On the maximal number of coprime subdegrees in finite primitive permutation groups

COPRIME SUBDEGREES FOR PRIMITIVE PERMUTATION GROUPS AND COMPLETELY REDUCIBLE LINEAR GROUPS

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Abstract. In this paper we answer a question of Gabriel Navarro about orbit sizes of a finite linear group $H \subseteq \mathrm{GL}(V)$ acting completely reducibly on a vector space V: if the H-orbits containing the vectors a and b have coprime lengths m and n, we prove that the H-orbit containing $a + b$ has length mn. Such groups H are always reducible if $n, m > 1$. In fact, if H is an irreducible linear group, we show that, for every pair of non-zero vectors, their orbit lengths have a non-trivial common factor.

In the more general context of finite primitive permutation groups G, we show that coprime non-identity subdegrees are possible if and only if G is of O'Nan-Scott type AS, PA or TW. In a forthcoming paper we will show that, for a finite primitive permutation group, a set of pairwise coprime subdegrees has size at most 2. Finally, as an application of our results, we prove that a field has at most 2 finite extensions of pairwise coprime indices with the same normal closure.

1. Introduction

1.1. **Completely reducible linear groups.** In this paper we are concerned with the orbit lengths of a *completely reducible* linear group and with the *subdegrees* of a primitive permutation group. Given a field k, a kH-module V is said to be completely reducible if V is a direct sum of irreducible kH-modules. Furthermore, the set of subdegrees of a finite transitive permutation group G is the set of orbit lengths of the stabilizer G_ω of a point ω.

Our first main result is a positive answer to a question of Gabriel Navarro [24] about actions of a finite linear group. Indeed, Navarro asked whether a finite completely reducible kH-module V with two H-orbits of relatively prime lengths m and n has an orbit of size mn.

Theorem 1.1. *Let k be a field, let H be a finite group, let V be a completely reducible kH-module and let a and b be elements of V. If the H-orbits a^H and b^H have sizes m and n, and m, n are relatively prime, then $\mathbf{C}_H(a+b) = \mathbf{C}_H(a) \cap \mathbf{C}_H(b)$ and the H-orbit $(a + b)^H$ has size mn.*

We note that Theorem 1.1 explicitly exhibits an H-orbit of size mn, namely the orbit containing $a + b$. Furthermore, Example 3.1 shows that the "completely

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2000 *Mathematics Subject Classification.* 20B15, 20H30.

Key words and phrases. coprime subdegrees; primitive permutation groups; completely reducible modules.

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The first author is supported by the MIUR project "Teoria dei gruppi ed applicazioni". The second author was partially supported by NSF grant DMS-1001962. The third author is supported by the ARC Federation Fellowship Project FF0776186. The fourth author is supported by the University of Western Australia as part of the Federation Fellowship project.

reducible" hypothesis in Theorem 1.1 is essential. Martin Isaacs [16, Theorem] has proved a similar result under stronger arithmetical conditions on m, n and the characteristic of k.

Our second main result is a somehow remarkable theorem (in our opinion) on irreducible linear groups. Theorem 1.2 shows that the groups arising in Theorem 1.1 are always reducible if $n, m > 1$.

Theorem 1.2. *Let k be a field, let H be a finite group, let V be a non-trivial irreducible kH-module and let a and b be in $V \setminus \{0\}$. Then the sizes of the H-orbits a^H and b^H have a non-trivial common factor.*

The strategy for proving Theorem 1.2 is to reduce the problem inductively to the case where H is a non-abelian simple group which admits a maximal factorization $H = AB$ with $|H : A|$ relatively prime to $|H : B|$. Table 1 contains all such triples (H, A, B). In the case where H is a sporadic simple group, we use the information in Table 1 for the proof of Theorem 1.2. Furthermore, we observe that as a consequence of Theorem 1.2, if (H, A, B) is one of the triples in Table 1, then there are no irreducible representations of H with A and B vector stabilizers.

A direct application of Theorem 1.2 gives the following corollary.

Corollary 1.3. *Let k be a field, let H be a finite group, let V be a non-trivial finite dimensional kH-module and let a and b be elements of V. If both H-orbits a^H and b^H span V, then $|a^H|$ and $|b^H|$ have a non-trivial common factor.*

In the same direction as Corollary 1.3, in the case of p-soluble groups, we prove the following theorem.

Theorem 1.4. *Let k be a field of characteristic $p \geq 0$, H a p-soluble finite group, V a kH-module and $a \in V$ fixed by a Sylow p-subgroup of H and with the H-orbit a^H spanning V. Then*

(a) $\dim \mathbf{C}_V(H) \leq 1$*; and*
(b) *if $b \in V$ and $\gcd(|a^H|, |b^H|) = 1$, then $b \in C_V(H)$.*

Here, by abuse of notation, "0-soluble finite group" means "finite group" and a "Sylow 0-subgroup" is the "identity subgroup". In the proof of Theorem 1.4 we do not make use of the Classification of the Finite Simple Groups. Moreover, since in an irreducible kH-module V every non-trivial H-orbit spans V, we see that Theorem 1.4 (b) generalizes (for the class of p-soluble groups) Theorem 1.2 and, in particular, offers an independent and more elementary proof. Note that if p does not divide the order of H (including the case $p = 0$), then H is p-soluble, the Sylow p-subgroups of H are trivial and, in particular, Theorem 1.4 applies in this situation.

1.2. **Coprime subdegrees in primitive permutation groups.** In the more general context of finite primitive permutation groups G, we investigate coprime subdegrees according to the O'Nan-Scott type of G. (We say that a subdegree d of G is *non-trivial* if $d \neq 1$.) In particular, Theorem 1.2 yields that the primitive permutation group $G = V \rtimes H$ acting on V has no pair of non-trivial coprime subdegrees. One of the most important modern methods for analyzing a finite primitive permutation group G is to study the *socle* N of G, that is, the subgroup generated by the minimal normal subgroups of G. The socle of an arbitrary finite group is isomorphic to a direct product of simple groups, and, for finite primitive groups these simple groups are pairwise isomorphic. The O'Nan-Scott theorem describes in detail the embedding of N in G and collects some useful information on the action of N. In [26] eight types of primitive groups are defined (depending on the structure and on the action of the socle), namely HA (*Holomorphic Abelian*), AS (*Almost Simple*), SD (*Simple Diagonal*), CD (*Compound Diagonal*), HS (*Holomorphic Simple*), HC (*Holomorphic Compound*), TW (*Twisted wreath*), PA (*Product Action*), and it is shown in [18] that every primitive group belongs to exactly one of these types.

Theorem 1.5. *Let G be a finite primitive permutation group. If G has two non-trivial coprime subdegrees, then G is of AS, PA or TW type. Moreover, for each of the O'Nan-Scott types AS, PA and TW, there exists a primitive group of this type with two non-trivial coprime subdegrees.*

It is possible, for a single primitive group to have several different pairs of non-trivial coprime subdegrees. We give a construction of groups of PA type with this property in Example 4.3. However, in the case of primitive groups of TW type, it is not possible to have as many as three pairwise coprime non-trivial subdegrees.

Theorem 1.6. *For a finite primitive permutation group of TW type, the maximal size of a set of pairwise coprime non-trivial subdegrees is at most* 2.

Using the Classification of the Finite Simple Groups, we have proved the following theorem in [8].

Theorem 1.7. *Let G be a finite primitive permutation group. The maximal size of a set of pairwise coprime non-trivial subdegrees of G is at most* 2.

Theorem 1.7 is related to a result on primitive groups first observed by Peter Neumann to be a consequence of a 1935 theorem of Marie Weiss. Its statement [25, Corollary 2, p. 93] is: if a primitive group has k pairwise coprime non-trivial subdegrees, then its rank is at least 2^k. Theorem 1.7 shows that this result can only be applied with $k = 1$ or $k = 2$. In light of Theorems 1.5 and 1.6, proving Theorem 1.7 reduces to consideration of primitive permutation groups of AS and PA type. In this paper we show that a proof of Theorem 1.7 reduces to a similar problem for transitive nonabelian simple permutation groups (it is this reduction that is used in [8] to prove Theorem 1.7).

Definition 1.8. Let T be a nonabelian simple group and L a subgroup of T. We say that L is *pseudo-maximal* in T if there exists an almost simple group H with socle T and a maximal subgroup M of H with $T \nsubseteq M$ and $L = T \cap M$.

We announce here a proof of the following theorem about nonabelian simple groups (which again will be proved in [8]).

Theorem 1.9. *Let T be a transitive nonabelian simple permutation group and assume that the stabilizer of a point is pseudo-maximal in T. Then the maximal size of a set of pairwise coprime non-trivial subdegrees of T is at most* 2.

Since a pseudo-maximal subgroup of T is not necessarily a maximal subgroup, we see that Theorem 1.9 is formally stronger than Theorem 1.7 for the class of nonabelian simple permutation groups. However, we prove in this paper that these two theorems are strongly related.

Theorem 1.10. *Theorem 1.7 follows from Theorem 1.9.*

We conclude with a problem on relatively prime subdegrees in primitive groups.

Problem 1.11. *Determine the finite primitive permutation groups G having two non-trivial coprime subdegrees m and n for which mn is not a subdegree of G.*

This problem is related to another classical result due to Marie Weiss [25, Theorem 3, p. 92]: if m and n are non-trivial coprime subdegrees of a primitive group G and $m < n$, then G has a subdegree d such that d divides mn and $d > n$. In Problem 1.11 we suggest that (apart a small list of exceptions) d can be chosen to be mn. Actually, we know only one almost simple group G where d cannot be taken to be nm.

Example 1.12. Let G be the sporadic simple group HS in its primitive permutation representation of degree 3850 on the cosets of $2^4.\mathrm{Sym}(6)$. Using the computational algebra system `magma` [4], it is easy to check that the subdegrees of G are $1, 15, 32, 90, 120, 160, 192, 240, 240, 360, 960, 1440$. In particular, we see that 15 and 32 are coprime but there is no subdegree of size 15×32=480.

We are grateful to Michael Giudici for providing this beutiful example.

We now give some examples which demonstrate that Theorem 1.7 is false for transitive groups that are not primitive. These examples show that there is no upper bound on the number of pairwise coprime non-trivial subdegrees for general transitive groups.

