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# On the Existence of Almost-Periodic Solutions for the 2D Dissipative Euler Equations

Luigi C. Berselli and Luca Bisconti

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**Abstract.** In this paper we study the two-dimensional dissipative Euler equations in a smooth and bounded domain. In presence of a large enough dissipative term (or equivalently a small enough external force) precise uniform estimates on the modulus of continuity of the vorticity are proved. These allow us to show existence of Stepanov almost-periodic solutions.

## 1. Introduction

In this paper we prove some results related with the long-time behavior of the Euler equations (with dissipation) for incompressible fluids in two space dimensions, aimed at proving existence of almost-periodic solutions. For the Euler equations, it is well-known that in the 2D case it is possible to prove –for smooth enough data– existence and uniqueness of smooth solution, for all positive times, see also the discussion in the next section for certain *less-standard* results. It is also clear that without any smoothing or dissipation, one cannot expect to have uniform boundedness of the energy and of other interesting quantities as the enstrophy or higher norms of the velocity. In order to study general properties as attractors or existence of almost-periodic solutions (where uniform bounds seem requested) we consider the so-called *dissipative Euler equations*

$$(1.1) \quad \begin{aligned} \partial_t u + \chi u + (u \cdot \nabla) u + \nabla \pi &= f && \text{in } ]0, +\infty[ \times \Omega, \\ \nabla \cdot u &= 0 && \text{in } ]0, +\infty[ \times \Omega, \\ u \cdot n &= 0 && \text{on } ]0, +\infty[ \times \Gamma, \\ u_0(0, x) &= u_0(x) && \text{in } \Omega, \end{aligned}$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded open set, with smooth boundary  $\Gamma$ ;  $n$  is the unit outward normal vector on  $\Gamma$ , the vector  $u(t, x) = (u_1(t, x), u_2(t, x))$  is the velocity

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of the fluid,  $\pi(t, x)$  is the kinematic pressure, and  $f = f(t, x)$  is the external force field.

The *damping* term  $\chi u$  (with a constant  $\chi > 0$ ) models the bottom friction in some 2D oceanic models (when the system is considered in a bounded domain; in that case, the system is called the viscous Charney-Stommel barotropic ocean circulation model of the gulf stream) or the Rayleigh friction in the planetary boundary layer (with space-periodic boundary conditions). The positive constant  $\chi$  is the Rayleigh friction coefficient (or the Ekman pumping/dissipation constant) or also the sticky viscosity, when the model is used to study motion in presence of rough boundaries, see for instance Gallavotti [23]. Early existence results can be found in Barcilon, Constantin, and Titi [3], while links between the driven and damped 2D Navier-Stokes, attractors, and statistical solutions are proved in Ilyin, Miranville, and Titi [25], Constantin and Ramos [18], and Constantin, Tarfulea, and Vicol [19]. In recent years the present model has been considered by a number of authors, see for instance [11, 13, 16, 17, 24]. The system (1.1) represents (probably) the “weakest” dissipative modification of the Euler equations and results on the long-time behavior of the damped/driven Navier-Stokes do not directly pass to the limit as the “viscosity goes to zero” hence, a completely different treatment is required to study the problem without viscosity. This is why here we use some specific topologies, which are not derived from the classical Hölder or Sobolev norms.

The main result we will prove is the existence of almost-periodic solutions in the sense of Stepanov, (cf. [10]) with values in  $L^2(\Omega)$ , under certain restrictions on the relative sizes of external force and dissipation term, see Theorem 5.1 for the precise statement.

To this end we need to show precise estimates, uniform in time, for the vorticity. The boundedness of the vorticity, beside being enough to show uniqueness of weak solutions, is not enough to prove results of “*asymptotic stability*,” which is one of the main points generally requested to prove existence of almost-periodic solutions, see Amerio and Prouse [1]. For dissipative equations this is now well-established (see also the recent results in [9] for an inverse problem) but the Euler system does not directly fit with the assumptions needed to use abstract results, and this is the motivation to resort to some stronger topology. In particular, the minimum amount of regularity needed to quantitatively estimate the difference of two solutions over large time-intervals seems to be the represented by the supremum (with respect to the  $x$  variable) norm of the gradient of velocity. The topology of Hölder spaces looks not adapted to this problem, hence we resort to something quite sharp, being the Dini-norm of the vorticity field. We point out that the use of this topology on continuous functions dates back to Beirão da Veiga [5] in the context of global well-posedness of the 2D Euler equations, while in questions of stability the role of Dini-continuous vorticity has been first recognised by Koch [29], even if the application to almost-periodic solutions and some of the techniques we apply here, are to our best knowledge original.

## 2. Notation and preliminary facts

Here and in the sequel, we suppose, without loss of generality, the diameter of the bounded set  $\Omega$  to be equal to one. To avoid technical complications, we assume also that  $\Omega$  is simply connected, referring to the cited bibliography how to modify the proofs to deal also with this case. With a standard notation in mathematical fluid mechanics, let  $\mathcal{V}$  denote the space of infinitely differentiable vector fields  $u$  on  $\Omega$  with compact support strictly contained in  $\Omega$ , and satisfying the constraint  $\nabla \cdot u = 0$ . We introduce the space  $H$  of measurable vector fields  $u : \Omega \rightarrow \mathbb{R}^2$  which are square integrable, divergence free, and tangential to the boundary  $\Gamma$ :

$$H := \{u \in [L^2(\Omega)]^2 : \nabla \cdot u = 0 \text{ in } \Omega, u \cdot n = 0 \text{ on } \Gamma\}.$$

In  $H$  the normal trace is well-defined in  $H^{-1/2}(\Gamma)$  and moreover  $H$  is a separable Hilbert space with the inner product of  $[L^2(\Omega)]^2$ , denoted in the sequel by  $\langle \cdot, \cdot \rangle$  and corresponding norm  $\|\cdot\|_2$ , see for instance [33]. (This space is also the closure of  $\mathcal{V}$  with respect to the norm  $\|\cdot\|_2$ ). As usual we will also denote by  $\|\cdot\|_p$  the  $L^p$ -norm with respect to the space variables belonging to  $\Omega$ . Let  $V \subset H$  be the following subspace:

$$V := \{u \in [H^1(\Omega)]^2 : \nabla \cdot u = 0 \text{ in } \Omega, u \cdot n = 0 \text{ on } \Gamma\}.$$

The space  $V$  is a separable Hilbert space with the inner product induced by  $[H^1(\Omega)]^2$  and its natural norm denoted by  $\|\cdot\|_{1,2}$ . Let us also introduce the tri-linear form on  $V$ , defined as

$$b(u, v, w) := \int_{\Omega} (u \cdot \nabla) v \cdot w \, dx.$$

Since we study time-evolution problems, given a Banach space  $X$ , we denote for  $p \in [1, +\infty]$  the usual Bochner spaces  $L^p(0, T; X)$  with associated norm  $\|f\|_{L^p(0, T; X)}^p := \int_0^T \|f(s)\|_X^p \, ds$ , (the lower upper bound of  $\|f(s)\|_X$  if  $p = +\infty$ ), while  $L_{loc}^p(X)$  is the space of measurable functions  $\mathbb{R} \mapsto X$  belonging to  $L^p(T_1, T_2; X)$ , for any couple  $T_1 \leq T_2 \in \mathbb{R}$ .

The definition of weak solution (see [13]) for the system (1.1), is the following:

**Definition 2.1.** *We say that a function  $u$  is a weak solution to (1.1) on  $[0, +\infty)$ , provided that the following four properties hold true:*

$$(2.1a) \quad u \in C([0, +\infty[; H) \cap L_{loc}^\infty(0, +\infty; V) \quad \text{with} \quad \partial_t u \in L_{loc}^2(0, +\infty; V'),$$

and a.e.  $t \geq t_0 \geq 0$  and for all  $v \in \mathcal{V}$

$$(2.1b) \quad \|u(t)\|_2^2 + 2\chi \int_{t_0}^t \|u(s)\|_2^2 \, ds \leq \|u(t_0)\|_2^2 + \int_{t_0}^t \langle f(s), u(s) \rangle \, ds,$$

$$(2.1c) \quad \|u(t)\|_{1,2}^2 \leq \|u(t_0)\|_{1,2}^2 e^{-\chi(t-t_0)} + \frac{1}{\chi} \int_{t_0}^t \|f(s)\|_{1,2}^2 e^{-\chi(t-s)} \, ds,$$

$$(2.1d) \quad \langle u(t) - u(t_0), v \rangle + \int_{t_0}^t (\chi \langle u(s), v \rangle + b(u(s), u(s), v)) \, ds = \int_{t_0}^t \langle f(s), v \rangle \, ds.$$

The following existence theorem is proved in [13], by adapting the well-known technique developed by Yudovich [35], which is based on approximation by a special Navier-Stokes system and by using *a-priori* estimates in  $L^2(\Omega)$  on both velocity and vorticity, obtained from the momentum equation and from (2.2).

**Theorem 2.1.** *Given  $u_0 \in V$  and  $f \in L^2_{loc}(0, +\infty; V)$ , there exists at least a weak solution for (1.1). Such a weak solution is unique if  $\text{curl } u_0 \in L^\infty(\Omega)$  and  $\text{curl } f \in L^1_{loc}(0, +\infty; L^\infty(\Omega))$ .*

In the context of existence and uniqueness of solutions in broader classes than with bounded vorticity we want also to recall the recent results by Bernicot and Hmidi [6], Azzam and Bedrossian [2], and references therein.

