A CHARACTERIZATION OF BALLS THROUGH OPTIMAL CONCAVITY FOR POTENTIAL FUNCTIONS

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ABSTRACT. In this short note two unconventional overdetermined problems are considered. Let $p \in (1, n)$; firstly, the following is proved: if Ω is a bounded domain in $\mathbf{R}^n$ whose p-capacitary potential function u has two homotetic convex level sets, then Ω is a ball. Then, as an application, we obtain the following: if Ω is a convex domain in $\mathbf{R}^n$ whose p-capacitary potential function u is $(1-p)/(n-p)$-concave (i.e. $u^{(1-p)/(n-p)}$ is convex), then Ω is a ball.

1. Introduction

Let $n \geq 3$, $\Omega \subset \mathbf{R}^n$ and $p \in (1, n)$. The p–capacity of Ω can be defined as follows (see for instance [10], §4.7):

$$(1.1) \qquad \mathrm{Cap}_p(\Omega) = \inf \left\{ \int_{\mathbf{R}^n} |\nabla v|^p \, dx \, : \, v \in C_c^\infty(\mathbf{R}^n) \, , \, v(x) \geq 1 \, \forall \, x \in \Omega \right\} \, ,$$

where $C_c^\infty(\mathbf{R}^n)$ denotes the set of functions from $C^\infty(\mathbf{R}^n)$ having compact support.

In the sequel Ω is a bounded open convex set, then the above infimum is in fact a minimum which is realized by the (classical) solution u of the following problem

$$(1.2) \qquad \begin{cases} \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0 \, , & \text{in} \quad \mathbf{R}^n \setminus \overline{\Omega} \\[2mm] u(x) = 1 \quad \text{in } \overline{\Omega} \, , \\[2mm] \lim_{|x| \to +\infty} u(x) = 0 \, . \end{cases}$$

The function u is called p–capacitary potential function of Ω and it holds

$$(1.3) \qquad \mathrm{Cap}_p(\Omega) = \int_{\mathbf{R}^n \setminus \overline{\Omega}} |\nabla u|^p \, dx \, .$$

It is well known that if Ω is (bounded, open and) convex, then u is *quasi-concave*, that is all its superlevel sets

$$\Omega(t) = \{x \in \mathbb{R}^n \, : \, u(x) \geq t\} \quad t \in (0, 1]$$

are convex, see [11, 17, 14]. In fact, if Ω is smooth and strictly convex, one could even expect u to satisfy some stronger concavity property, in particular u could be

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power concave. We recall here that a positive function v is said α-*concave*, for some $\alpha \in [-\infty, +\infty)$, if

$$\begin{cases} v^\alpha \text{ is concave, in case } \alpha > 0, \\ \log v \text{ is concave, in case } \alpha = 0, \\ v^\alpha \text{ is convex, in case } \alpha < 0, \\ \text{all the super level sets } \{x \in \mathbb{R}^n \, : \, v(x) \geq t\} \text{ are convex, in case } \alpha = -\infty. \end{cases}$$

In the cases $\alpha = -\infty$ and $\alpha = 0$, v is also (and more usually) said *quasi-concave* and *log-concave*, respectively, while when $\alpha = 1$ one is dealing with usual concave functions.

Notice that, if $v \in C^2(\mathbb{R}^n, [0, \infty))$ and $\alpha > -\infty$, then v is α-concave if and only if

$$(1.4) \qquad\qquad v\nabla^2 v \leq (1 - \alpha)\nabla v \otimes \nabla v \quad \text{in } \mathbb{R}^n$$

in the sense of symmetric matrices.

From Jensen's inequality (or from (1.4) is v is regular enough) it follows that if v is α-concave for some $\alpha > -\infty$, then v is β-concave for every $\beta \leq \alpha$; and it is clearly quasi-concave too. Given a quasi-concave function v, it is then natural to ask whether it satisfies some better concavity properties and following [16] it makes sense to define the *concavity number* of u as follows

$$\alpha(v) = \sup\{\beta \leq 1 \, : \, v \text{ is } \beta\text{-concave}\}.$$

It is easily also seen that if $\alpha(v) \in \mathbb{R}$, then the above supremum is in fact a maximum. Please refer to [16] for more details on α-concave functions.

