A PROOF OF A LOOMIS-WHITNEY TYPE INEQUALITY VIA OPTIMAL TRANSPORT

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ABSTRACT. The paper is devoted to exhibiting a proof of an analytical extension of the well-known Loomis-Whitney inequality. Such a proof is completely independent of the original one and it is based on the technique of optimal transport, which leads also to fully characterize the equality case.

1. Introduction

The Loomis-Whitney inequality is one of the most natural and powerful inequalities of geometric type. It gives a sharp upper estimate of the measure of a Borel set A in $\mathbb{R}^n$, $n \geq 2$, in terms of the $(n-1)$-contents of the orthogonal projections A^k of A on the coordinate hyperplanes $e_k^{\perp}$, being e_k, $k = 1, \dots, n$, the standard orthonormal basis for $\mathbb{R}^n$. Precisely,

$$\mathcal{H}_n(A)^{n-1} \leq \prod_{k=1}^{n} \mathcal{H}_{n-1}(A^k)\,, \tag{1}$$

where $\mathcal{H}_r$ denotes the r-dimensional Hausdorff measure. Clearly, in (1) equality holds when A is a coordinate box.

The original proof of (1) by Loomis and Whitney [18] goes back to 1949 and it is based on a discrete approach. Over the years numerous authors dealt with this inequality and gave suitable extensions, generalizations and variants. Moreover, the LW-inequality was used as a tool to apply in different contexts. The papers [6] and [7] contain a sufficiently comprehensive list of references of such a broad presence in literature.

The present paper is devoted to give a proof of the following analytical extension of (1):

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Theorem 1.1. *Let $f(x)$ be a bounded nonnegative measurable function with compact support in $\mathbb{R}^n$, $n \geq 2$, and define in $\mathbb{R}^{n-1}$, for $k = 1, \dots, n$, the functions*

$$f_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) = \sup_{x_k} f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n).$$

Then

$$\left(\int_{\mathbb{R}^n} f^{n/(n-1)}(x)\, dx \right)^{n-1} \leq \prod_{k=1}^{n} \int_{\mathbb{R}^{n-1}} f_k(x)\, dx \,. \tag{2}$$

Equality in (2) holds if and only if $f = \alpha 1_A$, where A is a Cartesian product of bounded measurable subsets of $\mathbb{R}$ and α is a nonnegative constant.

Note that if $f = 1_A$, then (2) gives the original LW-inequality (1).

Inequality (2) was proved by Bobkov and Nazarov in [2] (Proof of Lemma 3.1) by making a direct use of LW-inequality (1).

Instead, we give here a proof of (2), based on the technique of optimal transport, which does not depend of (1). Consequently in such a way we give also a new original proof of (1).

In addition, the technique we use permits to give a characterization of functions satisfying equality in (2), that was not given in [2]. These functions are multiple of the characteristic functions of the sets giving equality in (1), as expressed by Corollary 2 in [11].

In recent years the use of techniques based on optimal transport allowed to find simple and elegant proofs of important geometric and functional inequalities, like those of isoperimetric type. In this regard, see, for instance, [20], [10], [9], [12], [13], [8] and, for a general presentation of the subject, [23, Ch. 21]. At the same time the present paper enters in a current research area devoted to obtain functional inequalities which are extensions of classical geometric inequalities. For instance, in the recent paper [1], analytic versions of some local Loomis-Whitney inequalities are obtained. In this setting, classical inequalities as the uniform cover inequality by Bollobás and Thomason [3] are included. See also [4].