We now recall the definition of some further functional spaces that will be widely used in the sequel. We denote by  $L^p_{uloc}(X)$ , the Banach space of *uniformly locally  $p$ -integrable functions* on  $\mathbb{R}$ , defined for  $1 \leq p < +\infty$  by

$$L^p_{uloc}(X) := \left\{ u : \mathbb{R} \rightarrow X, u \in L^p_{loc}(\mathbb{R}; X) : \sup_{t \in \mathbb{R}} \int_t^{t+1} \|u(s)\|_X^p ds < \infty \right\},$$

endowed with the norm

$$\|u\|_{L^p_{uloc}(X)} := \left[ \sup_{t \in \mathbb{R}} \int_t^{t+1} \|u(s)\|_X^p ds \right]^{1/p}.$$

We give now the precise definition of the almost-periodic functions we will use.

**Definition 2.2.** *We say that a function  $f \in L^2_{uloc}(X)$  is Stepanov 2-almost-periodic (or simply Stepanov almost-periodic) if the set of its translates is relatively compact in the  $L^2_{uloc}(X)$ -topology. The space of Stepanov almost-periodic will be denoted by  $\mathcal{S}^2(X)$*

The condition that  $f \in \mathcal{S}^2(X)$  reads as follows:  $f \in L^2_{uloc}(X)$  and for any sequence  $\{r_m\}$  we can find a sub-sequence  $\{r_{m_k}\}$  and a function  $\tilde{f} \in L^2_{uloc}(X)$  such that

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s + r_{m_k}) - \tilde{f}(s)\|_2^2 ds \rightarrow 0.$$

Results and further properties of Stepanov spaces can be found for instance in the classical book by Besicovitch [10].

In the study of 2D Euler equations fundamental estimates are obtained by the analysis of the transport equation for the vorticity. In the case of the dissipative equations (1.1) the equation satisfied by the vorticity  $\xi = \text{curl } u := \partial_1 u_2 - \partial_2 u_1$  is the following

$$(2.2) \quad \partial_t \xi + (u \cdot \nabla) \xi + \chi \xi = \phi,$$

where  $\phi := \text{curl } f$ . By a change of variables we can also write

$$\partial_t \eta + (u \cdot \nabla) \eta = \phi e^{\chi t},$$

with  $\eta := \xi e^{\chi t}$ , coming back to a transport equations, without zero order terms.

Since we work with space-time functions we also define  $\overline{\Omega}_T := [0, T] \times \overline{\Omega}$  and we use the following notation: for a given  $T > 0$

$$\|f\|_{L^\infty} := \sup_{(x,t) \in \overline{\Omega}_T} |f(t, x)|.$$

What makes very special the two-dimensional Euler equations is also that the connection between velocity and vorticity can be made very explicit by the use of the stream-function, being particularly neat in the case of a simply-connected domain. Let  $\psi := -\Delta^{-1}\theta$  be the solution of the Poisson equation with homogeneous Dirichlet data

$$\begin{cases} -\Delta\psi = \theta & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma, \end{cases}$$

then the vector  $v := \nabla^\perp\psi := (-\partial_2\psi, \partial_1\psi)$  satisfies

$$\begin{cases} \operatorname{curl} v = \theta & \text{in } \Omega, \\ \nabla \cdot v = 0 & \text{in } \Omega, \\ v \cdot n = 0 & \text{on } \Gamma, \end{cases}$$

and for this reason we use the following notation  $v = \nabla^\perp(-\Delta^{-1}\theta) := \operatorname{curl}^{-1}\theta$ .

The use of the vorticity equation, being a non-local transport equation, is also at the basis of the classical existence results of classical solutions, dating back to Lichtenstein, Hölder, Wolibner, Leray, Schaeffer, and Kato. See also the historical account in Brezis and Browder [15, §11].

As will be clear later on, in order to prove some sharp estimates on the growth of the vorticity, we will use a particular topology, namely that of Dini-continuous functions  $C_D(\overline{\Omega}) \subset C(\overline{\Omega})$ . This space is the subset of continuous functions  $f : \overline{\Omega} \rightarrow \mathbb{R}^n$  such that

$$\|f\|_{C_D} := \|f\|_{L^\infty} + [f]_{C_D} := \|f\|_{L^\infty} + \int_0^1 \omega(f, \sigma) \frac{d\sigma}{\sigma} < +\infty,$$

where, for  $\sigma > 0$ , the quantity  $\omega(f, \sigma)$  denotes the *modulus of continuity* of  $f$ , defined as follows:

$$\omega(f, \sigma) := \sup \left\{ |f(x) - f(y)| \text{ with } x, y \in \overline{\Omega}, |x - y| < \sigma \right\}.$$

As it will be clear in the next section, the main reason for using this space is that the following potential-theoretic estimate holds true:

$$(2.3) \quad \exists C_0 = C_0(\Omega) > 0 : \quad \|\nabla u\|_\infty \leq C_0 \|\operatorname{curl} u\|_{C_D},$$

where  $\operatorname{curl} u$  is the vorticity. Some classical (well-known) results dating back to Dini [21] imply in fact that the second derivatives of  $-\Delta^{-1}f$  are bounded (but more precisely also continuous) if  $f \in C_D(\overline{\Omega})$ , while the simple boundedness of

$f$  is not enough (recall that  $-\Delta^{-1}f$  is the solution of the Poisson problem with vanishing Dirichlet data and right-hand side equal to  $f$ .) We do not exclude further extensions to other functional settings as Besov or multiplier spaces as in Vishik [34] or Koch and Sickel [30], however here we are not interested in these kind of technicalities, but rather focusing on a functional setting which is properly defined also in the case of a domain with boundary.

### 3. A basic estimate on the Dini-norm of the vorticity

We start proving existence and uniqueness of *strong* solutions to the Euler dissipative equations.

**Definition 3.1.** *We say that a vector field  $u$  is a strong solution to (1.1) in  $[0, T]$  if  $u \in C([0, T]; C(\overline{\Omega}))$  is divergence-free, tangential to the boundary, and with  $\operatorname{curl} u \in C([0, T]; C(\overline{\Omega}))$ ,  $\partial_t u \in L^1(0, T; L^2(\Omega))$ ,  $\pi \in L^1(0, T; W^{1,2}(\Omega))$ , is a weak solution and in addition*

$$\|\operatorname{curl} u(t)\|_\infty \leq \|\operatorname{curl} u_0\|_\infty + \int_0^t \|\operatorname{curl} f(s)\|_\infty e^{\chi s} ds \quad \forall t \in [0, T].$$

These solutions are called “strong solutions” since they are unique and continuously dependent on the data, but not classical, since *a priori*  $\nabla u \in C([0, T]; L^p(\Omega))$  for all  $p < +\infty$ , but  $\nabla u$  may be not point-wise defined. The proof is an easy adaption of the sharp results of Hadamard well-posedness proved in [5]. Nevertheless, since we will use these results (which are a sort of endpoint for the well-posedness of the Euler equations), which seem not to be very diffused in the literature, we sketch out the proof and we make some remarks in order to make the presentation self-contained.

The main theorem of existence and uniqueness for *strong* solutions is the following, which is proved below after some preliminary lemmas.

**Theorem 3.1.** *Let be given  $u_0 \in H$  with  $\operatorname{curl} u_0 \in C(\overline{\Omega})$ . Assume also that  $f \in L^1(0, T; H)$  with  $\operatorname{curl} f \in L^1(0, T; C(\overline{\Omega}))$ , and  $\chi > 0$ . Then, there exists a unique strong solution of the dissipative Euler equations in  $[0, T]$ .*

By using a classical approach (see the discussion in [15] and other remarks in [8]) the proof is based on a representation’s formula for the vorticity, by means of characteristics  $U(t, s, x)$ , which are solutions of the following Cauchy problem for ordinary differential equations

$$\begin{cases} \frac{dU(t, s, x)}{dt} = u(t, U(t, s, x)), \\ U(s, s, x) = x, \end{cases}$$

where  $(t, s, x) \in [0, T]^2 \times \overline{\Omega}$ , while  $u$  is the sought velocity field. From the solution of above family of Cauchy problems problem one can easily infer (cf. [29, Eq. (7)])

that

$$(3.1) \quad |\nabla U(t, s, x)| \leq e \left| \int_s^t \sup_{x \in \Omega} |\nabla u(\tau, x)| d\tau \right| \quad \forall (t, s, x) \in [0, T]^2 \times \bar{\Omega}.$$

Moreover, (cf. Kato [27]) the following potential-theoretic estimates for the characteristics hold true: If  $\xi \in L^\infty(\bar{\Omega}_T)$ , and if  $u = \text{curl}^{-1} \xi$ , then  $\exists c_1 > 0$  (depending only on  $\Omega$ , and hence independent of  $T$ ) such that

$$(3.2) \quad \begin{aligned} \|u\|_{L^\infty} &\leq c_1 \|\xi\|_{L^\infty}, \\ |u(t, x) - u(t, y)| &\leq c_1 \|\xi\|_{L^\infty} |x - y| \log \left( \frac{e}{|x - y|} \right) \quad \forall t \in [0, T], \forall x \neq y. \end{aligned}$$

Further, it is well-known that under boundedness of vorticity, characteristics are uniquely defined and are Hölder continuous, see e.g. [5], and they satisfy

$$(3.3) \quad \begin{aligned} |U(t, s, x) - U(t_1, s_1, x_1)| &\leq \\ &c_1 \|\xi\|_{L^\infty} |t - t_1| + e(1 + c_1 \|\xi\|_{L^\infty})(|x - x_1|^\alpha + |s - s_1|^\alpha), \end{aligned}$$

where the exponent is defined by  $\alpha := e^{-c_1 \|\xi\|_{L^\infty} T}$ .

