Lipschitz estimates for systems with ellipticity conditions at infinity

LIPSCHITZ ESTIMATES FOR SYSTEMS WITH ELLIPTICITY CONDITIONS AT INFINITY

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ABSTRACT. In the general vector-valued case $N \geq 1$ we prove the Lipschitz-continuity of local minimizers to some integrals of the Calculus of Variations of the form $\int_\Omega g(x, |Du|)\, dx$, with some p, q growth conditions only for $|Du| \to +\infty$ and without further structure conditions on the integrand $g = g(x, |Du|)$. We apply the regularity results to weak solutions to nonlinear elliptic systems of the form $\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i^\alpha (x, Du) = 0$, $\alpha = 1, 2, \dots, N$.

1. Assumptions and statement of the main result

Let Ω be an open bounded subset of $\mathbb{R}^n$, for $n \geq 2$. Let $u : \Omega \to \mathbb{R}^N$ $(N \geq 1)$, $u \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and consider the following functional of the Calculus of Variations

$$F(u) = \int_\Omega f(x, Du)\, dx. \tag{1.1}$$

We say that u is a *local minimizer* of F in (1.1) if $f(x, Du) \in L^1(\Omega)$ and

$$\int_{\mathrm{supp}\, w} f(x, Du)\, dx \leq \int_{\mathrm{supp}\, w} f(x, Du + Dw)\, dx, \tag{1.2}$$

for $w \in W^{1,1}(\Omega; \mathbb{R}^N)$ with $\mathrm{supp}\, w \subset\subset \Omega$. We assume that $f : \Omega \times \mathbb{R}^{Nn} \to [0, +\infty)$ and its derivatives $f_{\xi\xi}$, $f_{\xi x}$ are Carathéodory functions and f is represented in the form $f(x, \xi) = g(x, |\xi|)$ for a given function $g : \Omega \times [0, +\infty) \to [0, +\infty)$. Moreover, there exist positive constants t_0, λ, Λ such that, for all $\mu, \xi \in \mathbb{R}^{Nn}$, $\mu = \mu_i^\alpha, \xi = \xi_i^\alpha$, $i = 1, 2, \dots, n$, $\alpha = 1, 2, \dots N$, for $|\xi| \geq t_0$ and for a.e. $x \in \Omega$

$$\lambda\, |\xi|^{p-2} |\mu|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, \xi) \mu_i^\alpha \mu_j^\beta, \tag{1.3}$$

$$|f_{\xi_i^\alpha \xi_j^\beta}(x, \xi)| \leq \Lambda\, |\xi|^{q-2}, \tag{1.4}$$

$$|f_{\xi x}(x, \xi)| \leq h(x)\, |\xi|^{q-1}, \tag{1.5}$$

for some exponents $1 < p \leq q$ and $h \in L^r(\Omega)$ for some $r > n$. We observe that $g_t(x, 0) = 0$ since $f(x, \xi) = g(x, |\xi|)$ is a smooth function. We also assume $g(x, 0) = 0$. Throughout

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the paper we will denote by B_ρ and B_R balls of radii, respectively, ρ and R (with $\rho < R$) compactly contained in Ω and with the same center, let us say, $x_0 \in \Omega$.

Theorem 1.1. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the integral functional* (1.1)*, whose integrand f satisfies* (1.3)*,* (1.4)*,* (1.5) *with exponents p, q fulfilling*

$$\frac{q}{p} < 1 + \frac{\alpha}{n} \qquad \text{with} \quad \frac{\alpha}{n} = \frac{1}{n} - \frac{1}{r}. \tag{1.6}$$

Then u is locally Lipschitz continuous and for all $0 < \rho < R$ the following estimate holds

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \le C \left(\left(1 + \|h\|^2_{L^r(\Omega)}\right)^{\frac{n}{2\alpha}} \int_{B_R} \{1 + f(x, Du)\}\, dx \right)^\beta, \tag{1.7}$$

with $C \equiv C(n, r, p, q, \lambda, \Lambda, R, \rho)$ and $\beta \equiv \beta(n, p, q, \lambda, \Lambda, R, \rho)$.

As consequence of Theorem 1.1, under the stated assumptions *the Lavrentiev phenomenon* for the integral functional (1.1) cannot occur.

A further relevant consequence is the following regularity result for weak solutions to elliptic systems. In order to state it, we consider a nonlinear elliptic system of PDE's of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^\alpha (x, Du) = 0, \qquad \alpha = 1, 2, \ldots, N, \tag{1.8}$$

where $a_i^\alpha (x, \xi) = f_{\xi_i^\alpha} (x, \xi)$ and $f (x, \xi) = g (x, |\xi|)$. Under the assumptions (1.3), (1.4), (1.5), a solution *in the sense of distributions* to the elliptic system (1.8) is a map $u \in W^{1,p}_{\mathrm{loc}} \left(\Omega; \mathbb{R}^N\right)$ such that

$$\int_\Omega \sum_{i=1}^{n} a_i^\alpha (x, Du) \frac{\partial \varphi^\alpha}{\partial x_i} = 0, \qquad \alpha = 1, 2, \ldots, N, \tag{1.9}$$

for every $\varphi = (\varphi^\alpha)_{\alpha=1,2,\ldots,N} \in C^1_0 \left(\Omega; \mathbb{R}^N\right)$. Note that, in general, for differential problems under $p, q-$*growth conditions* (if p, q are not close enough, precisely, if (1.6) is not satisfied) the notion of solution to the elliptic system (1.8) *in the sense of distributions* may differ from the notion of *weak solution*, the difference being in the class of the allowed test functions φ, which in this second case is $W^{1,q}_{\mathrm{loc}} \left(\Omega; \mathbb{R}^N\right)$ (as necessary to treat variations). I.e., a *weak solution* to the elliptic (1.8) is a map $u \in W^{1,q}_{\mathrm{loc}} \left(\Omega; \mathbb{R}^N\right)$ which satisfies the integral condition (1.9) for every test function $\varphi = (\varphi^\alpha)_{\alpha=1,2,\ldots,N} \in W^{1,q}_{\mathrm{loc}} \left(\Omega; \mathbb{R}^N\right)$. By Theorem 1.1 we have

Corollary 1.2. *Every weak solution to system* (1.8) *is locally Lipschitz continuous in Ω.*

In general the elliptic system (1.8) may even lack a weak solution. Nevertheless, under the assumptions (1.3), (1.4), (1.5) the associated Dirichlet problem can be solved and the two notions of weak solution and solution in the sense of distributions turn out to be equivalent. We have in fact the following regularity results for systems. We consider below a Dirichlet problem, but a similar result could be stated for Neumann conditions, or for more general variational boundary value problems.