It is interesting to remark that, in spite of appearances, (2) implies an inequality of the same type as the one proved by Gagliardo [16] and Nirenberg [21]. Such an inequality, for every function $f(x) \in C^1(\mathbb{R}^n)$ with compact support, can be written in the following form:

$$\|f\|_{\frac{n}{n-1}} \leq \frac{1}{2} \prod_{k=1}^{n} \Big(\int_{\mathbb{R}^n} |\langle e_k, \nabla f(x) \rangle| \; dx \Big)^{1/n}, \tag{3}$$

where $\langle\cdot,\cdot\rangle$ stands for the scalar product and ∇ for the gradient. For every k, it turns out that the term $\int_{\mathbb{R}^n} |\langle e_k, \nabla f(x)\rangle|\, dx$ is the measure of the projection of the subgraph of f onto $e_k^\perp$ where every point of the projection must be counted according to its multiplicity. Thus, for every k, $\int\limits_{\mathbb{R}^{n-1}} f_k(x)\, dx \le \int_{\mathbb{R}^n} |\langle e_k, \nabla f(x)\rangle|\, dx$, so (2) implies (3). For more details see [24], where an extension to general directions of the Gagliardo-Nirenberg inequality is proved (Theorem 5.1) by the use of the Loomis-Whitney inequality.

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2. Proof of Theorem 1.1

Let us denote by Ω the support of $f(x)$ and assume, without loss of generality, that

$$\|f\|_{\frac{n}{n-1}} = 1. \tag{4}$$

Consider the probability measure μ in $\mathbb{R}^n$ with density $[f(x)]^{\frac{n}{n-1}}$ and support Ω and the uniform probability measure ν with density ℓ and support $R = [-\ell_1, \ell_1] \times \cdots \times [-\ell_n, \ell_n]$, where $\ell_1, \cdots, \ell_n$ must satisfy the condition

$$2^n \ell \prod_{k=1}^{n} \ell_k = 1. \tag{5}$$

It is well known (see, for instance, [5], [19], [20]) that there exists a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla\varphi : \mathbb{R}^n \to R$ transports μ onto ν and solves the Monge-Kantorovich minimization problem with quadratic cost between μ and ν. The monotone map $T = \nabla\varphi$ is referred to as the Brenier map.

The function φ satisfies the equation

$$\ell \det(D^2\varphi) = f^{\frac{n}{n-1}}, \text{ a.e. in } \Omega, \tag{6}$$

or equivalently

$$f^{-\frac{1}{n-1}} (\det(D^2\varphi))^{1/n} = \frac{1}{\ell^{1/n}}\,, \text{ a.e. in } \Omega\,, \tag{7}$$

where $D^2\varphi$ denotes the Hessian matrix of φ. We have to note that, without further assumptions, $D^2\varphi$ must be interpreted in the Alexandrov sense, i.e. as the absolutely continuous part of the distributional Hessian of the convex function φ, which is defined almost everywhere.

Analogously $\Delta\varphi$, as the trace of $D^2\varphi$, will denote throughout the absolutely continuous part of the distributed Laplacian of φ. This is in any case sufficient for our aims and guarantees the consistency of all integrals we are going to consider and the validity of integration by parts (for further details, see, for instance, [9]).

By the inequality between arithmetic and geometric mean, from (7) we deduce that

$$\frac{1}{n} f^{-\frac{1}{n-1}} \Delta\varphi \geq \frac{1}{\ell^{1/n}}, \text{ a.e. in } \Omega\,, \tag{8}$$

where equality holds if and only if $D^2\varphi$ is a multiple of the $n \times n$ identity matrix I_n, i.e. if and only if $D^2\varphi = \ell^{-1/n} f^{\frac{1}{n-1}} I_n$, a.e. in Ω, by the virtue of (6). Rewrite (8) as

$$\frac{1}{\ell^{1/n}} f^{\frac{n}{n-1}} \leq \frac{1}{n} f \Delta\varphi$$

and integrate both sides of this inequality on Ω to obtain:

$$\frac{1}{\ell^{1/n}} \leq \frac{1}{n} \int_\Omega f \Delta\varphi\, dx = \frac{1}{n} \sum_{k=1}^{n} \int_\Omega f \frac{\partial^2\varphi}{\partial x_k^2}\, dx_1 \dots dx_n\,. \tag{9}$$