Example 1.13. Let G be the direct product $F_1 \times \cdots \times F_\ell$ of ℓ Frobenius groups. For each $i \in \{1, \ldots, \ell\}$, let N_i be the Frobenius kernel of F_i and let K_i be a Frobenius complement for N_i in F_i. Assume that $|K_i|$ is coprime to $|K_j|$, for every two distinct elements i and j in $\{1, \ldots, \ell\}$. Write $N = N_1 \times \cdots \times N_\ell$ and $K = K_1 \times \cdots \times K_\ell$. Clearly, the group G acts on N as a holomorphic permutation group, that is, N acts on N by right multiplication and K acts on N by group conjugation. The stabilizer in G of the element 1 of N is K. Now, for each $i \in \{1, \ldots, \ell\}$, let $n_i \in N_i \setminus \{1\}$ and let $\omega_i = (1, \ldots, 1, n_i, 1, \ldots, 1)$ be the element of N with n_i in the i^{th} coordinate and 1 everywhere else. Clearly, $|\omega_i^K| = |K : \mathbf{C}_K(\omega_i)| = |K_i|$. Therefore G has a set of at least ℓ pairwise coprime non-trivial subdegrees.

However, if we restrict to *faithful subdegrees* of a transitive group G, that is, subdegrees d such that there exists an orbit of length d of a stabilizer G_α on which G_α acts faithfully, then in fact we can show that a conclusion analogous to the statement of Theorem 1.7 does hold. We note that, in particular, every primitive permutation group has a faithful subdegree [12, Theorem 3].

Theorem 1.14. *Let G be a finite transitive permutation group of degree $n > 1$. Assume that G is not regular and let H be the stabilizer of a point. Then a set of faithful subdegrees that are pairwise coprime has size at most 2. Moreover, if the Fitting subgroup of H is non-trivial, then any two faithful subdegrees of G have a non-trivial common factor.*

The proof of Theorem 1.14 also yields the following result about field extensions.

Theorem 1.15. *Let k be a field and let $k_1, \ldots, k_t$ be finite extensions of k all with the same normal closure K. Assume that the indices $[k_i : k]$ are pairwise coprime. Then $t \leq 2$.*

1.3. **Structure of the paper.** The structure of this paper is straightforward: we prove Theorem 1.2, Corollary 1.3 and Theorem 1.4 in Section 2; we prove Theorem 1.1 in Section 3; we prove Theorem 1.5 in Section 4; we prove Theorems 1.6 and 1.10 in Section 5; we prove Theorems 1.14 and 1.15 in Section 6; and we give Table 1 in Section 7.

2. Proofs of Theorems 1.2 and 1.4 and Corollary 1.3

We say that a factorization $H = AB$ is *coprime* if $|H : A|$ is relatively prime to $|H : B|$ and both A, B are proper subgroups of H. Also $H = AB$ is *maximal* if A and B are maximal subgroups of H. We start by proving a preliminary theorem on finite classical groups. We let τ denote the transpose inverse map of $\mathrm{GL}_n(q)$, that is, $x^\tau = (x^{tr})^{-1}$ where x^{tr} is the transpose matrix of x. We denote by $\mathrm{CSp}(2n, q)$ the conformal symplectic group, that is, the elements of $\mathrm{GL}_{2n}(q)$ preserving a given symplectic form up to a scalar multiple.

Theorem 2.1. *Let $n \geq 2$.*

(a): *Every element of* $\mathrm{GL}_n(q)$ *is conjugate to its inverse in* $\mathrm{GL}_n(q)\langle\tau\rangle$.
(b): *Every element of* $\mathrm{GU}_n(q)$ *is conjugate to its inverse in* $\mathrm{GU}_n(q)\langle\tau\rangle$.
(c): *Every element of* $\mathrm{Sp}(2n, q)$ *is conjugate to its inverse in* $\mathrm{CSp}(2n, q)$.
(d): *Every element of* $\mathrm{O}^\epsilon(n, q)$ *is conjugate to its inverse in* $\mathrm{O}^\epsilon(n, q)$*, for* $\epsilon \in \{\pm, \circ\}$.

Proof. We prove **(a)** and **(b)** first. Let $X = \mathrm{GL}_n(k)$ be the algebraic group obtained by taking the algebraic closure k of the finite field $\mathbb{F}_q$. Let $F : X \to X$ be the Lang-Steinberg map obtained by raising each entry of a matrix x of X to the qth power, and $G : X \to X$ the Lang-Steinberg map $F \circ \tau$. As usual, we denote by X^F and by X^G the fixed points of F and of G. In our case, we have $X^F = \mathrm{GL}_n(q)$ and $X^G = \mathrm{GU}_n(q)$. Let x be in X^F. Then x^{tr} and x are clearly conjugate in the algebraic group X and hence also $x^\tau = (x^{tr})^{-1}$ and x^{-1} are conjugate in X. Since the centralizer of any element of X is connected, it follows by the Lang-Steinberg theorem that x^τ and x^{-1} are conjugate in X^F. Therefore x and x^{-1} are conjugate in $\mathrm{GL}_n(q)\langle\tau\rangle$ and **(a)** is proved. Now, let x be in X^G. As we have noted in the proof of **(a)**, the elements x^τ and x^{-1} are conjugate in the algebraic group X. It follows by the Lang-Steinberg theorem that x^τ and x^{-1} are conjugate in X^G. Therefore x and x^{-1} are conjugate in $\mathrm{GU}_n(q)\langle\tau\rangle$ and **(b)** is proved.

(c) and **(d)**, when q is even, are the main theorem of [11]. Finally, **(c)** and **(d)**, when q is odd, are proved in [28]. □

Given a field k and a kH-module V, we let $V^* = \mathrm{Hom}_k(V, k)$ denote the *dual* kH-module of V. Furthermore, we denote by V_A the *restriction* of V to the subgroup A of H. Finally, if A is a subgroup of H and if V is a kA-module, then we denote by $V_A^H = V \otimes_{kA} kH$ the module *induced* by V from A to H.

Lemma 2.2. *Suppose that $H = AB$ is a factorization. If V is a non-trivial irreducible kH-module, then either A fixes no element of $V \setminus \{0\}$ or B fixes no element of $V^* \setminus \{0\}$.*

Proof. We argue by contradiction and we assume that A fixes $a \in V \setminus \{0\}$ and that B fixes $b \in V^* \setminus \{0\}$.

Let Ω (respectively Δ) be the set of right cosets of A (respectively B) in H. Clearly, H acts transitively on Ω and Δ, and as $H = AB$, the group B is transitive on Δ and A is transitive on Ω. Let k_A^H (respectively k_B^H) be the permutation module for the action of H on Ω (respectively Δ). Since A is transitive on Δ, the multiplicity of the trivial kA-module k in $(k_B^H)_A$ is 1, that is, $\dim \mathrm{Hom}_{kA}(k, (k_B^H)_A) = 1$. From Frobenius reciprocity, it follows that $\dim \mathrm{Hom}_{kH}(k_A^H, k_B^H) = \dim \mathrm{Hom}_{kA}(k, (k_B^H)_A) = 1$. Therefore, the only H-homomorphism of k_A^H to k_B^H is the homomorphism φ with $\mathrm{Ker}\,\varphi$ of codimension 1 in k_A^H and with $\mathrm{Im}\,\varphi$ the trivial submodule of k_B^H.

Since A fixes the non-zero vector a of V, we have $0 \neq \mathrm{Hom}_{kA}(k, V_A) \cong \mathrm{Hom}_{kH}(k_A^H, V)$ and hence k_A^H has a homomorphic image isomorphic to V. Similarly, since B fixes the non-zero vector b of V^*, we have $0 \neq \mathrm{Hom}_{kB}(k, V_B^*) \cong \mathrm{Hom}_{kH}(k_B^H, V^*)$ and hence k_B^H has a homomorphic image isomorphic to V^*. Using duality and the fact that $k^* \cong k$, we obtain that $(k_B^H)^* \cong (k^*)_B^H \cong k_B^H$ has a submodule isomorphic to $V^{**} \cong V$. This shows that there exists an H-homomorphism $\psi : k_A^H \to k_B^H$ with $k_A^H / \mathrm{Ker}\,\psi \cong V$ and with $\mathrm{Im}\,\psi \cong V$. Since V is non-trivial, we obtain that $\dim \mathrm{Hom}_{kH}(k_A^H, k_B^H) > 1$, a contradiction. □

Here we say that a factorization $H = AB$ is *exact* if $A \cap B = 1$.

Lemma 2.3. *Suppose that $H = AB$ is a coprime exact factorization. If V is a non-trivial irreducible kH-module, then either A or B fixes no element of $V \setminus \{0\}$.*

Proof. We argue by contradiction and we assume that both A and B fix some non-zero vector of V. Let r be the characteristic of the field k. Since $|H : A|$ is relatively prime to $|H : B|$ and $A \cap B = 1$, we have that either r does not divide $|A|$ or r does not divide $|B|$. Replacing A with B if necessary, we may assume that r does not divide $|B|$. Since the characteristic of k is coprime to the order of B, the module V_B is a completely reducible kB-module. Therefore, $V_B = W_1 \oplus \cdots \oplus W_s$ where W_i is an irreducible kB-module, for each $i \in \{1, \ldots, s\}$. Since B fixes a non-zero vector of V, we have that, for some $i \in \{1, \ldots, s\}$, W_i is a trivial kB-module. Now, $V_B^* = W_1^* \oplus \cdots \oplus W_s^*$ and hence W_i^* is a trivial submodule of V_B^*. This shows that B fixes a non-zero vector of V^*, but this contradicts Lemma 2.2. □

The following lemma is Lemma 5.1 in [13].