Corollary 1.3. *Let $u_0 \in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$ be a map such that $\int_\Omega f\left(x, Du_0\right)\,dx < +\infty$, with f satisfying the assumptions of Theorem 1.1. Then the Dirichlet problem*

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^\alpha\left(x, Du\right) = 0 \qquad \alpha = 1, 2, \ldots, N, & in\ \Omega\,, \\ u = u_0 & on\ \partial\Omega\,, \end{cases} \tag{1.10}$$

has a weak solution $u \in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$. Moreover $u \in W^{1,\infty}_{\mathrm{loc}}\left(\Omega;\mathbb{R}^N\right)$, that is u is locally Lipschitz continuous in Ω.

Corollary 1.4. *Let $f \in C^2\left(\Omega \times \mathbb{R}^{Nn}\right)$ with $f(x,\xi) = g(x,|\xi|)$ satisfying the assumptions of Theorem 1.1. Assume that there exist two positive constants m, M such that, for $0 < t \le 1$, for a.e. $x \in \Omega$*

$$m\left(\mu^2 + t^2\right)^{\frac{p-2}{2}} \le \frac{g_t(x,t)}{t} \le M\left(\mu^2 + t^2\right)^{\frac{p-2}{2}}, \tag{1.11}$$

$$m\left(\mu^2 + t^2\right)^{\frac{p-2}{2}} \le g_{tt}(x,t) \le M\left(\mu^2 + t^2\right)^{\frac{p-2}{2}}, \tag{1.12}$$

$$|g_{tx}(x,t)| \le M\left(\mu^2 + t^2\right)^{\frac{p-1}{2}}, \tag{1.13}$$

for some $\mu \in [0,1]$. Then every weak solution $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ to (1.8) is of class $C^{1,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$, for some $0 < \beta < 1$.

Further regularity of solutions to linear elliptic systems with continuous coefficients applies when we know that the gradient Du is locally in $C^{0,\beta}$ for some $0 < \beta < 1$. Indeed we state the following result.

Corollary 1.5. *Assume that $f \in C^{k-1,\beta}(\Omega;\mathbb{R}^N)$ with $f(x,\xi) = g(x,|\xi|)$ for some $k \ge 2$ and $g_{tt}(x,t) \ge m > 0$ for a.e. $x \in \Omega$, for all $t > 0$. Then every weak solution to elliptic system (1.8) is of class $C^{k,\beta}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.*

Finally we would like to focus on the fact that our assumptions allow us to consider a class of integrals of the Calculus of Variations with variable exponent, which can be typified by the model integral

$$I(u) = \int_\Omega a(x)\,|Du|^{p(x)}\,dx. \tag{1.14}$$

Theorem 1.6. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the integral functional (1.14) with $a(x), p(x)$ satisfying*

$$a(x) \ge a > 0, \qquad p(x) \ge p > 1, \qquad a, p \in W^{1,r}(\Omega), \text{ with } r > n. \tag{1.15}$$

Then u is locally Lipschitz continuous in Ω.

The model integral in (1.14) has been already studied by Coscia and Mingione in [8], where the Hölder continuity of the exponent $p(x)$ is assumed. However we emphasis that the integral in (1.14), in our context, is just a model example and our techniques permit to consider more general integrands as in (**??**).

The Lipschitz regularity for the case $f(x,Du) = a(x)h(|Du|)^{p(x)}$ is considered by the authors in [13].

2. A priori estimates

Let u be a local minimizer of functional (1.1) under the assumptions (1.3), (1.4), (1.5) for a given $t_0 > 0$. We can transform $f(x,\xi)$ into $f(x,t_0\xi)$, which satisfies the same assumptions (1.3), (1.4), (1.5) for $|\xi| \geq 1$ (with different constants depending on t_0). Then it is sufficient to obtain the a priori bound and the regularity results for $v = \frac{1}{t_0}u$. Therefore, for clarity of exposition and without loss of generality, we can assume $t_0 = 1$.

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In this section we make some supplementary assumption on f.

Assumption 2.1. *Assume that $f \in C^(\Omega.R^{nN}$ and there exist two positive constant K and H such that, for $\xi \in R^{nN}$ and a.e. $x \in \Omega$*

$$k\,(1+|\xi|^2)^{\frac{p-2}{2}}|\mu|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)\mu_i^\alpha \mu_j^\beta, \tag{2.1}$$

$$|f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)| \leq \; K\,(1+|\xi|)^{\frac{p-2}{2}}, \tag{2.2}$$

$$|f_{\xi x}(x,\xi)| \leq \; \bar{K}\,(1+|\xi|)^{\frac{p-2}{2}}, \tag{2.3}$$

The previous assumption permit to consider regular minimizer u Indeed,in the next theorem we obtain an a priori estimate for the L^∞-norm of the gradient of u, which are indipendent of K and H.

Proposition 2.2. *Let u be a local minimizer of the integral functional* (1.1)*, whose integrand f satisfies Assumption* (2.1) *and* (1.3)*,* (1.4)*,* (1.5)*, with exponents p, q fulfilling* (1.6)*.*

Then there exist constants $C \equiv C(n,r,p,q,\lambda,\Lambda)$ and $\beta \equiv \beta(n,r,p,q,\lambda,\Lambda)$ such that

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \leq \; C \left(\left[\frac{\left(1+\|h\|^2_{L^r(\Omega)}\right)^{\frac{1}{2}}}{(R-\rho)} \right]^{\frac{n}{\alpha}} \int_{B_R} \{1+f(x,Du)\}\,dx \right)^{\beta}. \tag{2.4}$$

Proof. Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a local minimizer of (1.1). We observe that Df has $p-1$ growth, then u satisfies the Euler's first variation

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(x,Du)\varphi^\alpha_{x_i}(x)\,dx = 0 \qquad \forall \varphi = (\varphi^\alpha)_{\alpha=1,\dots,N} \in W^{1,p}_0(\Omega;\mathbb{R}^N).$$