Finally, since, as we said, when Ω is convex its p-capacitary potential u is quasi-concave, we can set

$$\alpha(\Omega, p) = \alpha(u)$$

which we may call *the p-capacitary concavity number* of Ω.

If Ω is sufficiently regular and strictly convex, one can expect that $\alpha(\Omega, p) > -\infty$ (see Section 5 for a partial discussion of this and related topics). Indeed, when Ω is a ball of radius $R > 0$ centered at x_0 it is easy to find explicitly the solution of (1.2), that is

$$u(x) = R^q \, |x - x_0|^{-q}$$

where

$$q = \frac{n - p}{p - 1},$$

and it results to be $(-1/q)$-concave.

In this short note I prove that nothing better is possible and that this power concavity is optimal among convex sets, in the sense that the property of $u^{-1/q}$ *to be convex* characterizes balls. Precisely, the main result of this paper is the following.

Theorem 1.1. *Let $p \in (1, n)$ and Ω be a bounded convex domain in $\mathbb{R}^n$. Then*

$$\alpha(\Omega, p) \leq -\frac{p - 1}{n - p}$$

and equality holds if and only if Ω is a ball.

To prove this theorem we will use three main ingredients:
- the first one is the Brunn-Minkowski inequality for p-capacity and its equality condition, proved in [3, 4] for $p = 2$ and in [8] for a generic p;
- the second ingredient is an expression of p-capacity through the behavior at infinity of the potential function, see formula (2.4);
- the third ingredient is an easy relation existing between the p-capacity of a generic level set of u and the capacity of Ω, see formula (2.5).

In fact the second ingredient is needed to prove the following property, which has its own interest and it is new, to my knowledge.

Theorem 1.2. *If the solution u of* (1.2) *has two homothetic convex level sets, then Ω is a ball.*

In particular: *if u has a level set that is homothetic to Ω (and Ω is convex), then Ω is a ball.* We recall here that two sets $A, B \subset \mathbb{R}^n$ are said homothetic if there exist $\rho > 0$ and $\xi \in \mathbb{R}^n$ such that $B = \rho A + \xi$, i.e. if they are dilate and translate of each other.

Let me notice here that, even though in this presentation Theorem 1.1 is somewhat enlighten more than Theorem 1.2, the former can be in fact seen just as a nice application of the latter.

To some extent, both the problems considered in Theorem 1.1 and Theorem 1.2 fall in the framework of overdetermined problems. An overdetermined problem usually consists in a Dirichlet problem given in an unknown domain, whose solution is required to satisfy some extra condition (classically a Neumann boundary condition) which is so strong to determine univocally the shape of the domain itself. The most famous overdetermined problem is of course the one solved by Serrin [21], where the overdetermination writes as $u = 0$ and $|\nabla u| = $ constant on $\partial\Omega$, but also other kinds of overdetermined conditions have been considered in literature (see for instance [1, 2, 5, 6, 9, 13, 22, 23] and references therein). Here in Theorem 1.1 the overdetermination is given by the concavity property of the solution u of (1.2), while in Theorem 1.2 the overdetermination is given by the existence of two homotethic level sets.

As a final remark, it has to be noticed that the proof of Theorem 1.1 actually yields a much stronger result, namely Theorem 4.1 (see at the end of Section 4). The latter may look less attractive than Theorem 1.1, but in fact it is precisely what we prove and Theorem 1.1 could be seen a straightforward corollary of Theorem 4.1.

The paper is organized as follows. Firstly in Section 2 I recall some needed results and formulas (in particular the three main ingredients recalled above). Section 3 contains the proof of Theorem 1.2 while Theorem 1.1 is proved in Section 4. Finally Section 5 is devoted to some final comments and remarks; in particular it is showed that there exist convex sets different from a ball whose p-capacitary concavity number is greater than $-\infty$ and at the same time it is proved that there actually exist convex sets whose p-capacitary concavity number is precisely $-\infty$.

