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Global Kneser solutions to nonlinear equations with indefinite weight

Zuzana Došlá, Mauro Marini and Serena Matucci

Abstract. The paper deals with the nonlinear differential equation

$$(a(t)\Phi(x'))' + b(t)F(x) = 0, \quad t \in [1, \infty),$$

in the case when the weight b has indefinite sign. In particular, the problem of the existence of the so-called globally positive Kneser solutions, that is solutions x such that $x(t) > 0$, $x'(t) < 0$ on the whole closed interval $[1, \infty)$, is considered. Moreover, conditions assuring that these solutions tend to zero as $t \rightarrow \infty$ are investigated by a Schauder's half-linearization device jointly with some properties of the principal solution of an associated half-linear differential equation. The results cover also the case in which the weight b is a periodic function or it is unbounded from below.

Keywords. Second order nonlinear differential equation, boundary value problem on infinite intervals, globally positive solution, half-linear equation, disconjugacy, principal solution.

MSC 2010: Primary 34B40, Secondary 34B18.

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1 Introduction

In the paper, we consider the nonlinear differential equation for $t \in [1, \infty)$

$$(a(t)\Phi(x'))' + b(t)F(x) = 0, \tag{1}$$

where $\Phi(u) := |u|^\alpha \operatorname{sgn} u$, $\alpha > 0$, associated to the boundary conditions

$$x(1) = c > 0, \quad x(t) > 0 \text{ for } t \geq 1, \quad \lim_{t \rightarrow \infty} x(t) = 0. \quad (2)$$

Denote by Ψ the inverse function of Φ , that is $\Psi(u) := |u|^{1/\alpha} \operatorname{sgn} u$. We assume that the functions a, b are continuous functions on $[1, \infty)$, $a(t) > 0$, and

$$J_a = \int_1^\infty \Psi\left(\frac{1}{a(t)}\right) dt < \infty.$$

Moreover, the nonlinearity F is a continuous function on \mathbb{R} such that $uF(u) > 0$ for $u \neq 0$ and

$$\limsup_{u \rightarrow 0} \frac{F(u)}{\Phi(u)} < \infty. \quad (3)$$

The BVP (??)-(??) arises in the investigation of radial solutions in a fixed exterior domain for elliptic equations, see, e.g., [?]. Boundary value problems (BVPs), associated to equations of type (??), have attracted considerable attention in the last years, especially when they are examined on unbounded domains, see [?, ?, ?, ?, ?, ?]. Positive decreasing solutions of (??) are usually called *Kneser solutions* and have been investigated by many authors, see, e.g. [?, ?] and the references therein. Their asymptotic behavior is deeply studied when the weight b has fixed sign. When b changes sign, the structure of nonoscillatory solutions is more complicated, due to the possible presence of the so-called weakly oscillatory solutions, that is nonoscillatory solutions with changing-sign derivatives, see, e.g., [?, page 1248]. As far as we know, in this case very few results deal with the existence of Kneser solutions and with their decay at infinity, see, e.g., [?, ?, ?, ?, ?]. The investigated problem can be also viewed as an extension to the half-line of recent results on nonlinear BVPs on a compact interval, see, e.g., [?] or [?] and references therein, when the weight has indefinite sign or definite sign, respectively. The paper is motivated also by [?, ?] and completes some results there. More precisely, in [?] some asymptotic BVPs are studied for (??) when $F(u) = |u|^\beta \operatorname{sgn} u$, $\beta > 0$ and $b(t) \leq 0$ and in [?] equations with Sturm-Liouville operator, that is when $\alpha = 1$, are considered.

