Existence of strong minimizers for the Griffith static fracture model in dimension two

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We consider the Griffith fracture model in two spatial dimensions, and prove existence of strong minimizers, with closed jump set and continuously differentiable deformation fields. One key ingredient, which is the object of the present paper, is a generalization of the decay estimate by De Giorgi, Carriero, and Leaci to the vectorial situation. This is based on replacing the coarea formula by a method to approximate SBD^p functions with small jump set by Sobolev functions and is restricted to two dimensions. The other two ingredients are contained in companion papers and consist respectively in regularity results for vectorial elliptic problems of the elasticity type and in a method to approximate in energy $GSBD^p$ functions by SBV^p ones.

1 Introduction

The study of brittle fracture in solids is based on the Griffith model, which combines elasticity with a term proportional to the surface opened by the fracture. In its variational formulation one minimizes

$$E[\Gamma, u] := \int_{\Omega\setminus\Gamma} \Big(\frac{1}{2}\mathbb{C}e(u)\cdot e(u) + h(x,u)\Big)dx + 2\beta\mathcal{H}^{n-1}(\Gamma\cap\Omega) \tag{1.1}$$

over all closed sets $\Gamma \subset \overline{\Omega}$ and all deformations $u \in C^1(\Omega\setminus\Gamma, \mathbb{R}^n)$ subject to suitable boundary and irreversibility conditions. Here $\Omega \subset \mathbb{R}^n$ is the reference configuration, the function $h \in C^0(\Omega\times\mathbb{R}^n)$ represents external volume forces, $e(u) = (\nabla u + \nabla u^T)/2$ is the elastic strain, $\mathbb{C} \in \mathbb{R}^{(n\times n)\times(n\times n)}$ is the matrix of elastic coefficients, $\beta > 0$ the surface energy. The evolutionary problem of fracture can be modeled as a sequence of variational problems, in which one minimizes (1.1) subject to varying loads with a kinematic restriction representing the irreversibility of fracture, see [33, 8, 24].

Mathematically, (1.1) is a vectorial free discontinuity problem. Much better known is its scalar version, mechanically corresponding to the anti-plane case, in which one replaces the elastic energy by the Dirichlet integral,

$$E_{\mathrm{MS}}[\Gamma, u] := \int_{\Omega\setminus\Gamma} \Big(\frac{1}{2}|\nabla u|^2 + h(x,u)\Big) dx + 2\beta\mathcal{H}^{n-1}(\Gamma\cap\Omega)\,, \tag{1.2}$$

and one minimizes over all maps $u : \Omega \setminus \Gamma \to \mathbb{R}$. This scalar reduction coincides with the Mumford-Shah functional of image segmentation, and has been widely studied analytically and numerically [5, 25, 8]. The relaxation of (1.2) leads naturally to the space of special functions of bounded variation, and is given by

$$E^*_{\mathrm{MS}}[u] := \int_{\Omega} \Big(\frac{1}{2}|\nabla u|^2 + h(x,u)\Big) dx + 2\beta\mathcal{H}^{n-1}(J_u\cap\Omega)\,. \tag{1.3}$$

Here u belongs to the space $SBV^2(\Omega)$, which is the set of functions such that the distributional gradient Du is a bounded measure and can be written as $Du = \nabla u\mathcal{L}^n + [u]\nu_u\mathcal{H}^{n-1}\llcorner J_u$ with $\nabla u \in L^2(\Omega;\mathbb{R}^n)$, $[u]$ the jump of u, J_u the $(n-1)$-rectifiable jump set of u, which obeys $\mathcal{H}^{n-1}(J_u) < \infty$, and ν_u its normal. Existence of minimizers for the relaxed problem E^*_{MS} follows then from the general compactness properties of SBV^2, see [5] and references therein.

The breakthrough in the quest for an existence theory for the Mumford-Shah functional (1.2) came with the proof by De Giorgi, Carriero, and Leaci in 1989 [26] that the jump set of minimizers is essentially closed, in the sense that minimizers of the relaxed functional E^*_{MS} obey

$$\mathcal{H}^{n-1}(\Omega \cap J_u) = \mathcal{H}^{n-1}(\Omega \cap \overline{J_u}). \tag{1.4}$$

This permits to define Γ as the closure of J_u, and then to use regularity of local minimizers of the Dirichlet integral on the open set $\Omega \setminus \Gamma$ to prove smoothness of u. The essential closedness of the jump set stated in (1.4) is a property satisfied by several variants of the energy in (1.3), in particular also by some defined on vector-valued $SBV^2(\Omega,\mathbb{R}^N)$ functions. More precisely, the integrands dealt with in literature depend on the full gradient with some additional structure conditions: they are either convex and depending (essentially) on the modulus of the gradient (cf. [10, 32, 38]) or they are specific polyconvex integrands in two dimensions, i.e. $n = 2$ (cf. [1, 2]).

In this paper we study existence for (1.1) in two spatial dimensions; therefore the main difference with the results quoted above is the dependence of the bulk energy density on the linear elastic strain rather than on the full deformation gradient. Indeed, we assume that $\mathbb{C}$ is a symmetric linear map from $\mathbb{R}^{n\times n}$ to itself with the properties

$$\mathbb{C}(\xi - \xi^T) = 0 \text{ and } \mathbb{C}\xi\cdot\xi \geq c_0|\xi + \xi^T|^2 \text{ for all } \xi \in \mathbb{R}^{n\times n}. \tag{1.5}$$

This includes of course as a special case isotropic elasticity, $\mathbb{C}\xi\cdot\xi = \frac{1}{4}\lambda_1|\xi+\xi^T|^2 + \frac{1}{2}\lambda_2(\mathrm{Tr}\,\xi)^2$, where λ_1 and λ_2 are the Lamé constants.

Our main result is the following.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz set, $g\in L^\infty(\Omega;\mathbb{R}^2)$, let $\mathbb{C}$ obey the positivity condition (1.5), $\beta>0$, $h(x,z):=\kappa|z-g(x)|^2$ for some $\kappa>0$. Then the functional (1.1) has a minimizer in the class*

$$\mathcal{A} := \{(u,\Gamma) : \Gamma\subset\overline{\Omega} \text{ closed, } u\in C^1(\Omega\setminus\Gamma;\mathbb{R}^2)\}.$$

This result was announced in [18]. An extension to higher dimension will appear elsewhere [14].

We also consider a generalization of the basic model (1.1) with p-growth, which may be appropriate for the study of fracture models with nonlinear constitutive relations that account for damage and plasticity, see for example [41, Sect. 10 and 11] and references therein. We replace the quadratic energy density and the lower order term by the functions

$$\begin{aligned} f_\mu(\xi) &:= \frac{1}{p}\left((\mathbb{C}\xi\cdot\xi+\mu)^{p/2} - \mu^{p/2}\right), \\ h(x,z) &:= \kappa|z-g(x)|^p, \end{aligned} \tag{1.6}$$

where $\mu\geq 0$ and $\kappa>0$ are parameters and $g\in L^\infty(\Omega;\mathbb{R}^2)$. We remark that for $\mu>0$ and for small strains ξ this energy reduces to linear elasticity, $f_\mu(\xi) = \frac{1}{2}\mu^{p/2-1}\mathbb{C}\xi\cdot\xi + O(|\xi|^3)$. For large ξ it behaves, up to multiplicative factors, as $|\xi+\xi^T|^p$, which is for example appropriate for models that describe plastic deformation at large strains. We obtain the following.

Theorem 1.2. *Let $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz set, $p\in(1,\infty)$, $\mu\geq 0$, $\kappa,\beta>0$, $g\in L^\infty(\Omega;\mathbb{R}^2)$ if $p\in(1,2]$ and $g\in W^{1,p}(\Omega;\mathbb{R}^2)$ if $p\in(2,\infty)$, let $\mathbb{C}$ obey the positivity condition (1.5), and let f_μ be as in (1.6). Then the functional*

$$E_p[\Gamma,u] := \int_{\Omega\setminus\Gamma} (f_\mu(e(u)) + \kappa|u-g|^p)dx + 2\beta\mathcal{H}^1(\Gamma\cap\Omega) \tag{1.7}$$

has a minimizer in the class

$$\mathcal{A}_p := \{(u,\Gamma) : \Gamma\subset\overline{\Omega} \text{ closed, } u\in C^1(\Omega\setminus\Gamma;\mathbb{R}^2)\}. \tag{1.8}$$

Remark 1.3. *The assumption $g\in W^{1,p}(\Omega;\mathbb{R}^2)$ if $p>2$ is probably of technical nature and depends on the elliptic regularity results discussed in Section 2.1.*

In the last years several approaches have been proposed to show existence for E_{MS} after the seminal paper by De Giorgi, Carriero, and Leaci [26] in which the result has been first established (cf. [10, 32, 23, 43, 25, 27, 9], and

[31] for a recent review). Here we follow the general strategy of proof by De Giorgi, Carriero, and Leaci [26], although several new difficulties inherent to the dependence of the bulk energy density on the symmetrized gradient have to be faced.

We start off writing the relaxed formulation of (1.1), which for $\kappa > 0$ has a minimizer in the space $GSBD^p(\Omega)$ since no L^∞ bound is imposed (see below for the precise definition of the functional setting). This space and its companion SBD^p are, however, much less understood than the scalar analogues $(G)SBV^p$, though in the last few years there have been several contributions in this direction [22, 13, 20, 19, 16, 34, 35, 36]. In particular, since apart from trivial cases the Chain rule formula does not hold in SBD^p, the very definition of the generalized space $GSBD^p$ given in [22] requires a different approach with respect to the standard definition of $GSBV^p$ as the set of functions whose truncations belong to SBV^p.

The proof given in [26] of the closure condition (1.4) in the scalar case is based on a careful analysis of sequences of SBV^p (quasi-)minimizers with vanishing jump energy, for which a priori no control of any Lebsgue norm is available. The idea to circumvent this difficulty and to gain compactness in SBV^p introduced by De Giorgi, Carriero, and Leaci, however, makes substantial use of a Poincaré-type inequality for SBV functions that is proven via the coarea formula, which does not extend to the vectorial case. One key ingredient in our proof is then an approximation result for SBD^p functions with small jump set with $W^{1,p}$ functions, stated in Proposition 2.3 below, which permits to obtain an equivalent Poincaré-type inequality for SBD^p functions, however restricted to two spatial dimensions (see [19] for the proof).

We remark explicitly that this is the only issue in which we have to confine to two dimensions. Indeed, the other two key results of our approach have higher dimensional analogues. More precisely, the elliptic regularity of solutions to linear elasticity type systems in 2d stated in Theorem 2.2, has a partial regularity counterpart in dimension $n \geq 3$ with an estimate on the Hausdorff dimension of the singular set. Moreover, the decay estimate for elliptic energies with p-growth also holds in any dimension if $p = 2$, and in dimension $n = 3$ if $p \neq 2$ (cf. Proposition 2.1 and related comments). Finally, the strong approximation result of $GSBD^p$ functions with $SBV^p \cap L^\infty$ ones in Theorem 2.4 holds without any dimensional limitation (cf. [16]). The extension of the Poincaré-type inequality for SBD^p functions to higher dimensions has been obtained in [14] after the completion of the present paper. Together with the regularity results recalled in Section 2.1 and the density lower bound estimates proved in Section 3, this leads to corresponding generalizations of Theorems 1.1 and 1.2 in any dimension if $p = 2$, and in dimension $n = 3$ for $p \neq 2$.