2. Preliminaries

2.1. Ingredient 1: the Brunn-Minkowski inequality for p-capacity.

The original form of the Brunn–Minkowski inequality involves volumes of convex bodies (i.e. compact convex subsets of $\mathbf{R}^n$ with non–empty interior) and states that $\mathrm{Vol}_n(\cdot)^{1/n}$ is a concave function with respect to the Minkowski addition, i.e.

$$(2.1) \qquad [\mathrm{Vol}_n(\lambda K_1 + (1-\lambda)K_2)]^{\frac{1}{n}} \geq \lambda\,[\mathrm{Vol}_n(K_1)]^{\frac{1}{n}} + (1-\lambda)\,[\mathrm{Vol}_n(K_2)]^{\frac{1}{n}}$$

for every convex bodies K_1 and K_2 and $\lambda \in [0,1]$. Here Vol_n is the n-dimensional Lebesgue measure and the Minkowski addition of convex sets is defined as follows

$$A + B = \{x + y \mid x \in A,\ y \in B\},$$

while $\lambda A = \{\lambda x : x \in A\}$ for any $\lambda \in \mathbb{R}$, as usual.

Inequality (2.1) is one of the fundamental results in the modern theory of convex bodies; it can be extended to measurable sets and several other important inequalities, e.g. the isoperimetric inequality, can be deduced from it.

Suitable versions of the Brunn-Minkowski inequality hold also for the other quermassintegrals (see [20, 12]) and recently Brunn-Minkowski type inequalities have been proved for several important geometric and analytic functionals (see for instance the beautiful survey paper [12] by R. Gardner and possibly [19] for more recent references). Notice that in all the known cases, equality conditions are the same as in the classical Brunn-Minkowski inequality for the volume, i.e. equality holds if and only if the involved sets are (convex and) homothetic (i.e. translate and dilate of each other).

We will use the following theorem from [8].

Theorem 2.1 (Theorem 1, [8]). *Let K_1 and K_2 be n–dimensional convex bodies and let $p \in (1, n)$. Then*
(2.2)
$$\left[\mathrm{Cap}_p(\lambda K_1 + (1-\lambda)K_2)\right]^{\frac{1}{n-p}} \geq \lambda\,\left[\mathrm{Cap}_p(K_1)\right]^{\frac{1}{n-p}} + (1-\lambda)\,\left[\mathrm{Cap}_p(K_2)\right]^{\frac{1}{n-p}},$$

for every $\lambda \in [0,1]$. Moreover equality holds if and only if K_1 and K_2 are homothetic.

Roughly speaking (2.2) says that $\mathrm{Cap}_p(\cdot)^{\frac{1}{n-p}}$ is a concave function in the class of convex bodies endowed with the Minkowsky addition. But what is most relevant to the present paper is the equality condition: equality holds in (2.2) if and only if K_1 and K_2 are homothetic.

We recall that in the case of the Newton capacity, i.e. for $p = 2$ and $n \geq 3$, inequality (2.2) was proved by C. Borell [3] and more recently in [4] L.A. Caffarelli, D. Jerison and E.H. Lieb treated the equality case. In [8] the treatments of the inequality and of its equality case are unified and the results are extended to a generic $p \in (1, n)$.

2.2. Ingredient 2: an expression of p-capacity through the behavior at infinity of the potential.

In the case $p = 2$ $(n \geq 3)$ it is well known that the following relation between the Newton capacity of a convex domain and the behavior at infinity of the newtonian potential holds:

$$(2.3) \qquad\qquad \mathrm{Cap}_2(\Omega) = (n-2)\,\omega_n \lim_{|x|\to\infty} u(x)|x|^{n-2},$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$. An analogous relation holds in the generic case $p \in (1, n)$

$$(2.4) \qquad \mathrm{Cap}_p(\Omega) = c_{n,p} \left[\lim_{|x| \to \infty} u(x)|x|^{(n-p)/(p-1)} \right]^{p-1},$$

where

$$c_{n,p} = \left(\frac{n-p}{p-1} \right)^{p-1} \omega_n \, ;$$

refer to [8] for instance.