Our main scope is to state sufficient conditions for the existence of Kneser solutions of (??) subject to the conditions (??). Observe that equation (??) is a nonlinear perturbation of the equation

$$(a(t)\Phi(x'))' = 0,$$

whose solution x satisfying (??) is

$$x(t) = \frac{c}{J_a} \int_t^\infty \Psi \left(\frac{1}{a(t)} \right) dt.$$

Our method is based on a fixed point theorem for operators defined in a Fréchet space by a Schauder's linearization device, which does not require the explicit form of the fixed point operator, see [?, Theorem 1.3]. By means of this approach, the study of the topological properties (compactness and continuity) of the fixed-point operator, can be quite simplified because, very often, these properties become an immediate consequence of good *a-priori* bounds. These bounds are obtained using some properties of principal solutions of an associated half-linear equation, i.e. equation with $F(u) = \Phi(u)$. In Section 2 some properties of half-linear equations are recalled, and some new characterizations of the principal solution are established. A discussion on the assumptions, that are needed for the solvability of the BVP (??)-(??), is given in the final section, jointly with some examples. We point out that our existence result covers also the cases in which the weight b is a periodic function or it is unbounded from below.

We close the Introduction with some notations. For any solution x of (??), denote by $x^{[1]}$ the *quasiderivative* of x , i.e. the function

$$x^{[1]}(t) = a(t)\Phi(x'(t)). \quad (4)$$

Further, denote by b_+, b_- , respectively, the positive and the negative part of b , i.e., $b_+(t) = \max \{b(t), 0\}$, $b_-(t) = -\min \{b(t), 0\}$. Thus $b(t) = b_+(t) - b_-(t)$.

2 Half-linear equations

Consider the half-linear equation

$$(a(t)\Phi(y'))' + \beta(t)\Phi(y) = 0, \quad (5)$$

where β is a continuous function for $t \geq 1$. When (??) is nonoscillatory, the qualitative behavior of solutions of (??) is often studied via the associated Riccati equation, that is the equation

$$w' + \beta(t) + R(t, w) = 0, \quad (6)$$

where

$$R(t, w) = \alpha|w|\Psi\left(\frac{|w|}{a(t)}\right),$$

see, e.g., [?, Chapter 2.2.]

Recently, the notion of principal solution to (??) has been introduced in [?, ?, ?] by following the Riccati approach, see, also [?, Section 4.2.]. More precisely, among all eventually different from zero solutions of (??), there exists one, say w_x , which is continuable to infinity and is minimal in the sense that any other solution w of (??), which is continuable to infinity, satisfies $w_x(t) < w(t)$ as $t \rightarrow \infty$. This concept extends to the half-linear case the well-known notion of principal solution that was introduced in 1936 by W. Leighton and M. Morse for the linear case, see [?, Chapter XI. 6.].

Using the Sturmian separation theorem, the following comparison result holds. It is an easy consequence of [?, Theorem 4.2.2] and extends to the half-linear case a well-known criterion for the linear case, see [?, Corollary 6.5]. We recall the result in the form that will be needed in the sequel. Consider the half-linear equations

$$(a(t)\Phi(y'))' + \beta_2(t)\Phi(y) = 0, \tag{7}$$

$$(a(t)\Phi(z'))' + \beta_1(t)\Phi(z) = 0, \tag{8}$$

where β_i $i = 1, 2$ are continuous on $[1, \infty)$ and

$$\beta_1(t) \leq \beta_2(t) \text{ for } t \geq T \geq 1. \tag{9}$$

Lemma 1. [?, Theorem 4.2.2] *Let (??) be nonoscillatory. Assume (??) and denote by y_0, z_0 the principal solutions of (??) and (??), respectively, such that $y_0(t) > 0, z_0(t) > 0$ for $t \geq T_1 \geq T$ and $z_0(T_1) = y_0(T_1)$. Then we have for $t \geq T_1$*

$$0 < z_0(t) \leq y_0(t). \tag{10}$$

In addition, if $y_0'(t) < 0$ on $[T_1, \infty)$, then

$$z_0'(t) < 0 \text{ for } t \geq T_1. \tag{11}$$

Proof. Consider the associated Riccati equations to (??), (??), and denote by v_y, w_z their minimal solutions, respectively. Since y_0 and z_0 are positive