To construct the strong solution  $u$  to (1.1) we consider the following Banach space

$$X := \left\{ \theta : \bar{\Omega}_T \rightarrow \mathbb{R} : \theta \in C(\bar{\Omega}_T) \right\},$$

and we define a map  $J : X \rightarrow X$  by

$$[J\theta](t, x) := \xi_0(U[\theta](0, t, x)) e^{-\chi t} + \int_0^t \phi(s, U[\theta](s, t, x)) e^{-\chi(t-s)} ds,$$

where  $\xi_0 = \text{curl} u_0$  is the initial vorticity, while  $\phi = \text{curl} f$ ,  $u[\theta] = \text{curl}^{-1} \theta$ , and the characteristics  $U[\theta](t, s, x)$  are constructed tracing the trajectories by using the field  $u[\theta]$ .

We first show that this mapping has a fixed point, hence that this fixed point is the vorticity of a strong solution of the dissipative Euler equations. This solution is also a weak solution and uniqueness follows by standard results on weak solutions to the Euler equations with bounded vorticity. We split the proof in two lemmas, following step-by-step the approach in [5].

**Lemma 3.1.** *Let us define the convex set*

$$\mathcal{K} := \left\{ \theta \in X : \|\theta\|_{L^\infty} \leq \mathcal{R} \right\},$$

where  $\mathcal{R} := \|\text{curl} u_0\|_\infty + \int_0^T \|\text{curl} f(s)\|_\infty e^{\chi s} ds$ .

*Then, the inclusion  $J(\mathcal{K}) \subset \mathcal{K}$  holds true and moreover  $J(\mathcal{K})$  is a family of equicontinuous functions in  $\bar{\Omega}_T$ .*



*Proof.* The bound  $\|J\theta\|_{L^\infty} \leq \mathcal{R}$  is obvious as well as the equicontinuity of the family  $\xi_0(U[\theta](0, t, x)) e^{-\chi t}$  (in fact  $\xi_0$  is continuous on  $\overline{\Omega}$  and there is a composition with the uniformly continuous  $U[\theta]$  as follows by using (3.3)).

For the integral appearing in the definition of  $J\theta$  we write

$$\begin{aligned} & \left| \int_0^t \phi(s, U[\theta](s, t, x)) e^{-\chi(t-s)} ds - \int_0^{t_1} \phi(s, U[\theta](s, t_1, x_1)) e^{-\chi(t_1-s)} ds \right| \\ & \leq \left| \int_{t_1}^t \|\phi(s)\|_\infty e^{\chi s} ds \right| + |e^{-\chi t} - e^{-\chi t_1}| \int_0^{t_1} \|\phi(s)\|_\infty e^{\chi s} ds \\ & \quad + \int_0^{t_1} |\phi(s, U[\theta](s, t, x)) - \phi(s, U[\theta](s, t_1, x_1))| e^{\chi s} ds. \end{aligned}$$

The first and second term from the right-hand side clearly go to zero uniformly as  $t_1 \rightarrow t$ , by the absolute continuity of the integral and the continuity of the exponential function. For the last one observe that the function

$$\varpi(s, \epsilon) := \sup_{|z-z_1|<\epsilon} e^{\chi s} |\phi(s, z) - \phi(s, z_1)|,$$

satisfies the bound  $\varpi(s, \epsilon) \leq 2e^{\chi s} \|\phi(s)\|_\infty$  and  $\varpi(s, \epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$  for a.e.  $s \in [0, T]$ . Hence, we can apply Lebesgue dominated convergence theorem to show that its integral can be made as small as we wish. Combining this with the continuity of characteristics gives the thesis. For further details, see the proof of [5, Thm. 2.1].  $\square$

We then use some compactness results to employ a fixed-point argument.

**Lemma 3.2.** *The mapping  $J$  has a fixed point.*

*Proof.* In order to show existence of a fixed point we need to just show that the mapping  $J$  is continuous with respect to the  $L^\infty(\overline{\Omega}_T)$  topology. (This result – or some of its variations – seems to be part of the mathematical folklore and its proof is sketched in [5, Theorem 2.2] and [27, Lemma 2.8]. For the reader's convenience we include an elementary and complete alternative proof.)

By the compactness of the mapping – which is ensured by equicontinuity and the Ascoli-Arzelà theorem – it follows that all the other hypotheses of Schauder fixed point theorem are satisfied.

To this end, let be given  $\{\theta_m\}_m \subset \mathcal{K}$  such that  $\theta_m \rightarrow \theta$  uniformly in  $\overline{\Omega}_T$ . The unique function  $u_m$  such that  $u_m = \text{curl}^{-1} \theta_m$  satisfies  $u_m \rightarrow u$  uniformly in  $\overline{\Omega}_T$ . We show now that  $U_m(t, s, x) \rightarrow U(t, s, x)$  uniformly in  $[0, T]^2 \times \overline{\Omega}$ , where  $U_m$  is a solution of

$$\begin{cases} \frac{dU_m(t, s, x)}{dt} = u_m(t, U_m(t, s, x)), & t \in [0, T] \\ U_m(s, s, x) = x, & s \in [0, T]. \end{cases}$$

Fix some  $\epsilon \in ]0, 1[$ . Then, there exists  $N = N(\epsilon) \in \mathbb{N}$  such that

$$\sup_{(x,t) \in \overline{\Omega}_T} |u_n(t, x) - u(t, x)| < \epsilon, \quad \forall n > N.$$

Define then  $\zeta_{n,s}(t) := |U_n(t, s, x) - U(t, s, x)|^2$  and observe that, for  $n > N$ , and by using (3.2) and the bound  $\mathcal{R}$  on the elements of the set  $\mathcal{K}$

$$\begin{aligned} & \left| \frac{dU_n(t, s, x)}{dt} - \frac{dU(t, s, x)}{dt} \right| \\ & \leq |u_n(U_n(t, s, x)) - u(U(t, s, x))| + |u(U_n(t, s, x)) - u(U(t, s, x))| \\ & \leq \epsilon + c_1 \mathcal{R} |U_n(t, s, x) - U(t, s, x)| \log \left( \frac{e}{|U_n(t, s, x) - U(t, s, x)|} \right). \end{aligned}$$

Define now for some  $\lambda \in ]0, 1[$  (small enough in a way which will be fixed later on)

$$\tau_n := \inf\{t > s : \zeta_{n,s}(t) \geq \lambda^2\}.$$

Note that  $\tau_n$  is strictly larger than  $s$ , since  $\zeta_{n,s}(s) = 0$  and  $\zeta_{n,s}$  is a continuous function of its arguments. We then obtain in  $[s, \tau_n]$

$$\left| \frac{d\zeta_{n,s}}{dt} \right| \leq 2\lambda\epsilon + c_2 \zeta_{n,s} \log \left( \frac{e}{\zeta_{n,s}} \right).$$

We define then  $Z_{n,s}(t) := \frac{2\lambda\epsilon}{c_2} + \zeta_{n,s}(t)$  and, with simple calculations after optimization in  $\epsilon \in ]0, 1]$ , we can see that for

$$0 < \lambda < \lambda_0 := \sqrt{\frac{1}{e} + \frac{1}{c_2^2} - \frac{1}{c_2}},$$

it holds for  $s < t < \tau_n$

$$2\lambda\epsilon + c_2 \zeta_{n,s}(t) \log \left( \frac{e}{\zeta_{n,s}(t)} \right) \leq c_2 Z_{n,s}(t) \log \left( \frac{e}{Z_{n,s}(t)} \right).$$

We recall the fact that from the differential inequality

$$\begin{cases} y'(t) \leq C y(t) \log \left( \frac{e}{y(t)} \right), \\ y(0) = y_0, \end{cases}$$

we have by direct integration

$$y(t) \leq e \left( \frac{y_0}{e} \right)^{e^{-Ct}} \quad t \geq s,$$

consequently applying this to the function  $Z_{n,s}$  we have

$$Z_{n,s}(t) \leq e \left( \frac{2\lambda\epsilon}{c_2 e} \right)^{e^{-c_2 t}} \leq e \left( \frac{2\lambda\epsilon}{c_2 e} \right)^{e^{-c_2 T}} \quad \forall t \in [s, \min\{\tau_n, T\}],$$

provided that  $0 < \epsilon < \epsilon_0$ , where  $\epsilon_0 := \min\{1, \frac{c_2 e}{2\lambda_0}\}$ . Hence, we obtain

$$(3.4) \quad \zeta_{n,s}(t) \leq \frac{2\lambda\epsilon}{c_2} + e \left( \frac{2\lambda\epsilon}{c_2 e} \right)^{e^{-c_2 T}} \quad \forall t \in [s, \min\{\tau_n, T\}].$$

Since  $\frac{2\lambda\epsilon}{c_2} + e \left( \frac{2\lambda\epsilon}{c_2 e} \right) e^{-c_2 T}$  is monotonically increasing in  $\lambda$ , the quantity  $\zeta_{n,s}(t)$  is bounded by the value assumed at  $\lambda = \lambda_0$ . Hence we can choose  $0 < \epsilon_1 < \epsilon_0$  small enough such that

$$\frac{2\lambda_0\epsilon}{c_2} + e \left( \frac{2\lambda_0\epsilon}{c_2 e} \right) e^{-c_2 T} \leq \lambda^2 \quad \forall \epsilon \in ]0, \epsilon_1[.$$

This shows that, for small enough  $\epsilon > 0$  the same bound (3.4) holds for *all*  $s \in [0, T]$ , for *all*  $t \in [s, T]$ , and for *all*  $x \in \bar{\Omega}$ . Consequently,  $\zeta_{n,s}$  goes to zero uniformly when  $\epsilon$  goes to zero. The same reasoning can be used also for  $t \in [0, s]$ , hence we obtain that  $U_m$  converges uniformly to  $U$  in  $[0, T]^2 \times \bar{\Omega}$ .

Finally, from the definition of  $J$  (being composition of uniformly continuous functions) it follows that if  $\theta_m \rightarrow \theta$ , then  $J\theta_m \rightarrow J\theta$  uniformly.  $\square$

We can now give the proof of the existence result for strong solutions.