2.3. Ingredient 3: the p-capacity of a level set of the potential. Let u be the p-capacitary potential of a domain Ω and set

$$\Omega(t) = \{ x \in \mathbb{R}^n \, : \, u(x) \geq t \}$$

for $t \in (0, 1]$. Then it is easily seen that the following holds

$$(2.5) \qquad \mathrm{Cap}_p(\Omega(t)) = t^{1-p} \mathrm{Cap}_p(\Omega) \, .$$

Indeed, the p-capacitary potential u_t of $\Omega(t)$ is given by $u_t(x) = t^{-1}u(x)$, as it can be trivially verified, and (2.5) follows directly from (2.4). Notice however that (2.5) holds in more generality than (2.4), since it follows also from

$$\mathrm{Cap}_p(\Omega) = \int_{\partial\Omega} |\nabla u|^{p-1} d\sigma = \int_{\partial\Omega(t)} |\nabla u|^{p-1} d\sigma \, , \quad \text{for every } t \leq 1 \, ,$$

which can be obtained by integration by parts and does not need convexity of level sets, which is instead useful to prove (2.4).

3. Proof of Theorem 1.2

Let $0 < r < s \leq 1$, $\rho > 1$, $\xi \in \mathbb{R}^n$ such that

$$(3.1) \qquad \Omega(r) = \rho \, \Omega(s) + \xi \, ,$$

that is $\Omega(r)$ and $\Omega(s)$ are the homothetic superlevel sets of the statement.

Notice that, since $r < s$, it holds

$$\Omega(s) \subset \Omega(r) \, .$$

For $t \in (0, 1]$, let us denote by u_t the p-capacitary potential of $\Omega(t)$, i.e. the solution of

$$\begin{cases} \mathrm{div}(|\nabla u_t|^{p-2} \nabla u_t) = 0 \, , & \text{in} \quad \mathbf{R}^n \setminus \Omega(t) \\[2mm] u_t(x) = 1 & \text{in } \overline{\Omega}(t) \, , \\[2mm] \lim_{|x| \to +\infty} u_t(x) = 0 \, . \end{cases}$$

Then

$$u_t(x) = \frac{u(x)}{t} \, , \quad x \in \mathbf{R}^n \setminus \Omega(t) \, .$$

In particular

$$u_r(x) = \frac{u(x)}{r} \quad \text{for } x \in \mathbf{R}^n \setminus \Omega(r) \quad \text{and} \quad u_s(x) = \frac{u(x)}{s} \quad \text{for } x \in \mathbf{R}^n \setminus \Omega(s) \, .$$

On the other hand by (3.1) it holds

$$u_r(x) = u_s\left(\frac{x-\xi}{\rho}\right)$$

and we finally get

(3.2) $$u(x) = \frac{r}{s} u\left(\frac{x-\xi}{\rho}\right), \quad x \in \mathbf{R}^n \setminus \Omega(r).$$

Then by (2.4) the latter implies

$$\left(\frac{\mathrm{Cap}_p(\Omega)}{c_{n,p}}\right)^{\frac{1}{p-1}} = \lim_{|x|\to\infty} u(x)|x|^{\frac{n-p}{p-1}} = \frac{r}{s} \lim_{|x|\to\infty} u\left(\frac{x-\xi}{\rho}\right)|x|^{\frac{n-p}{p-1}}$$

$$= \frac{r}{s} \rho^{\frac{n-p}{p-1}} \lim_{|x|\to\infty} u\left(\frac{x-\xi}{\rho}\right) \left(\frac{|x-\xi|}{\rho}\right)^{\frac{n-p}{p-1}} \left(\frac{|x|}{|x-\xi|}\right)^{\frac{n-p}{p-1}}$$

$$= \frac{r}{s} \rho^{\frac{n-p}{p-1}} \left(\frac{\mathrm{Cap}_p(\Omega)}{c_{n,p}}\right)^{\frac{1}{p-1}},$$

whence

(3.3) $$\frac{r}{s} = \rho^{\frac{p-n}{p-1}}.$$

Moreover (3.2) entails

$$\Omega(t) = \rho\,\Omega\left(\frac{s}{r}t\right) + \xi \quad \text{for } t < r.$$