on $[T_1, \infty)$, the functions v_y, w_z exist on $[T_1, \infty)$. Using [?, Theorem 4.2.2] we get for $t \geq T_1$

$$w_z(t) = \frac{z_0^{[1]}(t)}{\Phi(z_0(t))} \leq \frac{y_0^{[1]}(t)}{\Phi(y_0(t))} = v_y(t) \quad (12)$$

or

$$\frac{z_0'(t)}{z_0(t)} \leq \frac{y_0'(t)}{y_0(t)}. \quad (13)$$

Integrating (??) on $[T_1, t]$ we get (??). Finally, (??) follows from (??). \square

Lemma ?? requires the positiveness of the principal solution in a *a-priori* fixed interval. To this end an important role is played by the disconjugacy property for (??). We recall that (??) is said to be *disconjugate* on an interval $I \subset [T, \infty)$ if any nontrivial solution of (??) has at most one zero on I . We refer to [?, Chapters 1.2 and 5.1] for basic properties of disconjugacy. If (??) is disconjugate on $[T, \infty)$, $T \geq 1$, then an easy consequence of [?, Theorem 4.2.3.] gives that the principal solution of (??) does not have zeros on (T, ∞) .

Lemma 2. *Equation (??) is disconjugate on $[T, \infty)$ if and only if (??) has the principal solution without zeros on (T, ∞) .*

Proof. Let (??) be disconjugate on $[T, \infty)$ and let y_0 be the principal solution of (??). By contradiction, assume that y_0 has a zero at some $T_1 > T$. Denote by y a nonprincipal solution of (??) with a zero point at some \bar{t} , with $T < \bar{t} < T_1$. Thus, by [?, Theorem 4.2.3.], the solution y has also a zero point on (T_1, ∞) , that is a contradiction with the disconjugacy of (??). The vice-versa follows from [?, Theorem 1.2.7.]. \square

Nevertheless, disconjugacy cannot be sufficient for the positiveness of the principal solution on the whole close half-line $[T, \infty)$, as it is shown in [?]. A sufficient condition is given by the following. Consider the equation

$$(a_1(t)\Phi(\xi'))' + \beta_1(t)\Phi(\xi) = 0, \quad (14)$$

where

$$a_1(t) \leq a(t), \beta(t) \leq \beta_1(t) \text{ on } t \geq T \geq 1, \quad (15)$$

i.e. equation (??) a Sturmian majorant of (??). The following holds.

Lemma 3. *If (??) is disconjugate on $[T, \infty)$ and at least one of the inequalities in (??) is strict on an interval of positive measure, then the principal solution of (??) is positive on the whole closed interval $[T, \infty)$.*

Proof. Since (??) is disconjugate on $[T, \infty)$, equation (??) is disconjugate too on the same interval. From Lemma ??, equation (??) has the principal solution y_0 without zeros on (T, ∞) . If $y_0(T) = 0$, in view of [?, Theorem 4.2.3.] every solution of (??) has a zero point on (T, ∞) , that is a contradiction. Hence $y_0(T) \neq 0$ and the assertion follows. \square

We close the section with some necessary and sufficient characterizations of the principal solution of (??), that are needed in the sequel. Define

$$J_\beta = \int_1^\infty |\beta(t)| \Phi \left(\int_t^\infty \Psi \left(\frac{1}{a(s)} \right) ds \right) dt. \quad (16)$$

Lemma 4. *Assume $J_\beta < \infty$. Then (??) is nonoscillatory and a solution y_0 of (??), $y_0(t) > 0$ for large t , is its principal solution if and only if any of the following conditions is satisfied.*

$i_1)$

$$\lim_{t \rightarrow \infty} \frac{y_0(t)}{y(t)} = 0 \quad (17)$$

for any nontrivial solution y of (??) such that $y \neq \lambda y_0$, $\lambda \in \mathbb{R}$.