Lemma 2.4. *Suppose that every element of H is conjugate to its inverse via an element of* $\mathrm{Aut}(H)$. *If V is an irreducible kH-module, then $V^* \cong V^x$ for some $x \in \mathrm{Aut}(H)$.*

Proof. Write $G = H \rtimes \mathrm{Aut}(H)$. We can view H as a subgroup of G. Since H is normal in G, from [23, Theorem 8.6] we see that the module $M = (V_H^G)_H$ is completely reducible with irreducible summands V^x, for $x \in G$. Furthermore, since every element of H is conjugate to its inverse via an element of G, we obtain that the Brauer character of M is real valued. Now, from [23, Theorem 1.19 and Lemma 2.2], we see that completely reducible modules with real Brauer characters are self dual and hence M is self dual, that is, $M^* \cong M$. Hence V^* is an irreducible direct summand of M, and so $V^* \cong V^x$ for some $x \in G$. □

Lemma 2.5. *Suppose that every element of H is conjugate to its inverse via an element of* $\mathrm{Aut}(H)$. *If $H = AB$ is a coprime factorization and V is a non-trivial irreducible kH-module, then either A fixes no element of $V \setminus \{0\}$ or B fixes no element of $V \setminus \{0\}$.*

Proof. From Lemma 2.4, $V^* \cong V^x$ for some $x \in \mathrm{Aut}(H)$. As $H = AB$ is a coprime factorization, we obtain $H = AB^x$.

We argue by contradiction and we assume that A fixes the non-zero vector a of V and that B fixes the non-zero vector b of V. So B^x fixes the vector b^x of V^x and, as $V^x \cong V^*$, the group B^x fixes some non-zero vector of V^*. This contradicts Lemma 2.2 applied to $G = AB^x$, and so the lemma is proven. □

In the following proposition we prove Theorem 1.2 in the case that the group H is a non-abelian simple group.

Proposition 2.6. *Let H be a non-abelian simple group, V be a non-trivial irreducible kH-module, and a and b be in $V \setminus \{0\}$. Then the sizes of the H-orbits a^H and b^H have a non-trivial common factor.*

Proof. We argue by contradiction and we assume that a^H and b^H have relatively prime sizes. Since $|a^H| = |H : \mathbf{C}_H(a)|$ and $|b^H| = |H : \mathbf{C}_H(b)|$ are coprime, $H = \mathbf{C}_H(a)\mathbf{C}_H(b)$ is a coprime factorization. Now we use the classification of the finite simple groups.

If H is a classical group, we see from Theorem 2.1 that every element of H is conjugate to its inverse via an element of $\mathrm{Aut}(H)$. Clearly, the same result holds true if H is an alternating group. Therefore, if H is a classical group or an alternating group, we obtain a contradiction from Lemma 2.5 (applied with $A = \mathbf{C}_H(a)$ and $B = \mathbf{C}_H(b)$). This shows that H is either an exceptional group of Lie type or a sporadic simple group. From Table 1, we see that exceptional groups of Lie type do not admit coprime factorizations. Therefore, H is a sporadic simple group. Again, using Table 1, we see that the only sporadic simple groups admitting a coprime factorization are M_{11}, M_{23} and M_{24}. In the rest of this proof we consider separately each of these groups. Note that Table 1 determines all possible coprime factorizations $H = AB$ with A and B maximal in H.

Case $H = M_{11}$. We first consider the case that $\mathbf{C}_H(a) \subseteq A = L_2(11)$ and $\mathbf{C}_H(b) \subseteq B = M_{10}$. We have $|H : A| = 12$, $|H : B| = 11$ and $A \cap B \cong \mathrm{Alt}(5)$. As 2 and 3 divide $|H : A|$ and as $B \cong \mathrm{Alt}(6).2$ has no subgroups of index 5, in order to have $\gcd(|H : A|, |H : \mathbf{C}_H(b)|) = 1$, we must have $B = \mathbf{C}_H(b)$. Since $|H : C_H(b)| = |H : B| = 11$ and $|H : \mathbf{C}_H(a)|$ is coprime to 11, the group $\mathbf{C}_H(a)$ has order divisible by 11 and hence it contains a Sylow 11-subgroup S. Now we have $H = S\mathbf{C}_H(b)$ with $S \cap \mathbf{C}_H(b) = 1$ and hence the result follows from Lemma 2.3.

Now we consider the case that $\mathbf{C}_H(a) \subseteq A = L_2(11)$ and $\mathbf{C}_H(b) \subseteq B = M_9.2$. We have $|H : A| = 12$, $|H : B| = 55$ and $A \cap B \cong \mathrm{Alt}(4)$. Since $|A \cap B| = |H : A| = 12$, by coprimality we have $B = \mathbf{C}_H(b)$. Therefore, $|H : \mathbf{C}_H(b)| = 55$ and 55 divides $|\mathbf{C}_H(a)|$. From the subgroup structure of $A = L_2(11)$, we see that $\mathbf{C}_H(a)$ contains a subgroup S of order 55. In particular, $H = S\mathbf{C}_H(b)$ and $S \cap \mathbf{C}_H(b) = 1$, and the result follows from Lemma 2.3.

Case $H = M_{23}$. In this case we have three maximal factorizations to consider. We start by studying the case that $\mathbf{C}_H(a) \subseteq A = M_{22}$ and $\mathbf{C}_H(b) \subseteq B = 23 : 11$. As $|H : A| = 23$ and $\gcd(|H : \mathbf{C}_H(a)|, |H : \mathbf{C}_H(b)|) = 1$, we have that 23 divides $\mathbf{C}_H(b)$. Let S be a Sylow 23-subgroup of $\mathbf{C}_H(b)$. Now, $H = \mathbf{C}_H(a)S$ and $\mathbf{C}_H(a) \cap S = 1$, and the result follows as usual from Lemma 2.3.

The other two maximal coprime factorizations of M_{23} in Table 1 are exact and hence the result follows again from Lemma 2.3.

Case $H = M_{24}$. We have $\mathbf{C}_H(a) \subseteq A = M_{23}$, $\mathbf{C}_H(b) \subseteq B = 2^6.3.\mathrm{Sym}(6)$, $|H : A| = 24$, $|H : B| = 1771 = 7 \cdot 11 \cdot 23$ and $|A \cap B| = 5760 = 2^7 \cdot 3^2 \cdot 5$. Since $|H : \mathbf{C}_H(a)|$ is divisible by 2 and 3, and $|H : \mathbf{C}_H(b)|$ is relatively prime to $|H:\mathbf{C}_H(a)|$, we have that $\mathbf{C}_H(b)$ contains a Sylow 2-subgroup and a Sylow 3-subgroup of H. Thus $|\mathbf{C}_H(b)|$ is divisible by $2^{10}\cdot 3^3$ and $|B:\mathbf{C}_H(b)|\leq 5$. Since B has no subgroup of index 5, we obtain $B=\mathbf{C}_H(b)$. With a similar argument applied to $\mathbf{C}_H(a)$, we get that $7\cdot 11\cdot 23$ divides $|\mathbf{C}_H(a)|$. From [6], we see that M_{23} has no proper subgroup of order divisible by 7, 11 and 23. Therefore $A=\mathbf{C}_H(a)$.

Let M be the permutation module k_A^H. (Thus M is the permutation module of the 2-transitive action of H on a set Ω of size 24. In particular, M is one of the modules investigated by Mortimer in [22].) Since A fixes the non-zero vector a of V, we have $\mathrm{Hom}_{kA}(k,V_A)\neq 0$ and so, from Frobenius reciprocity, we obtain $\mathrm{Hom}_{kH}(M,V)\neq 0$. Hence the kH-module V is isomorphic to M/W, for some maximal kH-submodule W of M. Let $(e_\omega)_{\omega\in\Omega}$ be the canonical basis of M and let p be the characteristic of k.

Let $e=\sum_{\omega\in\Omega}e_\omega$, $\mathcal{C}=\langle e\rangle$ and $\mathcal{C}^\perp=\{\sum_{\omega\in\Omega}c_\omega e_\omega\mid\sum_{\omega\in\Omega}c_\omega=0\}$. Clearly, $\mathcal{C}$ and $\mathcal{C}^\perp$ are submodules of M. Assume that $p\neq 2,3$. From [22, Table 1], we see that the module M is completely reducible, $M=\mathcal{C}\oplus\mathcal{C}^\perp$ and $\mathcal{C}^\perp$ is an irreducible kH-module. Since V is a non-trivial kH-module, we obtain $V\cong\mathcal{C}^\perp$. Since M is self dual, we obtain that $M^*\cong M$ and hence $V^*\cong V$. Therefore, since B fixes the non-zero vector b of V, it also fixes a non-zero vector of V^*, but this contradicts Lemma 2.2.

Now assume $p=3$. From [22, Lemma 2], we have that $\mathcal{C}\subseteq\mathcal{C}^\perp$ and $\mathcal{C}^\perp$ is the unique maximal submodule of M. Therefore $W=\mathcal{C}^\perp$ and $V\cong M/\mathcal{C}^\perp\cong\mathcal{C}$ is a trivial kH-module, a contradiction. Therefore it remains to consider the case $p=2$.

Since 2 divides $|\Omega|=24$, we have $\mathcal{C}\subseteq\mathcal{C}^\perp$. From [15, Beispiele 2 *b*)], we see that $\mathcal{C}^\perp$ is the unique maximal submodule of M and hence $V\cong M/\mathcal{C}^\perp\cong\mathcal{C}$ is a trivial kH-module, a contradiction. □

Now we are ready to prove Theorems 1.2, 1.4 and Corollary 1.3.

Proof of Theorem 1.2. We argue by contradiction and we let H be a minimal (with respect to the group order) counterexample. Let $a,b\in V\setminus\{0\}$ with $|a^H|$ relatively prime to $|b^H|$.

If H is a cyclic group of prime order p, then every H-orbit on $V\setminus\{0\}$ has size p, a contradiction. Similarly, from Proposition 2.6 we see that the group H is not a non-abelian simple group. Thus H has a non-identity proper normal subgroup N. From the Clifford correspondence, $V_N=W_1\oplus\cdots\oplus W_k$ with W_i a homogeneous kN-module, for each $i\in\{1,\ldots,k\}$, and with H acting transitively on the set of direct summands $\{W_1,\ldots,W_k\}$ of V. (A module is said to be *homogeneous* if it is the direct sum of pairwise isomorphic submodules.) Write $a=\sum_{i=1}^k a_i$ and $b=\sum_{i=1}^k b_i$ with $a_i,b_i\in W_i$, for each $i\in\{1,\ldots,k\}$. Let i and j be in $\{1,\ldots,k\}$ with $a_i\neq 0$ and $b_j\neq 0$. Since H acts transitively on $\{W_1,\ldots,W_k\}$, there exist h and k in H with $W_i^h=W_1$ and $W_j^k=W_1$. In particular, replacing a and b by a^h and b^k if necessary, we may assume that $i=j=1$. Since N is a normal subgroup of H, we get that $|a^N|$ divides $|a^H|$ (respectively $|b^N|$ divides $|b^H|$). Furthermore, since N acts trivially on $\{W_1,\ldots,W_k\}$, we obtain that $\mathbf{C}_N(a)\subseteq\mathbf{C}_N(a_1)$ and $\mathbf{C}_N(b)\subseteq\mathbf{C}_N(b_1)$, that is, $|a_1^N|$ divides $|a^N|$ and $|b_1^N|$ divides $|b^N|$. In particular, $|a_1^N|$ and $|b_1^N|$ are coprime.