Since by Assumption (2.1)$D^2 f$ has $p-2$ growth, standard difference quotient technique(see [19], Cap.8, sect.2) we have that

$$u \in W^{2,\min(2,p)}_{\text{loc}}(\Omega;\mathbb{R}^N) \;\text{ and }\; (1+|Du|^2)^{\frac{p-2}{2}}|D^2u| \in L^1_{\text{loc}}(\Omega). \tag{2.5}$$

and moreover the second variation system (see also [19], [22], see also [9, Proposition 3.1])[14])

$$\int_\Omega \left\{ \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)\varphi^\alpha_{x_i} u^\beta_{x_s x_j} + \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\varphi^\alpha_{x_i} \right\} dx = 0 \tag{2.6}$$

$$\forall s = 1,\dots,n, \qquad \forall \varphi = (\varphi^\alpha)_{\alpha=1,\dots,N} \in W^{1,p}_0(\Omega;\mathbb{R}^N).$$

ATTENZIONE DOBBIAMO TOGLIERE IL TEOREMA DI REGOLARITA (TEOREMA 4.4)

FINE-CORREZIONI-ELVIRA

Let $\eta \in C_0^1(\Omega)$. For any fixed $s \in \{1, \dots, n\}$, we choose

$$\varphi^\alpha = \eta^2 u^\alpha_{x_s} \Phi((|Du| - 1)_+)$$

for $\Phi : [0, +\infty) \to [0, +\infty)$ locally Lipschitz continuous function, with Φ and Φ' increasing and bounded on $[0, +\infty)$, such that $\Phi(0) = 0$ and

$$\Phi'(s)s \le c_\Phi \, \Phi(s) \tag{2.7}$$

for a suitable constant $c_\Phi > 0$. Here $(a)_+$ denotes the positive part of $a \in \mathbb{R}$; in the following we denote $\Phi((|Du| - 1)_+) = \Phi(|Du| - 1)_+$. We compute then

$$\varphi^\alpha_{x_i} = 2\eta\eta_{x_i} u^\alpha_{x_s} \Phi(|Du| - 1)_+ + \eta^2 u^\alpha_{x_s x_i} \Phi(|Du| - 1)_+ + \eta^2 u^\alpha_{x_s} \Phi'(|Du| - 1)_+ [(|Du| - 1)_+]_{x_i}.$$

Plugging this expression in (2.6) we obtain:

$$\begin{aligned}
0 \; = \; & \int_\Omega 2\eta\Phi(|Du| - 1)_+ \sum_{i,j,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du)\eta_{x_i} u^\alpha_{x_s} u^\beta_{x_s x_j} \, dx \\
& + \int_\Omega \eta^2 \Phi(|Du| - 1)_+ \sum_{i,j,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j} \, dx \\
& + \int_\Omega \eta^2 \Phi'(|Du| - 1)_+ \sum_{i,j,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) u^\alpha_{x_s} u^\beta_{x_s x_j} [(|Du| - 1)_+]_{x_i} \, dx \\
& + \int_\Omega 2\eta\Phi(|Du| - 1)_+ \sum_{i,\alpha} f_{\xi^\alpha_i x_s}(x, Du)\eta_{x_i} u^\alpha_{x_s} \, dx \\
& + \int_\Omega \eta^2 \Phi(|Du| - 1)_+ \sum_{i,\alpha} f_{\xi^\alpha_i x_s}(x, Du) u^\alpha_{x_s x_i} \, dx \\
& + \int_\Omega \eta^2 \Phi'(|Du| - 1)_+ \sum_{i,\alpha} f_{\xi^\alpha_i x_s}(x, |Du|) u^\alpha_{x_s} [(|Du| - 1)_+]_{x_i} \, dx \\
=: \; & I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{2.8}$$

In the following, constants will be denoted by C, regardless of their actual value.

We now sum the previous equation with respect to s from 1 to n and we denote by $\tilde I_1 - \tilde I_6$ the corresponding integrals.

Let us start with the estimate of the integral $\tilde I_1$. By the Cauchy-Schwartz inequality, the Young inequality and (1.4), we have

$$\begin{aligned}
|\tilde I_1| \; = \; & \left| \int_\Omega 2\eta\Phi(|Du| - 1)_+ \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du)\eta_{x_i} u^\alpha_{x_s} u^\beta_{x_s x_j} \, dx \right| \qquad (2.9) \\
\le \; & \int_\Omega 2\eta\Phi(|Du| - 1)_+ \left\{ \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du)\eta_{x_i} u^\alpha_{x_s} \eta_{x_j} u^\beta_{x_s} \right\}^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left\{ \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j} \right\}^{\frac{1}{2}} dx \\
\leq\ & C \int_\Omega |D\eta|^2 \Phi(|Du|-1)_+ \, |Du|^q \, dx \\
&+ \frac{1}{2} \int_\Omega \eta^2 \Phi(|Du|-1)_+ \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j} \, dx.
\end{aligned}
$$

Let us consider $\tilde{I}_3$. First of all we have that

$$
f_{\xi_i^\alpha \xi_j^\beta}(x,\xi) = \left(\frac{g_{tt}(x,|\xi|)}{|\xi|^2} - \frac{g_t(x,|\xi|)}{|\xi|^3} \right) \xi_i^\alpha \xi_j^\beta + \frac{g_t(x,|\xi|)}{|\xi|} \delta_{\xi_i^\alpha \xi_j^\beta}.
$$

At this point

$$
\begin{aligned}
& \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u^\alpha_{x_s} u^\beta_{x_s x_j} [(|Du|-1)_+]_{x_i} \\
=\ & \left(\frac{g_{tt}(x,|Du|)}{|Du|^2} - \frac{g_t(x,|Du|)}{|Du|^3} \right) \sum_{i,j,s,\alpha,\beta} u^\alpha_{x_s} u^\beta_{x_s x_j} u^\beta_{x_j} u^\alpha_{x_i} [(|Du|-1)_+]_{x_i} \\
& + \frac{g_t(x,|Du|)}{|Du|} \sum_{s,i,\alpha} u^\alpha_{x_s} u^\alpha_{x_s x_i} [(|Du|-1)_+]_{x_i} \qquad (2.10) \\
=\ & \left(\frac{g_{tt}(x,|Du|)}{|Du|} - \frac{g_t(x,|Du|)}{|Du|^2} \right) \sum_\alpha \left[\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right]^2 + g_t(x,|Du|) |D(|Du|-1)_+|^2,
\end{aligned}
$$

where we used the fact that

$$
[(|Du|-1)_+]_{x_i} = (|Du|)_{x_i} = \frac{1}{|Du|} \sum_{\alpha,s} u^\alpha_{x_i x_s} u^\alpha_{x_s} \qquad |Du| \geq 1.
$$