$i_2)$

$$y_0'(t) < 0 \text{ for large } t \text{ and } \lim_{t \rightarrow \infty} y_0(t) = 0, \quad 0 < \lim_{t \rightarrow \infty} |y_0^{[1]}(t)| < \infty. \quad (18)$$

$i_3)$

$$\lim_{t \rightarrow \infty} y_0(t) = 0, \quad \beta \Phi(y_0) \in L^1[1, \infty). \quad (19)$$

Proof. The nonoscillation of (??) follows from [?, Theorem 1]. The characterizations of the principal solution in claims $i_1)$ and $i_2)$ are in [?, Theorem 4] and [?, Theorem 3], respectively.

Now, let us show that (??) is a necessary and sufficient characterization of the principal solution of (??). If y_0 is the principal solution of (??), then (??) holds. Thus, from $(1 \leq t_1 \leq t_2)$

$$y_0^{[1]}(t_1) - y_0^{[1]}(t_2) = \int_{t_1}^{t_2} \beta(s) \Phi(y_0(s)) ds \quad (20)$$

we get $\beta \Phi(y_0) \in L^1[1, \infty)$ and so (??) is valid. Conversely, assume (??). By contradiction, suppose that y_0 is not the principal solution of (??) and let \bar{y} be the principal solution of (??) such that $\bar{y}(t) > 0$ for large t . From (??), the quasiderivative $y_0^{[1]}$ has a finite limit as t tends to infinity. Since y_0 is not the principal solution, from (??) we have $\lim_{t \rightarrow \infty} y_0^{[1]}(t) = 0$. Hence, we obtain

$$\lim_{t \rightarrow \infty} \frac{\bar{y}^{[1]}(t)}{y_0^{[1]}(t)} = \infty,$$

and, by the l'Hopital rule we get

$$\lim_{t \rightarrow \infty} \frac{\bar{y}(t)}{y_0(t)} = \infty,$$

which contradicts (??). □

Observe that the characterization (??), roughly speaking, means that the principal solution is the "smallest one" in a neighborhood of infinity.

When $\beta_+(t) = \max\{\beta(t), 0\}$ is identically zero for any large t , then the principal solution of (??) can be characterized without assuming $J_\beta < \infty$. The following result is an easy consequence of [?, Corollary 3.3].

Lemma 5. *Assume $\beta_+(t) = 0$ for $t \geq t_1 \geq 1$. Then (??) is nonoscillatory and a solution y_0 of (??), $y_0(t) > 0$ for large t , is its principal solution if and only if*

$$y'(t) < 0 \text{ for large } t \text{ and } \lim_{t \rightarrow \infty} y_0(t) = 0. \quad (21)$$

Proof. Since $\beta(t) \leq 0$ for any large t , equation (??) is nonoscillatory, see, e.g., [?, Lemma 4.1.2.]. Define

$$J_1 = \lim_{T \rightarrow \infty} \int_1^T \Psi\left(\frac{1}{a(r)}\right) \Psi\left(\int_1^r |\beta(s)| ds\right) dr,$$

$$J_2 = \lim_{T \rightarrow \infty} \int_1^T \Psi\left(\frac{1}{a(r)}\right) \Psi\left(\int_r^T |\beta(s)| ds\right) dr,$$

If $J_2 = \infty$, the assertion follows from [?, Corollary 3.3.]. If $J_2 < \infty$, we have for $1 < T_1 < T$

$$\int_1^T \Psi\left(\frac{1}{a(r)}\right) \Psi\left(\int_r^T |\beta(s)| ds\right) dr > \left(\int_1^{T_1} \Psi\left(\frac{1}{a(r)}\right)\right) \Psi\left(\int_{T_1}^T |\beta(s)| ds\right) dr.$$

Thus, since $J_a < \infty$, we obtain

$$\lim_{T \rightarrow \infty} \int_1^T |\beta(s)| ds < \infty,$$

and so $J_1 < \infty$. Hence, the assertion follows again from [?, Corollary 3.3].
□

Observe that the characterization (??) can be not true when $J_\alpha = \infty$. Indeed, in this case, if $J_1 = \infty$ and $J_2 < \infty$, then all nontrivial solutions y of (??), with $\beta_+(t) = 0$ for $t \geq t_1 \geq 1$, verify $\lim_{t \rightarrow \infty} |y(t)| > 0$, see [?, Theorem 4.1.4.].

