of the atoms dominated by a join-irreducible in the decomposition of the path). For instance, the red Dyck path in Figure 1 corresponds to the antichain of intervals $\{[2, 4], [4, 5], [6, 6], [8, 8], [9, 9]\}$ of the set $[9]$. For any two such antichains $\{I_1, \dots, I_n\}$ and $\{J_1, \dots, J_m\}$, it is $\{I_1, \dots, I_n\} \leqslant \{J_1, \dots, J_m\}$ in $\mathfrak{D}_n$ whenever, for every $i \leqslant n$, there exists $j \leqslant m$ such that $I_i \subseteq J_j$ (as we already noticed in the proof of Theorem 17). It is also useful to record an explicit expression for join and meet:

$$\{I_1, \ldots, I_n\} \vee \{J_1, \ldots, J_m\} = \{I_1, \ldots, I_n, J_1, \ldots, J_m\};$$
$$\{I_1, \ldots, I_n\} \wedge \{J_1, \ldots, J_m\} = \{I_i \cap J_j \mid i \leqslant n, j \leqslant m\},$$

where in both the r.h.s.'s we tacitly assume to discard all intervals that are not maximal (this is of course needed in order to get an antichain). We can also give a description of pseudonegation: if a path P is represented by the antichain of intervals $\{I_1, \ldots, I_m\}$, then $\sim P$ is represented by the (unique) family of maximal intervals constituting a partition of the set $[n-1] \setminus (I_1 \cup \cdots \cup I_m)$. Referring to the black path in Figure 2, its pseudonegation is represented by the antichain of intervals $\{[4,4],[12,14]\}$ of $[15]$.

We now state and prove a series of propositions which express some important combinatorial parameters on Dyck paths in terms of the above described "interval" representation of Dyck paths. For a classical reference on the enumerative combinatorics of Dyck paths, see the survey article [D]. Before starting we need to introduce a few notations and definitions.

For a given Dyck path $P \in D_n$, we denote with $\mathcal{F}_P$ the antichain of intervals of $[n-1]$ representing that path. If $\mathcal{F}_P = \{I_1, \ldots, I_m\}$, then the *cardinality* of $\mathcal{F}_P$ is $|\mathcal{F}_P| = m$, whereas the *weight* of $\mathcal{F}_P$ is $\|\mathcal{F}_P\| = |I_1 \cup \cdots \cup I_m|$. Moreover, we say that $I \in \mathcal{F}_P$ is *internal* when $1, n-1 \notin I$; the set of internal intervals of $\mathcal{F}_P$ is denoted with $\mathcal{F}_P^*$.

Proposition 19. *The number of peaks of a Dyck path $P \in D_n$ is given by $|\mathcal{F}_P| + \|\mathcal{F}_{\sim P}\| - |\mathcal{F}_{\sim P}^*|$.*

Proof. Each peak of P of height > 1 represents the contribution of a join-irreducible in the (unique) expansion of P as a join of join-irreducibles. Since join-irreducibles of P correspond to intervals of $\mathcal{F}_P$, the contributions of these peaks is exactly $|\mathcal{F}_P|$. As far as peaks at height 1 are concerned (i.e., hills), we observe that a bunch of s consecutive hills of P corresponds to an internal interval of cardinality $s+1$ of $\mathcal{F}_{\sim P}$, except when the bunch of hills is at the beginning or at the end of the path, in which cases it corresponds to a noninternal interval of cardinality s of $\mathcal{F}_{\sim P}$. This means that the number of hills of P is $\|\mathcal{F}_{\sim P}\| - |\mathcal{F}_{\sim P}^*|$, which concludes the proof. ∎

A byproduct of the above proof is the following.

Corollary 20. *The number of hills of a Dyck path $P \in D_n$ is given by $\|\mathcal{F}_{\sim P}\| - |\mathcal{F}_{\sim P}^*|$.*

Proposition 21. *The sum of the heights of the peaks of a Dyck path $P \in D_n$ is given by $\|\mathcal{F}_P\| + |\mathcal{F}_P| + \|\mathcal{F}_{\sim P}\| - |\mathcal{F}_{\sim P}^*| = n - 1 + |\mathcal{F}_P| - |\mathcal{F}_{\sim P}^*|$.*

Proof. Concerning peaks of height > 1, we observe that the height of each of them is the cardinality of the interval which correspond to it minus 1. Thus the contribution to the total heights sum of such peaks is $\|\mathcal{F}_P\| + |\mathcal{F}_P|$. On the other hand, the sum of the heights of the hills of P equals the number of hills of P, so (from the proof of the previous proposition), their contribution is given by $\|\mathcal{F}_{\sim P}\| - |\mathcal{F}_{\sim P}^*|$. Summing up the two quantities we have obtained gives the desired result. ∎