Thus, coming back to the estimate of $\tilde{I}_3$ from (2.10) we deduce

$$
\begin{aligned}
\tilde{I}_3 \ =\ & \int_\Omega \eta^2 \Phi'(|Du|-1)_+ \Bigg\{ \left(\frac{g_{tt}(x,|Du|)}{|Du|} - \frac{g_t(x,|Du|)}{|Du|^2} \right) \sum_\alpha \left[\sum_i u^\alpha_{x_i} (|Du|)_{x_i} \right]^2 \\
& + g_t(x,|Du|) |D(|Du|-1)_+|^2 \Bigg\} dx.
\end{aligned}
$$

Now we argue as in the proof of Lemma 4.1 of [29]. Using the inequality

$$
|D(|Du|-1)_+|^2 \leq |D^2 u|^2, \qquad |Du| \geq 1 \tag{2.11}
$$

we conclude that

$$
\tilde{I}_3 \geq \int_\Omega \eta^2 \Phi'(|Du|-1)_+ \frac{g_{tt}(x,|Du|)}{|Du|} \sum_\alpha \left(\sum_i u^\alpha_{x_i} [(|Du|-1)_+]_{x_i} \right)^2 dx \geq 0,
$$

where we used the fact that $g_{tt}(x,|Du|)\geq 0$ and $\Phi'(|Du|-1)_+\geq 0$. By (1.3), we have that

$$\begin{aligned}\frac{1}{2}\int_\Omega \eta^2\Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx &\leq \frac{1}{2}\tilde{I}_2+\tilde{I}_3\\ \leq\ |\tilde{I}_4|+|\tilde{I}_5|+|\tilde{I}_6|+C\int_\Omega |D\eta|^2\Phi(|Du|-1)_+\,|Du|^q\,dx.\end{aligned}$$

We now deal with $|\tilde{I}_4|$. We have

$$\begin{aligned}|\tilde{I}_4| &= \left|\int_\Omega 2\eta\Phi(|Du|-1)_+\sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\eta_{x_i}u^\alpha_{x_s}\,dx\right|\\ &\overset{(1.5)}{\leq} \int_\Omega 2\eta\Phi(|Du|-1)_+h(x)|Du|^{q-1}\sum_{i,s,\alpha}|\eta_{x_i}u^\alpha_{x_s}|\,dx\\ &\leq \int_\Omega(\eta^2+|D\eta|^2)h(x)\Phi(|Du|-1)_+|Du|^q\,dx.\end{aligned}$$

Consider $|\tilde{I}_5|$, we have

$$\begin{aligned}|\tilde{I}_5| &= \left|\int_\Omega \eta^2\Phi(|Du|-1)_+\sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x,Du)u^\alpha_{x_sx_j}\,dx\right|\\ &\overset{(1.5)}{\leq} \int_\Omega \eta^2\Phi(|Du|-1)_+h(x)|Du|^{q-1}|D^2u|\,dx\\ &\leq \int_\Omega \left[\eta^2\Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\right]^{1/2}\left[\eta^2\Phi(|Du|-1)_+|h(x)|^2|Du|^{2q-p}\right]^{1/2}\,dx\\ &\leq \varepsilon\int_\Omega \eta^2\Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx+C_\varepsilon\int_\Omega \eta^2\Phi(|Du|-1)_+|h(x)|^2|Du|^{2q-p}\,dx,\end{aligned}$$

where in the last line we used the Young inequality. Finally, for any $0<\delta<1$

$$\begin{aligned}|\tilde{I}_6| &= \left|\int_\Omega \eta^2\sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x,Du)u^\alpha_{x_s}\Phi'(|Du|-1)_+[(|Du|-1)_+]_{x_i}\,dx\right|\\ &\overset{(1.5)}{\leq} \int_\Omega \eta^2\Phi'(|Du|-1)_+\,h(x)|Du|^{q-1}|Du||D(|Du|-1)_+|\,dx\\ &\overset{(2.11)}{\leq} \int_\Omega \eta^2\Phi'(|Du|-1)_+h(x)|Du|^q\,|D^2u|\,dx\\ &= \int_\Omega \eta^2\Phi'(|Du|-1)_+h(x)[(|Du|-1)_++\delta]\,[(|Du|-1)_++\delta]^{-1}|Du|^q\,|D^2u|\,dx\\ &\leq \int_\Omega \eta^2\left\{\frac{1}{c_\Phi}\Phi'(|Du|-1)_+[(|Du|-1)_++\delta]|Du|^{p-2}|D^2u|^2\right\}^{1/2}\\ &\quad\times\left\{c_\Phi\Phi'(|Du|-1)_+|h(x)|^2|Du|^{2q-p+2}[(|Du|-1)_++\delta]^{-1}\right\}^{1/2}\,dx\\ &\leq \frac{\varepsilon}{c_\Phi}\int_\Omega \eta^2\Phi'(|Du|-1)_+[(|Du|-1)_++\delta]|Du|^{p-2}|D^2u|^2\,dx\end{aligned}$$

$$+ C_\varepsilon \, c_\Phi \int_\Omega \eta^2 \Phi'(|Du|-1)_+ |h(x)|^2 |Du|^{2q-p+2} [(|Du|-1)_+ + \delta]^{-1} \, dx.$$

We concentrate our attention on the first term in the last inequality. We split the set Ω as $\Omega = \{x : |Du(x)| \geq 2\} \cup \{x : |Du(x)| \leq 2\}$ and we observe that in the set $\{x : |Du(x)| \geq 2\}$ we also have $(|Du|-1)_+ \geq 1$ which in turn implies

$$(|Du|-1)_+ + \delta \leq 2(|Du|-1)_+ \tag{2.12}$$

as long as we have chosen $\delta < 1$.