3 The main result

In this section we prove the existence of solutions for the BVP (??)-(??). The solvability is based on a general fixed point theorem for operators defined in a Fréchet space by a Schauder's linearization device ([?, Theorem 1.4]). In particular, this result does not require the explicit form of the fixed point associated operator. Moreover, it seems particularly useful when the BVP is considered in a noncompact interval. In this case, it permit us to overcome difficulties which originate from the check of topological properties, like the compactness, of the fixed point associated operator. Roughly speaking, it reduces the solvability of the BVP to the existence of *a-priori* bounds for solutions of another, possibly nonlinear, BVP. We recall it in the form that will be used in the following.

Theorem 1. *Consider the BVP on $[1, \infty)$,*

$$(a(t)\Phi(x'))' + b(t)F(x) = 0, \quad x \in S, \quad (22)$$

where S is a nonempty subset of the Fréchet space $C[1, \infty)$ of the continuous functions defined in $[1, \infty)$ endowed with the topology of uniform convergence on compact subsets of $[1, \infty)$. Let G be a continuous function on \mathbb{R}^2 , such that $F(d) = G(d, d)$ for any $d \in \mathbb{R}$. Assume that there exist a nonempty, closed, convex and bounded subset $\Omega \subseteq C[1, \infty)$ and a bounded closed subset $S_1 \subseteq S \cap \Omega$ such that for any $u \in \Omega$ the BVP on $[1, \infty)$

$$(a(t)\Phi(x'))' + b(t)G(u(t), x(t)) = 0, \quad x \in S_1 \quad (23)$$

admits a unique solution. Then the BVP (??) has at least a solution.

Denote by \tilde{F} the function

$$\tilde{F}(v) = \frac{F(v)}{\Phi(v)} \text{ on } (0, \infty). \quad (24)$$

In view of (??), fixed $c > 0$, there exists M_c such that

$$\tilde{F}(v) \leq M_c \text{ on } [0, c]. \quad (25)$$

The following holds.

Theorem 2. *Let $c > 0$ be fixed and M_c be given by (??). Consider the half-linear differential equation (??), where*

$$a_1(t) \leq a(t), \quad \beta_1(t) \geq M_c b_+(t) \text{ on } t \geq 1, \quad (26)$$

and at least one of the inequalities in (??) is strict on an interval of positive measure. Assume that (??) is disconjugate on $[1, \infty)$ and its principal solution ξ_0 is positive on $(1, \infty)$ and decreasing for any $t \geq 1$.

Then, the BVP (??)-(??) has at least one solution x if any of the following conditions holds.

i₁)

$$\lim_{T \rightarrow \infty} \int_1^T |b(t)| \Phi \left(\int_t^\infty \Psi \left(\frac{1}{a(s)} \right) ds \right) dt < \infty. \quad (27)$$

i₂) There exists $\bar{t} \geq 1$ such that $b_+(t) = 0$ for any $t \geq \bar{t}$.

*Moreover, the solution x is decreasing for any t and, if *i₁*) holds, the limit*

$$\lim_{t \rightarrow \infty} \frac{x(t)}{\int_t^\infty \Psi(a^{-1}(s)) ds} \quad (28)$$

is finite and different from zero.