Going back to commenting the proof, we note that rather than extending the quoted Poincaré-type inequality for SBD^p functions to $GSBD^p$ ones, we argue by approximating $GSBD^p$ functions by SBD^p ones in energy. The latter issue is discussed in [42] for $p = 2$ and any dimension, see Section 2.2 below. The case of a general exponent $p \in (1, \infty)$ is established in a companion paper [16] without dimensional restrictions and requires a nontrivial modification of the original arguments in [11, 12, 42]. Since the SBD^p-$GSBD^p$ approximation does not preserve the boundary values, one additionally needs to suitably combine the two approximation results carefully.

Let us also stress that under the working assumption that g is bounded, by the maximum principle, i.e. by truncations, the fidelity term in the scalar case is a lower order perturbation that originates and justifies the more general regularity theory developed in literature for Mumford-Shah quasi-minimizers. In the vector valued setting of interest here instead, for the above mentioned lack of truncation techniques, such a term plays a nontrivial role in the asymptotic analysis of sequences with infinitesimal jump energy and has to be taken into account (cf. Proposition 3.4).

In any case, the asymptotics of such sequences in the framework under investigation is related, similarly to the scalar setting, to minimizers of an elliptic problem. In the scalar case, standard elliptic regularity directly gives the necessary decay estimates for the energy (cf. [5, 32]). The case of the system of linearized elasticity is also well-known in literature. Instead, for systems of linearized elasticity type with $p \neq 2$ the regularity is less standard, and we summarize the results we need in Section 2.1. Details and extensions to higher dimensions are discussed elsewhere [17]. In particular, partial regularity with an explicit estimate on the Hausdorff dimension of the potential singular set are established in [17]. We remark that it is a major open problem to prove or disprove full regularity in the case $p \neq 2$. Despite this, the mentioned Hausdorff dimension estimate is particularly relevant in view of the possible extensions of the existence of minimizers of the energy in (1.1) in higher dimensions.

Our main contribution is a statement on the regularity of weak local minimizers (cf. (3.2) for the precise definition). In particular, we show (see Theorem 3.11 below) that if $u \in GSBD^p(\Omega)$ is a local minimizer for the weak formulation then $\mathcal{H}^1(\Omega \cap \overline{J_u} \setminus J_u) = 0$ and $u \in C^1(\Omega \setminus \overline{J_u}; \mathbb{R}^2)$. The presence of the fidelity term, that is the condition $\kappa > 0$, is only required for establishing the existence of a weak minimizer in $GSBD^p(\Omega)$ via [22, Theorem 11.3], independently of the dimension. If $\kappa = 0$ and if Dirichlet boundary conditions are imposed, the existence of a weak minimizer in $GSBD^p(\Omega)$ in dimension 2 is guaranteed by [36, Theorem 4.15], while in dimension $n > 2$ this is still an open problem.

Let us conclude the introduction by outlining the organization of the paper. We first provide the technical preliminaries: in Section 2.1 we state the needed elliptic decay estimates, then in Section 2.2 we introduce the spaces SBD^p and $GSBD^p$ and discuss the quoted approximation results.

In Section 3 we first define the total energy G, and also an appropriate homogeneous version G_0 (see (3.1) and (3.3)), that is obtained by relaxing E in (1.1) on $GSBD^p(\Omega)$ as done for E^*_{MS} from E_{MS}. In Propositions 3.2 and 3.4 we investigate the compactness and the asymptotics of minimizing sequences for G_0 with vanishing jump energy. We show that they converge, up to the addition of affine functions, to a local minimizer of G_0 on Sobolev spaces. This result is instrumental to obtain the decay of the energy G_0 for functions whose deviation from minimality and measure of the jump set are small (Lemma 3.6). We conclude the section by proving the density lower bound and the essential closedness of the jump set for local minimizers of the total energy G (Lemma 3.8, Corollary 3.9, and Theorem 3.11). Finally, in Section 3.2 we prove the main results Theorem 1.1 and Theorem 1.2.

2 Preliminaries

2.1 Regularity for generalized linear elasticity systems

In this section we investigate the regularity properties of minimizers of elastic type energies. Despite several related contributions present in literature (see [37, 29, 30] and references therein), we have not found the exact statements needed for our purposes. We summarize here the results of interest, and provide elsewhere [17] a self-contained proof of the elliptic decay estimates as well as of full and partial regularity for local minimizers according to the dimensional setting of the problem, following the techniques of [3, 29, 30, 39, 40].

We first present a decay property of the L^p-norm of $e(u)$, with u a local minimizer of $v \mapsto \int_\Omega f_0(e(v))dx$, i.e.,

$$\int_\Omega f_0(e(u))dx \leq \int_\Omega f_0(e(v))dx,$$

for all $v \in W^{1,p}(\Omega;\mathbb{R}^n)$ satisfying $\{v \neq u\} \subset\subset \Omega$. Such a result is necessary to prove the density lower bound inequality in Section 3. Since in this paper the decay property will be applied to the blow-ups of minimizers, there are no lower order terms, therefore we state the result only for the functional with $\kappa = \mu = 0$ (cf. [17] for the proof given in the general case).

Proposition 2.1 ([17, Propositions 3.4 and 4.3])**.** *Let $n = 2$ or $n = 3$, and $p \in (1, \infty)$. Let $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ be a local minimizer of*

$$v \mapsto \int_\Omega f_0(e(v))dx.$$

Then, there exists $\gamma_0 = \gamma_0(n, p)$, with $\gamma_0 = 0$ if $n = 2$ and $\gamma_0 \in [0, 1)$ if $n = 3$, such that for all $\gamma \in (\gamma_0, 1)$ there is a constant $c = c(\gamma, p, n) > 0$ such *that if* $B_{R_0}(x_0) \subset\subset \Omega$*, then for all* $\rho < R \le R_0 \le 1$

$$\int_{B_\rho(x_0)} |f_0(e(u))|^2 dx \le c\left(\frac{\rho}{R}\right)^{n-\gamma} \int_{B_R(x_0)} |f_0(e(u))|^2 dx,$$

with $c = c(\gamma, p, n) > 0$*.*

In the quadratic case $p = 2$ it is well-known that the minimizer u is $C^\infty(\Omega;\mathbb{R}^n)$ in any dimension as long as g is smooth (see for instance [40, Theorem 10.14] or [39, Theorem 5.14, Corollary 5.15]). Below we state a partial regularity result in the n-dimensional setting when $p \ne 2$ (see [17, Section 4] for the proof).

Theorem 2.2 ([17, Proposition 4.4 and Theorem 4.9])**.** *Let* $n \ge 2$*,* $p \in (1,\infty)$*,* κ *and* $\mu \ge 0$*,* $g \in L^s(\Omega;\mathbb{R}^n)$*, with* $s > p$*, if* $p \in (1,2]$ *and* $g \in W^{1,p}(\Omega;\mathbb{R}^n)$ *if* $p \in (2,\infty)$*. Let* $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ *be a local minimizer of*

$$v \mapsto \int_\Omega f_\mu(e(v))dx + \kappa\int_\Omega |v-g|^p dx.$$

If $n = 2$*, then* $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ *for all* $\alpha \in (0,1)$ *when* $1 < p < 2$ *and* $\mu > 0$ *or when* $p \ge 2$*, and for some* $\alpha(p) \in (0,1)$ *when* $1 < p < 2$ *and* $\mu = 0$*.*
If $n \ge 3$*,* κ *and* $\mu > 0$*, and* g *also satisfies* $g \in L^\infty(\Omega;\mathbb{R}^n)$*, then there exists an open set* $\Omega_u \subseteq \Omega$ *such that* $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega_u;\mathbb{R}^n)$ *for all* $\beta \in (0, 1/2)$*. Moreover,*

$$\dim_{\mathcal{H}}(\Omega\setminus\Omega_u) \le (n-\widetilde{p}) \vee 0,$$

where $\widetilde{p} := p^* \wedge 2$*,* $p^* := \frac{np}{n-p}$ *if* $p \in (1,n)$ *and* ∞ *otherwise.*

2.2 Approximation of SBD^p and $GSBD^p$ functions

We start by briefly collecting the main properties of GBD and $GSBD^p$ of interest to us. Let Ω be a bounded open set in $\mathbb{R}^n$. If $u : \Omega \to \mathbb{R}^n$ is a Borel function, we say that $x \in \Omega$ is a point of approximate continuity for u if there is $a \in \mathbb{R}^n$ such that for each $\varepsilon > 0$

$$\lim_{r\to 0}\frac{1}{r^n}\mathcal{L}^n\left(B_r(x)\cap\{|u-a|\ge\varepsilon\}\right) = 0. \tag{2.1}$$

We say that x is a jump point, and we write $x \in J_u$, if there exist two distinct vectors $a^\pm \in \mathbb{R}^n$ and a unit vector $\nu \in \mathbb{R}^n$ such that the approximate limit of the restriction of u to $\{y \in \Omega : \pm(y-x)\cdot\nu > 0\}$ is $a^\pm$.

The space $BD(\Omega)$ of functions with bounded deformation in Ω and its subspace $SBD(\Omega)$ have been widely studied due to their role in the variational formulation of many problems in plasticity and fracture mechanics. Let us recall that the jump set J_u of a function $u \in BD(\Omega)$ is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable and that for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$ the function u has one-sided approximate limits $u^{\pm}(x)$ with respect to a suitable direction $\nu_u(x)$ normal to J_u at x. We denote by S_u the set of approximate discontinuity points, in the sense of the set of points where (2.1) does not hold. Moreover one can define the approximate symmetric gradient $e(u) \in L^1(\Omega;\mathbb{R}^{n\times n})$. For further details and properties see [44, 4, 6, 28, 22].

The subspace $SBD^p(\Omega)$, $p>1$, contains all functions $u \in BD(\Omega)$ whose symmetric distributional derivative can be decomposed as

$$Eu = e(u)\mathcal{L}^n \llcorner \Omega + (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

with $e(u) \in L^p(\Omega;\mathbb{R}^{n\times n})$ and $\mathcal{H}^{n-1}(J_u) < \infty$. Fine properties and rigidity properties of SBD^p have been highlighted in [7, 11, 15, 13, 34, 35, 20].

The generalized space $GSBD(\Omega)$ introduced in [22] has proved to be the correct space where setting a number of problems in linearized elasticity, see [42, 36]. An $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}^n$ belongs to $GSBD(\Omega)$ if there exists a bounded positive Radon measure $\lambda_u \in \mathcal{M}_b^+(\Omega)$ such that the following condition holds for every $\xi \in \mathbb{S}^{n-1}$: for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ the function u_y^ξ defined by $u_y^\xi(t) := u(y+t\xi)\cdot\xi$ belongs to $SBV_{\mathrm{loc}}(\Omega_y^\xi)$, where $\Omega_y^\xi := \{t \in \mathbb{R} : y + t\xi \in \Omega\}$, and for every Borel set $B \subset \Omega$ it satisfies

$$\int_{\Omega^\xi} \Big(|Du_y^\xi|(B_y^\xi \setminus J^1_{u_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{u_y^\xi}) \Big) d\mathcal{H}^{n-1} \le \lambda_u(B), \tag{2.2}$$

where $J^1_{u_y^\xi} := \{t \in J_{u_y^\xi} : |[u_y^\xi](t)| \ge 1\}$.