*Proof of Theorem 3.1.* By calling  $\xi \in \mathcal{K}$  the fixed point of the map  $J$ , it satisfies  $\xi = J\xi$ , i.e.,

$$(3.5) \quad \xi(t, x) = \xi_0(U(0, t, x)) e^{-\chi t} + \int_0^t \phi(s, U(s, t, x)) e^{-\chi(t-s)} ds \quad t \in [0, T].$$

By a standard argument (adapting for instance that in [5, Lemmas 2.3-2.4] and [27, Lemma 2.4]) we obtain that  $x \mapsto U(t, s, x)$  is measure preserving (since  $\nabla \cdot u = 0$ , where  $u := \text{curl}^{-1} \xi$ ). By multiplying (3.5) by a smooth test function  $\Psi$ , by integrating over  $]0, T[ \times \Omega$ , and with a change of variables in the multiple integrals it follows that the scalar  $\xi$  satisfies

$$\int_0^T \int_{\Omega} \left[ \xi \frac{\partial \Psi}{\partial t} + (\xi u) \cdot \nabla \Psi - \chi \xi \Psi + \phi \Psi \right] dx dt = 0 \quad \forall \Psi \in C_0^\infty(]0, T[ \times \Omega)$$

and  $u$  is (also) a weak solution of the dissipative Euler equations. Finally the basic uniqueness results as in Yudovich [35] and Bardos [4] (see also Bessaih and Flandoli [12, 13] for the dissipative case) show that the solution is unique.  $\square$

We prove now the fundamental estimate needed to prove existence of almost-periodic solutions to the dissipative Euler equations. The main point is a uniform (in time) bound for the Dini-norm of the vorticity. To this end we recall that the existence of *classical* (since now all terms are point-wise defined) solutions to the Euler equations such that  $\xi \in C([0, T]; C_D(\bar{\Omega}))$  is not new. This appeared first in Beirão da Veiga [5, Thm. 1.4] and again and in an independent way (with a slightly-different proof) in Koch [29, Thm. 2]. We do not reproduce here the proof, which is also in this case based on the representation formula (3.5) and Schauder fixed point theorem, but we give just the main point, which is a uniform estimate for the Dini-norm of the vorticity.

**Theorem 3.2.** *In addition to the hypotheses of Theorem 3.1 let us assume that  $\operatorname{curl} u_0 \in C_D(\overline{\Omega})$  and that  $\operatorname{curl} f \in L^1_{loc}(0, +\infty; C_D(\overline{\Omega}))$ . Then, for any  $T > 0$ , the unique strong solution of the dissipative Euler equations is such that  $\operatorname{curl} u \in C([0, T]; C_D(\overline{\Omega}))$ .*

*Moreover, if  $\operatorname{curl} f \in L^\infty(0, +\infty; C_D(\overline{\Omega}))$ , then there exists  $\chi_0 > 0$  (depending on the initial datum  $\xi_0$ , the force  $f$ , and the domain  $\Omega$ , cf. (3.8)) such that if  $\chi > \chi_0$ , then*

$$(3.6) \quad \sup_{t \geq 0} \|\operatorname{curl} u(t)\|_{C_D} \leq C < +\infty.$$

where  $C = C(\xi_0, f, \chi, \Omega)$ .

*Proof.* We know yet from the previous Theorem 3.1, for any  $\chi > 0$ , the existence and uniqueness of a *strong* solution corresponding to the data  $(u_0, f)$ . In particular, adapting [5, Thm. 1.4] and [29, Thm. 2] it is straightforward to show that the solution is such that  $\operatorname{curl} u \in C([0, T]; C_D(\overline{\Omega}))$ . For the reader's convenience we recall that the main point is to check that the fixed point of the mapping  $J$  satisfies  $\xi = J\xi \in C([0, T]; C_D(\overline{\Omega}))$ . This allows to use on the Schauder fixed point argument in the topology of  $L^\infty(\overline{\Omega}_T)$ .

We show now that the presence of a large-enough dissipative constant  $\chi$  the representation formula allows us to obtain uniform bounds on the Dini-norm of the vorticity over all positive times. For any given  $T > 0$ ,  $\xi$  is the fixed point of the mapping  $J$ , hence it satisfies (3.5). We show now an uniform bound for the Dini-norm of  $\xi$ . First the  $L^\infty(\Omega)$  bound is straightforward

$$\|\xi(t)\|_\infty \leq \|\xi_0\|_\infty e^{-\chi t} + \sup_{t \geq 0} \|\phi(t)\|_\infty \frac{1 - e^{-\chi t}}{\chi} \quad \forall t \geq 0,$$

and it shown also in [13]. In the calculations which follow we are assuming that we have a unique solution such that  $\xi \in C([0, T]; C_D(\overline{\Omega}))$ . This implies that  $U$  is Lipschitz continuous (especially in the space variable) and that its Lipschitz-norm is bounded by the Dini-norm of  $\xi$ . We will work on a given interval  $[0, T]$  and then we will show that the estimates are independent of  $T$ , for large enough  $\chi > 0$ .

We estimate the Dini-continuity of  $\eta = \xi e^{\chi t}$ , where  $\xi$  is the vorticity of the solution, hence such that  $\xi = J\xi$  on  $[0, T]$ . Observe that, from the equation satisfied by  $\eta$  we have the representation formula

$$\eta(t, x) = \xi_0(U(0, t, x)) + \int_0^t \phi(s, U(s, t, x)) e^{\chi s} ds,$$

and clearly

$$\|\eta(t)\|_\infty \leq \|\xi_0\|_\infty + \sup_{t \geq 0} \|\phi(t)\|_\infty \frac{e^{\chi t} - 1}{\chi}.$$

Moreover, we observe that  $[\eta(t)]_{C_D} = [\xi(t)]_{C_D} e^{\chi t}$ , as easily follows from the defi-

dition. We estimate the Dini-semi-norm of  $\eta$  as follows:

$$\begin{aligned}
(3.7) \quad [\eta(t)]_{C_D} &:= \int_0^1 \sup_{|x-y| \leq \rho} |\eta(t, x) - \eta(t, y)| \frac{d\rho}{\rho} \\
&\leq \int_0^1 \sup_{|x-y| \leq \rho} |\xi_0(U(0, t, x)) - \xi_0(U(0, t, y))| \frac{d\rho}{\rho} \\
&\quad + \int_0^t \int_0^1 \sup_{|x-y| \leq \rho} |\phi(s, U(s, t, x)) - \phi(s, U(s, t, y))| e^{\chi s} \frac{d\rho}{\rho} ds \\
&=: B_1 + B_2.
\end{aligned}$$

Next, we estimate separately  $B_1$  and  $B_2$ . For the first term, making a change of variable by means of the unitary diffeomorphism  $U(0, t, x)$ , we have that

$$\begin{aligned}
B_1 &\leq \int_0^1 \sup_{|x-y| \leq \rho \|\nabla U(0, t, \cdot)\|_\infty} |\xi_0(x) - \xi_0(y)| \frac{d\rho}{\rho} \\
&\leq \int_0^1 \sup_{|x-y| \leq \rho} |\xi_0(x) - \xi_0(y)| \frac{d\rho}{\rho} + 2\|\xi_0\|_\infty \int_1^{\|\nabla U(0, t, \cdot)\|_\infty} \frac{d\rho}{\rho} \\
&\leq [\xi_0]_{C_D} + 2\|\xi_0\|_\infty \log \|\nabla U(0, t, \cdot)\|_\infty,
\end{aligned}$$

(where the term  $2\|\xi_0\|_\infty \log \|\nabla U(0, t, \cdot)\|_\infty$ , is set to zero if  $\|\nabla U(0, t, \cdot)\|_\infty < 1$ ) and, by appealing to (3.1), we obtain

$$\begin{aligned}
B_1 &\leq [\xi_0]_{C_D} + 2\|\xi_0\|_\infty \int_0^t \|\nabla u(s)\|_\infty ds \\
&\leq [\xi_0]_{C_D} + 2C_0 \|\xi_0\|_\infty \int_0^t \|\xi(s)\|_{C_D} ds \\
&\leq [\xi_0]_{C_D} + 2C_0 \|\xi_0\|_\infty \int_0^t \|\eta(s)\|_{C_D} e^{-\chi s} ds.
\end{aligned}$$

For the term  $B_2$ , by making the change of variables by means of  $U(s, t, x)$ , we have that

$$\begin{aligned}
B_2 &\leq \int_0^t \int_0^1 \sup_{|x-y| \leq \rho \|\nabla U(s, t, \cdot)\|_\infty} |\phi(s, x) - \phi(s, y)| \frac{d\rho}{\rho} e^{\chi s} ds \\
&\leq \int_0^t [\phi(s)]_{C_D(\bar{\Omega})} e^{\chi s} ds + 2\|\phi(s)\|_\infty \int_0^t \int_1^{\|\nabla U(s, t, \cdot)\|_\infty} \frac{d\rho}{\rho} e^{\chi s} ds \\
&\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \int_0^t e^{\chi s} ds + 2 \sup_{t \geq 0} \|\phi(t)\|_\infty \int_0^t \log \|\nabla U(s, t, \cdot)\|_\infty e^{\chi s} ds \\
&\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \int_0^t e^{\chi s} ds + 2 \sup_{t \geq 0} \|\phi(t)\|_\infty \int_0^t \log \|\nabla U(s, t, \cdot)\|_\infty e^{\chi s} ds \\
&\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \int_0^t e^{\chi s} ds + 2 \sup_{t \geq 0} \|\phi(t)\|_\infty \int_0^t \int_s^t \|\nabla u(\tau)\|_\infty e^{\chi s} d\tau ds.
\end{aligned}$$