Proof. Consider the equations

$$(a(t)\Phi(y'))' + M_c b_+(t)\Phi(y) = 0, \quad (29)$$

$$(a(t)\Phi(z'))' - M_c b_-(t)\Phi(z) = 0. \quad (30)$$

Since (??) is disconjugate on $[1, \infty)$, from (??) and Lemma ?? we have that the principal solution y_0 of (??) started at $y_0(1) = c$ is positive on the whole interval $[1, \infty)$. Moreover, we have $\lim_{t \rightarrow \infty} y_0(t) = 0$ as it follows from Lemma

?? or from Lemma ??, according to i_1) or i_2) holds. Since the principal solution of the Sturmian majorant (??) is positive decreasing, in view of Lemma ?? we have $y_0'(t) < 0$ for any $t \geq 1$. Since $-M_c b_-(t) \leq 0 \leq M_c b_+(t)$, equation (??) is a Sturmian majorant for (??). Thus (??) is disconjugate on $[1, \infty)$ too and its principal solution z_0 , $z_0(1) = c$, is positive for $t \geq 1$. Using the comparison result stated in Lemma ??, we get on $[1, \infty)$

$$0 < z_0(t) \leq y_0(t), \quad z_0'(t) < 0, \quad \lim_{t \rightarrow \infty} z_0(t) = 0.$$

Case i_1). Now, assume (??). Let Ω and S be the subsets of the Fréchet space $C[1, \infty)$ given by

$$\begin{aligned} \Omega &= \{u \in C[1, \infty), z_0(t) \leq u(t) \leq y_0(t)\}, \\ S &= \left\{x \in C[1, \infty), x(1) = c, x(t) > 0, \lim_{t \rightarrow \infty} x(t) = 0, b \Phi(x) \in L^1[1, \infty)\right\}. \end{aligned} \quad (31)$$

Notice that $\Omega \subset S$. Indeed, if (??) holds, from (??) a positive constant M_y exists such that

$$y_0(t) \leq M_y \int_t^\infty \Psi\left(\frac{1}{a(s)}\right) ds, \quad t \geq 1. \quad (32)$$

Thus, we have for every $u \in \Omega$

$$\begin{aligned} \int_1^\infty |b(t)| \Phi(u(t)) dt &\leq \int_1^\infty |b(t)| \Phi(y_0(t)) dt \\ &\leq \Phi(M_y) \int_1^\infty |b(t)| \Phi\left(\int_t^\infty \Psi\left(\frac{1}{a(s)}\right) ds\right) dt < \infty. \end{aligned}$$

Hence, taking into account that $z_0(1) = y_0(1) = c$, and $\lim_{t \rightarrow \infty} y_0(t) = 0$, we obtain $\Omega \subset S$. For any $u \in \Omega$, consider the half-linear equation

$$(a(t)\Phi(x'))' + b(t)\tilde{F}(u(t))\Phi(x) = 0, \quad (33)$$

where \tilde{F} is given in (??). In view of Lemma ??, for any $u \in \Omega$, equation (??) has a unique solution $x_u \in S$, which is the principal solution. In view of (??), equations (??) and (??) are a Sturmian majorant and a Sturmian minorant for (??), respectively. Thus, using again the comparison result in Lemma ??, we obtain on $[1, \infty)$

$$0 < z_0(t) \leq x_u(t) \leq y_0(t), \quad (34)$$

i.e., $x_u \in \Omega$. By applying Theorem ??, with $S_1 = \Omega \cap S = \Omega$, we get the existence of a solution x_0 of (??) on $[1, \infty)$ belonging to the set $\Omega \cap S$. Clearly x_0 satisfies all the conditions in (??). Moreover, since x_0 is the principal solution of (??) for $u = x_0$, from Lemma ?? we get

$$-\infty < \lim_{t \rightarrow \infty} x_0^{[1]}(t) < 0,$$

and, using the l'Hopital rule, we obtain that the limit (??) is finite and different from zero.