If $u \in GSBD(\Omega)$, the aforementioned quantities $e(u)$ and J_u are still well-defined, and are respectively integrable and rectifiable in the previous sense. In analogy to $SBD^p(\Omega)$, the subspace $GSBD^p(\Omega)$ includes all functions in $GSBD(\Omega)$ satisfying $e(u) \in L^p(\Omega;\mathbb{R}^{n\times n})$ and $\mathcal{H}^{n-1}(J_u) < \infty$.

Next proposition states that a $GSBD^p$-function with a small jump set can be approximated by Sobolev functions. This is a minor reformulation of the result of [19] (see [13, 34, 35] for related works in SBD^p). Its proof is based on first covering the jump set with countably many balls with finite overlap and properties (i) and (ii), and then in each ball B constructing w as a piecewise affine approximation to u on a suitably chosen triangular grid, which refines towards ∂B in such a way that grid segments do not intersect J_u, following a strategy developed in [21].

Proposition 2.3. *Let $p \in (1,\infty)$, $n=2$. There exist universal constants $c, \eta, \xi > 0$ such that if $u \in SBD^p(B_\rho)$, $\rho > 0$, satisfies*

$$\mathcal{H}^1(J_u \cap B_\rho) < \eta\,(1-s)\frac{\rho}{2}$$

for some $s \in (0,1)$, then there are a countable family $\mathcal{F} = \{B\}$ of closed balls overlapping at most ξ times of radius $r_B < (1-s)\rho/2$ and center $x_B \in \overline{B}_{s\rho}$, and a field $w \in SBD^p(B_\rho)$ such that *(i)* $\eta r_B \le \mathcal{H}^1(J_u \cap B) \le 2\eta r_B$ *for all* $B \in \mathcal{F}$*;*

(ii) $\mathcal{H}^1\big(J_u \cap \cup_{\mathcal{F}} \partial B\big) = \mathcal{H}^1\big((J_u \cap B_{s\rho}) \setminus \cup_{\mathcal{F}} B\big) = 0$*;*

(iii) $w = u$ $\mathcal{L}^2$*-a.e. on* $B_\rho \setminus \cup_{\mathcal{F}} B$*;*

(iv) $w \in W^{1,p}(B_{s\rho};\mathbb{R}^2)$ *and* $\mathcal{H}^1(J_w \setminus J_u) = 0$*;*

(v) for each $B \in \mathcal{F}$ *one has* $w \in W^{1,p}(B;\mathbb{R}^2)$ *with*

$$\int_B |e(w)|^p dx \le c \int_B |e(u)|^p dx; \tag{2.3}$$

and there exists a skew-symmetric matrix A *such that*

$$\int_{B_{s\rho} \setminus \cup_{\mathcal{F}} B} |\nabla u - A|^p dx \le c \int_{B_\rho} |e(u)|^p dx; \tag{2.4}$$

(vi) $\cup_{\mathcal{F}} B \subset B_{\frac{1+s}{2}\rho}$ *and* $\sum_{\mathcal{F}} \mathcal{L}^2(B) \le \frac{c}{\eta}\,\rho\,\mathcal{H}^1(J_u \cap B_\rho)$*;*

(vii) if, additionally, $u \in L^\infty(B_\rho;\mathbb{R}^2)$ *then* $w \in L^\infty(B_\rho;\mathbb{R}^2)$ *with*

$$\|w\|_{L^\infty(B_\rho;\mathbb{R}^2)} \le \|u\|_{L^\infty(B_\rho;\mathbb{R}^2)}.$$

The next result is an approximation in energy of $GSBD^p$ functions with SBV^p functions, which was proven in [42] for $p = 2$ and for any dimension, building upon ideas developed in [11, 12] for SBD^2 functions. The extension to $p \ne 2$ is discussed in details elsewhere [16]. Let us only mention that despite we still follow the ideas in [11, 12], in the nonquadratic case a different definition of the piecewise affine approximants is needed. Indeed, it requires the use of a different interpolation scheme and a different finite-element grid for the actual construction.

Theorem 2.4 ([16, Theorem 3.1])**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz set,* $u \in GSBD^p(\Omega) \cap L^p(\Omega;\mathbb{R}^n)$*. Then there is a sequence* $v_j \in L^\infty \cap SBV^p(\Omega;\mathbb{R}^n)$ *such that*

$$\lim_{j\to\infty} \big(\|e(v_j) - e(u)\|_{L^p(\Omega;\mathbb{R}^{n\times n})} + \|v_j - u\|_{L^p(\Omega;\mathbb{R}^n)} + \big|\mathcal{H}^{n-1}(J_{v_j}) - \mathcal{H}^{n-1}(J_u)\big|\big) = 0\,.$$

3 Proof of existence of strong minimizers

We prove that weak minimizers (in $GSBD^p$) have an essentially closed jump set, and therefore can be identified with strong minimizers. The general strategy is similar to the one by De Giorgi, Carriero, and Leaci [26]; the key new ingredients are the approximation results for $GSBD^p$ functions with Sobolev functions discussed in Section 2.2 and corresponding rigidity estimates for treating the lower-order term.

3.1 Density lower bound

In this section we assume that $\kappa \geq 0$, $\beta > 0$, $p > 1$, $g \in L^\infty(\Omega; \mathbb{R}^n)$, $\mu \geq 0$ are given, and that $\Omega \subset \mathbb{R}^n$ is a bounded, open, Lipschitz set. For all $u \in GSBD(\Omega)$ and all Borel sets $A \subset \Omega$ we define the functional

$$G(u, \kappa, \beta, A) := \int_A f_\mu(e(u))dx + \kappa \int_A |u - g|^p dx + 2\beta \mathcal{H}^{n-1}(J_u \cap A). \quad (3.1)$$

Moreover, we say that $u \in GSBD^p(\Omega)$ is a local minimizer of $G(\cdot, \kappa, \beta, \Omega)$ provided

$$G(u, \kappa, \beta, \Omega) \leq G(v, \kappa, \beta, \Omega), \quad (3.2)$$

for all $v \in GSBD^p(\Omega)$ satisfying $\{v \neq u\} \subset\subset \Omega$.

In order to prove the main result of the paper, Theorem 1.2, we use an homogeneous version of G

$$G_0(u, \kappa, \beta, A) := \int_A f_0(e(u))dx + \kappa \int_A |u|^p dx + 2\beta \mathcal{H}^{n-1}(J_u \cap A) \quad (3.3)$$

to get an appropriate decay estimate (Lemma 3.6) and then density lower bounds for the full energy G_0 and the jump energy alone (cf. Lemma 3.8 and Corollary 3.9 respectively). For convenience, we introduce for open sets $A \subset \Omega$ the deviation from minimality

$$\Psi_0(u, \kappa, \beta, A) := G_0(u, \kappa, \beta, A) - \Phi_0(u, \kappa, \beta, A),$$

where

$$\Phi_0(u, \kappa, \beta, A) := \inf\{G_0(v, \kappa, \beta, A) : \ v \in GSBD(\Omega), \ \{v \neq u\} \subset\subset A\}. \quad (3.4)$$

The functions f_μ with $\mu > 0$ and f_0 are both convex and with p-growth. The p-homogeneous function f_0 captures the asymptotic behavior of f_μ at infinity,

$$f_0(\xi) = \lim_{t \to \infty} \frac{f_\mu(t\xi)}{t^p} = \frac{1}{p}(\mathbb{C}\xi \cdot \xi)^{p/2}. \quad (3.5)$$

Before proceeding with the proofs we state an auxiliary result that will be repeatedly used in what follows (see [20, Lemma 4.3] for the elementary proof).

Lemma 3.1. *Let $\omega \subseteq B_r(y)$ satisfy*

$$\mathcal{L}^n(\omega) \leq \frac{1}{4}\mathcal{L}^n(B_r(y)),$$

and let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be an affine function. Then

$$\mathcal{L}^n(B_r(y))\|\varphi\|_{L^\infty(B_r(y),\mathbb{R}^n)} \leq \bar{c}\|\varphi\|_{L^1(B_r(y)\setminus\omega,\mathbb{R}^n)},$$

where the constant $\bar{c}$ depends only on the dimension n.

We investigate first the compactness properties of sequences having vanishing jump energy. We show that a sequence u_h in SBD^p with vanishing jump energy converges, up to the addition of affine functions a_h, to a function having no jump. To see this, we regularize the given functions using Proposition 2.3 and then we use standard compactness results for bounded sequences in Sobolev spaces.

Proposition 3.2. *Let $n = 2$, $p \in (1, \infty)$, $B_\rho \subset \mathbb{R}^2$ a ball, $u_h \in SBD^p(B_\rho)$ and*

$$\sup_h \int_{B_\rho} f_0(e(u_h))dx < \infty, \qquad \mathcal{H}^1(J_{u_h}) \to 0. \tag{3.6}$$

Then there is a function $u \in W^{1,p}(B_\rho; \mathbb{R}^2)$ such that for any sequence $s_j \uparrow 1$ there is a subsequence h_j, a sequence of affine functions $a_j : \mathbb{R}^2 \to \mathbb{R}^2$ with $e(a_j) = 0$, a sequence $z_j \in SBD^p(B_\rho)$ with

(i) $\{z_j \neq u_{h_j}\} \subset\subset B_\rho$ *and* $\mathcal{L}^2(\{z_j \neq u_{h_j}\}) \to 0$;

(ii) $z_j \in W^{1,p}(B_{s_j\rho}; \mathbb{R}^2)$, *and*

$$\int_{B_\rho} |e(z_j)|^p dx \leq c \int_{B_\rho} |e(u_{h_j})|^p dx \tag{3.7}$$

for a universal constant c;

(iii) $\mathcal{H}^1(J_{z_j} \setminus J_{u_{h_j}}) = 0$;

(iv) $\|u - (z_j - a_j)\|_{L^p(B_{s_j\rho};\mathbb{R}^2)} \to 0$.

Moreover $u_{h_j} - a_j \to u$ $\mathcal{L}^2$-a.e. on B_ρ and

$$\int_{B_\rho} f_0(e(u))dx \leq \liminf_{h\to\infty} \int_{B_\rho} f_0(e(u_h))dx\,. \tag{3.8}$$

Proof. Up to the extraction of a subsequence, we may assume that the inferior limit in (3.8) is actually a limit.