Hence, by setting

$$s_0 = s, \quad , s_1 = r, \quad s_k = \left(\frac{r}{s}\right)^k s = \rho^{\frac{k(p-n)}{p-1}} s, \quad k = 2, 3, \ldots,$$

it holds

$$\lim_{k\to\infty} s_k = 0$$

and

$$\Omega(s_k) = \rho\,\Omega(s_{k-1}) + \xi = \rho^2 \Omega(s_{k-2}) + \rho\xi + \xi = \ldots$$

(3.4)

$$= \rho^k \Omega(s_0) + \xi \sum_{i=0}^{k-1} \rho^i = \rho^k \Omega(s) + \xi \frac{\rho^k - 1}{\rho - 1}.$$

Now let $x, y \in \partial\Omega(s)$, i.e.

$$u(x) = u(y) = s,$$

and set

$$x_k = \rho^k x + \xi \frac{\rho^k - 1}{\rho - 1},$$

$$y_k = \rho^k y + \xi \frac{\rho^k - 1}{\rho - 1}.$$

Then

$$\lim_{k\to\infty} |x_k| = \lim_{k\to\infty} |y_k| = \infty$$

and (2.4) yields

(3.5) $$\lim_{k\to\infty} u(x_k)|x_k|^{\frac{p-1}{n-p}} = \left(\frac{\mathrm{Cap}_p(\Omega)}{c_{n,p}}\right)^{1/(p-1)} = \lim_{k\to\infty} u(y_k)|y_k|^{\frac{p-1}{n-p}}.$$

On the other hand

$$u(x_k) = u(y_k) = s_k \,,$$

hence (3.5) reads

$$\lim_{k\to\infty} s_k |x_k|^{\frac{p-1}{n-p}} = \lim_{k\to\infty} s_k |y_k|^{\frac{p-1}{n-p}} \,,$$

that is

$$\lim_{k\to\infty} \left(\rho^{\frac{k(p-n)}{p-1}} s \left| \rho^k x + \xi \frac{\rho^k - 1}{\rho - 1} \right|^{\frac{p-1}{n-p}} \right) = \lim_{k\to\infty} \left(\rho^{\frac{k(p-n)}{p-1}} s \left| \rho^k y + \xi \frac{\rho^k - 1}{\rho - 1} \right|^{\frac{p-1}{n-p}} \right) \,,$$

which implies

$$\lim_{k\to\infty} \left| x + \xi \frac{1 - \rho^{-k}}{\rho - 1} \right|^{\frac{p-1}{n-p}} = \lim_{k\to\infty} \left| y + \xi \frac{1 - \rho^{-k}}{\rho - 1} \right|^{\frac{p-1}{n-p}} \,.$$

Since $\rho > 1$, the latter finally implies

$$\left| x + \xi \frac{1}{\rho - 1} \right| = \left| y + \xi \frac{1}{\rho - 1} \right| = R \,,$$

which means that $\Omega(s)$ is a ball or radius R centered at the point $\xi/(1 - \rho)$, i.e.

$$\Omega(s) = B\left(\frac{\xi}{1 - \rho}, R\right) \,,$$

and from (3.4) we easily obtain

$$\Omega(s_k) = B\left(\frac{\xi}{1 - \rho}, R\rho^k\right) \quad \text{for } k = 1, 2, \dots \,.$$

Then u is radial in $\mathbb{R}^n \setminus \overline{\Omega(s)}$ and, by analytic continuation, it is radial in $\mathbb{R}^n \setminus \overline{\Omega}$ and Ω is a ball.

4. Proof of Theorem 1.1

Let $u \in C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \overline{\Omega})$ be the solution of (1.2) and

$$q = (1 - p)/(n - p) \,.$$

Firstly we notice that, thanks to the monotonicity of the α-concavity property, if u is α-concave for some $\alpha \geq q$, then it is q-concave. Next we will proceed by proving that, if u is q-concave, then all its level sets are homothetic (and finally the proof will be concluded thanks to Theorem 1.2).