Case i_2). Assume that $\bar{t} \geq 1$ exists such that $b_+(t) = 0$ for any $t \geq \bar{t}$. Let Ω be the subset of $C[1, \infty)$ given by (??). As before, for any $u \in \Omega$, equation (??) has a unique principal solution x_u . In view of Lemma ??, we have $x_u \in S_2$, where

$$S_2 = \left\{ x \in C[1, \infty), x(1) = c, x(t) > 0, \lim_{t \rightarrow \infty} x(t) = 0 \right\},$$

and, clearly, if x is a solution of (??) and $x \in S_2$, then x is the principal solution. A similar argument to the one given in case i_1) gives (??) and so $\Omega \subseteq S_2$. From Theorem ??, with $S_1 = \Omega \cap S_2 = \Omega$, there exists a solution x_0 of (??) on $[1, \infty)$. Since x_0 is also principal solution of (??) with $u = x_0$, and $\lim_{t \rightarrow \infty} y_0(t) = 0$, in view of (??) the solution x_0 satisfies the boundary conditions (??). \square

As follows from the proof of Theorem ??, when (??) holds, then (??) has a positive solution x such that $\lim_{t \rightarrow \infty} x(t) = 0$, $-\infty < \lim_{t \rightarrow \infty} x^{[1]}(t) < 0$. The following result illustrates, in some sense, the necessity of assumption (??) for obtaining solutions with this kind of asymptotic growth at infinity.

Theorem 3. *If there exist a solution x of (??) which is not identically zero for large t and*

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} x^{[1]}(t) = 0, \quad (35)$$

then

$$\lim_{T \rightarrow \infty} \int_1^T |b(t)| \Phi \left(\int_t^\infty \Psi(a^{-1}(s)) ds \right) dt = \infty. \quad (36)$$

Proof. By contradiction, assume (??). By (??) there exists $M > 0$ such that for $u \in [-1, 1]$

$$|F(u)| \leq M|\Phi(u)|.$$

Choose t_0 large so that

$$\int_{t_0}^{\infty} |b(s)| \Phi \left(\int_s^{\infty} \Psi(a^{-1}(r)) dr \right) ds < (2M)^{-1}. \quad (37)$$

Without loss of generality, suppose $|x(t)| \leq 1$ on $[t_0, \infty)$. From (??) we get for $t \geq t_0$

$$|x^{[1]}(t)| \leq \int_t^{\infty} |b(s)| |F(x(s))| ds \leq M \int_t^{\infty} |b(s)| \Phi(|x(s)|) ds. \quad (38)$$

Since $|x^{[1]}|$ is bounded for $t \geq t_0$ and (??) holds, there exists a point T of maximum, $t_0 \leq T \leq \infty$, that is

$$\max_{t \geq t_0} |x^{[1]}(t)| = |x^{[1]}(T)|. \quad (39)$$

Since x is not identically zero for large t we have $|x^{[1]}(T)| > 0$. Moreover, integrating (??) we get

$$|x(t)| \leq \int_t^{\infty} \Psi(a^{-1}(r)) \Psi(|x^{[1]}(r)|) dr.$$

Hence, from (??), (??) and (??) we have

$$\begin{aligned} |x^{[1]}(t)| &\leq M \int_t^{\infty} |b(s)| \Phi \left(\int_s^{\infty} \Psi(a^{-1}(r)) \Psi(|x^{[1]}(r)|) dr \right) ds \leq \\ &\leq M |x^{[1]}(T)| \int_{t_0}^{\infty} |b(s)| \Phi \left(\int_s^{\infty} \Psi(a^{-1}(r)) dr \right) ds \leq \frac{|x^{[1]}(T)|}{2} \end{aligned}$$

which gives a contradiction for $t = T$. □

Observe that in Theorem ?? the monotonicity of the solution x for large t , is not required. Hence, if (??) has a (nontrivial) oscillatory solution x which satisfies (??), then (??) is valid.

4 Concluding remarks

Theorem ?? requires that there exists a Sturmian majorant of the half-linear equation (??) satisfying suitable properties.