For each $h \in \mathbb{N}$ and for any $s \in [1/2, 1)$ let $w_h^{(s)} \in SBD^p(B_\rho)$ and $\mathcal{F}_h^s$ be the function and the family of balls obtained by Proposition 2.3 applied to u_h. By (2.3) and Korn's inequality we can choose affine functions $a_h^{(s)}$: $\mathbb{R}^2 \to \mathbb{R}^2$ such that $e(a_h^{(s)}) = 0$ and for a universal constant c

$$\|\nabla w_h^{(s)} - \nabla a_h^{(s)}\|_{L^p(B_{s\rho};\mathbb{R}^{2\times 2})} \leq c\|e(w_h^{(s)})\|_{L^p(B_{s\rho};\mathbb{R}^{2\times 2})} \leq c\|e(u_h)\|_{L^p(B_\rho;\mathbb{R}^{2\times 2})},$$

and

$$\|w_h^{(s)} - a_h^{(s)}\|_{L^p(B_{s\rho};\mathbb{R}^2)} \leq c\rho\|e(u_h)\|_{L^p(B_\rho;\mathbb{R}^{2\times 2})}.$$

Now notice that for h large $\mathcal{L}^2(B_{\rho/2} \cap \{w_h^{(s)} = w_h^{(1/2)} = u_h\}) \geq \frac{1}{4}\mathcal{L}^2(B_\rho)$ in view of item (vi) in Proposition 2.3 and since $\mathcal{H}^1(J_{u_h}) \to 0$ as $h \uparrow \infty$ (cf.

(3.6)). Thus, for h large, Lemma 3.1 and the triangular inequality give for a universal constant c

$$\begin{aligned}
&\|\nabla a_h^{(s)} - \nabla a_h^{(1/2)}\|_{L^p(B_{s\rho};\mathbb{R}^{2\times 2})} \\
&\leq c\|\nabla a_h^{(s)} - \nabla a_h^{(1/2)}\|_{L^p(B_{\rho/2}\cap\{w_h^{(s)}=w_h^{(1/2)}=u_h\};\mathbb{R}^{2\times 2})} \leq c\|e(u_h)\|_{L^p(B_\rho;\mathbb{R}^{2\times 2})},
\end{aligned} \tag{3.9}$$

and

$$\|a_h^{(1/2)} - a_h^{(s)}\|_{L^p(B_{s\rho};\mathbb{R}^2)} \leq c\rho\|e(u_h)\|_{L^p(B_\rho;\mathbb{R}^{2\times 2})}.$$

Therefore, the sequence $w_h^{(s)} - a_h^{(1/2)}$ is bounded in $W^{1,p}(B_{s\rho};\mathbb{R}^2)$ and a subsequence (depending on s and not relabeled) converges to some $w^{(s)}$ weakly in $W^{1,p}(B_{s\rho};\mathbb{R}^2)$, strongly in $L^q(B_{s\rho};\mathbb{R}^2)$ for all $q \in [1, p^*)$ and pointwise $\mathcal{L}^2$-a.e. on $B_{s\rho}$.

Note that $\mathcal{L}^2(\cup_{\mathcal{F}_h^s} B) \leq \frac{c}{\eta}\rho\mathcal{H}^1(J_{u_h})$ for all $s \in [1/2, 1)$ by item (vi) in Proposition 2.3, c a universal constant. Therefore, by (3.6) we conclude that $w^{(s)} = w^{(t)}$ $\mathcal{L}^2$-a.e. on $B_{s\rho}$ if $1/2 \leq s \leq t < 1$. Thus, we may define a limit function u on B_ρ such that $u = w^{(s)}$ $\mathcal{L}^2$-a.e. on $B_{s\rho}$ for all $s \in [1/2, 1)$. In particular, $u \in W^{1,p}_{\mathrm{loc}}(B_\rho;\mathbb{R}^2)$.

Moreover, recalling that we have assumed the inferior limit in (3.8) to be a limit, we obtain for all $s \in [1/2, 1)$

$$\begin{aligned}
\liminf_{h\to\infty} \int_{B_\rho} f_0(e(u_h))dx &\geq \liminf_{h\to\infty} \int_{B_{s\rho}\setminus\cup_{\mathcal{F}_h^s} B} f_0(e(w_h^s))dx \\
&= \liminf_{h\to\infty} \int_{B_{s\rho}} f_0\big(e(w_h^s)\chi_{B_\rho\setminus\cup_{\mathcal{F}_h^s} B}\big)dx \geq \int_{B_{s\rho}} f_0(e(u))dx.
\end{aligned} \tag{3.10}$$

Indeed, for the first inequality we used the positivity of f_0 and that $e(w_h^s) = e(u_h)\chi_{B_\rho\setminus\cup_{\mathcal{F}_h^s} B}$ $\mathcal{L}^2$-a.e. on B_ρ (see [5, Prop. 3.73]), and for the subsequent equality we used that $f_0(\underline{0}) = 0$; so that we may conclude by Reshetnyak lower semicontinuity result in view of the convexity of f_0 and the weak convergence of $e(w_h^s)\chi_{B_\rho\setminus\cup_{\mathcal{F}_h^s} B}$ to $e(u)$ in $L^p(B_{s\rho};\mathbb{R}^2)$, consequence of the weak convergence of $w_h^{(s)} - a_h^{(1/2)}$ to u in $W^{1,p}(B_{s\rho};\mathbb{R}^2)$ and of $\mathcal{L}^2(\cup_{\mathcal{F}_h^s} B) \leq \frac{c}{\eta}\rho\mathcal{H}^1(J_{u_h}) \to 0$.

Thus, the lower semicontinuity estimate in (3.8) follows at once by letting $s \uparrow 1$.

Eventually, given any sequence $s_j \uparrow 1$, for every $j \in \mathbb{N}$ let $h_j \geq j$ be such that

$$\|w_{h_j}^{(s_j)} - a_{h_j}^{1/2} - u\|_{L^p(B_{s_j\rho};\mathbb{R}^2)} \leq 1/j,$$

set $z_j := w_{h_j}^{(s_j)}$ and $a_j := a_{h_j}^{(1/2)}$, then properties (i)-(iv) follow by construction.

Finally, the sequence $u_{h_j} - a_j$ converges in measure to u by item (iii) in Proposition 2.3 and since $\mathcal{L}^2(\cup_{\mathcal{F}_{h_j}^{s_j}} B)$ is infinitesimal as already noticed. $\square$

Remark 3.3. *The result above extends to sequences* $u_h \in GSBD^p(B_\rho) \cap L^p(B_\rho;\mathbb{R}^2)$ *by using the approximation argument that will be employed in Proposition 3.4 below.*

We investigate next the asymptotics of sequences with vanishing jump energy. More precisely, we show that asymptotically locally minimizing sequences u_h for G_0 with vanishing jump energy in fact converge, up to the addition of affine functions a_h, to a local minimizer u of G_0 on Sobolev spaces. The convergence to a Sobolev function is guaranteed by Proposition 3.2. The local minimality of the limit follows from a standard variational argument: a competitor v for u is modified close to the boundary to obtain a competitor for u_h. However, a technical difficulty arises since the volume term in the transition zone is estimated by a term of the form $|u_h - a_h - u|^p$ that is infinitesimal $\mathcal{L}^n$ a.e. but not in L^1. Hence we need an intermediate interpolation step.

Proposition 3.4. *Let* $n = 2$, $p \in (1,\infty)$. *Let* B_r *be a ball,* $u_h \in GSBD^p(B_r)$ *and* $\kappa_h \in [0,\infty)$, $\beta_h \in (0,\infty)$ *be two sequences with* $\kappa_h \to 0$ *as* $h \to \infty$, *and such that*

$$\sup_h G_0(u_h,\kappa_h,\beta_h,B_r) < \infty, \text{ and } \lim_{h\to\infty} \Psi_0(u_h,\kappa_h,\beta_h,B_r) = \lim_{h\to\infty} \mathcal{H}^1(J_{u_h}) = 0\,.$$

Then there exists $u \in W^{1,p}(B_r;\mathbb{R}^2)$, $\overline{a} : \mathbb{R}^2 \to \mathbb{R}^2$ *affine with* $e(\overline{a}) = 0$ *and a subsequence* h_j *such that*

(i) for all $\rho \in (0,r)$

$$\lim_{j\to\infty} G_0(u_{h_j},\kappa_{h_j},\beta_{h_j},B_\rho) = \int_{B_\rho} f_0(e(u))dx + \int_{B_\rho} |\overline{a}|^p dx;$$

(ii) for all $v \in u + W^{1,p}_0(B_r;\mathbb{R}^2)$

$$\int_{B_r} f_0(e(u))dx \le \int_{B_r} f_0(e(v))dx;$$

(iii) $u_{h_j} - a_j \to u$ *pointwise* $\mathcal{L}^2$*-a.e. on* B_r *for some affine functions* a_j, $e(u_{h_j}) \to e(u)$ *in* $L^p(B_\rho;\mathbb{R}^{2\times 2})$, $\beta_{h_j}\mathcal{H}^1(J_{u_{h_j}} \cap B_\rho) \to 0$, *and* $\kappa_{h_j}^{1/p} u_{h_j} \to \overline{a}$ *in* $L^p(B_\rho;\mathbb{R}^2)$ *for all* $\rho \in (0,r)$.

Proof. Theorem 2.4 provides $v_h \in SBV^p \cap L^\infty(B_r;\mathbb{R}^2)$, for every $h \in \mathbb{N}$, such that

$$\|e(u_h) - e(v_h)\|_{L^p(B_r;\mathbb{R}^{2\times 2})} + |\mathcal{H}^1(J_{u_h}) - \mathcal{H}^1(J_{v_h})| + \|u_h - v_h\|_{L^p(B_r;\mathbb{R}^2)} \le (h + \beta_h^2)^{-1}. \quad (3.11)$$

In particular, for all $\rho \in (0, r]$

$$\limsup_{h\to\infty} G_0(u_h, \kappa_h, \beta_h, B_\rho) = \limsup_{h\to\infty} G_0(v_h, \kappa_h, \beta_h, B_\rho), \tag{3.12}$$

and

$$\lim_{h\to\infty} \mathcal{H}^1(J_{v_h}) = 0\,.$$

Hence, $(v_h)_{h\in\mathbb{N}}$ satisfies (3.6) in Proposition 3.2. Let a_{h_j} and u be the functions obtained by Proposition 3.2, then $v_{h_j} - a_{h_j} \to u$ pointwise $\mathcal{L}^2$-a.e. on B_r. Recall that Proposition 3.2 and (3.11) imply that

$$\int_{B_\rho} f_0(e(u))dx \le \liminf_{h\to\infty} \int_{B_\rho} f_0(e(u_h))dx. \tag{3.13}$$

Additionally, up to extracting a further subsequence we may assume that $u_{h_j} - a_{h_j} \to u$ pointwise $\mathcal{L}^2$-a.e. on B_r by (3.11). Here and henceforth we denote h_j by h for simplicity.

Since $s \mapsto G_0(u_h, \kappa_h, \beta_h, B_s)$ is nondecreasing and uniformly bounded, by Helly's theorem we can extract a subsequence, not relabeled for convenience, such that the pointwise limit

$$\lim_{h\to\infty} G_0(u_h, \kappa_h, \beta_h, B_s) =: \Lambda(s) \tag{3.14}$$

exists finite for all $s \in I := (0, r)$, and Λ is a nondecreasing function.