As W_1 is a homogeneous kN-module, there exists an irreducible kN-module U such that $W_1=U_1\oplus\cdots\oplus U_r$ with $U_i\cong U$, for each $i\in\{1,\ldots,r\}$. Assume that U is the trivial kN-module. So, N acts trivially on W_1. Since H permutes transitively the set of direct summands $\{W_1, \dots, W_k\}$ and since N is normal in H, we obtain that N acts trivially on V and $N = 1$, a contradiction. Therefore U is a non-trivial irreducible kN-module and, in particular, $|a_1^N|, |b_1^N| > 1$.

Write $a_1 = \sum_{i=1}^r x_i$ and $b_1 = \sum_{i=1}^r y_i$ with $x_i, y_i \in U_i$, for each $i \in \{1, \dots, r\}$. Since N stabilizes the direct summands $\{U_1, \dots, U_r\}$ of U, we obtain that $\mathbf{C}_N(a_1) = \cap_{i=1}^r \mathbf{C}_N(x_i)$ and $\mathbf{C}_N(b_1) = \cap_{i=1}^r \mathbf{C}_N(y_i)$. In particular, for each $i \in \{1, \dots, r\}$, we have that $|x_i^N|$ divides $|a_1^N|$ and $|y_i^N|$ divides $|b_1^N|$. Since $a_1, b_1 \neq 0$, there exist $i, j \in \{1, \dots, r\}$ with $x_i \neq 0$ and $y_j \neq 0$. Fix $\varphi_i : U_i \to U$ and $\varphi_j : U_j \to U$ two N-isomorphisms and write $x = \varphi_i(x_i)$ and $y = \varphi_j(y_j)$. In particular, x and y are non-zero elements of the non-trivial irreducible kN-module U. Furthermore, since φ_i and φ_j are N-isomorphisms, we obtain $\mathbf{C}_N(x_i) = \mathbf{C}_N(x)$ and $\mathbf{C}_N(y_j) = \mathbf{C}_N(y)$ and thus $|x^N|$ and $|y^N|$ are coprime. This contradicts the minimality of H and hence the theorem is proved. □

Proof of Corollary 1.3. We argue by contradiction and we assume that V is a non-trivial finite dimensional kH-module and that a and b are elements of V with $V = \langle a^h \mid h \in H \rangle = \langle b^h \mid h \in H \rangle$ and with $\gcd(|a^H|, |b^H|) = 1$. Now we argue by induction on $\dim_k V$. If V is irreducible, then the result follows from Theorem 1.2. So, we assume that this is not the case. Let W be a minimal submodule of V and suppose that V/W is non-trivial. Clearly, $(a + W)^H$ and $(b + W)^H$ span V/W and hence, by induction, the lengths of the orbits of $(a + W)^H$ and $(b + W)^H$ are not coprime. As $|(a + W)^H|$ divides $|a^H|$ and $|(b + W)^H|$ divides $|b^H|$, we have that $|a^H|$ and $|b^H|$ are not coprime.

Suppose now that V/W is the trivial kH-module. We claim that in this case V splits over W, that is, $V = \langle v \rangle \oplus W$ for some element v of V fixed by H. If the characteristic of V is zero, then V is semisimple and our claim is immediate. Suppose that V has characteristic $p > 0$. Replacing a by b if necessary, we may assume that $p \nmid |a^H|$ and hence $\mathbf{C}_H(a)$ contains a Sylow p-subgroup P of H. We claim that $V \cong k \oplus W$, that is, V splits over W. The module V corresponds to an element δ of $\mathrm{Ext}_G^1(k, W) \cong H^1(G, W^*)$ (see [3, Section (III) 2] for the last isomorphism). On the other hand, V splits over W as a kP-module because $P \subseteq \mathbf{C}_H(a)$ and $a \notin W$. Thus $\delta = 0$ in $H^1(P, W^*)$. However, from [3, Theorem 10.3], we see that the restriction map $\mathrm{res} : H^1(G, W^*) \to H^1(P, W^*)$ is injective. So $\delta = 0$ is $H^1(G, W^*)$ and V splits over W. In particular, H fixes a vector $v \in V \setminus W$ and $V = \langle v \rangle \oplus W$.

Write $a = \lambda v + a'$ and $b = \mu v + b'$ with $\lambda, \mu \in k$, $a' \in W$ and $b' \in W$. Clearly, $a', b' \neq 0$ because a^H and b^H span V and V is not the trivial module. Similarly, W is not the trivial kH-module. Since H fixes v, we have $\mathbf{C}_H(a) = \mathbf{C}_H(a')$ and $\mathbf{C}_H(b) = \mathbf{C}_H(b')$ and hence $|a'^H|, |b'^H|$ are relatively prime. This contradicts Theorem 1.2 applied to the irreducible module W and to the vectors a', b'. □

Proof of Theorem 1.4. Write $A = \mathbf{C}_H(a)$. Since H is p-soluble and A contains a Sylow p-subgroup of H, the group H contains a p'-subgroup L with $H = AL$. (For example, $H = AL$ for each Hall p'-subgroup L of H.) Now, let L be any p'-subgroup of H with $H = AL$ and define $\psi_L : V \to V$ by setting

$$\psi_L(v) = \sum_{x \in L} v^x.$$

We claim that $\mathbf{C}_V(L) = \psi_L(V)$. For $v \in V$ and $y \in L$, we have

$$\psi_L(v)^y = \left(\sum_{x\in L} v^x\right)^y = \sum_{x\in L} v^{xy} = \sum_{x\in L} v^x = \psi_L(v).$$

So $\psi_L(V) \subseteq \mathbf{C}_V(L)$. Conversely, if $v \in \mathbf{C}_V(L)$, then

$$\psi_L(v) = \sum_{x\in L} v^x = \sum_{x\in L} v = |L|v.$$

As $|L|$ is coprime to p, we have $v = \psi_L(v/|L|) \in \psi_L(V)$.

We now show that $\mathbf{C}_V(H) = \mathbf{C}_V(L)$. As $L \subseteq H$, we have $\mathbf{C}_V(H) \subseteq \mathbf{C}_V(L)$. As $H = AL$, we have $a^H = a^{AL} = a^L$. So, for every $v \in a^H$, the image $\psi_L(v)$ is a multiple of the sum of the elements of $a^L = a^H$. We deduce that $\psi_L(v)$ is H-invariant, that is, H fixes $\psi_L(v)$. Since a^H spans V, we obtain that H fixes every element of $\psi_L(V) = \mathbf{C}_V(L)$, that is, $\mathbf{C}_V(L) \subseteq \mathbf{C}_V(H)$.

Now we are ready to prove (a). Since $a^H = a^L$ and a^H spans V, the vector space V is generated by a as a kL-module. Thus, the map $\pi : kL \to V$, given by $\pi(\sum_{x\in L}\alpha_x x) = \sum_{x\in L}\alpha_x a^x$, defines a kL-homomorphism of kL onto V. Since p is coprime to $|L|$, by Maschke's theorem the kL-module V is isomorphic to a direct summand of the group-algebra kL. Therefore $\dim \mathbf{C}_V(H) = \dim \mathbf{C}_V(L) \leq \dim \mathbf{C}_{kL}(L) = 1$.

We now prove (b). Let $b \in V$ with $\gcd(|a^H|, |b^H|) = 1$. Write $B = \mathbf{C}_V(b)$ and observe that $H = AB$. Since a is fixed by a Sylow p-subgroup of H, we see that p does not divide $|H : A| = |B : (A\cap B)|$ and so $A\cap B$ contains a Sylow p-subgroup of B. As H is p-soluble, we get that B is p-soluble and that B contains a p-complement L, say. So, $B = (A\cap B)L$ and $H = AB = AL$. In particular, we are in the position to apply the first part of the proof to L. Thus $b \in \mathbf{C}_V(B) \subseteq \mathbf{C}_V(L) = \mathbf{C}_V(H)$. □

3. Proof of Theorem 1.1

In this section we use Theorem 1.2 to prove Theorem 1.1. We start by showing that the hypothesis "completely reducible" is essential.

Example 3.1. Let p be an odd prime, V be the 2-dimensional vector space of row vectors over a field $\mathbb{F}_p$ of size p, λ be a generator of the multiplicative group $\mathbb{F}_p \setminus \{0\}$ and

$$H = \langle g, h\rangle \text{ with } g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, h = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}.$$

The group H has order $p(p-1)$ and has $p+1$ orbits on V. Namely, for each $a \in \mathbb{F}_p \setminus \{0\}$, the set $\{(x, a) \mid x \in \mathbb{F}_p\}$ is an H-orbit of size p. Furthermore, $\{(0, 0)\}$ and $\{(a, 0) \mid a \in \mathbb{F}_p \setminus \{0\}\}$ are H-orbits of size 1 and $p-1$, respectively. Write $e_1 = (\lambda, 0)$ and $e_2 = (0, 1-\lambda)$. We have $\mathbf{C}_H(e_1) = \langle g\rangle$, $\mathbf{C}_H(e_2) = \langle h\rangle$ and $\mathbf{C}_H(e_1+e_2) = \langle gh\rangle \neq \mathbf{C}_H(e_1) \cap \mathbf{C}_H(e_2) = 1$.