Assume that

$$v = u^q \quad \text{is convex in } \mathbb{R}^n \,.$$

Hence for every $v_0, v_1 \in \mathbb{R}$ and for every $\mu \in (0, 1)$ it holds
(4.1)
$$\{x \in \mathbb{R}^n \,:\, v(x) \leq (1-\mu)v_0 + \mu v_1\} \supseteq (1-\mu)\{x \in \mathbb{R}^n \,:\, v(x) \leq v_0\} + \mu\{x \in \mathbb{R}^n \,:\, v(x) \leq v_1\} \,.$$

Now take $r, s \in (0, 1]$, fix $\lambda \in (0, 1)$ and set

$$(4.2) \qquad\qquad t = [(1 - \lambda)r^q + \lambda s^q]^{1/q} \,.$$

By setting $v_0 = r^q$ and $v_1 = s^q$, we have

$$t^q = (1 - \lambda)v_0 + \lambda v_1$$

and

$$\Omega(r) = \{v \leq r^q\} \,,$$

$$\Omega(s) = \{v \leq s^q\},$$
$$\Omega(t) = \{v \leq t^q\}.$$

Then (4.1) reads

$$\Omega(t) \supseteq (1-\lambda)\,\Omega(r) + \lambda\,\Omega(s).$$

Thanks to the monotonicity of p-capacity with respect to set inclusion, the latter implies

$$\mathrm{Cap}_p(\Omega(t)) \geq \mathrm{Cap}_p\big((1-\lambda)\,\Omega(r) + \lambda\,\Omega(s)\big)$$

and by chaining with the Brunn-Minkowski inequality for p-capacity (see Theorem 2.1) we obtain

$$(4.3) \qquad \mathrm{Cap}_p(\Omega(t))^{1/(n-p)} \geq (1-\lambda)\,\mathrm{Cap}_p(\Omega(r))^{1/(n-p)} + \lambda\,\mathrm{Cap}_p(\Omega(s))^{1/(n-p)}$$

On the other hand, by (2.5) we have

$$\mathrm{Cap}_p(\Omega(r)) = r^{1-p}\mathrm{Cap}_p(\Omega),$$

$$\mathrm{Cap}_p(\Omega(s)) = s^{1-p}\mathrm{Cap}_p(\Omega)$$

and

$$\mathrm{Cap}_p(\Omega(t)) = t^{1-p}\mathrm{Cap}_p(\Omega).$$

Substituting in (4.3) and taking into account (4.2), we finally get

$$\mathrm{Cap}_p(\Omega(t))^{1/(n-p)} = (1-\lambda)\,\mathrm{Cap}_p(\Omega(r))^{1/(n-p)} + \lambda\,\mathrm{Cap}_p(\Omega(s))^{1/(n-p)},$$

i.e. equality holds in (4.3), and consequently equality must hold in the Brunn-Minkowski inequality for p-capacity for $\Omega(r)$ and $\Omega(s)$. Then Theorem 2.1 tells that $\Omega(r)$ and $\Omega(s)$ must be homothetic. This concludes the proof thanks to Theorem 1.2, as already said.

$\square$

As a final remark of this section, we notice that in fact in the previous proof, we did not use the full strength of q-concavity of u; that is we do not use 4.1 for every combination of μ, v_0 and v_1, but it is actually sufficient to have it for a fixed triplet λ, r, s in order to obtain the full result. Then we actually proved the following strengthened version of Theorem 1.1.

Theorem 4.1. *Let $p \in (1, n)$, $q = (1-p)/(n-p)$, Ω be a bounded domain in $\mathbb{R}^n$ (not necessarily convex) and u be the solution of (1.2). If u has three convex super level sets*

$$\Omega(r) = \{x \in \mathbb{R}^n \,:\, u(x) \geq r\}, \ \ \Omega(s) = \{x \in \mathbb{R}^n \,:\, u(x) \geq s\}, \ \ \Omega(t) = \{x \in \mathbb{R}^n \,:\, u(x) \geq t\}$$

(say $0 < s < r \leq 1$) such that

$$t \geq [(1-\lambda)r^q + \lambda\,s^q]^{1/q}$$

and

$$\Omega(t) \supseteq (1-\lambda)\,\Omega(r) + \lambda\,\Omega(s)$$

for some $\lambda \in (0, 1)$, then Ω is a ball.