Being $(\kappa_h^{1/p} u_h)_h$ bounded in $L^p(B_r;\mathbb{R}^2)$, it has a subsequence (not relabeled) converging to some $\overline{a} \in L^p(B_r;\mathbb{R}^2)$ weakly in $L^p(B_r;\mathbb{R}^2)$. At the same time $\kappa_h^{1/p}(u_h - a_h) \to 0$ pointwise $\mathcal{L}^2$-a.e. in B_r, as $\kappa_h \downarrow 0$ as $h \to \infty$, therefore $(\kappa_h^{1/p} a_h)_h$ is bounded in $L^p(B_r;\mathbb{R}^2)$ by Lemma 3.1. Hence, by the Urysohn property, by the weak L^p-convergence of $(\kappa_h^{1/p} u_h)_h$ and by the equi-integrability of $(\kappa_h^{1/p}(u_h - a_h))_h$ we obtain that in turn $(\kappa_h^{1/p} a_h)_h$ converges weakly to $\bar{a}$ in $L^p(B_r;\mathbb{R}^2)$. Since $\kappa_h^{1/p} a_h$ are affine functions, and the space of affine functions is finite dimensional, convergence is actually strong, and $\overline{a}$ is affine on B_r, with $e(\bar{a}) = 0$.

Fixed $\rho \in I$ a continuity point of Λ satisfying (3.14) we apply Proposition 3.2 again to B_ρ and obtain a subsequence of h not relabeled, a sequence $(z_h^{(\rho)})_h \in SBD^p(B_\rho)$, and a sequence $a_h^{(\rho)} : \mathbb{R}^2 \to \mathbb{R}^2$ of affine functions with $e(a_h^{(\rho)}) = 0$, such that $v_h - a_h^{(\rho)} \to u^{(\rho)}$ $\mathcal{L}^2$-a.e. on B_ρ, $z_h^{(\rho)} - a_h^{(\rho)} \to u^{(\rho)}$ in $L^p_{\mathrm{loc}}(B_\rho;\mathbb{R}^2)$ and $\{z_h^{(\rho)} \ne v_h\} \subset\subset B_\rho$, for some $u^{(\rho)} \in W^{1,p}(B_\rho;\mathbb{R}^2)$. Thus, we may consider $z_h^{(\rho)}$ as a function in $SBD^p(B_r)$ by extending it equal to v_h on $B_r \setminus B_\rho$.

Next note that $z_h^{(\rho)} - a_h \to u$ in $L^p_{\mathrm{loc}}(B_\rho;\mathbb{R}^2)$, where a_h and u are the globally chosen functions introduced above. This claim easily follows from the convergences $v_h - a_h \to u$ $\mathcal{L}^2$-a.e. on B_r and $v_h - a_h^{(\rho)} \to u^{(\rho)}$ $\mathcal{L}^2$-a.e. on B_ρ. Indeed, from these we deduce that $a_h^{(\rho)} - a_h \to u - u^{(\rho)}$ in $L^p(B_\rho;\mathbb{R}^2)$. Hence, the claim follows at once by taking into account this and the convergence $z_h^{(\rho)} - a_h^{(\rho)} \to u^{(\rho)}$ in $L^p_{\mathrm{loc}}(B_\rho;\mathbb{R}^2)$.

Let $v \in W^{1,p}(B_r;\mathbb{R}^2)$ be such that $\{u \neq v\} \subset\subset B_\rho$ and let $0 < \rho''' < \rho'' < \rho' < \rho < \overline{\rho} < r$, with $\rho''', \overline{\rho} \in I$ and assume in addition that $\{u \neq v\} \subseteq B_{\rho''}$.

Let $\zeta \in C_c^\infty(B_{\rho'};[0,1])$, $\varphi \in C_c^\infty(B_{\overline{\rho}};[0,1])$ be cut-off functions such that $\zeta = 1$ on $B_{\rho''}$, $\varphi = 1$ on B_ρ, and $\|\nabla\zeta\|_{L^\infty(B_{\rho'};\mathbb{R}^2)} \leq 2(\rho' - \rho'')^{-1}$, $\|\nabla\varphi\|_{L^\infty(B_{\overline{\rho}};\mathbb{R}^2)} \leq 2(\overline{\rho} - \rho)^{-1}$. Define

$$\overline{u}_h := \zeta(v + a_h) + (1-\zeta)\big(\varphi\, z_h^{(\rho)} + (1-\varphi)u_h\big)$$

and note that

$$\overline{u}_h = \begin{cases} \zeta(v + a_h) + (1-\zeta)z_h^{(\rho)} & \text{on } B_{\rho'} \\ \varphi\, z_h^{(\rho)} + (1-\varphi)u_h & \text{on } B_r \setminus B_{\rho'}. \end{cases}$$

Since $\{\overline{u}_h \neq u_h\} \subset\subset B_{\overline{\rho}}$, by the very definition of Ψ_0 we have

$$G_0(u_h, \kappa_h, \beta_h, B_{\overline{\rho}}) \leq G_0(\overline{u}_h, \kappa_h, \beta_h, B_{\overline{\rho}}) + \Psi_0(u_h, \kappa_h, \beta_h, B_r). \tag{3.15}$$

We estimate separately the contributions on $B_{\rho'}$ and $B_{\overline{\rho}} \setminus B_{\rho'}$ for the first summand on the right hand side above as follows. First, for some $c = c(p) > 0$ we have

$$\begin{aligned} G_0\left(\overline{u}_h, \kappa_h, \beta_h, B_{\rho'}\right) &\leq G_0(v + a_h, \kappa_h, \beta_h, B_{\rho''}) + c\, G_0(v + a_h, \kappa_h, \beta_h, B_{\rho'} \setminus B_{\rho''}) \\ &\quad + c\, G_0(z_h^{(\rho)}, \kappa_h, \beta_h, B_{\rho'} \setminus B_{\rho''}) + \frac{c}{(\rho' - \rho'')^p} \int_{B_{\rho'}\setminus B_{\rho''}} |v + a_h - z_h^{(\rho)}|^p dx \\ &= \int_{B_{\rho''}} f_0(e(v))dx + \kappa_h \int_{B_{\rho''}} |v + a_h|^p dx \\ &\quad + c \int_{B_{\rho'}\setminus B_{\rho''}} f_0(e(v))dx + c\,\kappa_h \int_{B_{\rho'}\setminus B_{\rho''}} |v + a_h|^p dx \\ &\quad + c\, G_0(z_h^{(\rho)}, \kappa_h, \beta_h, B_{\rho'} \setminus B_{\rho''}) + \frac{c}{(\rho' - \rho'')^p} \int_{B_{\rho'}\setminus B_{\rho''}} |v + a_h - z_h^{(\rho)}|^p dx. \end{aligned}$$

Moreover, since $\{z_h^{(\rho)} \neq v_h\} \subset\subset B_\rho$, and $\overline{u}_h = z_h^{(\rho)}$ on $B_\rho \setminus B_{\rho'}$ we have

$$\begin{aligned} G_0(\overline{u}_h, \kappa_h, \beta_h, B_{\overline{\rho}} \setminus B_{\rho'}) \leq{}& c\, G_0(z_h^{(\rho)}, \kappa_h, \beta_h, B_{\overline{\rho}} \setminus B_{\rho'}) + c\, G_0(u_h, \kappa_h, \beta_h, B_{\overline{\rho}} \setminus B_\rho) \\ &+ \frac{c}{(\overline{\rho} - \rho)^p} \int_{B_{\overline{\rho}}\setminus B_\rho} |v_h - u_h|^p dx. \end{aligned}$$

Therefore, since $u = v$ on $B_\rho \setminus B_{\rho''}$ we deduce that

$$
\begin{aligned}
&G_0\left(\overline{u}_h, \kappa_h, \beta_h, B_{\overline{\rho}}\right) \leq \int_{B_{\rho''}} f_0(e(v))dx + \kappa_h \int_{B_{\rho''}} |v + a_h|^p dx \\
&+c \int_{B_{\rho'} \setminus B_{\rho''}} f_0(e(v))dx + c\,\kappa_h \int_{B_{\rho'} \setminus B_{\rho''}} |u + a_h|^p dx \\
&+c\,G_0(z_h^{(\rho)}, \kappa_h, \beta_h, B_{\overline{\rho}} \setminus B_{\rho''}) + c\,G_0(u_h, \kappa_h, \beta_h, B_{\overline{\rho}} \setminus B_\rho) \\
&+\frac{c}{(\rho' - \rho'')^p} \int_{B_{\rho'} \setminus B_{\rho''}} |u + a_h - z_h^{(\rho)}|^p dx + \frac{c}{(\overline{\rho} - \rho)^p} \int_{B_{\overline{\rho}} \setminus B_\rho} |v_h - u_h|^p dx.
\end{aligned}
\tag{3.16}
$$

Note that for h sufficiently large by Proposition 3.2 we have

$$
G_0(z_h^{(\rho)}, \kappa_h, \beta_h, B_{\overline{\rho}} \setminus B_{\rho''}) \leq c\,G_0(v_h, \kappa_h, \beta_h, B_{\overline{\rho}} \setminus B_{\rho''}) + \kappa_h \int_{B_\rho \setminus B_{\rho''}} |z_h^{(\rho)}|^p dx.
\tag{3.17}
$$

To estimate the last addend, denote by $\mathcal{F}_h = \{B^h\}$ the family of balls with finite overlap in the construction of $z_h^{(\rho)}$ in Proposition 3.2, and by $\mathcal{F}'_h$ the subfamily of those contained in $B_\rho \setminus B_{\rho'''}$, thus for large h

$$
\begin{aligned}
\int_{B_\rho \setminus B_{\rho''}} |z_h^{(\rho)}|^p dx &\leq \int_{B_\rho \setminus B_{\rho''}} |v_h|^p dx + \int_{\cup_{\mathcal{F}'_h} B^h} |z_h^{(\rho)}|^p dx \\
&\leq \int_{B_\rho \setminus B_{\rho''}} |v_h|^p dx + c \sum_{B^h \in \mathcal{F}'_h} \int_{B^h} \big(|z_h^{(\rho)} - a_{B^h}|^p + |a_{B^h}|^p\big) dx \\
&\leq \int_{B_\rho \setminus B_{\rho''}} |v_h|^p dx + c \sum_{B^h \in \mathcal{F}'_h} \int_{B^h} \big(r_{B^h}^p |e(z_h^{(\rho)})|^p + |a_{B^h}|^p\big) dx \\
&\leq \int_{B_\rho \setminus B_{\rho''}} |v_h|^p dx + c\rho^p \int_{B_\rho \setminus B_{\rho'''}} |e(v_h)|^p dx + \sum_{B^h \in \mathcal{F}'_h} \int_{B^h} |a_{B^h}|^p dx,
\end{aligned}
\tag{3.18}
$$

where a_{B^h} is an affine function with $e(a_{B^h}) = 0$ in the Poincaré-Korn inequality on B^h. In the last inequality we used property (ii) in Proposition 3.2 and the fact that the balls B^h's have finite overlap. Since the center of B^h

belongs to $B_{s_h\rho}$, by Lemma 3.1 we infer that

$$\begin{aligned}\sum_{B^h\in\mathcal{F}'_h}\int_{B^h}|a_{B^h}|^p dx &\le \bar c\sum_{B^h\in\mathcal{F}'_h}\int_{B^h\cap B_{s_h\rho}}|a_{B^h}|^p dx\\ &\le c\sum_{B^h\in\mathcal{F}'_h}\int_{B^h\cap B_{s_h\rho}}\big(|a_{B^h}-z_h^{(\rho)}|^p+|z_h^{(\rho)}|^p\big)dx\\ &\le c\rho^p\int_{B_\rho\setminus B_{\rho'''}}|e(v_h)|^p dx + c\int_{B_{s_h\rho}}|z_h^{(\rho)}-a_h|^p dx + c\int_{B_\rho\setminus B_{\rho'''}}|a_h|^p dx,\end{aligned}\tag{3.19}$$

where $c=c(n,p)$. Therefore, property (iv) of Proposition 3.2, (3.12), (3.14), (3.17)-(3.19), and the choices of the radii $\rho''',\overline{\rho}\in I$ yield

$$\begin{aligned}\limsup_{h\to\infty}\big(G_0(z_h^{(\rho)},\kappa_h,\beta_h,B_{\overline{\rho}}\setminus B_{\rho''})+G_0(u_h,\kappa_h,\beta_h,B_{\overline{\rho}}\setminus B_\rho)\big)\\ \le c(\Lambda(\overline{\rho})-\Lambda(\rho'''))+c\int_{B_{\overline{\rho}}\setminus B_{\rho'''}}|\overline{a}|^p dx.\end{aligned}$$