Here is an example for the prime $p = 2$. Let $H = \mathrm{Sym}(4)$ be the symmetric group of degree 4 and M the permutation module with basis e_1, e_2, e_3, e_4 over a field k of size 2. It is easy to see that the only kH-submodules of M are 0, $M_1 = \langle e_1 + e_2 + e_3 + e_4\rangle$, $M_2 = \langle e_1 + e_2, e_1 + e_3, e_1 + e_4\rangle$ and M, and that $0 \subset M_1 \subset M_2 \subset M$, that is, M is uniserial. Let V be the kH-module M/M_1. Clearly, H acts faithfully on V and, as M_2/M_1 is the unique proper submodule of V, we have that V is not completely reducible. Write $a = e_1 + M_1$ and $b = e_1 + e_2 + M_1$. We have $\mathbf{C}_H(a) = \langle (2, 3), (3, 4)\rangle$ and $\mathbf{C}_H(b) = \langle (1, 2), (1, 3, 2, 4)\rangle$ and so a^H has size

4 and b^H has size 3. Finally, $\mathbf{C}_H(a+b) = \mathbf{C}_H(e_2+M_1) = \langle (1,3),(3,4)\rangle \neq \langle (3,4)\rangle = \mathbf{C}_H(a) \cap \mathbf{C}_H(b)$. Furthermore, the orbits of H on V have sizes $1, 3$ and 4.

We note that an example similar to Example 3.1 is in [16, Example 1].

Proof of Theorem 1.1. As V is completely reducible, we have $V = \mathbf{C}_V(H) \oplus W$ for some direct summand W of V. Clearly, replacing V by W if necessary, we may assume that $\mathbf{C}_V(H) = 0$, that is, H fixes no non-zero vector of V. There is also no loss in assuming that V is generated by a and b as a kH-module. Let $V(a)$ and $V(b)$ denote the kH-submodules generated by a and b, respectively. We claim that $V(a) \cap V(b) = 0$, whence $V = V(a) \oplus V(b)$ and $\mathbf{C}_H(a+b) = \mathbf{C}_H(a) \cap \mathbf{C}_H(b)$ as required.

Suppose not. Let S be a simple kH-submodule of $V(a)\cap V(b)$. Since $\mathbf{C}_H(V) = 0$, H does not act trivially on S. Since V is completely reducible, $V = S \oplus T$ as kH-modules. Let π denote the projection of V onto S with kernel T. Clearly, $|a^H|$ is a multiple of $|\pi(a)^H|$ and similarly for b. Since $S \leq V(a)$, $\pi(a) \neq 0$ (and similarly for b). Thus, the lengths of the H-orbits in S of $\pi(a)$ and $\pi(b)$ are coprime contradicting Theorem 1.2. □

We point out that from Theorem 1.1 we can easily deduce the following well-known result of Yuster (see [29] or [17, 3.34]).

Corollary 3.2. *Let H and A be finite groups with $|H|$ relatively prime to $|A|$ and with H acting as a group of automorphisms on A. If, for $a, b \in A$, the H-orbits a^H and b^H have relatively prime size, then H has an orbit of size $|a^H||b^H|$.*

Proof. As $|H|$ is relatively prime to $|A|$, from [14, Lemma 2.6.2] we see that we may assume that A is a direct product of elementary abelian groups. In particular, from Maschke's theorem, A is a completely reducible H-module, possibly of mixed characteristic. Now the result follows from Theorem 1.1. □

4. Proof of Theorem 1.5

The main ingredient in the proof of Theorem 1.5 is Theorem 1.2 and the positive solution of Fisman and Arad [9] of Szep's conjecture.

Theorem 4.1. [9] *Let $G = AB$ be a finite group such that A and B are both subgroups of G with non-trivial centres. Then G is not a non-abelian simple group.*

We start by considering some examples.

Example 4.2. Primitive groups of AS type. From [21], we see that the sporadic simple group $G = J_1$ has a primitive permutation representation of rank 5 on a set Δ of size 266. The subdegrees of G are $1, 11, 12, 110$ and 132. In particular, G has two coprime subdegrees. No primitive group of smaller rank has this property: the proof of this assertion requires the classification of the finite simple groups [5, Remark, p. 33].

Now we give an infinite family of examples. Let p be a prime with $p \equiv \pm 1 \mod 5$ and with $p \equiv \pm 1 \mod 16$, and let $G = \mathrm{PSL}(2,p)$. From [27, Chapter 3, Section 6], we see that G contains a maximal subgroup H with $H \cong \mathrm{Alt}(5)$. Consider G as a primitive permutation group acting on the set Δ of right cosets of H in G. Let K be a maximal subgroup of H with $K \cong \mathrm{Alt}(4)$. As 8 divides $|G|$, we see from [27, Chapter 3, Section 6] that $N_G(K) \cong \mathrm{Sym}(4)$. Let $g \in N_G(K) \setminus H$.

Then $K = H \cap H^g$, $|H : H \cap H^g| = 5$ and so G has a suborbit of size 5 on Δ. Similarly, let now K be a Sylow 5-subgroup of H. Using the generators of H given in [27, Chapter 3, Section 6], we see, with a direct computation, that there exists $g \in N_G(K) \setminus H$ with $K = H \cap H^g$. Therefore $|H : H \cap H^g| = 12$ and so G has a suborbit of size 12. Furthermore, another explicit computation with the generators of H shows that there exists $g \in G$ with $H \cap H^g = 1$. So G has a suborbit of size $60 = 5 \cdot 12$.

Example 4.3. Primitive groups of PA type. Let G be a primitive group of AS type on Δ with non-trivial coprime subdegrees a and b. Let δ, δ_1 and δ_2 be in Δ with $a = |\delta_1^{G_\delta}|$, $b = |\delta_2^{G_\delta}|$. For each $n \geq 2$, the wreath product $W = G \operatorname{wr} \operatorname{Sym}(n)$ endowed with its natural product action on $\Omega = \Delta^n$ is a primitive group of PA type. Consider the elements $\alpha = (\delta, \ldots, \delta)$, $\beta = (\delta_1, \ldots, \delta_1)$ and $\gamma = (\delta_2, \ldots, \delta_2)$ of Ω. We have $|\beta^{W_\alpha}| = a^n$ and $|\gamma^{W_\alpha}| = b^n$ and so a^n and b^n are two coprime subdegrees of W. In many cases there are several pairs of coprime non-trivial subdegrees of W. For example, if $n \geq 3$ and n is coprime to b, then the point $\beta' = (\delta_1, \delta, \ldots, \delta)$ lies in a W_α-orbit of size na and we have also na and b^n as coprime non-trivial subdegrees.

In particular, this construction can be applied with G and Δ as in Example 4.2.

Example 4.4. Primitive groups of TW type. In this example we construct a primitive group of TW type with two non-trivial coprime subdegrees. We start by recalling the structure and the action of a primitive group of twisted wreath type. We follow [7, Section 4.7]. Let T be a non-abelian simple group, H be a group, L be a subgroup of H and $\varphi : L \to \operatorname{Aut}(T)$ be a homomorphism with the image of φ containing the inner automorphisms of T. Let R be a set of left coset representatives of L in H and T^H be the set of all functions $f : H \to T$ from H to T. Clearly, T^H is a group under pointwise multiplication, and H acts as a group of automorphisms on T^H by setting $f^x(z) = f(xz)$, for $f \in T^H$ and for $x, z \in H$. Write $N = \{f \in T^H \mid f(zl) = f(z)^{\varphi(l)} \text{ for all } z \in H \text{ and } l \in L\}$. It is easy to verify that N is an H-invariant subgroup of T^H isomorphic to T^R. In fact, the restriction mapping $f \mapsto f|_R$ is an isomorphism of N onto T^R. The semidirect product $G = N \rtimes H$ is said to be the *twisted wreath product* determined by H and φ. The group G acts on $\Omega = N$ by setting $\alpha^{nh} = (\alpha n)^h$, for each $\alpha \in \Omega$, $n \in N$ and $h \in H$. (In particular, N acts on Ω by right multiplication and H acts on Ω by conjugation.) From [7, Lemma 4.7B], we see that if H is a primitive permutation group with point stabilizer L and if the image of φ is not a homomorphic image of H, then G acts primitively on Ω and the socle of G is N.

Write $H = \operatorname{PSL}(2,7)^2 \rtimes \langle \iota \rangle$ where ι is the involutory automorphism of $\operatorname{PSL}(2,7)^2 = \operatorname{PSL}(2,7) \times \operatorname{PSL}(2,7)$ defined by $(x,y)^\iota = (y,x)$. Write $L = \{(x,x) \mid x \in \operatorname{PSL}(2,7)\}\langle \iota \rangle$. The group H is a primitive group of SD type in its action on the right cosets of L. Let T be $\operatorname{PSL}(2,7)$ and let $\varphi : L \to \operatorname{Aut}(T)$ be the mapping sending $(x,x)\iota^i$ to the inner automorphism of T given by x, for each $x \in \operatorname{PSL}(2,7)$ and $i \in \{0,1\}$. Clearly, φ is a homomorphism whose image contains the inner automorphisms of T. Let G be the twisted wreath product determined by H and φ and let N be its socle as described above. Since H has no homomorphic image isomorphic to T, we see that G is a primitive permutation group on $\Omega = N$. The group H is the stabilizer G_f of the function $f : H \to T$ mapping every element of H to the identity of T.

Write

$$\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(regarded as an element of T). Let $C = (\langle\gamma\rangle \times \langle\gamma\rangle) \rtimes \langle\iota\rangle$ as a subgroup of H and define $g : H \to T$ by

$$g(z) = \begin{cases} \gamma^{\varphi(l)} & \text{if } z = cl, \text{ for some } c \in C \text{ and } l \in L, \\ 1 & \text{if } z \in H \setminus CL. \end{cases}$$

We claim that the function g is well-defined. In fact, if $z = c_1 l_1 = c_2 l_2$ for $c_1, c_2 \in C$ and $l_1, l_2 \in L$, then $l_2 l_1^{-1} \in C \cap L = \langle(\gamma,\gamma)\rangle\langle\iota\rangle$. Hence $l_2 = ul_1$ with $u = (\gamma^k, \gamma^k)\iota^i$ for some $k \in \{0, \ldots, 6\}$ and $i \in \{0, 1\}$. In particular, since γ^k centralizes γ, we obtain $\gamma^{\varphi(l_1)} = \gamma^{\varphi(u)\varphi(l_1)} = \gamma^{\varphi(ul_1)} = \gamma^{\varphi(l_2)}$ and hence the image $\gamma^{\varphi(l)}$ is independent of the representation $z = c_i l_i$ of z.