The proof is precisely the same as above, as already said. Let me just point out that here we do not need to assume the convexity of Ω; we have however to assume the convexity of $\Omega(r)$, which yields the convexity of $\Omega(s)$ and $\Omega(t)$ (in fact of $\Omega(\tau)$ for every $\tau \leq r$, but not of Ω itself), in order to apply Theorem 2.1, that has been proved only for convex sets up to now.

5. Final comments and remarks.

So far we know that for every convex set Ω the associated p-potential u is quasi-concave, which allow us to define the p-concavity number $\alpha(\Omega, p)$ and in the case of a ball we can explicitly calculate $\alpha(B, p) = (1 - p)/(n - p)$. Then we have proved here that this value is the maximum possible and that it is realized only by the balls. On the other hand, one could wonder whether the ball is the only one set such that $\alpha(\Omega, p) > -\infty$, or even, on the contrary, if it happens that $\alpha(\Omega, p) > -\infty$ for every convex set. The answer to both these questions is negative.

Proposition 5.1. *Let $n = 3$. Then there exist infinitely many convex sets such that $\alpha(\Omega, 2) \geq -2$.*

Proof. Let $n \geq 3$, $p > 1$, $q = (n - p)/(p - 1)$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball and $u(x) = |x|^{-q}$ its p-potential. Then we notice that $v(x) = u(x)^{-2/q} = |x|^2$ is uniformly convex in $\mathbb{R}^n \setminus B$ since

$$(5.1) \qquad\qquad D^2 v = 2I \quad \text{for every } x \in \mathbb{R}^n \setminus B,$$

where I denotes the $n \times n$ identity matrix.

Now let $n = 3$ and $p = 2$ (whence $q = 1$), take $a, b, c > 0$ and denote by $\Omega(a, b, c)$ the ellipsoid with axes of length a, b and c, i.e.

$$\Omega(a, b, c) = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1 \right\}.$$

Then the Newton potential u_{abc} of $\Omega(a, b, c)$ can be computed as in [15, Section 6 - formula (33)] and it can be seen to depend smoothly (together with all its derivatives) on a, b and c. Then chosen any $\epsilon \in (0, 1)$, by (5.1) for a, b, c close enough to 1 we have $D^2(u_{abc}^{-2}) \geq \epsilon I$, which means that u_{abc} is (-2)-concave, i.e. $\alpha(\Omega(a, b, c), 2) \geq -2$. $\qquad \square$

Actually, even for generic $n \geq 3$ and $p > 1$, thanks to (5.1) if a smooth convex domain Ω is close to a ball in some strong sense so that its potential u is C^2-close enough to the potential of the ball, then $\alpha(\Omega, p) \geq -2q$.

Proposition 5.2. *Let $n \geq 3$ and $p > 1$. There exist infinitely many convex set such that the associated p-capacitary potential is genuinely quasi-concave and nothing more, that is $\alpha(\Omega, p) = -\infty$.*

Proof. Consider an hypercube $Q = \{x = (x_1, \ldots, x_n) : |x_i| < 1 \ i = 1, \ldots, n\}$ and let u be its p-capacitary potential. Then it is easily seen by a barrier argument (see for instance [18, Section 4.2]) that ∇u blows up on the vertices and edges of Q (in fact at every singular point of ∂Q). This prevents u to be α-concave for any $\alpha > -\infty$, since the gradient of u^α has to blow up at the same points (and this contradicts the monotonicity property of the gradient of a convex function).

The argument obviously works for every convex polytopes. $\qquad \square$

According to the above argument, the main obstacle to power concavity of the potential seems to be the lack of regularity of $\partial \Omega$. Then let me state the following conjecture, after recalling that a convex domain in said of class C_+^2 if it has a C^2 boundary whose Gauss curvature never vanishes.

Conjecture: $\alpha(\Omega, p) > -\infty$ for every Ω of class C_+^2 (and for every $p > 1$).

In conclusion, it remains open a final natural question: whether for every $\alpha \in [-\infty, -1/q]$ (and for every $p > 1$) there exist convex sets whose p-capacitary concavity number is precisely α. To investigate in more detail this question is beyond the scope of this short note, but I would guess the answer is affirmative (and Propositions 5.1 and 5.2 give some hints in this direction).

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