Moreover, recalling the convergences $u_h-v_h\to 0$ in $L^p(B_r;\mathbb{R}^2)$, $z_h^{(\rho)}-a_h\to u$ $L^p(B_{\rho'};\mathbb{R}^2)$, $\kappa_h^{1/p}a_h\to\overline{a}$ in $L^p(B_r;\mathbb{R}^2)$ and $\kappa_h\to 0$ as $h\to\infty$, we infer

$$\begin{aligned}\lim_{h\to\infty}\Big(\kappa_h\int_{B_{\rho'}\setminus B_{\rho''}}|u+a_h|^p dx+\frac{1}{(\rho'-\rho'')^p}\int_{B_{\rho'}\setminus B_{\rho''}}|u+a_h-z_h^{(\rho)}|^p dx\\ +\frac{1}{(\overline{\rho}-\rho)^p}\int_{B_{\overline{\rho}}\setminus B_\rho}|v_h-u_h|^p dx\Big)=\int_{B_{\rho'}\setminus B_{\rho''}}|\overline{a}|^p dx,\end{aligned}$$

Hence, by taking the superior limit as $h\to\infty$ in (3.15), in view of (3.16) and the last two inequalities we get

$$\begin{aligned}\Lambda(\overline{\rho})\le\int_{B_{\rho''}}f_0(e(v))dx+\int_{B_{\rho''}}|\overline{a}|^p dx+c\int_{B_{\rho'}\setminus B_{\rho''}}f_0(e(v))dx\\ +c(\Lambda(\overline{\rho})-\Lambda(\rho'''))+c\int_{B_{\overline{\rho}}\setminus B_{\rho'''}}|\overline{a}|^p dx.\end{aligned}$$

On the other hand, the weak convergence of $(\kappa_h^{1/p}u_h)_h$ to $\overline{a}$ and (3.13) yield

$$\int_{B_\rho}f_0(e(u))dx+\int_{B_\rho}|\overline{a}|^p dx\le\liminf_{h\to\infty}\int_{B_\rho}\big(f_0(e(u_h))+\kappa_h|u_h|^p\big)\,dx\le\Lambda(\rho).\tag{3.20}$$

Therefore, from the last two inequalities we conclude as $\rho''',\overline{\rho}\to\rho$

$$\begin{aligned}\int_{B_\rho}f_0(e(u))dx+\int_{B_\rho}|\overline{a}|^p dx\\ \le\Lambda(\rho)\le\int_{B_\rho}f_0(e(v))dx+\int_{B_\rho}|\overline{a}|^p dx,\end{aligned}\tag{3.21}$$

and thus in particular

$$\int_{B_\rho} f_0(e(u))dx \le \int_{B_\rho} f_0(e(v))dx \tag{3.22}$$

for all $v \in W^{1,p}(B_r;\mathbb{R}^2)$ such that $\{u \ne v\} \subset\subset B_\rho$ and for $\mathcal{L}^1$ a.e. $\rho \in (0,r)$. Clearly, a simple approximation argument yields that the inequality (3.22) holds for all $v \in u + W_0^{1,p}(B_r;\mathbb{R}^2)$, i.e. item (ii) is established.

Finally, setting $v = u$ in (3.21), we deduce that for $\mathcal{L}^1$ a.e. $\rho \in (0,r)$

$$\int_{B_\rho} f_0(e(u)) + |\overline{a}|^p dx = \Lambda(\rho).$$

Being the left-hand side there continuous as a function of ρ, Λ turns out to be continuous as well, and recalling its very definition and the monotonicity of the integral we conclude that convergence in (3.14) holds for all $\rho \in (0,r)$, i.e. item (i) is established as well. Furthermore, from this and (3.21) above one deduces that equality holds in (3.20), and therefore that the convergence of $e(u_h)$ and $\kappa_h^{1/p} u_h$ is strong, which concludes the proof of (iii). □

We are now ready to prove a fundamental decay property of G_0 by following the ideas in [10, Lemma 3.9]. Nevertheless, we note explicitly that contrary to [10, Lemma 3.9] the lack of truncation arguments forces to take also into account the fidelity term in the decay process, since a priori we have no L^∞ bound on local minimizers. As part of the argument extends directly to higher dimension, we give a proof of the density lower bound that depends only on the decay property. However, the decay property has been proven using the regularity of Sobolev minimizers as well as Propositions 3.2 and 3.4, which have only been established in dimension $n = 2$.

Definition 3.5. *Let $n \ge 2$, $p \in (1,\infty)$, $\kappa \ge 0$, $\beta > 0$. We say that the decay property holds for the functional G_0 in dimension n if the following is true. There exists $\gamma_0 \in [0,1)$ such that for any $\gamma \in (\gamma_0, 1)$ there is $\tau_\gamma > 0$ such that for all $\tau \in (0, \tau_\gamma]$ there exist $\varepsilon \in (0,1)$, $\vartheta \in (0,1)$, and $R > 0$, such that if $u \in GSBD^p(\Omega)$ satisfies*

$$\mathcal{H}^{n-1}(J_u \cap B_\rho(x)) \le \varepsilon\rho^{n-1} \text{ and } G_0(u,\kappa,\beta,B_\rho(x)) \le (1+\vartheta)\Phi_0(u,\kappa,\beta,B_\rho(x))$$

for some $B_\rho(x) \subset\subset \Omega$ with $0 < \rho < R$, then

$$G_0(u,\kappa,\beta,B_{\tau\rho}(x)) \le \tau^{n-\gamma} G_0(u,\kappa,\beta,B_\rho(x)).$$

The decay lemma shows that the energy G_0 of a function with a small jump set and a small deviation from minimality decays as the energy of a local minimizer of G_0 on Sobolev spaces. This property follows straightforwardly from a compactness argument and the results in Propositions 2.1 and 3.4.

Lemma 3.6 (Decay)**.** *The decay property holds in dimension $n = 2$ for any $p \in (1, \infty)$, $\kappa \geq 0$, $\beta > 0$.*

In what follows c_γ denotes the constant in Proposition 2.1 having chosen $\gamma \in (\gamma_0, 1)$, and $\bar{c}$ that of Lemma 3.1.

Proof of Lemma 3.6. Let $\tau_\gamma > 0$ be such that $\max\{c_{\gamma/2}\tau_\gamma^{\gamma/2}, \bar{c}^p\,\tau_\gamma^\gamma, \tau_\gamma\} = 1/2$.

By contradiction suppose the statement false. Then there would be $\tau \in (0, \tau_\gamma]$ and three sequences $\varepsilon_h \to 0$, $\vartheta_h \to 0$, $\rho_h \to 0$, a sequence $u_h \in GSBD^p(\Omega)$, and a sequence of balls $B_{\rho_h}(x_h) \subset\subset \Omega$ such that

$$\mathcal{H}^1(J_{u_h} \cap B_{\rho_h}(x_h)) = \varepsilon_h\, \rho_h,$$
$$G_0(u_h, \kappa, \beta, B_{\rho_h}(x_h)) = (1 + \vartheta_h)\Phi_0(u_h, \kappa, \beta, B_{\rho_h}(x_h)),$$

with

$$G_0(u_h, \kappa, \beta, B_{\tau\rho_h}(x_h)) > \tau^{2-\gamma} G_0(u_h, \kappa, \beta, B_{\rho_h}(x_h)).$$

We define

$$\sigma_h := \frac{\rho_h}{G_0(u_h, \kappa, \beta, B_{\rho_h}(x_h))} \quad \text{and} \quad v_h(y) := \frac{(\sigma_h \rho_h)^{1/p}}{\rho_h} u_h(x_h + \rho_h y)$$

so that $v_h \in GSBD^p(B_1)$ satisfies $\mathcal{H}^1(J_{v_h}) = \varepsilon_h$, $G_0(v_h, \kappa\rho_h^p, \beta\sigma_h, B_1) = 1$, $\Psi_0(v_h, \kappa\rho_h^p, \beta\sigma_h, B_1) = \vartheta_h/(1 + \vartheta_h)$, and

$$G_0(v_h, \kappa\rho_h^p, \beta\sigma_h, B_\tau) > \tau^{2-\gamma}. \tag{3.23}$$

By Proposition 3.4 there exist a subsequence h not relabeled, a function $v \in W^{1,p}(B_1; \mathbb{R}^2)$, and affine functions a_h such that $v_h - a_h \to v$ $\mathcal{L}^2$-a.e. on B_1, and for some affine function $\overline{a}$ with $e(\overline{a}) = 0$

$$\int_{B_\rho} f_0(e(v))dx + \int_{B_\rho} |\overline{a}|^p dx = \lim_{h\to\infty} G_0(v_h, \kappa\rho_h^p, \beta\sigma_h, B_\rho) \leq 1 \tag{3.24}$$

for all $\rho \in (0, 1)$, with v a minimizer of $w \mapsto \int_{B_1} f_0(e(w))dx$ among all $w \in v + W_0^{1,p}(B_1; \mathbb{R}^2)$.

Hence, by Proposition 2.1, applied with the exponent $\gamma/2$, by Lemma 3.1 and (3.24)

$$\begin{aligned}\lim_{h\to\infty} G_0(v_h, \kappa\rho_h^p, \beta\sigma_h, B_\tau) &= \int_{B_\tau} f_0(e(v))dx + \int_{B_\tau} |\overline{a}|^p dx \\ &\leq c_{\gamma/2}\tau^{2-\gamma/2} + \|\overline{a}\|^p_{L^\infty(B_\tau;\mathbb{R}^2)} \mathcal{L}^2(B_1)\tau^2 \leq \big(c_{\gamma/2}\tau^{\gamma/2} + \bar{c}^p\tau^\gamma\big)\tau^{2-\gamma} < \tau^{2-\gamma},\end{aligned}$$

where the last inequality follows by the definition of τ_γ. This contradicts (3.23). □

Remark 3.7. *A suitable version of Proposition 2.1 holds in any dimension (cf. [17, Proposition 4.3]), though such a statement is not enough to deduce the decay property for $n \geq 4$ (cf. Definition 3.5). In the physical dimension $n = 3$ the decay property follows as in Lemma 3.6 using [14, Theorem 3] rather than Proposition 3.4.*

We finally establish the density lower bound for the homogeneous energy G_0 and for the jump term of a local minimizer u for G. The proof of the next result follows the lines of [32, Lemma 4.3].