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Therefore we have, using (2.7) and the fact that Φ' is increasing

$$\begin{aligned}
&\int_\Omega \eta^2 \Phi'(|Du|-1)_+ [(|Du|-1)_+ + \delta] |Du|^{p-2} |D^2u|^2 \, dx \\
\leq \quad & \int_{|Du|\geq 2} \eta^2 \Phi'(|Du|-1)_+ [(|Du|-1)_+ + \delta] |Du|^{p-2} |D^2u|^2 \, dx \\
& + \int_{1<|Du|\leq 2} \eta^2 \Phi'(|Du|-1)_+ [(|Du|-1)_+ + \delta] |Du|^{p-2} |D^2u|^2 \, dx \\
\overset{(2.12)}{\leq} \quad & 2 \int_{|Du|\geq 2} \eta^2 \Phi'(|Du|-1)_+ (|Du|-1)_+ |Du|^{p-2} |D^2u|^2 \, dx \\
& + \int_{1<|Du|\leq 2} \eta^2 \Phi'(|Du|-1)_+ (|Du|-1)_+ |Du|^{p-2} |D^2u|^2 \, dx \\
& + \delta \int_{1<|Du|\leq 2} \eta^2 \Phi'(|Du|-1)_+ |Du|^{p-2} |D^2u|^2 \, dx \\
\overset{(2.7)}{\leq} \quad & 2 \, c_\Phi \int_\Omega \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2} |D^2u|^2 \, dx + \delta \int_{1<|Du|\leq 2} \eta^2 \Phi'(|Du|-1)_+ |Du|^{p-2} |D^2u|^2 \, dx.
\end{aligned}$$

Now, choosing ε sufficiently small and putting together all the estimates obtained for $|\tilde{I}_4|, |\tilde{I}_5|, |\tilde{I}_6|$, we deduce

$$\begin{aligned}
&\int_\Omega \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2} |D^2u|^2 \, dx \\
\leq \quad & C \, c_\Phi \int_\Omega (\eta^2 + |D\eta|^2)(1 + h^2(x)) |Du|^{2q-p} \\
& \times \left[\Phi(|Du|-1)_+ |Du|^{2q-p} + \Phi'(|Du|-1)_+ |Du|^2 [(|Du|-1)_+ + \delta]^{-1} \right] \, dx \\
& + \delta \int_{1<|Du|\leq 2} \eta^2 \Phi'(|Du|-1)_+ |Du|^{p-2} |D^2u|^2 \, dx. ,
\end{aligned} \tag{2.13}$$

with a constant C depending on n, r, p, q.

Now we define

$$\Phi(s) := (1+s)^{\gamma-2} s^2 \qquad \gamma \geq 0; \tag{2.14}$$

we have

$$\Phi'(s) = (\gamma s + 2) s (1+s)^{\gamma-3}. \tag{2.15}$$

This function satisfies (2.7) with $c_\Phi = 2(1+\gamma)$.
We now approximate this function Φ by a sequence of functions Φ_h, each of them being equal to Φ in the interval $[0,h]$, and then extended to $[h,+\infty)$ with the constant value $\Phi(h)$.Moreover Φ_h and Φ'_h converge monotonically to Φ and Φ' respectively. The expression of Φ_h) can be inserted in (2.13) and then it is possible to pass to the limit as $h \to +\infty$ by the monotone convergence theorem. Therefore we obtain for every $0<\delta<1$

$$\begin{aligned} & \int_\Omega \eta^2(1+(|Du|-1)_+)^{\gamma-2}(|Du|-1)_+^2|Du|^{p-2}|D^2u|^2\,dx \\ \leq\ & C\,(1+\gamma)^2\int_\Omega(\eta^2+|D\eta|^2)(1+h(x)^2)(1+(|Du|-1)_+)^{\gamma+2q-p}\,dx \\ & +\delta\, C(\gamma)\int_{1<|Du|<2}\eta^2|Du|^{p-2}|D^2u|^2\,dx, \end{aligned}$$

where we used the fact that

$$\frac{(|Du|-1)_+}{(|Du|-1)_++\delta}\leq 1 \qquad \forall\,\delta>0.$$

and $\Phi'(t-1)_+ \leq C(\gamma)$ when $1<t<2$. Moreover, by the following Lemma 2.3, (2.5) implies

$$\int_{1<|Du|<2}\eta^2|Du|^{p-2}|D^2u|^2\,dx \leq C\int_{1<|Du|<2}\eta^2(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|\,dx<+\infty$$

so we can pass to the limit for $\delta\to 0$ and the last term in the previous inequality vanishes.

——FINE-CORREZIONI-ELVIRA——

Since $h\in L^r(\Omega)$, by the Hölder inequality and by denoting

$$m := \left(\frac{r}{2}\right)' = \frac{r}{r-2}, \tag{2.16}$$

we have, using (2.11)

$$\begin{aligned} & \int_\Omega \eta^2(1+(|Du|-1)_+)^{\gamma-2}(|Du|-1)_+^2|Du|^{p-2}|D((|Du|-1)_+)|^2\,dx \\ \leq\ & C\,(1+\gamma)^2\,H\left[\int_\Omega(\eta^2+|D\eta|^2)^m(1+(|Du|-1)_+)^{(\gamma+2q-p)m}\,dx\right]^{\frac{1}{m}}, \end{aligned} \tag{2.17}$$

by denoting, from now on

$$H := (1+\|h\|^2_{L^r(\Omega)}) \tag{2.18}$$

and where C now depends also on r and $|\Omega|$ (and so on n).

Let us introduce

$$G(t) = 1+\int_0^t\sqrt{\Phi(s)}(1+s)^{\frac{p-2}{2}}\,ds = 1+\int_0^t(1+s)^{\frac{\gamma}{2}+\frac{p}{2}-2}\,s\,ds \tag{2.19}$$

and we obtain the following upper bound for $[G(t)]^2$

$$[G(t)]^2 \leq\ 4\,(1+t)^{\gamma+p} \leq\ 4\,(1+t)^{\gamma+2q-p}, \tag{2.20}$$

where we used the fact that $p \leq q \leq 2q-p$. On the other hand

$$G_t(t) = \sqrt{\Phi(t)}\,(1+t)^{\frac{p-2}{2}} \stackrel{(2.14)}{=} (1+t)^{\frac{\gamma}{2}+\frac{p}{2}-2} t \tag{2.21}$$

which in turn allows us to give the following estimate for the gradient of the function $w = \eta G((|Du|-1)_+)$

$$\begin{aligned} & \int_\Omega |D(\eta G((|Du|-1)_+))|^2\,dx \qquad (2.22)\\ \leq\ & 2\int_\Omega |D\eta|^2 |G((|Du|-1)_+)|^2\,dx \\ & + 2\int_\Omega \eta^2 [G_t((|Du|-1)_+)]^2 [D((|Du|-1)_+)]^2\,dx \\ \stackrel{(2.17),(2.20),(2.21)}{\leq}\ & C\,(1+\gamma)^2\, H\, \left[\int_\Omega (\eta^2+|D\eta|^2)^m [1+(|Du|-1)_+^{(\gamma+2q-p)m}]\,dx\right]^{\frac{1}{m}}. \end{aligned}$$