Proposition 22. *The number of returns of a Dyck path $P \in D_n$ is given by $\|\mathcal{F}_{\sim P}\| + 1$.*

Proof. The total number of returns of P is given by the number of its hills plus the number of its nontrivial factors. As we have already proved, the number of hills of P is given by $\|\mathcal{F}_{\sim P}\| - |\mathcal{F}^*_{\sim P}|$. Moreover we observe that the number of nontrivial factors of P is "approximately equal" to the number of nontrivial factors of $\sim P$. They are indeed equal if and only if P either starts or ends with a hill (but not both); in this case, P has precisely $|\mathcal{F}_{\sim P}|$ nontrivial factors, and so the total number of returns of P is $\|\mathcal{F}_{\sim P}\| - |\mathcal{F}^*_{\sim P}| + |\mathcal{F}_{\sim P}| = \|\mathcal{F}_{\sim P}\| + 1$ (since in this case $\sim P$ has precisely one nontrivial factor either at the beginning or at the end, which corresponds to a single noninternal interval). Otherwise, P has one more (resp., less) nontrivial factor than $\sim P$ if and only if P both starts and ends with a nontrivial factor (resp., with a hill); in this case P has precisely $|\mathcal{F}_{\sim P}|+1$ (resp., $|\mathcal{F}_{\sim P}|-1$) nontrivial factors, and so the total number of returns of P is $\|\mathcal{F}_{\sim P}\| - |\mathcal{F}^*_{\sim P}| + |\mathcal{F}_{\sim P}| + 1$ (resp., $\|\mathcal{F}_{\sim P}\| - |\mathcal{F}^*_{\sim P}| + |\mathcal{F}_{\sim P}| - 1$), which equals $\|\mathcal{F}_{\sim P}\| + 1$ (the reader is invited to check all the details). ∎

All the results illustrated so far concern statistics which can be directly expressed in terms of global parameters. We give below a few simple examples in which it is necessary to take into account some local information. The last example is especially interesting, being an instance of a kind of "pattern occurrence" statistic. Since the proofs are quite easy, we leave most of them to the reader. Recall that the "interval" representation of a generic Dyck path P is written $\{I_1, \dots, I_m\}$, where each I_i is an interval of $[n-1]$, and the intervals are listed in increasing order of their minima. Moreover, we say that two consecutive intervals I_i and I_{i+1} are *distanced* when $\max I_i < \min I_{i+1} - 1$.

Proposition 23. *The height of the first peak of a Dyck path $P \in D_n$ is given by*

$$\begin{cases} |I_1| + 1 & \text{, if } 1 \in I_1 \\ 1 & \text{, otherwise} \end{cases}.$$

Proposition 24. *The number of peaks before the first return of a Dyck path $P \in D_n$ is given by*

$$\begin{cases} \max\{k \mid I_{i-1} \text{ and } I_i \text{ are not distanced, for all } i \leqslant k\} & \text{, if } 1 \in I_1 \\ 1 & \text{, otherwise} \end{cases}.$$

Proposition 25. *The number of occurrences of the (consecutive) factor duu in a Dyck path $P \in D_n$ is given by*

$$|\{i \leqslant n-1 \mid \text{either } I_{i-1} \text{ and } I_i \text{ are distanced or } |I_i \backslash I_{i-1}| > 1\}.$$

Proof. Each occurrence of *duu* in P corresponds to the occurrence of a valley not immediately followed by a peak. For any such valley we have two distinct possibilities. If the valley is not on the x-axis, then it corresponds to a transition between two consecutive join-irreducibles such that the rightmost one dominates at least two atoms which are not dominated by the leftmost one. In terms of the "interval" representation of the path, this corresponds to a consecutive pair of non-distanced intervals I_{i-1} and I_i such that $|I_i \backslash I_{i-1}| > 1$. On the other hand, if the valley lies on the x-axis, then it is immediately followed by a nontrivial factor, and the first interval I_i corresponding to such a factor is clearly distanced from the previous one I_{i-1}. ∎

6. Conclusions and further work. The main aim of the present paper is to build a bridge between a certain fragment of the Halpern-Shoham logic (sometimes called the logic of sub-intervals) and the combinatorics (and algebra) of Dyck paths. Hopefully this should result in a better and more effective description of the logic of sub-intervals, which can now benefit from a powerful combinatorial machinery. We claim that this link is also relevant from a strictly combinatorial point of view, since it provides one more example (and a nice one, in our opinion) of how the combinatorics of lattice paths can play an important role in apparently unrelated fields.