Changing the order of integration in the last integral we have

$$\begin{aligned}
B_2 &\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \int_0^t e^{\chi s} ds + 2 \sup_{t \geq 0} \|\phi(t)\|_\infty \int_0^t \int_0^\tau \|\nabla u(\tau)\|_\infty e^{\chi s} ds d\tau \\
&\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \frac{e^{\chi t} - 1}{\chi} + 2C_0 \sup_{t \geq 0} \|\phi(t)\|_\infty \int_0^t \int_0^\tau \|\xi(\tau)\|_{C_D} e^{\chi s} ds d\tau \\
&\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \frac{e^{\chi t} - 1}{\chi} + 2C_0 \sup_{t \geq 0} \|\phi(t)\|_{L^\infty} \int_0^t \|\xi(\tau)\|_{C_D} \frac{e^{\chi \tau} - 1}{\chi} d\tau \\
&\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \frac{e^{\chi t}}{\chi} + \frac{2C_0}{\chi} \sup_{t \geq 0} \|\phi(t)\|_\infty \int_0^t \|\eta(\tau)\|_{C_D} d\tau.
\end{aligned}$$

Collecting all the estimates we get the following inequality, where  $\Phi := \sup_t \|\phi(t)\|_{C_D}$

$$\|\eta(t)\|_{C_D} \leq \|\xi_0\|_{C_D} + \frac{2\Phi}{\chi} e^{\chi t} + 2C_0 \left[ \|\xi_0\|_{C_D} + \frac{\Phi}{\chi} \right] \int_0^t \|\eta(s)\|_{C_D} ds.$$

By using Gronwall lemma we get

$$\begin{aligned}
\|\eta(t)\|_{C_D} &\leq \left[ \|\xi_0\|_{C_D} + \frac{2\Phi}{\chi} - \frac{2\Phi\chi}{\chi^2 - 2C_0(\Phi + \|\xi_0\|_{C_D}\chi)} \right] e^{\frac{2C_0 t(\Phi + \|\xi_0\|_{C_D}\chi)}{\chi}} \\
&\quad + \frac{2\Phi\chi}{\chi^2 - 2C_0(\Phi + \|\xi_0\|_{C_D}\chi)} e^{t\chi}
\end{aligned}$$

and consequently

$$\begin{aligned}
\|\xi(t)\|_{C_D} &\leq \left[ \|\xi_0\|_{C_D} + \frac{2\Phi}{\chi} - \frac{2\Phi\chi}{\chi^2 - 2C_0(\Phi + \|\xi_0\|_{C_D}\chi)} \right] e^{t \frac{2C_0(\Phi + \|\xi_0\|_{C_D}\chi)}{\chi} - \chi} \\
&\quad + \frac{2\Phi\chi}{\chi^2 - 2C_0(\Phi + \|\xi_0\|_{C_D}\chi)},
\end{aligned}$$

which is uniformly bounded on  $[0 + \infty[$  if

$$2C_0\Phi + 2C_0\|\xi_0\|_{C_D}\chi - \chi^2 < 0,$$

that is if

$$(3.8) \quad \chi > \chi_0 := C_0\|\xi_0\|_{C_D} + \sqrt{C_0^2\|\xi_0\|_{C_D}^2 + 2C_0\Phi}.$$

□

**Remark 3.1.** *In order to obtain directly continuity of the mapping  $J$  and also uniform estimates, the Hölder topology seems not suitable. The reader can also compare with [5, Rem. 2.2] and also the related observation on the non-continuity in [29, p. 494] of  $C^{1,\alpha}$ -under simple rigid rotations. The fixed point and other arguments require also to handle these topologies, especially when looking for properties valid for arbitrary positive times. The connection between continuity of the*

mapping  $t \mapsto u(t)$ , the growth in a critical way of different norms (Dini, Hölder and Sobolev), and the long-time behavior is especially addressed in [29]. Moreover, in a recent work of Kiselev and Šverák [28] it is shown that for the Euler equations (that is in the case  $\chi = 0$ ) it is possible to find smooth initial data producing solutions with sharp growth in derivatives of the vorticity, such that an exponential growth for  $\|\nabla u\|_{L^\infty}$  follows.

**Remark 3.2.** Especially in connection with the existence of attractors, hence with uniform bounds together with a condition of semi-group a similar approach is used in [8], using other arguments, strictly related with the Hadamard well-posedness. Results concerning the existence of certain strong global-attractors are announced in [7]

#### 4. Existence of solutions defined on the whole real line

This section is devoted to prove the existence of weak solutions to (1.1) defined on the whole real axis. To do so, we follow the analysis carried out in [31, § 3] and in [13, § 1], to obtain the following result.

**Theorem 4.1.** Assume that  $f \in L^2_{loc}(\mathbb{R}; V)$  and that  $\text{curl } f \in L^\infty(\mathbb{R}; C_D(\overline{\Omega}))$ . Then, if  $\chi > \chi_1(f, \Omega) := \sqrt{2C_0\Phi} > 0$  (cf. (3.8), with  $\Phi := \|\text{curl } f\|_{L^\infty(\mathbb{R}; C_D)}$ ), there exists a weak solution  $\tilde{u}$  to (1.1), defined on  $\mathbb{R}$ , which verifies

$$(4.1) \quad \sup_{t \in \mathbb{R}} \|\nabla \tilde{u}(t)\|_\infty \leq C_2,$$

with  $C_2 := C_0 C_1$ , and the constants  $C_0, C_1$  are given in (2.3) and (4.3), respectively.

*Proof.* We consider the system (1.1) in  $[-k, +\infty)$ ,  $k \in \mathbb{N}$ , with initial datum  $u_k(-k) = 0$  (and so  $\xi_k(-k) \equiv 0$ ). Arguing as in the proof of Theorem 3.2, we get the existence of a unique strong solution  $u_k$  to (1.1), on the interval  $[-k, +\infty)$ , such that  $\text{curl } u_k \in C([-k, +\infty); C_D(\overline{\Omega}))$ .

As a further consequence of the results in Theorem 3.2, it follows that if  $\chi > \chi_1$ , then

$$(4.2) \quad \sup_{t \geq -k} \|\text{curl } u_k(t)\|_{C_D} \leq C_1 < +\infty,$$

where

$$(4.3) \quad C_1(f, \chi, \Omega) := \frac{2\Phi\chi}{\chi^2 - 2C_0\Phi}$$

is the constant  $C = C(\xi_0, f, \chi, \Omega)$ , introduced in (3.6), in the case when  $\xi_0 = 0$ .

Let us set now

$$\tilde{u}_k(t) := \begin{cases} u_k(t) & \text{for } t \in [-k, \infty), \\ 0 & \text{for } t \in (-\infty, -k]. \end{cases}$$

Clearly, relation (4.2) still remains true for  $\tilde{u}_k$ ,  $k \in \mathbb{N}$ , and, by appealing to the inequality (2.3), we get

$$\frac{1}{C_0} \sup_{t \in \mathbb{R}} \|\nabla \tilde{u}_k(t)\|_\infty \leq \sup_{t \in \mathbb{R}} \|\operatorname{curl} \tilde{u}_k(t)\|_{C_D} \leq C_1.$$

In particular, we have that  $\nabla \tilde{u}_k$  is uniformly bounded in  $\mathbb{R} \times \Omega$  by  $C_2$ . Therefore, there is a sub-sequence of  $\tilde{u}_k$  (labeled again  $\tilde{u}_k$ ) and a function with  $\nabla \tilde{u} \in L^\infty(\mathbb{R}; L^\infty(\Omega))$  such that

$$(4.4) \quad \nabla \tilde{u}_k \rightharpoonup \nabla \tilde{u} \text{ in } L^\infty(\mathbb{R}; L^\infty(\Omega))\text{-weak}^*,$$

and, due to the weak\* lower semi-continuity of the norm, we also get

$$\sup_{t \in \mathbb{R}} \|\nabla \tilde{u}(t)\|_\infty \leq C_2.$$

Next, we show that  $\tilde{u}$  is a solution to (1.1) in the distributional sense and that verifies relations (2.1a)-(2.1d). In such a way, we will retrieve the existence of a weak solution to (1.1), defined on  $\mathbb{R}$ , with the property that  $\|\nabla \tilde{u}(t)\|_\infty$  is uniformly bounded on the whole real line. This latter fact will be crucial in order to prove the existence of  $\mathcal{S}^2(H)$ -almost-periodic solutions to (1.1) (see below for details).

Let  $L > 0$  be an arbitrary number. By using (2.1d) for the sequence  $\tilde{u}_k$  we get

$$\begin{aligned} |\langle \tilde{u}_k(t) - \tilde{u}_k(s), \varphi \rangle| &\leq \chi \int_s^t |\langle \tilde{u}_k(\tau), \varphi \rangle| d\tau + \int_s^t |b(\tilde{u}_k(\tau), \tilde{u}_k(\tau), \varphi)| d\tau \\ &\quad + \int_s^t |\langle f(\tau), \varphi \rangle| d\tau, \end{aligned}$$

for all  $\varphi \in \mathcal{V}$ , and for all  $-k \leq s \leq t \leq L$ . By the boundedness of  $\nabla \tilde{u}_k$  in  $L^\infty(\mathbb{R}; L^\infty(\Omega))$ , and the hypotheses on  $f$ , it follows that  $\tilde{u}_k(t) - \tilde{u}_k(s)$  is bounded in  $L^2_{loc}(-\infty, L; V')$ . In particular, the sequence  $\tilde{u}_k$  is bounded in  $L^2_{loc}(-\infty, L; V) \cap W^{1,2}_{loc}(-\infty, L; V')$ . By using classical compactness arguments, we can extract a sub-sequence (still labeled as  $\tilde{u}_k$ ) such that

$$\tilde{u}_k \rightarrow \tilde{u} \text{ in } L^2(-L, L; H)\text{-strong},$$

$$\tilde{u}_k \rightharpoonup \tilde{u} \text{ in } L^\infty(-L, L; V)\text{-weak}^*,$$

$$\tilde{u}_k \rightharpoonup \tilde{u} \text{ in } L^2(-L, L; V)\text{-weak},$$

$$\partial_t \tilde{u}_k \rightharpoonup \partial_t \tilde{u} \text{ in } L^2(-L, L; V')\text{-weak},$$

$$\exists E \subset [-L, L] \text{ of zero Lebesgue meas. s.t. } \forall t \in [-L, L] \setminus E, \tilde{u}_k(t) \rightarrow \tilde{u}(t) \text{ in } H,$$

and the limit  $\tilde{u}$  coincides with that in (4.4), due to the uniqueness of the limit for the convergence in distribution. Moreover, by using the standard interpolation theory (see, e.g. [13, 33]), it follows that  $\tilde{u} \in C(\mathbb{R}; H)$ , and so condition (2.1a) is verified.