Now, let $2^* = \frac{2n}{n-2}$ for $n>2$, while 2^* equal to any fixed real number greater than 2, if $n=2$. By Sobolev's inequality there exists a constant C such that

$$\left\{\int_\Omega [\eta G((|Du|-1)_+)]^{2^*}\,dx\right\}^{\frac{2}{2^*}} \leq C\int_\Omega |D(\eta G((|Du|-1)_+))|^2\,dx. \tag{2.23}$$

Moreover, since $r>n$, we have

$$1 \leq m \stackrel{(2.16)}{:=} \frac{r}{r-2} < \frac{n}{n-2} = \frac{2^*}{2}. \tag{2.24}$$

Observe that

$$(2q-p)m = 2(q-p)m + pm; \tag{2.25}$$

moreover, in view of the strict inequality in (1.6), we infer the existence of $0<\epsilon<1$ such that

$$(q-p) + \epsilon\left(\frac{1}{n}-\frac{1}{r}\right) \leq p\left(\frac{1}{n}-\frac{1}{r}\right). \tag{2.26}$$

We also set

$$\tilde{M} := 2(q-p)m + p(m-1) + \epsilon \qquad \tilde{N} := p-\epsilon. \tag{2.27}$$

We remark that $\tilde{M}>0$ because $q \geq p$, $m \geq 1$, $\epsilon>0$ and $\tilde{N}>0$ since $\epsilon<1<p$; moreover we observe that

$$\tilde{M}+\tilde{N} = (2q-p)m \tag{2.28}$$

and

$$\tilde{M} > (2q-p)m - p. \tag{2.29}$$

Now we prove that

$$\frac{1}{(\gamma+p)^2}\left[\int_\Omega \eta^{2^*}[1+(|Du|-1)_+]^{\left(\gamma+\frac{\tilde{M}}{m}\right)\frac{2^*}{2}+\tilde{N}}\,dx\right]^{\frac{2}{2^*}} \leq 4\left(\int_\Omega [\eta G((|Du|-1)_+)]^{2^*}\right)^{\frac{2}{2^*}}. \tag{2.30}$$

In view of (2.19), by setting $t := (|Du|-1)_+$, (2.30) is proved if

$$\frac{1}{\gamma+p}\,(1+t)^{\left(\frac{\gamma}{2}+\frac{\tilde{M}}{2m}+\frac{\tilde{N}}{2^*}\right)} \le 2\left(1+\int_0^t (1+s)^{\frac{\gamma}{2}+\frac{p}{2}-2}\,s\,ds\right). \tag{2.31}$$

Now, if $t \le 1$, then easily

$$\frac{1}{\gamma+p}(1+t)^{\left(\frac{\gamma}{2}+\frac{\tilde{M}}{2m}+\frac{\tilde{N}}{2^*}\right)} \le \frac{2}{\gamma+p} \le 2 \le 2\left(1+\int_0^t (1+s)^{\frac{\gamma}{2}+\frac{p}{2}-2}\,s\,ds\right)$$

so that (2.31) is achieved. Let now $t \ge 1$, then (2.31) becomes, after differentiation

$$\frac{\frac{\gamma}{2}+\frac{\tilde{M}}{2m}+\frac{\tilde{N}}{2^*}}{\gamma+p}(1+t)^{\frac{\gamma}{2}+\frac{\tilde{M}}{2m}+\frac{\tilde{N}}{2^*}-1} \le 2(1+t)^{\frac{\gamma}{2}+\frac{p}{2}-2}t. \tag{2.32}$$

If we are able to show that

$$\frac{\tilde{M}}{2m}+\frac{\tilde{N}}{2^*} \le \frac{p}{2}, \tag{2.33}$$

then we would have

$$\frac{\frac{\gamma}{2}+\frac{\tilde{M}}{2m}+\frac{\tilde{N}}{2^*}}{\gamma+p}(1+t)^{\frac{\tilde{M}}{2m}+\frac{\tilde{N}}{2^*}} \quad\le\quad \frac{1}{2}\,(1+t)^{\frac{p}{2}} = \frac{1}{2}\,(1+t)^{\frac{p-2}{2}}(1+t) \overset{t\ge 1}{\le} (1+t)^{\frac{p-2}{2}}t$$

and so also (2.32) is satisfied.

Thus all is reduced to prove (2.33), which is equivalent to

$$(q-p)+\frac{p}{2}-\frac{p}{2m}+\frac{\epsilon}{2m}+\frac{p}{2^*}-\frac{\epsilon}{2^*} \le \frac{p}{2}.$$

Since

$$\frac{1}{2m}-\frac{1}{2^*} = \frac{r-2}{2r}-\frac{n-2}{2n} = \frac{n(r-2)-r(n-2)}{2nr} = \frac{r-n}{nr} = \frac{1}{n}-\frac{1}{r} = \frac{\alpha}{n},$$

then, by (2.26), the claim (2.30) is satisfied.

By collecting (2.22), (2.23) and (2.30), we obtain

$$\begin{aligned}
&\left[\int_\Omega \eta^{2^*}[1+(|Du|-1)_+]^{\left(\gamma+\frac{\tilde{M}}{m}\right)\frac{2^*}{2}}\,[1+(|Du|-1)_+]^{\tilde{N}}\,dx\right]^{\frac{2}{2^*}} \\
\le\ & C\,H\,(\gamma+2q-p)^4\left[\int_\Omega (\eta^2+|D\eta|^2)^m[1+(|Du|-1)_+]^{(\gamma+2q-p)m}\,dx\right]^{\frac{1}{m}},
\end{aligned} \tag{2.34}$$

where the constant C only depends on $n, r, p, q, \lambda, \Lambda$ but is independent of γ.

Now, let η to be equal to 1 in B_ρ, with $\operatorname{supp}\eta \subset B_R$ and such that $|D\eta| \le \frac{1}{(R-\rho)}$. Let us denote by

$$\kappa := \gamma m + \tilde{M} \overset{(2.28)}{=} (\gamma+2q-p)m - \tilde{N}.$$

We notice that $\kappa \ge \tilde{M}$ since $\gamma \ge 0$; moreover $\frac{2^*}{2m} > 1$ due to (2.24). Therefore, from (2.34) we now have

$$\left\{\int_{B_\rho} [1+(|Du|-1)_+]^{\kappa\frac{2^*}{2m}}[1+(|Du|-1)_+]^{\tilde{N}}\,dx\right\}^{\frac{2m}{2^*}} \tag{2.35}$$

$$\leq \; C\, H^m \left(\frac{(\kappa + \tilde{N})^2}{R-\rho} \right)^{2m} \int_{B_R} [1 + (|Du| - 1)_+]^{\kappa} [1 + (|Du| - 1)_+]^{\tilde{N}} \, dx,$$

where the constant C only depends on $n, r, p, q, \lambda, \Lambda$.