For any $t \geq 0$, define $\Omega_t = \{x \in \Omega : f(x) \geq t\}$. By the layer cake representation of f, we have that

$$\int_\Omega f \frac{\partial^2\varphi}{\partial x_k^2}\, dx_1 \dots dx_n = \int_0^{+\infty} \int_{\Omega_t} \frac{\partial^2\varphi}{\partial x_k^2}\, dx_1 \dots dx_n\, dt\,, \tag{10}$$

for every k, $1 \leq k \leq n$. Let Ω_t^k be the orthogonal projection of Ω_t along e_k. For any $y = (y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n) \in \Omega_t^k$, denote by $\omega_t^k(y)$ the chord of Ω_t obtained as the intersection of Ω_t and the line issued from y and parallel to e_k, i.e.

$$\omega_t^k(y) = \{s \in \mathbb{R} : (y_1, \dots, y_{k-1}, s, y_{k+1}, \dots, y_n) \in \Omega_t\}\,.$$

Thus we have that

$$\int_{\Omega_t} \frac{\partial^2\varphi}{\partial x_k^2}\, dx_1 \dots dx_n = \int_{\Omega_t^k} \int_{\omega_t^k(y)} \frac{\partial^2\varphi}{\partial x_k^2}\, dx_k\, dy \leq \int_{\Omega_t^k} \int_{\hat\omega_t^k(y)} \frac{\partial^2\varphi}{\partial x_k^2}\, dx_k dy, \tag{11}$$

where $\hat\omega_t^k(y)$ stands for the convex hull of $\omega_t^k(y)$, i.e.

$$\hat\omega_t^k(y) = [\alpha_k(y;t), \beta_k(y;t)]\,,$$

where

$$\alpha_k(y;t) = \inf \omega_t^k(y)\,, \quad \beta_k(y;t) = \sup \omega_t^k(y)\,.$$

The inequality follows from the fact that $\frac{\partial^2\varphi}{\partial x_k^2} \geq 0$.

Then

$$\int\limits_{\hat\omega_t^k(y)} \frac{\partial^2\varphi}{\partial x_k^2}\,dx_k = \frac{\partial\varphi}{\partial x_k}\Big|_{\alpha_k(y;t)}^{\beta_k(y;t)} \le 2\ell_k, \tag{12}$$

since $\nabla\varphi : \mathbb{R}^n \to R$. Coupling (7) and (8) yields

$$\int\limits_{\Omega_t} \frac{\partial^2\varphi}{\partial x_k^2\,dx_1\dots dx_n} \le 2\ell_k\mathcal{H}_{n-1}(\Omega_t^k), \tag{13}$$

that in turn, through (10), gives

$$\int\limits_{\Omega} f\frac{\partial^2\varphi}{\partial x_k^2\,dx_1\dots dx_n} \le 2\ell_k\int\limits_0^{+\infty}\mathcal{H}_{n-1}(\Omega_t^k)\,dt. \tag{14}$$

Let us recall that, for $k = 1,\dots,n$, the function $f_k : \mathbb{R}^{n-1} \to \mathbb{R}$ is defined by

$$f_k(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n) = \sup_{x_k} f(x_1,\dots,x_{k-1},x_k,x_{k+1},\dots,x_n),$$

where we identify $(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)$ with $(x_1,\dots,x_{k-1},0,x_{k+1},\dots,x_n)$ and $\mathbb{R}^{n-1}$ with $e_k^\perp$.