As a consequence of the strong convergence of  $\tilde{u}_k$  to  $\tilde{u}$  in  $L^2_{loc}(\mathbb{R}; H)$ , for any compact interval  $[-L, L] \subseteq \mathbb{R}$ , we can pass to the limit in equation (2.1d), proving that  $\tilde{u}$  is solution to (1.1) in the space of distributions  $\mathcal{D}'(\mathbb{R}; V')$ .



Now, take inequality (2.1b) for  $\tilde{u}_k$ , i.e.,

$$(4.5) \quad \|\tilde{u}_k(t)\|_2^2 + \chi \int_{-k}^t \|u_k(s)\|_2^2 ds \leq \int_{-k}^t |\langle f(s), \tilde{u}_k \rangle| ds \quad \text{for a.e. } t \in [-k, L].$$

Using again the strong convergence of  $\tilde{u}_k$  to  $\tilde{u}$  in  $L_{loc}^2(\mathbb{R}; H)$ , passing to the limit on both sides of the above inequality, it follows that the left-hand side of (4.5) converges to  $\|\tilde{u}(t)\|_2^2 + \chi \int_{-k}^t \|\tilde{u}(s)\|_2^2 ds$  and the right-hand-side converges to  $\int_{-k}^t \langle f(s), \tilde{u}(s) \rangle ds$ , and then for all  $k \in \mathbb{N}$

$$\|\tilde{u}(t)\|_2^2 + \chi \int_{-k}^t \|\tilde{u}(s)\|_2^2 ds \leq \int_{-k}^t |\langle f(s), \tilde{u}(s) \rangle| ds \quad \text{for a.e. } t \in [-k, +\infty].$$

Thus,  $\tilde{u}$  verifies (2.1b).

Finally, relation (2.1c) easily follows by exploiting the same argument used in [13, § 3] and the solution to elliptic problem for  $\tilde{u} = \text{curl}^{-1} \tilde{\xi}$ , as explained in the previous section.  $\square$

The previous result lead to the definition below.

**Definition 4.1.** *Provided that  $f \in L_{loc}^2(\mathbb{R}; V)$ , we say that a weak solution  $u$  to the dissipative Euler equation (1.1) is “global” if it verifies (2.1a) on  $\mathbb{R}$ , and the properties (2.1b)-(2.1d) hold for a.e.  $t, t_0 \in \mathbb{R}$ , with  $t \geq t_0$ .*

**Remark 4.1.** *Since  $u$  is tangential to the boundary, and  $\Omega$  is bounded then the Poincarè inequality holds, consequently, from  $\nabla u \in L^\infty(\mathbb{R} \times \Omega)$  it follows that also  $u$  is uniformly bounded.*

#### 4.1. Some remarks on uniform bounds in Hilbert spaces

Simpler and more standard techniques can be used to show the following uniform bounds, which are nevertheless too weak for the existence of almost-periodic solutions. We report them for the reader convenience and also to show in a different way some related estimates, which (contrary to the previous section) hold true for any positive  $\chi$ . We point out that they are useless to show certain asymptotic equivalence properties, that is to quantitatively estimate the difference of two solutions starting from different initial data, explaining the critical role of the functional setting we use and of the restrictions on the dissipation constant  $\chi$ .

**Lemma 4.1.** *In addition to the hypotheses of Theorem 2.1 assume that  $f \in L_{uloc}^2(0, +\infty; H)$ . Then, weak solutions  $u$  to (1.1) are defined for all  $t \geq 0$ , they belong to  $L^\infty(0, +\infty; H)$ , and the following estimate holds true*

$$(4.6) \quad \|u(t)\|_2^2 \leq \|u_0\|_2^2 e^{-\chi t} + \frac{3}{\chi^2} \|f\|_{L_{uloc}^2(0, +\infty; H)}^2 \quad t \in [0, \infty).$$

*Proof.* Consider the dissipative Euler equation (1.1) and using  $u$  as test function, we get the following inequality

$$\frac{d}{dt} \|u\|_2^2 \leq -\chi \|u\|_2^2 + \frac{1}{\chi} \|f\|_2^2.$$

Notice that the calculations can be made rigorous by considering the same equations along Galerkin approximate functions, or using the fact that the solution is a weak solution over  $[0, T]$ , for all positive  $T$ . Set  $z(t) := \|u(t)\|_2^2$  and  $\beta(t) := \|f(t)\|_2^2$  (in particular  $\beta \in L^1_{uloc}(0, +\infty)$ ). Now, to estimate  $z$  in  $L^\infty(0, +\infty)$ , we follow a more or less classical argument as in [31, Prop. 2.1]. Suppose there exists  $\bar{t} \in [0, +\infty[$  such that  $z(\bar{t}) \leq z(\bar{t} + 1)$ . Then, it follows that

$$0 \leq z(\bar{t} + 1) - z(\bar{t}) = \int_{\bar{t}}^{\bar{t}+1} \partial_t z(s) ds \leq -\chi \int_{\bar{t}}^{\bar{t}+1} z(s) ds + \frac{1}{\chi} \int_{\bar{t}}^{\bar{t}+1} \beta(s) ds,$$

that is

$$\chi \int_{\bar{t}}^{\bar{t}+1} z(s) ds \leq \frac{1}{\chi} \int_{\bar{t}}^{\bar{t}+1} \beta(s) ds \leq \frac{1}{\chi} \|\beta\|_{L^1_{uloc}(\mathbb{R}^+)}.$$

Observe now that for every  $\tau, \sigma \in [\bar{t}, \bar{t} + 1]$ , it holds that

$$|z(\tau) - z(\sigma)| \leq \int_{\bar{t}}^{\bar{t}+1} \left| -\chi z(s) + \frac{1}{\chi} \beta(s) \right| ds \leq \frac{2}{\chi} \|\beta\|_{L^1_{uloc}(\mathbb{R}^+)}.$$

By the integral mean-value theorem, it follows that there exists  $\zeta \in (\bar{t}, \bar{t} + 1)$  such that  $z(\zeta) = \int_{\bar{t}}^{\bar{t}+1} z(s) ds$ , so we obtain

$$\begin{aligned} z(\bar{t}) \leq z(\bar{t} + 1) &\leq |z(\bar{t} + 1) - z(\zeta)| + \int_{\bar{t}}^{\bar{t}+1} z(s) ds \\ &\leq \frac{2}{\chi} \|\beta\|_{L^1_{uloc}(\mathbb{R}^+)} + \frac{1}{\chi^2} \|\beta\|_{L^1_{uloc}(\mathbb{R}^+)} \leq \frac{3}{\chi^2} \|\beta\|_{L^1_{uloc}(\mathbb{R}^+)}, \end{aligned}$$

and the above estimate holds for every  $\bar{t} \in [0, \infty)$  such that  $z(\bar{t}) \leq z(\bar{t} + 1)$ . Instead, in the case when  $z(\bar{t}) > z(\bar{t} + 1)$ , one repeats the same procedure for  $z(\bar{t} - 1)$  and  $z(\bar{t})$ . Continuing in this manner, we need to estimate  $z(t)$  on  $[0, 1]$ . The estimate in  $[0, 1]$  follows by applying the Gronwall inequality. Hence, we find (4.6).  $\square$

By using the same approach, one can easily show also the following result.

**Lemma 4.2.** *In addition to the hypotheses of Theorem 2.1 assume that  $f \in L^2_{uloc}(0, +\infty; V)$ . Then, weak solutions  $u$  to (1.1) belong to  $L^\infty(0, \infty; V)$ , and the following estimate holds true*

$$\|u(t)\|_{1,2}^2 \leq \|u_0\|_{1,2}^2 e^{-\chi t} + \frac{3}{\chi^2} \|f\|_{L^2_{uloc}(\mathbb{R}^+; V)}^2 \quad t \in [0, \infty).$$

If  $f \in L^2_{uloc}(V)$ , these two lemmas are then enough to show, by the same argument with the initial value problem in  $[-k, +\infty)$  and letting then  $k \rightarrow +\infty$  that

$$\exists C > 0 : \quad \|u(t)\|_{1,2} \leq C \quad \forall t \in \mathbb{R}.$$

The same argument as before implies then the following result

**Theorem 4.2.** *In addition to the hypotheses of Theorem 2.1 assume that  $f \in L^2_{uloc}(\mathbb{R}; V)$ . Then, here exists a weak solution  $\tilde{u}$  to (1.1), defined on  $\mathbb{R}$ , such that*

$$\sup_{t \in \mathbb{R}} \|\tilde{u}(t)\|_{1,2} \leq C < +\infty.$$

The reason why this latter result is useless is that the estimate for  $u \in V$  do not imply any kind of uniqueness. A bounded vorticity is enough to obtain uniqueness, but to estimate in a uniform way the difference of two solutions a bound on the gradient in  $L^\infty(\Omega)$  seems necessary and the larger space in which we are able to prove this result is that of Dini-continuous vorticities.