Fixed $\bar{R}$ and $\bar{\rho}$, with $\bar{R} > \bar{\rho}$, we define the decreasing sequence of radii $\{\rho_i\}_{i \geq 0}$

$$\rho_i = \bar{\rho} + \frac{\bar{R} - \bar{\rho}}{2^i} \qquad \forall i \geq 0.$$

We observe that $\rho_0 = \bar{R} > \rho_i > \rho_{i+1} > \bar{\rho}$. We also define the increasing sequence of exponents $\{\kappa_i\}_{i \geq 0}$ such that

$$\kappa_0 := \tilde{M} \qquad \kappa_{i+1} = \kappa_i \frac{2^*}{2m} \qquad i \geq 0.$$

We notice that $\kappa_0 > 0$ because $\tilde{M} > 0$. We rewrite (2.35) with $R = \rho_i$, $\rho = \rho_{i+1}$, $\kappa = \kappa_i$; then, after observing that

$$R - \rho := \rho_i - \rho_{i+1} = \frac{\bar{R} - \bar{\rho}}{2^{i+1}}$$

we obtain for every $i \geq 0$

$$\left\{ \int_{B_{\rho_{i+1}}} [1 + (|Du| - 1)_+]^{\kappa_{i+1}} [1 + (|Du| - 1)_+]^{\tilde{N}} \, dx \right\}^{\frac{1}{\kappa_{i+1}}}$$

$$\leq \; \left[C\, H^m \left(\frac{(\kappa_i + \tilde{N})^{\frac{3}{2}} 2^{i+1}}{\bar{R} - \bar{\rho}} \right)^{2m} \right]^{\frac{1}{\kappa_i}} \left(\int_{B_{\rho_i}} [1 + (|Du| - 1)_+]^{\kappa_i} [1 + (|Du| - 1)_+]^{\tilde{N}} \, dx \right)^{\frac{1}{\kappa_i}}.$$

The last inequality can be rewritten as

$$A_{i+1} \leq \; C_i \, A_i \tag{2.36}$$

having set

$$A_i := \left(\int_{B_{\rho_i}} [1 + (|Du| - 1)_+]^{\kappa_i} [1 + (|Du| - 1)_+]^{\tilde{N}} \, dx \right)^{\frac{1}{\kappa_i}}$$

$$C_i := \left[C\, H^m \left(\frac{(\kappa_i + \tilde{N})^{\frac{3}{2}} 2^{i+1}}{\bar{R} - \bar{\rho}} \right)^{2m} \right]^{\frac{1}{\kappa_i}}.$$

By iteration of (2.36), we deduce

$$\left\{ \int_{B_{\bar{\rho}}} [1 + (|Du| - 1)_+]^{\kappa_0 \left(\frac{2^*}{2m}\right)^{i+1}} [1 + (|Du| - 1)_+]^{\tilde{N}} \, dx \right\}^{\left(\frac{2m}{2^*}\right)^{i+1}}$$

$$\leq \; \tilde{C} \int_{B_{\bar{R}}} [1 + (|Du| - 1)_+]^{(2q-p)m} \, dx, \tag{2.37}$$

where, by taking into account that $m > 1$, we have

$$
\begin{aligned}
\tilde{C} &\leq \prod_{k=0}^{\infty} \left[C\, H^m \left(\frac{(\kappa_k + \tilde{N})^{\frac{3}{2}} 2^{k+1}}{\bar{R} - \bar{\rho}} \right)^{2m} \right]^{\left(\frac{2m}{2^*}\right)^k} \\
&= \prod_{k=0}^{\infty} \left[C\, H^m \left(\frac{\left[\tilde{M} \left(\frac{2^*}{2m}\right)^k + \tilde{N} \right]^{\frac{3}{2}} 2^{k+1}}{\bar{R} - \bar{\rho}} \right)^{2m} \right]^{\left(\frac{2m}{2^*}\right)^k} \\
&\leq \prod_{k=0}^{\infty} \left[\frac{2\, C\, H^m \, [(2q-p)m]^{3m}}{(\bar{R} - \bar{\rho})^{2m}} \right]^{\left(\frac{2m}{2^*}\right)^k} \\
&\leq \frac{C\, H^{\frac{1}{2\left(\frac{1}{n} - \frac{1}{r}\right)}}}{(\bar{R} - \bar{\rho})^{\frac{2\, 2^*\, m}{2^* - 2m}}},
\end{aligned}
$$

with a constant $C = C(n, r, p, q)$. Let us denote

$$
\tau := \frac{2\, 2^*\, m}{2^* - 2m} = \frac{1}{\frac{1}{n} - \frac{1}{r}}; \tag{2.38}
$$

thus (2.37) implies

$$
\left\{ \int_{B_{\bar{\rho}}} [1 + (|Du| - 1)_+]^{\kappa_0 \left(\frac{2^*}{2m}\right)^{i+1}} dx \right\}^{\left(\frac{2m}{2^*}\right)^{i+1}} \leq C \left[\frac{\sqrt{H}}{(\bar{R} - \bar{\rho})} \right]^{\tau} \int_{B_{\bar{R}}} [1 + (|Du| - 1)_+]^{(2q-p)m} \, dx. \tag{2.39}
$$

At this point we pass to the limit as $i \to +\infty$, obtaining

$$
\begin{aligned}
\sup \left\{ [1 + (|Du| - 1)_+]^{\tilde{M}} : x \in B_{\bar{\rho}} \right\} &= \lim_{i \to +\infty} \left\{ \int_{B_{\bar{\rho}}} [1 + (|Du| - 1)_+]^{\tilde{M} \left(\frac{2^*}{2m}\right)^{i+1}} \right\}^{\left(\frac{2m}{2^*}\right)^{i+1}} \\
&\leq C \left[\frac{\sqrt{H}}{(\bar{R} - \bar{\rho})} \right]^{\tau} \int_{B_{\bar{R}}} [1 + (|Du| - 1)_+]^{(2q-p)m} \, dx.
\end{aligned} \tag{2.40}
$$

Let us now set

$$
V(x) := 1 + (|Du|(x) - 1)_+ \qquad \text{and} \qquad s := (2q - p)m; \tag{2.41}
$$

then estimate (2.40) becomes

$$
\sup_{x \in B_\rho} |V(x)| \leq C \left(\left[\frac{\sqrt{H}}{(\bar{R} - \bar{\rho})} \right]^{\frac{\tau}{s}} \|V\|_{L^s(B_R)} \right)^{\frac{s}{\tilde{M}}} \tag{2.42}
$$

for every ρ, R such that $0 < \rho < R \leq \rho + 1$ and where $C = C(n, r, p, q)$.