We first show that, on a ball $B_\rho(x)$ centred at a point of the jump set with $\mathcal{H}^{n-1}$ density one, the ratio of the energy G_0 and of ρ^{n-1} can be estimated from below by a constant depending only on the data. Using the decay lemma one first shows that if the energy G of u is small in a ball $B_\rho(x)$, then it is also small in dyadic concentric balls. Then, assuming that the lower bound estimate for G_0 is false on $B_\rho(x)$ implies that G itself is small on the same ball. Hence, the previous argument applies, concluding that the $\mathcal{H}^{n-1}$ density of J_u at x is strictly less than one, giving a contradiction.

As this argument does not depend on dimension except for the decay property we formulate it for general n. We denote by J_u^* the set of points $x \in J_u$ with density one, namely

$$J_u^* := \left\{ x \in J_u : \lim_{\rho \to 0} \frac{\mathcal{H}^{n-1}(J_u \cap B_\rho(x))}{\omega_{n-1}\rho^{n-1}} = 1 \right\}, \tag{3.25}$$

where ω_{n-1} is the $(n-1)$-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.

Lemma 3.8 (Density lower bound for G_0)**.** *Let $n \geq 2$, $p > 1$, $\kappa \geq 0$, $\beta > 0$, $\mu \geq 0$, $g \in L^\infty(\Omega;\mathbb{R}^n)$. Assume the decay property holds for G_0 in dimension n. If $u \in GSBD^p(\Omega)$ is a local minimizer of $G(\cdot,\kappa,\beta,\Omega)$ defined in (3.1), then there exist ϑ_0 and R_0, depending only on n, p, $\mathbb{C}$, κ, β, μ, and $\|g\|_{L^\infty(\Omega;\mathbb{R}^n)}$, such that if $0 < \rho < R_0$, $x \in \Omega \cap \overline{J_u^*}$, and $B_\rho(x) \subset\subset \Omega$, then*

$$G_0(u,\kappa,\beta,B_\rho(x)) \geq \vartheta_0 \rho^{n-1}. \tag{3.26}$$

Proof. Let us first assume that $x \in J_u^*$.
Step 1. We choose $\gamma \in (\gamma_0, 1)$ in the decay property (Definition 3.5) and choose $\tau \in (0, 2^{8/(\gamma-1)} \wedge \tau_\gamma)$, with τ_γ as in Definition 3.5. Let $\varepsilon := \omega_{n-1} \wedge \varepsilon(\tau)$, where $\varepsilon(\tau) \in (0,1)$, $\vartheta = \vartheta(\tau) \in (0,1)$, and $R = R(\tau) > 0$, are as in the decay property.

We claim that there exists a radius $R_1 = R_1(n,\tau,\mu,p,\|g\|_{L^\infty(\Omega;\mathbb{R}^n)}) > 0$ such that if

$$G(u,\kappa,\beta,B_\rho(x)) < \beta\,\varepsilon\rho^{n-1} \tag{3.27}$$

for some $0 < \rho < R_1$, then one of the following inequalities holds

$$G(u,\kappa,\beta,B_{\tau\rho}(x)) < \tau^{n-1}\rho^{n-(1+\gamma)/2}, \tag{3.28}$$

$$G(u,\kappa,\beta,B_{\tau\rho}(x)) < \tau^{n-(1+\gamma)/2} G(u,\kappa,\beta,B_\rho(x)). \tag{3.29}$$

We distinguish two cases. If

$$G(u,\kappa,\beta,B_{\tau\rho}(x)) < \rho^{n-\gamma}, \tag{3.30}$$

then (3.28) holds provided we choose $R_1 \le \tau^{2(n-1)/(1-\gamma)}$.

To deal with the remaining case we state two elementary inequalities: for any $\sigma > 0$ there is $k_\sigma > 1$ (implicitly depending also on p) such that

$$|z+\zeta|^p \le (1+\sigma)|z|^p + k_\sigma|\zeta|^p \qquad \text{for all } z,\, \zeta \in \mathbb{R}^n \tag{3.31}$$

and

$$f_0(\xi) - \mu^{p/2} \le f_\mu(\xi) \le (1+\sigma)f_0(\xi) + k_\sigma\mu^{p/2} \qquad \text{for all } \xi \in \mathbb{R}^{n\times n}. \tag{3.32}$$

Using (3.32) and (3.31) with $\sigma = 1$, and the fact that $g \in L^\infty(\Omega;\mathbb{R}^n)$, we get

$$G(u,\kappa,\beta,B_{\tau\rho}(x)) \le 2G_0(u,\kappa,\beta,B_{\tau\rho}(x)) + \omega_n k_1(\mu^{p/2} + \|g\|^p_{L^\infty(\Omega;\mathbb{R}^2)})\rho^n\,.$$

Since (3.30) does not hold, choosing $R_1 \le R$ such that

$$8\omega_n k_1(\mu^{p/2} + \|g\|^p_{L^\infty(\Omega;\mathbb{R}^n)})R_1^\gamma \le 1 \tag{3.33}$$

we obtain

$$G(u,\kappa,\beta,B_{\tau\rho}(x)) \le 4G_0(u,\kappa,\beta,B_{\tau\rho}(x)). \tag{3.34}$$

Suppose now that

$$G_0(u,\kappa,\beta,B_\rho(x)) \le (1+\vartheta)\Phi_0(u,\kappa,\beta,B_\rho(x)). \tag{3.35}$$

Then, by (3.27) and (3.35), the decay property, (3.32), (3.34) and $g \in L^\infty(\Omega;\mathbb{R}^n)$ yield

$$\begin{aligned} G(u,\kappa,\beta,B_{\tau\rho}(x)) &\le 4\tau^{n-\gamma}G_0(u,\kappa,\beta,B_\rho(x)) \\ &\le 8\tau^{n-\gamma}G(u,\kappa,\beta,B_\rho(x)) + 4\omega_n(\mu^{p/2} + k_1\|g\|^p_{L^\infty(\Omega;\mathbb{R}^n)})\rho^n. \end{aligned}$$

Since (3.30) is not satisfied, as above we can absorb the last term in the left-hand side by taking into account the condition in (3.33) to obtain

$$G(u,\kappa,\beta,B_{\tau\rho}(x)) \le 16\tau^{n-\gamma}G(u,\kappa,\beta,B_\rho(x)).$$

The proof of (3.29) is concluded since $16\tau^{(1-\gamma)/2} < 1$.

Hence, we are left with proving (3.35) assuming that (3.30) is violated. To this aim we first fix $\sigma = \sigma(\tau) \in (0, 1/2)$ such that

$$(1+2\sigma)^2 = 1+\vartheta. \tag{3.36}$$

By (3.31), (3.32), and $g \in L^\infty(\Omega;\mathbb{R}^n)$ we obtain

$$G_0(u,\kappa,\beta,B_\rho) \le (1+\sigma)G(u,\kappa,\beta,B_\rho) + \omega_n(\mu^{p/2} + k_\sigma\|g\|^p_{L^\infty(\Omega;\mathbb{R}^n)})\rho^n \le (1+2\sigma)G(u,\kappa,\beta,B_\rho), \quad (3.37)$$

provided

$$\omega_n(\mu^{p/2} + k_\sigma\|g\|^p_{L^\infty(\Omega;\mathbb{R}^n)})R_1^\gamma \le \sigma. \quad (3.38)$$

Now, for any field $v \in GSBD^p(\Omega)$ with $\{v \neq u\} \subset\subset B_\rho$, being u a local minimizer of G, (3.31) and (3.32) give

$$G(u,\kappa,\beta,B_\rho) \le G(v,\kappa,\beta,B_\rho) \le (1+\sigma)G_0(v,\kappa,\beta,B_\rho) + \omega_n k_\sigma(\mu^{p/2} + \|g\|^p_{L^\infty(\Omega;\mathbb{R}^n)})\rho^n,$$

as (3.30) is violated, we infer

$$\big(1-\omega_n k_\sigma(\mu^{p/2} + \|g\|^p_{L^\infty(\Omega;\mathbb{R}^n)})R_1^\gamma\big)G(u,\kappa,\beta,B_\rho) \le (1+\sigma)G_0(v,\kappa,\beta,B_\rho). \quad (3.39)$$

We choose $R_1 \in (0,R)$ such that (3.33), (3.38) and

$$\frac{1+\sigma}{1+2\sigma} \le 1-\omega_n\, k_\sigma(\mu^{p/2} + \|g\|^p_{L^\infty(\Omega;\mathbb{R}^n)})R_1^\gamma \quad (3.40)$$

are satisfied. Then (3.39) becomes $G(u,\kappa,\beta,B_\rho) \le (1+2\sigma)G_0(v,\kappa,\beta,B_\rho)$, so that recalling (3.37) and the choice of $\sigma \in (0,1/2)$ made in (3.36), we get

$$G_0(u,\kappa,\beta,B_\rho) \le (1+\vartheta)G_0(v,\kappa,\beta,B_\rho).$$

We finally deduce (3.35) from the latter inequality by taking the infimum on the class of admissible v introduced above (cf. the definition of Φ_0 in (3.4)).

Step 2. Fix $R_2 > 0$ such that

$$R_2 < R_1 \wedge (\beta\,\varepsilon)^{2/(1-\gamma)} \wedge \tau, \quad (3.41)$$

where $0 < R_1 \le R$ satisfies (3.33), (3.38) and (3.40). For any $\rho < R_2$ set $\rho_i := \tau^i\rho$, $i \in \mathbb{N}$. Let us show by induction that (3.27) implies for all $i \in \mathbb{N}$

$$G(u,\kappa,\beta,B_{\rho_i}(x)) < \beta\,\varepsilon\rho_i^{n-1}. \quad (3.42)$$

The first inductive step $i = 0$ is exactly (3.27). Suppose now that (3.42) holds for some i, then by Step 1 either (3.28) or (3.29) holds. In the former case by (3.41) we have

$$G(u,\kappa,\beta,B_{\rho_{i+1}}(x)) < \tau^{n-1}\rho_i^{n-(1+\gamma)/2} = \rho_i^{(1-\gamma)/2}\rho_{i+1}^{n-1} < \beta\,\varepsilon\rho_{i+1}^{n-1}.$$

Instead, in the second instance by the inductive assumption we infer, since $\tau \leq 1$,

$$\begin{aligned} G(u, \kappa, \beta, B_{\rho_{i+1}}(x)) &< \tau^{n-(1+\gamma)/2} G(u, \kappa, \beta, B_{\rho_i}(x)) \\ &< \tau^{n-(1+\gamma)/2} \beta\, \varepsilon \rho_i^{n-1} < \beta\, \varepsilon \rho_{i+1}^{n-1}. \end{aligned}$$

Step 3. Let $\sigma = \sigma(\tau) > 0$ be as in (3.36), and fix $R_0 > 0$ such that $R_0 \leq \frac{\beta\varepsilon}{2\omega_n k_\sigma(\mu^{p/2} + \|g\|^p_{L^\infty(\Omega;\mathbb{R}^n)})} \wedge R_2$, with R_2 defined in (3.41). We claim that for all $\rho \in (0, R_0)$

$$G_0(u, \kappa, \beta, B_\rho(x)) \geq \vartheta_0\, \rho^{n-1} \tag{3.43}$$

with $\vartheta_0 := \frac{\beta\varepsilon}{2(1+\sigma)}$.