Fix z in H. Distinguishing the cases $z \in CL$ and $z \notin CL$, it is easy to verify that $g(zl) = g(z)^{\varphi(l)}$ for each $l \in L$ and $z \in H$, and hence $g \in \Omega$. For each $c \in C$ and $z \in H$, we have $g^c(z) = g(cz) = g(z)$, and hence $C \subseteq \mathbf{C}_H(g)$. We claim that $C = \mathbf{C}_H(g)$. Let $h = (h_1, h_2)\iota^j$ be in $\mathbf{C}_H(g)$. Suppose that $h \notin CL$. As $g(1) = \gamma \neq 1$ and $g^h(1) = g(h) = 1$, we obtain $g \neq g^h$. Thus $h \in CL$ and $\mathbf{C}_H(g) \subseteq CL$. As $C \subseteq \mathbf{C}_H(g)$, replacing h by ch for a suitable element $c \in C$, we may assume that $h \in L$, that is, $h = (x, x)\iota^i$ for some $x \in T$ and $i \in \{0, 1\}$. Now $\gamma = g(1) = g^h(1) = g(h) = \gamma^{\varphi(x)}$. Hence $x \in \mathbf{C}_T(\gamma) = \langle\gamma\rangle$, $h \in C \cap L$ and our claim is proved. Thus the H-orbit containing g has size $|H : C| = 24^2 = 576$, and 576 is a subdegree of G.

Write

$$a = \begin{bmatrix} 0 & 4 \\ 5 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$$

(again thought of as elements of T). The element a has order 4, the element b has order 3, and $\langle a, b\rangle \cong \mathrm{Sym}(4)$. Let $D = (\langle a, b\rangle \times \langle a, b\rangle) \rtimes \langle\iota\rangle$ and let $t = (\gamma, 1)$. A direct computation shows that $D^t \cap L$ is a dihedral group of size 8, namely

$$\langle (a^2, a^2), (r, r)\iota\rangle \quad \text{with } r = \begin{bmatrix} 3 & 5 \\ 4 & 0 \end{bmatrix}$$

and with centre

$$\langle(\eta, \eta)\rangle \quad \text{where } \eta = r^2 = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}.$$

Define

$$h(z) = \begin{cases} \eta^{\varphi(l)} & \text{if } z = dtl, \text{ for some } d \in D \text{ and } l \in L, \\ 1 & \text{if } z \notin DtL. \end{cases}$$

We claim that the function h is well-defined. In fact, if $z = d_1 t l_1 = d_2 t l_2$ for $d_1, d_2 \in D$ and $l_1, l_2 \in L$, then $l_2 l_1^{-1} = t^{-1} d_2^{-1} d_1 t \in D^t \cap L$. Hence $l_2 = ul_1$ with $u \in D^t \cap L$. In particular, since u centralizes (η, η), we obtain $\eta^{\varphi(l_1)} = \eta^{\varphi(u)\varphi(l_1)} = \eta^{\varphi(ul_1)} = \eta^{\varphi(l_2)}$ and so the image $\eta^{\varphi(l)}$ is independent of the representation $z = d_i t l_i$ of z.

Fix z in H. As above, by distinguishing the cases $z \in DtL$ and $z \notin DtL$, it is easy to verify that $h(zl) = h(z)^{\varphi(l)}$ for each $l \in L$, and hence $h \in \Omega$. From the definition of h, we see that for each $d \in D$ and $z \in H$, we have $h^d(z) = h(dz) = h(z)$. Hence $D \subseteq \mathbf{C}_H(h)$. Since D is a maximal subgroup of H, we obtain $D = \mathbf{C}_H(h)$. Thus the H-orbit containing h has size $|H : D| = 7^2 = 49$, and 49 is a subdegree of G.

This shows that G has two coprime subdegrees, namely 576 and 49. Now, using the computational algebra system GAP [10], it is easy to show that $\mathbf{C}_H(fg) = \mathbf{C}_H(f) \cap \mathbf{C}_H(g)$. In particular, the suborbit of G containing fg has size $576 \cdot 49$.

Proof of Theorem 1.5. From Examples 4.2, 4.3 and 4.4 and Theorem 1.2, it suffices to show that if G is a primitive group of type HS, HC, SD or CD, then G has no non-trivial coprime subdegrees. We argue by contradiction and we assume that G is a primitive group on Ω of HS, HC, SD or CD type with two non-trivial coprime subdegrees.

We first consider the case that G is of HS or HC type. Let $N = T_1 \times \cdots \times T_\ell$ be the socle of G, with $T_i \cong T$ for some non-abelian simple group T, for each $i \in \{1, \ldots, \ell\}$. From the description of the O'Nan-Scott types in [26], we see that ℓ is even (with $\ell = 2$ if G is of HS type and with $\ell > 2$ if G is of HC type). Furthermore, relabelling the indices $\{1, \ldots, \ell\}$ if necessary, $M_1 = T_1 \times \cdots \times T_{\ell/2} \cong T^{\ell/2}$ and $M_2 = T_{\ell/2+1} \times \cdots \times T_\ell = T^{\ell/2}$ are minimal normal regular subgroups of G, and Ω can be identified with $T^{\ell/2}$. Namely, the action of N on Ω is permutation isomorphic to the action of $T^{\ell/2} \times T^{\ell/2}$ on $T^{\ell/2}$ given by

$$z^{(a,b)} = a^{-1}zb = (a_1^{-1}z_1b_1, \ldots, a_{\ell/2}^{-1}z_{\ell/2}b_{\ell/2}),$$

for $a = (a_1, \ldots, a_{\ell/2}), b = (b_1, \ldots, b_{\ell/2}), z = (z_1, \ldots, z_{\ell/2}) \in T^{\ell/2}$. Under this identification, the stabilizer in $T^{\ell/2} \times T^{\ell/2}$ of the element $(1, \ldots, 1)$ of $T^{\ell/2}$ is the set $\{(a, a) \mid a \in T^{\ell/2}\}$ acting on $T^{\ell/2}$ by conjugation, that is, $z^{(a,a)} = a^{-1}za$.

Let ω_1, ω_2 be elements of $\Omega \setminus \{\omega\}$ with $m = |\omega_1^{G_\omega}|$ coprime to $n = |\omega_2^{G_\omega}|$. Since N is normal in G, we have that $m' = |\omega_1^{N_\omega}| = |N_\omega : N_{\omega,\omega_1}|$ divides m and $n' = |\omega_2^{N_\omega}| = |N_\omega : N_{\omega,\omega_2}|$ divides n. In particular, m' and n' are coprime. Identifying Ω with $T^{\ell/2}$ (as above), ω with $(1, \ldots, 1)$, N_ω with $T^{\ell/2}$ (as above), ω_1 with $(x_1, \ldots, x_{\ell/2})$ and ω_2 with $(y_1, \ldots, y_{\ell/2})$, we get

$$\begin{aligned} N_{\omega,\omega_1} &\cong \mathbf{C}_{T^{\ell/2}}((x_1, \ldots, x_{\ell/2})) = \mathbf{C}_T(x_1) \times \cdots \times \mathbf{C}_T(x_{\ell/2}) \\ N_{\omega,\omega_2} &\cong \mathbf{C}_{T^{\ell/2}}((y_1, \ldots, y_{\ell/2})) = \mathbf{C}_T(y_1) \times \cdots \times \mathbf{C}_T(y_{\ell/2}). \end{aligned}$$

In particular, $m' = \prod_{i=1}^{\ell/2} |T : \mathbf{C}_T(x_i)|$ and $n' = \prod_{i=1}^{\ell/2} |T : \mathbf{C}_T(y_i)|$. As m' is coprime to n', for each $i, j \in \{1, \ldots, \ell/2\}$, the indices $|T : \mathbf{C}_T(x_i)|$ and $|T : \mathbf{C}_T(y_j)|$ are coprime and hence $T = \mathbf{C}_T(x_i)\mathbf{C}_T(y_j)$. As $\omega_1, \omega_2 \neq 1$, there exist $i, j \in \{1, \ldots, \ell/2\}$ with $x_i \neq 1$ and $y_j \neq 1$. So $T = \mathbf{C}_T(x_i)\mathbf{C}_T(y_j)$ is a coprime factorization and Theorem 4.1 yields that T is not a non-abelian simple group, a contradiction.

It remains to consider the case that G is of SD or CD type. From the description of the O'Nan-Scott types in [26], we may write the socle N of G as

$$N = (T_{1,1} \times \cdots \times T_{1,r}) \times (T_{2,1} \times \cdots \times T_{2,r}) \times \cdots \times (T_{s,1} \times \cdots \times T_{s,r}),$$

with $T_{i,j} \cong T$ for some non-abelian simple group T, for each $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$: where $r \geq 2$ and $s \geq 1$ (with $s = 1$ if G is of SD type and with $s \geq 2$ if G is of CD type). The set Ω can be identified with $(T_{1,1} \times \cdots \times T_{1,r-1}) \times (T_{2,1} \times \cdots \times T_{2,r-1}) \times \cdots \times (T_{s,1} \times \cdots \times T_{s,r-1}) \cong T^{s(r-1)}$ and, for the point $\omega \in \Omega$ identified with $(1, \ldots, 1)$, the stabilizer N_ω is $D_1 \times \cdots \times D_s \cong T^s$ where D_i is the diagonal subgroup $\{(t, \ldots, t) \mid t \in T\}$ of $T_{i,1} \times \cdots \times T_{i,r}$. That is to say, the action of N_ω on Ω is permutation isomorphic to the action of

T^s on $T^{s(r-1)}$ by "diagonal" componentwise conjugation: the image of the point $(x_{1,1},\dots,x_{1,r-1},\dots,x_{s,1},\dots,x_{s,r-1})$ under the permutation $(t_1,\dots,t_s)$ is

$$(x_{1,1}^{t_1},\dots,x_{1,r-1}^{t_1},x_{2,1}^{t_2},\dots,x_{2,r-1}^{t_2},\dots,x_{s,1}^{t_s},\dots,x_{s,r-1}^{t_s}).$$