## 5. Existence of almost-periodic solutions

We finally prove existence of almost-periodic solutions, under the natural assumption that the external force field  $f \in \mathcal{S}^2(H)$  and is also such that  $\text{curl } f \in L^\infty(\mathbb{R}; C_D(\bar{\Omega}))$ . With these hypotheses we will show that the global weak solution built up in Theorem 4.1 is  $\mathcal{S}^2(H)$ -almost-periodic as well, but a restriction on the size of  $\chi$  is needed.

To reach this goal, some preliminary facts will be provided first. Let  $f$  and  $\hat{f}$  be two external force fields satisfying the hypotheses of Theorem 4.1, and take  $u$  and  $\hat{u}$  the associated global weak solutions constructed as in Theorem 4.1. Denote the differences  $w := u - \hat{u}$  and  $g := f - \hat{f}$ . Taking the difference between the two equations satisfied by  $u$  and  $\hat{u}$  we get

$$\partial_t w + \chi w + \nabla(\pi - \hat{\pi}) = -(u \cdot \nabla) w - (w \cdot \nabla) \hat{u} + g.$$

Now, taking the  $L^2$ -product with  $w$  we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 + \chi \|w(t)\|_2^2 &\leq |b(w(t), \hat{u}(t), w(t))| + \|w(t)\| \|g(t)\| \\ &\leq \|w(t)\|_2^2 \|\nabla \hat{u}(t)\|_\infty + \frac{\chi}{2} \|w(t)\|_2^2 + \frac{1}{2\chi} \|g(t)\|_2^2 \\ &\leq C_2 \|w(t)\|_2^2 + \frac{\chi}{2} \|w(t)\|_2^2 + \frac{1}{2\chi} \|g(t)\|_2^2, \end{aligned}$$

where we used the inequality (4.1). Hence, we get with standard manipulations

$$(5.1) \quad \|w(t)\|_2^2 \leq \|w_0\|_2^2 e^{(2C_2 - \chi)(t - t_0)} + \frac{1}{\chi} \int_{t_0}^t \|g(\tau)\|_2^2 e^{(2C_2 - \chi)(t - \tau)} d\tau,$$

where  $w_0 = u_0 - \hat{u}_0$ ,  $t, t_0 \in \mathbb{R}$  with  $t \geq t_0$ .

**Remark 5.1.** *Let  $u$  be a global weak solutions constructed as in Theorem 4.1. Given the external force field  $f$  as in the hypotheses, it is always possible to choose the parameter  $\chi$  large enough such that  $\chi > \sqrt{2C_0\Phi}$ , in order that from the existence result it follows*

$$\sup_{t \in \mathbb{R}} \|\nabla u(t)\|_\infty \leq C_2.$$

Thus, in order to have  $2C_2 - \chi < 0$ , it is sufficient to take

$$\chi > \sqrt{6C_0\Phi} > \sqrt{2C_0\Phi}.$$

For the remainder of this section we always assume that  $2C_2 - \chi$  is strictly negative. We are now ready to proceed to the proof of our main result.

**Theorem 5.1.** *Suppose that the hypotheses of Theorem 4.1 are verified and also that  $f \in \mathcal{S}^2(H)$ . Moreover, suppose that  $\chi > \chi_2 := \sqrt{6C_0\Phi}$ . Then, there exists a weak solution  $u$  to (1.1) such that  $u \in \mathcal{S}^2(H)$ .*

*Proof.* We prove that the global solution  $u$  to (1.1), constructed as in the previous section, belongs to  $\mathcal{S}^2(H)$ . As usual we argue by contradiction, see for instance Foias [22], for early results on the Navier-Stokes equations with “large viscosity”, instead of the large dissipation used here (Notice that in that case the condition on the viscosity is used to ensure global regularity for the three-dimensional problem). Therefore, there is a sequence  $\{h_m\}$  and a function  $\tilde{f}$  such that

$$(5.2) \quad \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s + h_m) - \tilde{f}(s)\|_2^2 ds \rightarrow 0,$$

and there exist a sequence  $\{t_k\}$ , two sub-sequences  $\{h_{m_k}\}, \{h_{n_k}\}$  (of  $\{h_m\}$ ), and a constant  $\delta_0 > 0$  such that

$$(5.3) \quad 0 < \delta_0 \leq \int_{t_k}^{t_k+1} \|u(s + h_{m_k}) - u(s + h_{n_k})\|_2^2 ds \quad \forall k \in \mathbb{N}.$$

Since  $f$  is  $\mathcal{S}^2(H)$ -almost-periodic, by relation (5.2), one has that (up to a sub-sequence  $k'$  still denoted by  $k$ ) there exist  $f_1^*$  and  $f_2^*$  such that

$$(5.4) \quad \begin{aligned} \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s + t_k + h_{m_k}) - f_1^*(s)\|_2^2 ds &\rightarrow 0, \\ \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s + t_k + h_{n_k}) - f_2^*(s)\|_2^2 ds &\rightarrow 0. \end{aligned}$$

Applying the triangle inequality twice, it can be easily proved that  $f_1^* = f_2^* =: f^*$  (for details see [31, Thm 4.1]).

For any  $k \in \mathbb{N}$ , and corresponding to the external force fields  $f_1^k(r) := f(r + t_k + h_{m_k})$  and  $f_2^k(r) := f(r + t_k + h_{n_k})$ , we can construct two global solutions

$u_1^k(r) := u(r + t_k + h_{m_k})$  and  $u_2^k(r) := u(r + t_k + h_{n_k})$ , with  $r \in \mathbb{R}$ . Hence, relation (5.3) can be rewritten as follows

$$(5.5) \quad \delta_0 \leq \int_{t_k}^{t_k+1} \|u_1^k(s - t_k) - u_2^k(s - t_k)\|_2^2 ds = \int_0^1 \|u_1^k(s) - u_2^k(s)\|_2^2 ds,$$

and observe that under our hypotheses

$$\sup_{t \in \mathbb{R}} \|\nabla u_i^k(t)\|_\infty \leq C_2 < +\infty, \quad \text{for } i = 1, 2,$$

where  $C_2$  is given in (4.1).

Following the lines of reasoning in the proof of Theorem 4.1, from  $u_1^k$  and  $u_2^k$  we can extract sub-sequences (still labeled  $u_1^k$  and  $u_2^k$ ) strongly converging in  $L_{loc}^2(\mathbb{R}; H)$  to the global weak solutions  $u_1$  and  $u_2$ , respectively. Thus, passing to the limit in (5.5), we get

$$(5.6) \quad \delta_0 \leq \int_0^1 \|u_1(s) - u_2(s)\|_2^2 ds.$$

On the other hand, exploiting inequality (5.1), we get (recall that  $\chi - 2C_2 > 0$ )

$$\begin{aligned} & \int_0^1 \|u_1^k(s) - u_2^k(s)\|_2^2 ds \\ & \leq \|u_1^k(t_0) - u_2^k(t_0)\|_2^2 \int_0^1 e^{(2C_2 - \chi)(s - t_0)} ds \\ & \quad + \frac{1}{\chi} \int_0^1 ds \int_{t_0}^s \|f_1^k(\tau) - f_2^k(\tau)\|_2^2 e^{(2C_2 - \chi)(s - \tau)} d\tau \\ & \leq \frac{1}{\chi - 2C_2} \|u_1^k(t_0) - u_2^k(t_0)\|_2^2 e^{(\chi - 2C_2)t_0} (1 - e^{-(\chi - 2C_2)}) \\ & \quad + \frac{1}{\chi} \int_0^1 e^{-(\chi - 2C_2)s} ds \int_{t_0}^1 \|f_1^k(\tau) - f_2^k(\tau)\|_2^2 e^{(\chi - 2C_2)\tau} d\tau, \end{aligned}$$

and consequently

$$(5.7) \quad \begin{aligned} \int_0^1 \|u_1^k(s) - u_2^k(s)\|_2^2 ds & \leq \frac{1}{\chi - 2C_2} \|u_1^k(t_0) - u_2^k(t_0)\|_2^2 e^{(\chi - 2C_2)t_0} \\ & \quad + \frac{1}{\chi} \int_{t_0}^1 \|f_1^k(s) - f_2^k(s)\|_2^2 e^{(\chi - 2C_2)s} ds. \end{aligned}$$

Here, without loss of generality, we can assume that  $t_0 \leq 0$ , and recall that  $\|u_i^k\|_2$  is bounded uniformly. Then, fix  $t_0 < 0$  small enough, such that it holds

$$\frac{1}{\chi - 2C_2} \|u_1^k(t_0) - u_2^k(t_0)\|_2^2 e^{(\chi - 2C_2)t_0} < \frac{\delta_0}{4}.$$

In order to estimate the second term on the right-hand side of (5.7), we use a well-known argument used for instance in [26, Lemma 4.1]. Given  $t_0 \leq 0$  determined

from the previous inequality, let  $M \in \mathbb{N}$  such that  $t_0 + (M - 1) \leq 1 \leq t_0 + M$ . Therefore, we have that

$$\begin{aligned} & \int_{t_0}^1 \|f_1^k(s) - f_2^k(s)\|_2^2 e^{(\chi - 2C_2)s} ds \\ & \leq \sum_{m=1}^M \int_{t_0+m-1}^{t_0+m} \|f_1^k(s) - f_2^k(s)\|_2^2 e^{(\chi - 2C_2)s} ds \\ & \leq \sum_{m=1}^M \int_{t_0+m-1}^{t_0+m} \|f_1^k(s) - f_2^k(s)\|_2^2 e^{(\chi - 2C_2)(m+2-M)} ds \\ & = e^{(\chi - 2C_2)(2-M)} \sum_{m=1}^M e^{(\chi - 2C_2)m} \int_{t_0+m-1}^{t_0+m} \|f_1^k(s) - f_2^k(s)\|_2^2 ds, \end{aligned}$$

where we used that  $\chi - 2C_2 > 0$  and also that from the definition of  $M$  it follows  $t_0 \leq 2 - M$ . Hence, adding to both sides  $m \in \mathbb{N}$  the upper bound for the exponential in the interval  $[t_0 + (m - 1), t_0 + m]$  follows.