By contradiction, if (3.43) does not hold, we find by (3.31), (3.32), and since $\rho < R_0$

$$\begin{aligned} &G(u, \kappa, \beta, B_\rho(x)) \\ &\quad \leq (1+\sigma) G_0(u, \kappa, \beta, B_\rho(x)) + \omega_n k_\sigma(\mu^{p/2} + \|g\|^p_{L^\infty(\Omega;\mathbb{R}^n)})\rho^n < \beta\varepsilon\rho^{n-1}. \end{aligned}$$

Hence (3.27) holds true, and therefore by Step 2 inequality (3.42) yields

$$\liminf_{\rho\to 0} \frac{1}{\rho^{n-1}} G(u, \kappa, \beta, B_\rho(x)) \leq \beta\varepsilon,$$

in turn implying

$$\liminf_{\rho\to 0} \frac{\mathcal{H}^{n-1}(J_u \cap B_\rho(x))}{\omega_{n-1}\rho^{n-1}} \leq \frac{\varepsilon}{2\omega_{n-1}} < 1$$

by the definition of ε, so contradicting (3.25). This concludes the proof of (3.43) for points in J_u^*.

Finally, since the definitions of R_0 and ϑ_0 are independent of the particular point $x \in J_u^*$, (3.43) readily extends to $\Omega \cap \overline{J_u^*}$ and (3.26) is proven. □

Under the same assumptions of Lemma 3.8, the density lower bound for the $\mathcal{H}^{n-1}$ measure of the jump set follows straightforwardly. Indeed, by taking into account the density lower bound for G_0 one first proves that the deviation from minimality of u is small on balls $B_\rho(x)$ centred at jump points. Then, if the measure of the jump set were small in the same ball, the decay property would provide a decay of G_0 on concentric dyadic balls that would contradict the density lower bound for G_0 (a uniform energy upper bound for G_0 on $B_\rho(x)$ holds by an elementary comparison argument)

Corollary 3.9 (Density lower bound for the jump)**.** *Under the same assumptions as Lemma 3.8, there exist ϑ_1 and R_1, depending only on n, p, $\mathbb{C}$, κ, β, μ, and $\|g\|_{L^\infty(\Omega;\mathbb{R}^n)}$, such that if $0 < \rho < R_1$, $x \in \Omega \cap \overline{J_u^*}$, and $B_\rho(x) \subset\subset \Omega$, then*

$$\mathcal{H}^{n-1}(J_u \cap B_\rho(x)) \geq \vartheta_1 \rho^{n-1}. \tag{3.44}$$

Proof. Let $x \in \Omega \cap \overline{J_u^*}$ and $B_\rho(x) \subset\subset \Omega$. Denoting by ϑ_0 and R_0 the constants in Lemma 3.8, if $\rho \in (0, R_0]$ we have both

$$G_0(u, \kappa, \beta, B_\rho(x)) \geq \vartheta_0 \rho^{n-1} \tag{3.45}$$

by Lemma 3.8 itself, and the energy upper bound

$$G(u, \kappa, \beta, B_\rho(x)) \leq 2n\omega_n \beta \rho^{n-1} + \omega_n \kappa \|g\|^p_{L^\infty(B_\rho(x);\mathbb{R}^n)} \rho^n.$$

The latter easily follows by the local minimality of u and comparing its energy with that of $u\chi_{B_\rho(x)\setminus B_{\rho-\delta}(x)}$ and then letting $\delta \downarrow 0$. Moreover, by taking into account the first inequality in (3.32), we have that

$$G_0(u, \kappa, \beta, B_\rho(x)) \leq 2^{p-1} G(u, \kappa, \beta, B_\rho(x)) + \omega_n \rho^n \big(\mu^{p/2} + 2^{p-1}\kappa \, \|g\|^p_{L^\infty(B_\rho(x);\mathbb{R}^n)}\big).$$

Hence, for all $\rho \in (0, 1 \wedge R_0]$ we conclude that

$$\begin{aligned} G_0(u, \kappa, \beta, B_\rho(x)) \leq & 2^p n \omega_n \beta \rho^{n-1} \\ & + \omega_n \big(\mu^{p/2} + 2^p \kappa \, \|g\|^p_{L^\infty(B_\rho(x);\mathbb{R}^n)}\big) \rho^n \leq c_* \rho^{n-1}, \end{aligned} \tag{3.46}$$

where c_* depends on n, p, κ, β, μ, and $\|g\|_{L^\infty(\Omega;\mathbb{R}^n)}$.

We fix $\gamma \in (0,1)$ and choose $\tau \in (0, \tau_\gamma]$ in the decay property such that $c_* \tau^{1-\gamma} < \vartheta_0$. Let $\varepsilon = \varepsilon(\tau) > 0$, $\vartheta = \vartheta(\tau)$ and $R = R(\tau) > 0$ be the constants provided by the decay property. We now show that

$$\mathcal{H}^{n-1}(J_u \cap B_\rho(x)) > \varepsilon \rho^{n-1} \tag{3.47}$$

for all $\rho \in (0, R_1]$, with $R_1 := 1 \wedge R_0 \wedge R$. Indeed, arguing as in (3.36)-(3.40), but using (3.45) in place of the negation of (3.30), we deduce that for $\rho \leq R_1$

$$G_0(u, \kappa, \beta, B_\rho(x)) \leq (1 + \vartheta) \Phi_0(u, \kappa, \beta, B_\rho(x)).$$

If (3.47) were false we would conclude using (3.45), the decay property and (3.46) for some $\bar\rho \in (0, R_1]$ that

$$\begin{aligned} \vartheta_0 (\tau\bar\rho)^{n-1} & \leq G_0(u, \kappa, \beta, B_{\tau\bar\rho}(x)) \\ & \leq \tau^{n-\gamma} G_0(u, \kappa, \beta, B_{\bar\rho}(x)) \leq c_* \tau^{n-\gamma} \bar\rho^{n-1}, \end{aligned}$$

contradicting the choice of τ. □

Corollary 3.10. *Under the same assumptions as Lemma 3.8, the set*

$$\Omega_u := \{x \in \Omega : \, G_0(u, \kappa, \beta, B_\rho(x)) < \vartheta_0 \rho^{n-1} \quad \textit{for some } \rho \in (0, R_0 \wedge \operatorname{dist}(x, \partial\Omega))\}$$

is open and obeys $\Omega_u \cap \overline{J_u^*} = \emptyset$*. Moreover,* $\mathcal{H}^{n-1}(\Omega_u \cap J_u) = 0$.

Proof. Let $x \in \Omega_u$. Then there is $\rho \in (0, R_0)$ with $G_0(u, \kappa, \beta, B_\rho(x)) < \vartheta_0 \rho^{n-1}$, and therefore there is $\delta \in (0, \rho)$ such that

$$G_0(u, \kappa, \beta, B_\rho(x)) < \vartheta_0 (\rho - \delta)^{n-1}.$$

The inclusion $B_\delta(x) \subset \Omega_u$ follows straightforwardly. Indeed, let $y \in B_\delta(x)$, we have

$$G_0(u, \kappa, \beta, B_{\rho-\delta}(y)) \leq G_0(u, \kappa, \beta, B_\rho(x)) < \vartheta_0 (\rho - \delta)^{n-1}.$$

Therefore Ω_u is open.

By Lemma 3.8 and the definition we immediately obtain $\Omega_u \cap \overline{J_u^*} = \emptyset$.

Since $\mathcal{H}^{n-1}(J_u \setminus J_u^*) = 0$, by the $(n-1)$-rectifiability of J_u, and $\Omega_u \cap J_u^* = \emptyset$, we infer that $\mathcal{H}^{n-1}(\Omega_u \cap J_u) = 0$. □

In dimension 2 the assumptions of Lemma 3.8 hold true and Sobolev minimizers are regular everywhere, therefore we may conclude the following result.

Theorem 3.11. *Let $n = 2$, $\Omega \subset \mathbb{R}^2$ open, $p \in (1, \infty)$, $\kappa \geq 0$, $\beta > 0$, $\mu \geq 0$, $g \in L^\infty(\Omega; \mathbb{R}^2)$ if $p \in (1, 2]$ and $g \in W^{1,p}(\Omega; \mathbb{R}^2)$ if $p > 2$.*

Let $u \in GSBD^p(\Omega)$ be a local minimizer of G according to (3.2), then $\Omega \cap \overline{S_u} = \Omega \cap \overline{J_u} = \Omega \setminus \Omega_u$,

$$\mathcal{H}^1(\Omega \cap \overline{J_u} \setminus J_u) = 0 \tag{3.48}$$

and $u \in C^1(\Omega \setminus \overline{J_u}; \mathbb{R}^2)$.

Proof. Since $GSBD^p$ is defined via slices and Ω_u is open, from $\mathcal{H}^1(\Omega_u \cap J_u) = 0$ we deduce $u \in W^{1,p}_{\mathrm{loc}}(\Omega_u; \mathbb{R}^2)$. Thus, by elliptic regularity of Theorem 2.2 we obtain $u \in C^1(\Omega_u; \mathbb{R}^2)$. Hence, $S_u \subseteq \Omega \setminus \Omega_u$ and actually $\Omega \cap \overline{S_u} \subseteq \Omega \setminus \Omega_u$, as Ω_u is open.

On the other hand, if $x \in \Omega \setminus \overline{J_u}$, then $u \in W^{1,p}(B_\rho(x); \mathbb{R}^2)$ for some $\rho > 0$, as $GSBD^p$ is defined via slices and again by elliptic regularity $u \in C^1(B_\rho(x); \mathbb{R}^2)$. Thus, $x \in \Omega_u$, and since $J_u \subseteq S_u$ we conclude $\Omega \setminus \Omega_u = \Omega \cap \overline{S_u} = \Omega \cap \overline{J_u}$.

Eventually, (3.48) is a straightforward consequence of (3.44) and [5, Theorem 2.56]. □

3.2 Proof of the main results

We are finally ready to establish existence of strong minimizers for the Griffith static fracture model. For simplicity of notation we write the functional G appearing in (3.1) as $G(\cdot) = G(\cdot, \kappa, \beta, \Omega)$.

Proof of Theorem 1.2. By the compactness and lower semicontinuity result [22, Theorem 11.3], G has a minimizer u in $GSBD(\Omega)$. By Theorem 3.11 we obtain $u \in C^1(\Omega \setminus \overline{J_u}; \mathbb{R}^2)$ so that $E_p(\overline{J_u}, u) = G(u)$, E_p being defined in (1.7). Now, if $\Gamma \subset \overline{\Omega}$ is closed and $v \in W^{1,p}_{\mathrm{loc}}(\Omega \setminus \Gamma; \mathbb{R}^2)$ with $E_p(\Gamma, v) < \infty$, then $v \in GSBD(\Omega)$ with $\mathcal{H}^1(J_v \setminus \Gamma) = 0$, again arguing by slicing. We conclude that

$$E_p(\overline{J_u}, u) = G(u) \leq G(v) \leq E_p(\Gamma, v).$$

□

The proof of Theorem 1.1 is analogous.

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