Note that the graph of f_k is nothing but the projection of the graph of f along e_k and that

(15)

$$\int\limits_{\Omega^k} f_k(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)\,dx_1\dots dx_{k-1}dx_{k+1}\dots dx_n = \int\limits_0^{+\infty}\mathcal{H}_{n-1}(\Omega_t^k)\,dt.$$

If we set, for simplicity,

$$F_k = \int\limits_{\Omega^k} f_k(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)\,dx_1\dots dx_{k-1}dx_{k+1}\dots dx_n,$$

then (9), (14) and (15) give

$$\frac{1}{\ell^{1/n}} \le \frac{1}{n}\sum_{k=1}^n\int\limits_\Omega f\frac{\partial^2\varphi}{\partial x_k^2}\,dx_1\dots dx_n \le \frac{2}{n}\sum_{k=1}^n \ell_k F_k. \tag{16}$$

Now, the parameters $\ell_1,\dots,\ell_n$ minimizing the right-hand side of (16) under the condition (5) are

$$\ell_i = \frac{\prod\limits_{k=1}^n F_k^{1/n}}{2\ell^{1/n}F_i},\ i = 1,\dots,n.$$

Thus

$$\frac{2\ell^{1/n}}{n}\sum_{k=1}^{n}\ell_k F_k=\prod_{k=1}^{n}F_k^{1/n}. \tag{17}$$

By (16) and (17) we deduce that $1\le\prod_{k=1}^{n}F_k^{1/n}$, that is inequality (2), owing to the normalization condition (4).

To prove the second part of the Theorem, assume that in (2) the equality sign holds.

The first step consists in showing that f is a multiple of a characteristic function. To do this, denote by f_σ the function whose superlevel sets $\Omega_{\sigma,t}$, for every nonnegative t, are obtained by a sequence of successive Steiner symmetrizations of the superlevel sets Ω_t of f, with respect to the coordinate hyperplanes. It is well known that

$$\mathcal{H}_{n-1}(\Omega_{\sigma,t})=\mathcal{H}_{n-1}(\Omega_t)\,, \tag{18}$$

$$\mathcal{H}_{n-2}(\Omega^k_{\sigma,t})\le\mathcal{H}_{n-2}(\Omega^k_t)\,,\ k=1,\dots,n. \tag{19}$$

By (18) and (19) we deduce that f_σ satisfies equality in (2), too. This imply, in particular, that for $k=1,\dots,n$, inequality in (12) is just an equality:

$$\frac{\partial\varphi}{\partial x_k}(\beta_k(y;t))-\frac{\partial\varphi}{\partial x_k}(\alpha_k(y;t))=2\ell_k\ , \tag{20}$$

for almost all $y\in\Omega^k_t$ and for almost all $t\ge 0$. Let us take y and $t_1<t_2$, satisfying (20).

Thus

$$\frac{\partial\varphi}{\partial x_k}(\beta_k(y;t_i))=\ell_k\ ,\ \frac{\partial\varphi}{\partial x_k}(\alpha_k(y;t_i))=-\ell_k\ ,\ i=1,2. \tag{21}$$

Denote, for simplicity, $\beta_k(y;t_2)=p$, $\beta_k(y;t_1)=q$ and $p_s=sp+(1-s)q$, where $s\in[0,1]$. The convexity of φ implies that

$$\langle\nabla\varphi(p_s)-\nabla\varphi(q),p_s-q\rangle\ge 0\ ,\quad\langle\nabla\varphi(p)-\nabla\varphi(p_s),p-p_s\rangle\ge 0.$$

Hence, since $p_s-q=s(p-q)$, $p-p_s=(1-s)(p-q)$ and $p-q$ is parallel to e_k, by (21) we conclude that

$$\langle\nabla\varphi(p_s),e_k\rangle=\ell_k\ ,$$

for every $s\in[0,1]$. This means that the set $I_k=(\Omega_{\sigma,t_1}\setminus\Omega_{\sigma,t_2})\bigcap\Gamma_{\sigma,t_2,k}$, being $\Gamma_{\sigma,t_2,k}$ the cylinder based on Ω^k_{σ,t_2} and parallel to e_k, is mapped by $\nabla\varphi$ on the boundary of R. Thus, I_k has zero measure, for every $k=1,\dots,n$, because $\nabla\varphi$ is a mass transport, which implies that $\Omega_{\sigma,t_1}\setminus\Omega_{\sigma,t_2}$ also has zero measure.