The fact that the results obtained in the present paper are so neat and effective is certainly due in part to the nice combinatorial structure of Dyck paths and to the nice logical structure of the logic of sub-intervals. Nevertheless, neither of these two structures is trivial. In particular, the logic of sub-intervals is especially interesting since, in a sense, it constitutes a sort of intermediate step between genuine interval logics and classical temporal logics (of time instants rather than time intervals). Indeed, with point-based temporal logics it shares the feature that, if a statement is true over a certain interval, then it is true over all of its sub-intervals (and so, in particular, over all singleton intervals). This can be expressed by saying that the logic of sub-intervals has the same behavior of point-based interval logics when "looking downwards". On the other hand, it is not true in general that a statement which is true over two distinct intervals is also true over the union of those intervals. Therefore, the logic of sub-intervals is sensibly different from classical temporal logics when "looking upwards".

The study initiated in the present paper is amenable of extensions and generalizations in several directions. From a combinatorial point of view, it seems natural to replace Dyck paths with other families of paths, provided that the resulting posets are in fact distributive lattices. The first, obvious candidates are Motzkin and Schröder paths, which have been considered in Section 2.1 for what concerns their Heyting algebra structure. In both cases, an investigation of the logic-theoretic counterparts of such algebras is likely to be done along similar lines. From a logic-theoretic point of view, there are at least two possible directions for further research. First, it is natural to ask whether it is possible to find analogous results and similar combinatorial descriptions for other fragments, assuming that the underlying order of time instants still is a finite total order. To give a more concrete hint, the fragment associated with the binary relation "*begins*" between time intervals (the pair of intervals (I, J) is in this relation whenever I and J have the same starting point and I is contained in J) is likely to have a nice combinatorial description, though probably expressible using objects that are not lattice paths, and it is conceivable that algebraic structures similar to those studied in the present paper can arise. A second kind of logic-theoretic generalization concerns the structure of the space of time instants. Instead of considering a finite total order, one can consider either an infinite (but discrete) total order or a more general partial order. In the former case, the combinatorial structure that is involved could be some notion of *infinite Dyck path*, that is a Dyck path *having infinite length*. There are at least two different ways to formalize this suggestion: either working in the set of *all Dyck paths* (of any length), which is naturally partially ordered, and can be probably interpreted as some kind of infinitary limit object for the family of Dyck lattices; or introducing Dyck paths *having infinite support*, i.e. Dyck paths unbounded on the right which do not have a suffix consisting of the concatenation of an infinite number of peaks lying on the x-axis. Both frameworks seem to have never been considered, at least with respect to their combinatorial structure. If instead the order structure of time instants is not a total order, the logic involved becomes considerably more complicated. Some sources suggest to consider (discrete) partial orders having the so-called *linear interval property*, which means that every interval is totally ordered (this is essentially equivalent to have a sort of tree-like structure). This seems in fact an interesting way to relax the request of having a total order without losing too much. In this situation, one should probably look for a definition of what we could call a *fuzzy Dyck path*, i.e. a Dyck path some of whose steps are not univocally determined.

There are also a few more hints for further work that are suggested by our research, which will be briefly illustrated below as a conclusion of our paper.

As already illustrated in section 4, the case of Dyck algebras investigated here is just an instance of a more general situation. The study of the complete distributive lattices (Heyting algebras) of the down-sets of the poset of intervals of a generic poset is a totally unexplored subject, which seems interesting to be pursued both from the algebraic and the logic-theoretic point of view. We remark that the relevance of posets of intervals in certain logical framework has already been noticed, see [CM]. In particular, the case in which the starting poset is a Boolean algebra (example 3 in section 4) is related to the logic of the n-cube, initiated in [RM] and recently explored in [Mun]

It would be nice to have a purely algebraic characterization of Dyck lattices and of Dyck algebras. Even if they are not a variety (in the sense of universal algebra), they show some interesting features. For instance, Dyck lattices are projective distributive lattices (this follows from a result of Balbes [B], which asserts that a finite distributive lattice is projective if and only if the poset of its join-irreducibles is a meet-semilattice).

While in Section 5 we give some instances of how the Heyting algebra structures of Dyck algebras can be useful to recover combinatorial properties of Dyck paths, it is conceivable that also the opposite point of view might yield interesting results. Namely, one can ask whether certain combinatorial statistics on Dyck paths give some information on how a path sits inside the associated Dyck algebra.

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