Next, by using explicit expression for the summation of a geometric sum, we obtain

$$\begin{aligned} & e^{(\chi - 2C_2)(2-M)} \sum_{m=1}^M e^{(\chi - 2C_2)m} \int_{t_0+m-1}^{t_0+m} \|f_1^k(s) - f_2^k(s)\|_2^2 ds, \\ & \leq e^{(\chi - 2C_2)(2-M)} \max_{m=1, \dots, M} \int_{t_0+m-1}^{t_0+m} \|f_1^k(s) - f_2^k(s)\|_2^2 ds \cdot \sum_{m=1}^M e^{(\chi - 2C_2)m} \\ & \leq \frac{e^{(\chi - 2C_2)(M+1)} - 1}{e^{\chi - 2C_2} - 1} e^{(\chi - 2C_2)(2-M)} \sup_{t \geq t_0} \int_t^{t+1} \|f_1^k(s) - f_2^k(s)\|_2^2 ds \\ & \leq \frac{e^{3(\chi - 2C_2)(M+1)}}{e^{\chi - 2C_2}} \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f_1^k(s) - f_2^k(s)\|_2^2 ds \end{aligned}$$

Next, recall that, due to (5.4),  $f_i^k$ , for  $i = 1, 2$ , converges to  $f^*$  in  $L_{uloc}^2(H)$ , as  $k$  goes to  $+\infty$ . Hence, by collecting the estimates and for fixed  $t_0 \leq 0$  and for  $k$  large enough, we obtain

$$\begin{aligned} & \frac{1}{\chi} \int_{t_0}^1 \|f_1^k(s) - f_2^k(s)\|_2^2 e^{(\chi - 2C_2)s} ds \\ & \leq \frac{1}{\chi} \frac{e^{3(\chi - 2C_2)(M+1)}}{e^{\chi - 2C_2}} \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f_1^k(s) - f_2^k(s)\|_2^2 ds < \frac{\delta_0}{4}. \end{aligned}$$

Hence, by collecting all the estimates, we get

$$\int_0^1 \|u_1^k(s) - u_2^k(s)\|_2^2 ds < \frac{\delta_0}{2},$$

and since  $u_i^k$  converges strongly in  $L^2$  to  $u_i$  (and also a.e. up to a redefinition on a subset  $E \subset \mathbb{R}$  of Lebesgue measure zero), we obtain that

$$\int_0^1 \|u_1(s) - u_2(s)\|_2^2 ds \leq \frac{\delta_0}{2},$$

contradicting (5.6) and the assertion is proved.  $\square$

### 5.1. Further regularity of almost-periodic solutions

By using a classical characterization of Stepanov almost-periodic functions and a Theorem of Dafermos [20] we can prove also the following easy corollary.

**Corollary 5.1.** *Suppose that the hypotheses of Theorem 5.1 are verified. Then, there exists a weak solution  $u$  to (1.1) such that  $u \in \mathcal{S}^2(V \cap W^{1,q}(\Omega))$ , for all  $q < \infty$ , and  $\operatorname{curl} u \in \mathcal{S}(C(\bar{\Omega}))$ .*

The proof of this corollary is based first on a characterization of Stepanov almost-periodicity in terms of Bohr-Bochner almost-periodicity, see Bochner [14]. To this end, recall that if we set  $t \mapsto u^t(s) = u(t+s)$ , with  $s \in [0,1]$ , then for  $u \in L^2_{\text{uloc}}(\mathbb{R}; X)$  we can define the map  $t \mapsto u_* := u^t$ , which belongs to  $C(\mathbb{R}; L^2(0,1; X))$ . Then,  $u \in \mathcal{S}^2(X)$  (Stepanov almost-periodic with values in  $X$ ), if and only if  $u_* \in AP(\mathbb{R}; L^2(0,1; X))$ , that is Bohr-Bochner almost-periodic with values in  $L^2(0,1; X)$  (Recall that a function is Bohr-Bochner almost-periodic if it is continuous and its translates are relatively compact in the  $C^0$ -topology.)

Further, we will apply the following lemma due to Dafermos [20].

**Lemma 5.1.** *Let  $Y, Z$  be complete metric spaces, continuously embedded in a Hausdorff space  $W$ . Suppose that*

$$u : \mathbb{R} \rightarrow Y \cap Z$$

*is almost-periodic in  $Y$  and its range is relatively compact in  $Z$ . Then,  $u$  is almost-periodic in  $Z$ . (Here almost-periodicity is in the sense of Bohr-Bochner)*

Next, we will need the following compactness result à la Aubin-Lions (in particular we use a version valid for non-reflexive spaces as proved by Dubinskiĭ, see Simon [32])

**Lemma 5.2.** *Let be given three Banach spaces  $Y_1 \hookrightarrow X \hookrightarrow Y_2$  (that is the first inclusion is compact and the second continuous) the set  $F$  functions  $f : [0, T] \rightarrow Y_2$  such that there exists  $C > 0$*

$$F := \{f \in L^2(0, T; Y_1), f_t \in L^2(0, T; Y_2) : \|f\|_{L^2(0, T; Y_1)} + \|f_t\|_{L^2(0, T; Y_2)} \leq C\}$$

*is relatively compact in  $L^2(0, T; X)$ .*

*Proof of Corollary 5.1.* We observe that by easy computations we have

$$u_t \in L_{uloc}^2(\mathbb{R}; L^2(\Omega)).$$

In fact, by testing the equation (1.1) by  $u_t$  and by Young inequality we get

$$\frac{1}{2} \|u_t\|_2^2 + \frac{\chi}{2} \frac{d}{dt} \|u\|_2^2 \leq \|u\|_2^2 \|\nabla u\|_\infty^2 + \|f\|_2^2,$$

hence, by using the previously obtained bounds for  $\|u\|_2$  and  $\|\nabla u\|_\infty$ , and integrating over a generic interval  $[t, t+1]$  we obtain that

$$u_t \in L_{uloc}^2(\mathbb{R}; H),$$

since  $\nabla \cdot u_t = 0$  and  $(u_t \cdot n)|_\Gamma = 0$ .

By defining the following Banach space

$$\mathcal{E}(\Omega) := \{v : \Omega \rightarrow \mathbb{R}^2 : v \in V \cap C(\overline{\Omega}), \nabla v \in C(\overline{\Omega}), \text{curl } v \in C_D(\overline{\Omega})\},$$

we can observe that we are in the following situation about the time-translates

$$v_*(s) \in C(\mathbb{R}; L^2(0, 1; \mathcal{E}(\Omega))) \quad (v_*)_t(s) \in C(\mathbb{R}; L^2(0, 1; H)).$$

We use the compactness result from Lemma 5.2 with  $Y_1 = \mathcal{E}(\Omega)$ ,  $X = \mathcal{F}(\Omega)$  and  $Y_2 = H$ , where

$$\mathcal{F}(\Omega) := \{v : \Omega \rightarrow \mathbb{R}^2 : v \in H \cap C(\overline{\Omega}), \nabla v \in L^q(\Omega) \forall q < \infty, \text{curl } v \in C(\overline{\Omega})\}.$$

Let us briefly show the compactness of the inclusion  $\mathcal{E}(\Omega) \hookrightarrow \mathcal{F}(\Omega)$ . Let be given a sequence  $\{f_n\}$  bounded in  $\mathcal{E}(\Omega)$ , hence

$$\exists C : \|f_n\|_{V \cap L^\infty} + \|\nabla f_n\|_{L^\infty} + \|\text{curl } f_n\|_{C_D} \leq C \quad \forall n \in \mathbb{N}.$$

We recall now that the embedding of  $C_D(\overline{\Omega})$  into  $C(\overline{\Omega})$  is compact, since (cf. [29, p. 498]) for  $x$  close enough to  $y$

$$|\phi(x) - \phi(y)| \leq \frac{\|\phi\|_{C_D}}{|\log|x-y||} \quad \forall \phi \in C_D(\Omega),$$

hence we have equicontinuity and Ascoli-Arzelà theorem applies. Thus with this observation and by using classical Rellich-Kondrachov results on Sobolev spaces, we can extract a sub-sequence (relabelled as  $\{f_n\}$ ) and find  $f \in \mathcal{F}(\Omega)$  such that

$$\begin{aligned} f_n &\rightharpoonup f && V \cap W^{1,q}(\Omega), \forall q < \infty \\ f_n &\xrightarrow{*} f && W^{1,\infty}(\Omega), \\ f_n &\rightarrow f && H \cap C^{0,\alpha}(\overline{\Omega}), \forall \alpha < 1 \\ \text{curl } f_n &\rightarrow \text{curl } f && L^\infty(\Omega) \\ \nabla f_n &\rightarrow \nabla f && L^q(\Omega), \forall q < \infty \end{aligned}$$

where in particular, we used that the  $L^q$ -norm of the gradient, for all  $q < \infty$  can be controlled with that of the curl and with the divergence (which is vanishing), for functions tangential to the boundary. This is a by product of the representation formulas coming from the potential theory. Hence, by recalling that  $u \in \mathcal{S}^2(H)$  by Theorem 5.1, all the hypotheses of Lemma 5.1 are satisfied with  $Y = X = L^2(0, 1; H)$  and  $Z = L^2(0, 1; \mathcal{F}(\Omega))$ , ending the proof.  $\square$



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