Since t_1 and t_2 are arbitrary, the conclusion is that f_σ coincides a.e. with a multiple of a characteristic function, and consequently f is so, i.e. $f(x) = c\,\chi_\Omega(x)$, for some $c > 0$ and for almost all $x \in \Omega$.

The fact that $f(x) = c\,\chi_\Omega(x)$ satisfies equality in (2) implies that Ω is one of the sets satisfying equality in the classical Loomis-Whitney inequality (1). Thus, we could use the characterization of those sets given in [EFKY16] and end the proof by concluding that Ω must be the Cartesian product of bounded measurable subsets of $\mathbb{R}$.

Nevertheless, to get the same conclusion, we show that one can use an alternative and independent argument which still relies on the mass transport.

The equality condition we are assuming implies that (8) holds with equality sign. We already noted that this happens if and only if $D^2\varphi$ is a multiple of the identity matrix. That is, by (7), if and only if $D^2\varphi = \ell^{-1/n} f^{\frac{1}{n-1}} I_n$, a.e. in Ω. We already know that $f = c\,\chi_\Omega$. Thus, $D^2\varphi = \rho I_n$ a.e. in Ω, where $\rho = \ell^{-1/n} c^{\frac{1}{n-1}}$, and the map of the mass transport on each connected component of Ω is $\nabla\varphi(x) = \rho x + v$, where v depends on the component. Thanks to homogeneity, without loss of generality, we can assume $\rho = 1$.

Since equality must hold in (12),

$$\int_{\hat{\omega}_t^k(y)} dx_k = 2\ell_k\,,$$

for almost every $t > 0$, $y \in \Omega^k$ and for every k. Therefore, for every coordinate direction, almost all the chords of Ω have constant length and, by Fubini's theorem, we obtain that

$$\mathcal{H}_n(\Omega) = \mathcal{H}_{n-1}(\Omega^k) \cdot 2\ell_k$$

and an induction argument yields

$$\mathcal{H}_n(\Omega) = \prod_{k=1}^{n} 2\ell_k\,. \tag{22}$$

Let P_k be the projection of Ω on the e_k-axis, $2p_k = \mathcal{H}_1(P_k)$ and $S_k(x) = \Omega \bigcap (e_k^\perp + x)$. Arguing as before for the section $S_k(x)$, we have that, for every k, $\mathcal{H}_{n-1}(S_k(x)) = \prod_{i=1, i\neq k}^{n} 2\ell_i$ and $\mathcal{H}_n(\Omega) = 2p_k \prod_{i=1, i\neq k}^{n} 2\ell_i$. Thus, by (22), $p_k = \ell_k$, for every k, and

$$\mathcal{H}_n(\Omega) = \prod_{k=1}^{n} 2p_k = \mathcal{H}_n(P_1 \times \cdots \times P_n)\,.$$

Since $\Omega \subseteq P_1 \times \cdots \times P_n$, we conclude that $\Omega = P_1 \times \cdots \times P_n$, up to a set of zero measure.

The proof is concluded.

□

As a final remark we recall that the paper [11, Corollary 2] contains the following stability result for the LW inequality (1):

if a body A in $\mathbb{R}^n$ satisfies

$$\mathcal{H}_n(A) \geq (1-\varepsilon) \prod_{k=1}^{n} \mathcal{H}_{n-1}(A^k)^{1/(n-1)},$$

then there exists a body B which is the Cartesian product of bounded measurable subsets of $\mathbb{R}$ such that $\mathcal{H}_n(A \Delta B) \leq c\mathcal{H}_n(B)\varepsilon$, where c is a constant depending only on n.

The proof of this result is based on a clever use of some concepts from information theory, like the entropy of a random variable.

On the other hand, mass transport techniques have been often used to recover stability results for classical inequalities (see, for example, [13], [14], [22], [15], [17]). Unfortunately, the use of these techniques here, when simply restricted to characteristic functions, seems to lead to a stability of order $\sqrt{\varepsilon}$ and we can only conjecture that also for inequality (2) a stability of order ε holds.

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