We now use the classical interpolation inequality

$$
\|V\|_{L^s(B_\rho)} \leq \|V\|_{L^p(B_\rho)}^{\frac{p}{s}} \|V\|_{L^\infty(B_\rho)}^{1 - \frac{p}{s}}, \tag{2.43}
$$

which permits to estimate the essential supremum of $|Du|$ in terms of its L^p-norm. In fact (2.42) and (2.43) give

$$\|V\|_{L^s(B_\rho)} \leq C^{1-\frac{p}{s}} \|V\|^{\frac{p}{s}}_{L^p(B_\rho)} \left(\left[\frac{\sqrt{H}}{(\bar{R}-\bar{\rho})} \right]^{\frac{\tau}{s}} \|V\|_{L^s(B_R)} \right)^{\theta} \tag{2.44}$$

where

$$\theta := \frac{s}{\tilde{M}} \left(1 - \frac{p}{s}\right) = \frac{1}{\tilde{M}}(s-p) \overset{(2.41)}{=} \frac{1}{\tilde{M}}[(2q-p)m - p] \overset{(2.29)}{<} 1. \tag{2.45}$$

For $0 < \bar{\rho} < \bar{R}$ and for every $k \geq 0$, let us define

$$\rho_k := \bar{R} - (\bar{R} - \bar{\rho})2^{-k} \qquad B_k := \|V\|_{L^s(B_{\rho_k})}.$$

By inserting in (2.44) $\rho = \rho_k$ and $R = \rho_{k+1}$ (so that $R - \rho = (\bar{R} - \bar{\rho})2^{-(k+1)}$) we have for every $k \geq 0$

$$B_k \leq C^{1-\frac{p}{s}} \|V\|^{\frac{p}{s}}_{L^p(B_{\bar{R}})} \left(2^{\frac{\tau}{s}(k+1)} \left[\frac{\sqrt{H}}{(\bar{R}-\bar{\rho})} \right]^{\frac{\tau}{s}} B_{k+1} \right)^{\theta}. \tag{2.46}$$

By iteration of (2.46), we deduce for $k \geq 0$

$$B_0 \leq \left(C^{1-\frac{p}{s}} \left[\frac{\sqrt{H}}{(\bar{R}-\bar{\rho})} \right]^{\frac{\tau}{s}\theta} \|V\|^{\frac{p}{s}}_{L^p(B_{\bar{R}})} \right)^{\sum_{i=0}^{k} \theta^i} 2^{\frac{\tau}{s}\sum_{i=0}^{k+1} i\theta^i} (B_{k+1})^{\theta^{k+1}}. \tag{2.47}$$

By (2.45), the series appearing in (2.47) are convergent.

Since B_k is bounded independently of k, i.e.

$$B_{k+1} \leq \|V\|_{L^s(B_{\bar{R}})},$$

we can pass to the limit as $k \to +\infty$ and we obtain for every $0 < \rho < R$ with a constant $C = C(n, r, p, q)$ independent of k

$$\|V\|_{L^s(B_\rho)} \leq C \left(\left[\frac{\sqrt{H}}{(R-\rho)} \right]^{\frac{\tau}{s}\theta} \|V\|^{\frac{p}{s}}_{L^p(B_R)} \right)^{\frac{1}{1-\theta}}. \tag{2.48}$$

Combining (2.42) and (2.48), by setting $\rho' = \frac{(R+\rho)}{2}$ we have

$$\begin{aligned}
\|V\|_{L^\infty(B_\rho)} &\leq C \left(\left[\frac{\sqrt{H}}{(\rho'-\rho)} \right]^{\frac{\tau}{s}} \|V\|_{L^s(B_{\rho'})} \right)^{\frac{s}{\tilde{M}}} \\
&\leq C \left(\left[\frac{\sqrt{H}}{(\rho'-\rho)} \right]^{\frac{\tau}{s}(1-\theta)} \left[\frac{\sqrt{H}}{(R-\rho')} \right]^{\frac{\tau}{s}\theta} \|V\|^{\frac{p}{s}}_{L^p(B_R)} \right)^{\frac{s}{\tilde{M}}\frac{1}{1-\theta}};
\end{aligned}$$

now, since

$$(\rho' - \rho) = (R - \rho') = \frac{R-\rho}{2},$$

this implies

$$\|Du\|_{L^\infty(B_\rho)} \le C \left[\frac{\sqrt{H}}{(R-\rho)}\right]^{\tilde{\beta}} \left(\int_{B_R} (1+|Du|^p)\, dx\right)^{\beta},$$

with

$$\beta := \frac{1}{\tilde{M}(1-\theta)} \overset{(2.45)}{=} \frac{1}{\tilde{M}\left(1-\frac{s}{M}+\frac{p}{M}\right)} = \frac{1}{\tilde{M}-s+p} \overset{(2.25),(2.27)}{=} \frac{1}{\epsilon} > 1 \tag{2.49}$$

$$\tilde{\beta} := \frac{\tau}{s}\frac{s}{\tilde{M}}\frac{1}{1-\theta} \overset{(2.38),(2.49)}{=} \frac{1}{\frac{1}{n}-\frac{1}{r}}\frac{1}{\epsilon} = \frac{n}{\alpha}\frac{1}{\epsilon} \tag{2.50}$$

Since $f(x,\xi) \ge C|\xi|^p$ for every $|\xi| \ge 1$, (2.4) follows. □

We need in the proof above the following elementary result:

In the following we need of the elementary lemma:

Lemma 2.3. *For every $t \in R$, with $t \ge t_0 > 0$ for some $t_0 > 0$ then for all $p \ge 1$ we have:*

$$\min\left\{\left(\frac{t_0^2}{t_0^2+1}\right)^{\frac{p-2}{2}}, 1\right\}(1+t^2)^{\frac{p-2}{2}} \le t^{p-2} \le \max\left\{\left(\frac{t_0^2}{t_0^2+1}\right)^{\frac{p-2}{2}}, 1\right\}(1+t^2)^{\frac{p-2}{2}} \tag{2.51}$$

with $C > 0$ depending on t_0.

Per la dimostrazione, vedi lo scanner che ti ho inviato

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