Let ω_1,ω_2 be elements of $\Omega\setminus\{\omega\}$ with $m=|\omega_1^{G_\omega}|$ coprime to $n=|\omega_2^{G_\omega}|$. Since N is normal in G, we have that $m'=|\omega_1^{N_\omega}|=|N_\omega:N_{\omega,\omega_1}|$ divides m and $n'=|\omega_2^{N_\omega}|=|N_\omega:N_{\omega,\omega_2}|$ divides n. In particular, m' and n' are coprime. Identifying ω_1 with $(x_{1,1},\dots,x_{1,r-1},\dots,x_{s,1},\dots,x_{s,r-1})$ and ω_2 with $(y_{1,1},\dots,y_{1,r-1},\dots,y_{s,1},\dots,y_{s,r-1})$, we get

$$\begin{aligned} N_{\omega,\omega_1} &\cong \mathbf{C}_T(\langle x_{1,1},\dots,x_{1,r-1}\rangle)\times\cdots\times\mathbf{C}_T(\langle x_{s,1},\dots,x_{s,r-1}\rangle)\\ N_{\omega,\omega_2} &\cong \mathbf{C}_T(\langle y_{1,1},\dots,y_{1,r-1}\rangle)\times\cdots\times\mathbf{C}_T(\langle y_{s,1},\dots,y_{s,r-1}\rangle). \end{aligned}$$

In particular, $m'=\prod_{i=1}^s|T:\mathbf{C}_T(\langle x_{i,1},\dots,x_{i,r-1}\rangle)|$ and $n'=\prod_{i=1}^s|T:\mathbf{C}_T(\langle y_{i,1},\dots,y_{i,r-1}\rangle)|$.

As $\omega_1,\omega_2\neq\omega$, there exist $i_1,i_2\in\{1,\dots,r-1\}$ and $j_1,j_2\in\{1,\dots,s\}$ with $x_{i_1,j_1}\neq 1$ and $y_{i_2,j_2}\neq 1$. Now, since $\mathbf{C}_T(\langle x_{i_1,1},\dots,x_{i_1,r-1}\rangle)\subseteq\mathbf{C}_T(x_{i_1,j_1})\subsetneq T$ and $\mathbf{C}_T(\langle x_{i_2,1},\dots,x_{i_2,r-1}\rangle)\subseteq\mathbf{C}_T(x_{i_2,j_2})\subsetneq T$ and since m' is relatively prime to n', we obtain $T=\mathbf{C}_T(x_{i_1,j_1})\mathbf{C}_T(x_{i_2,j_2})$. From Theorem 4.1, T is a not non-abelian simple group, a contradiction.

□

5. Proofs of Theorems 1.6 and 1.10

Before proving Theorem 1.6 and 1.10 we need the following definition and lemmas.

Definition 5.1. If G is a finite group, we let $\mu(G)$ denote the maximal size of a set $\{G_i\}_{i\in I}$ of proper subgroups of G with $|G:G_i|$ and $|G:G_j|$ relatively prime, for each two distinct elements i and j of I.

Lemma 5.2. *If K is a direct product of isomorphic nonabelian simple groups, then $\mu(K)\leq 2$.*

Proof. We have $K=T_1\times\cdots\times T_\ell$ with $T_i\cong T$, for some nonabelian simple group T and for some $\ell\geq 1$. We argue by contradiction and we assume that $\mu(K)\geq 3$, that is, K has three proper subgroups A_1, A_2 and A_3 with $|K:A_1|$, $|K:A_2|$ and $|K:A_3|$ relatively prime. Write $a_i=|K:A_i|$ for $i\in\{1,2,3\}$. Replacing A_i with a maximal subgroup of K containing A_i if necessary, we may assume that A_i is maximal in K, for $i\in\{1,2,3\}$. In particular, (up to relabelling the simple direct summands of K) we have that either $A_1=M_1\times T_2\times\cdots\times T_\ell$ (for some maximal subgroup M_1 of T_1) or $A_1=\{(t_1,t_1^\eta)\mid t_1\in T_1\}\times T_3\times\cdots\times T_\ell$ (for some isomorphism $\eta:T_1\to T_2$). In the latter case we have that $a_1=|T|$ is not relatively prime to a_2 and to a_3, a contradiction. Therefore, up to relabeling the indices, we may assume that $A_i=M_i\times T_2\times\cdots\times T_\ell$ with M_i a maximal subgroup of T_1, for $i\in\{1,2,3\}$. This gives that the nonabelian simple group T_1 admits three coprime factorizations $T_1=M_1M_2=M_1M_3=M_2M_3$ with $|T_1:M_1|,|T_1:M_2|$ and $|T_1:M_3|$ relatively prime. A simple inspection in Table 1 shows that this is impossible. The same conclusion can be obtained using [2], where the authors determine all multiple factorizations of finite nonabelian simple groups $T=M_iM_j$, for i and j distinct elements of $\{1,2,3\}$. In particular, it is readily checked that in none of the multiple factorizations in [2] are the indices $|T : M_1|$, $|T : M_2|$ and $|T : M_3|$ pairwise coprime. □

Lemma 5.3. *Let G be a transitive permutation group on Ω and let ω be in Ω. Suppose that N is normal in G_ω and N fixes a unique point on Ω. Then the number of coprime subdegrees of G is at most $\mu(N)$.*

Proof. Let $O_1, \dots, O_r$ be orbits of G_ω on $\Omega \setminus \{\omega\}$ of pairwise coprime sizes and let $\omega_i \in O_i$, for $i \in \{1, \dots, r\}$. Now the orbits of N on O_i have all the same size, m_i say, and m_i divides $|O_i|$. Since N fixes only the point ω of Ω, we have that $m_i > 1$. Therefore $\{N_{\omega_i}\}_{i\in\{1,\dots,r\}}$ is a set of proper subgroups of N with pairwise coprime indices. Thus $r \leq \mu(N)$. □

Proof of Theorem 1.6. Let G be a primitive group of TW type. We use the notation introduced in the first paragraph of Example 4.4: so $G = N \rtimes H$ is the twisted wreath product determined by H and $\varphi : L \to \mathrm{Aut}(T)$. Recall that G acts primitively on N, with N acting on itself by right multiplication and with H acting on N by conjugation. In particular, H is the stabilizer of the point $1 \in N$. Let K be a minimal normal subgroup of H. From [7, Theorem 4.7B (*i*)], K is a direct product of nonabelian simple groups. Write $\ell = |H : L|$. Hence $N = T_1 \times \cdots \times T_\ell$ with $T_i \cong T$, for each $i \in \{1, \dots, \ell\}$. Furthermore, $L = N_H(T_i)$ for some $i \in \{1, \dots, \ell\}$. Relabeling the T_j if necessary, we may assume that $i = 1$. From [7, Theorem 4.7B (*ii*)], the group L is a core-free subgroup of H and hence $K \nsubseteq L$.

We claim that $K \cap L$ acting by conjugation on the simple group T_1 induces all the inner automorphisms. If not, then $K \cap L \subseteq \mathbf{C}_H(T_1)$ because $K \cap L$ is a normal subgroup of L and T_1 is nonabelian simple. Thus the homomorphism $\varphi : L \to \mathrm{Aut}(T)$ can be extended to a homomorphism $\hat{\varphi} : KL \to \mathrm{Aut}(T)$ of the group KL by setting $\hat{\varphi}(kl) = \varphi(l)$, for each $l \in L$ and $k \in K$. As $L \subsetneq KL$, this contradicts the maximality condition of H in G given in [1, Lemma 3.1 (*b*)], and the claim is proved. In particular, since K is a normal subgroup of H and since H acts transitively on the ℓ simple direct summands $\{T_1, \dots, T_\ell\}$, we obtain that $K \cap N_H(T_i)$ induces by conjugation all the inner automorphisms of T_i, for each $i \in \{1, \dots, \ell\}$. This gives $\mathbf{C}_N(K) = 1$ and so K fixes a unique point of N. Now the proof follows from Lemmas 5.2 and 5.3. □

Before concluding this section we show that coprime subdegrees in primitive groups G of AS or PA type determine coprime subdegrees in transitive non-abelian simple groups T, and we give a strong relation between G and T. Let G be a primitive group of AS or PA type. We recall that from the description of the O'Nan-Scott types in [26] the group G is a subgroup of the wreath product $H \,\mathrm{wr}\, \mathrm{Sym}(\ell)$ endowed with its natural product action on Δ^ℓ, with H an almost simple primitive group on Δ (we have $\ell = 1$ and $G = H$ if G is of AS type, and $\ell > 1$ if G is of PA type). Furthermore, if T is the socle of H, then $N = T_1 \times \cdots \times T_\ell$ is the socle of G, where $T_i \cong T$ for each $i \in \{1, \dots, \ell\}$. Write $G_i = N_G(T_i)$ and denote by $\pi_i : G_i \to H$ the natural projection. From [26], we see that replacing H by $\pi_i(G_i)$ if necessary, we may assume that $\pi_i(G_i)$ is surjective. In this case, we say that H is the *component subgroup* of G. In particular, if G is of AS type, the component subgroup of G is G itself.

(We recall that the definition of pseudo-maximal subgroup is in Definition 1.8.)

Proposition 5.4. *Let G be a primitive permutation group of AS or PA type acting on Δ^ℓ with component subgroup $H \subseteq \mathrm{Sym}(\Delta)$ and let T be the socle of H. For $\delta \in \Delta$, the stabilizer T_δ is a pseudo-maximal subgroup of T. Furthermore, if G has k non-trivial coprime subdegrees, then T acting on Δ has at least k non-trivial coprime subdegrees.*

Proof. As H is a primitive group of AS type on Δ, we have that H_δ is a maximal subgroup of the almost simple group H with $T \nsubseteq H_\delta$, for each $\delta \in \Delta$. Therefore $T_\delta = T \cap H_\delta$ is a pseudo-maximal subgroup of T and, in particular, $N_H(T_\delta) = H_\delta$. Let Λ be the set of fixed points of T_δ on Δ and let $\delta_1, \delta_2 \in \Lambda$. By transitivity, there exists $h \in H$ with $\delta_1^h = \delta_2$, that is, $T_\delta = T_{\delta_2} = T_{\delta_1^h} = T_{\delta_1}^h = T_\delta^h$. Therefore $h \in N_H(T_\delta) = H_\delta$ and Λ is the H_δ-orbit containing δ, that is, $\Lambda = \{\delta\}$ and T_δ fixes a unique point of Δ.

Let $\delta \in \Delta$ and write $\alpha = (\delta, \ldots, \delta) \in \Delta^\ell$. Let $N = T_1 \times \cdots \times T_\ell$ be the socle of G. Clearly, $G_\alpha \subseteq H_\delta \operatorname{wr} \mathrm{Sym}(\ell)$ and, as G_α is a maximal subgroup of G and as $N \subseteq G$, we obtain $N_\alpha = (T_1)_\delta \times \cdots \times (T_\ell)_\delta$.

Assume that G has k non-trivial coprime subdegrees $n_1, \ldots, n_k$. Now, there exist $\beta_i = (\delta_{i,1}, \ldots, \delta_{i,\ell})$ with $n_i = |\beta_i^{G_\alpha}|$, for $i \in \{1, \ldots, k\}$. Since N is a normal subgroup of G, we obtain that $n_i' = |\beta_i^{N_\alpha}|$ divides n_i, and so $n_1', \ldots, n_k'$ are pairwise coprime. Furthermore

$$\beta_i^{N_\alpha} = (\delta_{i,1}, \ldots, \delta_{i,\ell})^{(T_1)_\delta \times \cdots \times (T_\ell)_\delta} = \delta_{i,1}^{(T_1)_\delta} \times \cdots \times \delta_{i,\ell}^{(T_\ell)_\delta}$$

for each $i \in \{1, \ldots, k\}$, and so $n_i' = \prod_{j=1}^{\ell} |\delta_{i,j}^{T_\delta}|$.

As n_i' is coprime with n_j', for each distinct i and j in $\{1, \ldots, k\}$, the subdegrees $|\delta_{i,x}^{T_\delta}|$ and $|\delta_{j,y}'^{T_\delta}|$ of T acting on Δ are coprime, for each $x, y \in \{1, \ldots, \ell\}$. Since for each $i \in \{1, \ldots, k\}$ we have $\beta_i \neq \alpha$, there exists $j_i \in \{1, \ldots, \ell\}$ with $\delta_{i,j_i} \neq \delta$. Since T_δ fixes only the element δ of Δ, we have $|\delta_{i,j_i}^{T_\delta}| > 1$. Thus $|\delta_{1,j_1}^{T_\delta}|, \ldots, |\delta_{k,j_k}^{T_\delta}|$ are k non-trivial coprime subdegrees of T acting on Δ. □

Proof of Theorem 1.10. Assume that Theorem 1.9 holds true. Let G be a primitive permutation group on Ω with three non-trivial coprime subdegrees. From Theorems 1.5 and 1.6, G is of AS or PA type. Since Theorem 1.9 holds true, Proposition 5.4 yields a contradiction. □

6. Proofs of Theorems 1.14 and 1.15

As usual, we denote by $\mathbf{F}(G)$ the *Fitting subgroup* of the finite group G, that is, the largest normal nilpotent subgroup of G. The proof of Corollary 1.14 and 1.15 will follow from Lemma 6.1 and from the results in Section 5.

Lemma 6.1. *Let H be a finite permutation group. Let $O_1, \ldots, O_t$ be H-orbits having pairwise coprime size, with $|O_i| > 1$ and with H faithful on O_i for each $i \in \{1, \ldots, t\}$. Then $t \leq 2$. Moreover, if $\mathbf{F}(H) \neq 1$, then $t = 1$*

Proof. For each $i \in \{1, \ldots, t\}$, let ω_i be an element of O_i and set $H_i = H_{\omega_i}$. By hypothesis, H_i is a proper core-free subgroup of H. Let N be a minimal normal subgroup of H and set $N_i = H_i \cap N$. As H_i is core-free in H, we have $N_i \subsetneq N$. Note that $|H_i N : H_i| = |N : (H_i \cap N)| = |N : N_i|$ and hence $|H : H_i|$ is a multiple of $|N : N_i|$. This shows that $\{N_i\}_{i=1,\ldots,t}$ is a family of proper subgroups of N with $|N : N_i|$ relatively prime to $|N : N_j|$, for each two distinct elements i and j in

$\{1,\dots,t\}$. Hence $t \leq \mu(N)$. If N is a p-group (for some prime p) then $\mu(N) = 1$ and if N is non-abelian then $\mu(N) \leq 2$ by Lemma 5.2. □

Proof of Theorem 1.14. Let G be a non-regular finite transitive permutation group on Ω and let α be in Ω. Set $H = G_\alpha$. The result now follows from Lemma 6.1. □

Proof of Theorem 1.15. If $t = 1$, then there is nothing to prove. So we may assume that $t \geq 2$. We claim that K/k is a separable extension (and so, as K/k is normal, a Galois extension). This is clear if k has characteristic 0. Suppose then that k has characteristic $p > 0$. Now, if for some $i \in \{1,\dots,t\}$, the extension k_i/k is separable, then K/k is separable (being the normal closure of a separable extension). Therefore we may assume that k_i/k is inseparable, for each $i \in \{1,\dots,t\}$. This gives that p divides $[k_i : k]$, for each $i \in \{1,\dots,t\}$. As $t \geq 2$, we obtain a contradiction and the claim is proved.

Let H be the Galois group $\mathrm{Gal}(K/k)$ and set $H_i = \mathrm{Gal}(K/k_i)$, for $i \in \{1,\dots,t\}$. Since the normal closure of k_i is K, we obtain that H_i is core-free in H. Therefore $H_1,\dots,H_t$ is a family of core-free subgroups of H of pairwise coprime index. Now apply Lemma 6.1 to obtain $t \leq 2$. □

7. Coprime factorizations of non-abelian simple groups

Liebeck, Praeger and Saxl [19, 20] completely classified the maximal factorizations of all finite almost simple groups. Tables 1–6 and Theorem D of [19] determine all the triples (G, A, B) where G is a nonabelian simple group, and A and B are maximal subgroups of G with $G = AB$.

Now, if A' and B' are subgroups of G with $|G : A'|$ relatively prime to $|G : B'|$, then $G = A'B'$. In particular, A' and B' give rise to a coprime factorization of G. Moreover, if A (respectively B) is a maximal subgroup of G with $A' \subseteq A$ (respectively $B' \subseteq B$), then $G = AB$ is a maximal coprime factorization. Therefore, the list of all non-abelian simple groups admitting a *coprime factorization* can be easily obtained with some elementary arithmetic from [19]. Namely, for each triple (G, A, B) in Tables 1–6 and in Theorem D of [19], it suffices to check whether $|G : A|$ is relatively prime to $|G : B|$. Table 1 in this paper gives all possible maximal coprime factorizations (G, A, B) and, in particular, the list of all non-abelian simple groups admitting a coprime factorization. The notation we use is standard and follows the notation in [19, Section 1.2].

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G	A	B	Remark
$\mathrm{Alt}(p^\ell)$	$\mathrm{Alt}(p^\ell - 1)$	max. imprimitive	p prime, $\ell \geq 2$, $i \in \{1, \ldots, \ell - 1\}$
$\mathrm{Alt}(p)$	$\mathrm{Alt}(p-1)$	max. primitive	p prime
$\mathrm{Alt}(8)$	$\mathrm{Alt}(8)_x$	AGL(3,2)	$x \subseteq \{1, \ldots, 8\}, \lvert x\rvert \in \{1, 2, 3\}$
M_{11}	$L_2(11)$	M_{10}	
M_{11}	$L_2(11)$	$M_9.2$	
M_{23}	$23 : 11$	M_{22}	
M_{23}	$23 : 11$	$M_{21}.2$	
M_{23}	$23 : 11$	$2^4.\,\mathrm{Alt}(7)$	
M_{24}	M_{23}	$2^6.3.\,\mathrm{Sym}(6)$	
$L_4(q)$	$\mathrm{PSp}(4, q)$	P_i	$i \in \{1, 3\}$, q odd, $q \not\equiv 1 \mod 8$
$L_n(q)$	$\mathrm{PSp}(n, q)$	P_i	$i \in \{1, n-1\}$, $n = 2^r$, $r \geq 2$, q even
$L_n(q)$	$\widehat{\mathrm{GL}}_{b^{r-1}}(q^b).b$	P_i	$i \in \{1, n-1\}$, $n = b^r$, $r \geq 1$, b prime,
			$r = 1$ if $b = 2$ and $q \equiv 3 \mod 4$,
			$b > 2$ if $q \equiv 1 \mod 4$
$L_2(q)$	P_1	$\mathrm{Sym}(4)$	$q \in \{7, 23\}$
$L_2(q)$	P_1	$\mathrm{Alt}(5)$	$q \in \{11, 19, 29, 59\}$
$L_5(2)$	P_i	$31 : 5$	$i \in \{2, 3\}$
$U_{2^r}(2^k)$	N_1	$P_{2^{r-1}}$	$r \geq 2$, $k \geq 1$
$U_4(2)$	$3^3.\,\mathrm{Sym}(4)$	P_2	
$\mathrm{PSp}_{2m}(q)$	$O^-_{2m}(q)$	P_m	m odd and q even
$\mathrm{PSp}_4(3)$	$2^4.\mathrm{Alt}(5)$	P_i	$i \in \{1, 2\}$
$\Omega_{2m+1}(q)$	N_1^-	P_m	q odd, $m \geq 3$ odd
$\Omega^+_{2m}(q)$	N_1	P_i	$i \in \{m-1, m\}$, $m \geq 5$ odd,
			q even or $q \equiv 3 \mod 4$
$\Omega_7(3)$	$\mathrm{Sp}_6(2)$	P_3	
$\Omega_7(3)$	$2^6.\,\mathrm{Alt}(7)$	P_3	
$\mathrm{P}\Omega^+_8(3)$	$\Omega^+_8(2)$	P_i	$i \in \{1, 3, 4\}$

Table 1. Maximal coprime factorizations of a finite non-abelian simple group G

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