A prototype of an equation which satisfies these properties can be easily obtained from the well-known Euler equation

$$\left(|h'|^{p-1} \operatorname{sgn} h'\right)' + \left(\frac{p-1}{p}\right)^p t^{-p} |h| \operatorname{sgn} h = 0, \quad (40)$$

where $p > 1$. It is well-known, see, e.g., [?, page 146], that the function

$$h_0(t) = t^{(p-1)/p}$$

is the principal solution of equation (??). Using the transformation $y(t) = |h'(t)|^{p-1} \operatorname{sgn} h'(t)$ and setting $\alpha = (p-1)^{-1}$, we obtain the half-linear equation

$$\left(t^{1+\alpha} \Phi(y')\right)' + \left(\frac{1}{1+\alpha}\right)^{1+\alpha} \Phi(y) = 0. \quad (41)$$

From [?, Theorem 4.2.4.], the function

$$y_0(t) = \left(\frac{1}{1+\alpha}\right)^{1/\alpha} t^{-1/(1+\alpha)}$$

is the principal solution of (??). Moreover, y_0 is positive decreasing on the closed interval $[1, \infty)$ and so (??) is disconjugate on the same interval. Then the following holds.

Lemma 6. *Consider equation (??), with $\Phi(u) = |u|^\alpha \operatorname{sgn} u$. Assume that*

$$a(t) \geq t^{1+\alpha}, \quad M_c b_+(t) \leq \left(\frac{1}{1+\alpha}\right)^{1+\alpha}, \quad t \in [1, \infty), \quad (42)$$

where M_c is given in (??) and at least one of the inequalities in (??) is strict on an interval of positive measure. Then equation (??) has a Sturmian majorant whose principal solution is positive decreasing on $[1, \infty)$.

From here and Theorem ?? we get

Corollary 1. *Let $c > 0$ be fixed and M_c be given by (??). Assume that (??) holds and at least one of the inequalities in (??) is strict on an interval of positive measure. If (??) holds or $b_+(t) = 0$ for large t , then the BVP (??)-(??) has a Kneser solution.*

Our result is illustrated by the following examples.

Example 1. Consider the equation

$$(a(t)(x')^3)' + b(t) F(x) = 0, \quad t \geq 1, \quad (43)$$

where

$$a(t) = 1 + t^4, \quad b(t) = 16^{-2} \exp(1 - t) \cos t \quad (44)$$

and $F(u) = |u|^\beta \operatorname{sgn} u$, $\beta > 3$. Let $c \in (0, 1]$ be fixed. Then $M_c = c^\beta \leq 1$ and for $t \geq 1$

$$M_c b_+(t) \leq 16^{-2} \exp(1 - t) \leq 16^{-2}.$$

Moreover, the condition (??) is valid. In view of Corollary ??, equation (??) has a Kneser solution x satisfying (??) for any $c \in (0, 1]$, and by Theorem ??

$$\lim_{t \rightarrow \infty} \frac{x(t)}{\int_t^\infty a^{-1/3}(s) ds} = \ell_x, \quad 0 < \ell_x < \infty.$$

Example 2. Consider the equation

$$(t^2|x'|^{1/2} \operatorname{sgn} x')' + b(t) F(x) = 0, \quad t \geq 1, \quad (45)$$

where

$$b(t) = \sqrt{\frac{2}{27}} ((1 - \operatorname{sgn}(t - 10)) \sin t - |\sin t|)$$

and $F(u) = u$. For any $c \in (0, 1]$ fixed, we have $M_c = c \leq 1$ and for $t \geq 1$ it holds

$$M_c b_+(t) \leq 2\sqrt{\frac{2}{27}}.$$

Moreover, $b(t) \leq 0$ for $t \geq 10$. Thus by Corollary ??, equation (??) has a Kneser solution x satisfying (??) for any $c \in (0, 1]$.

Remark. When $b_+(t) = 0$ for any $t \geq \bar{t}$ and (??) holds, then Theorem ?? gives the existence of a positive solution x of (??) such that the limit (??) is finite and different from zero. When (??) does not hold, then the precise asymptotic behavior of decaying solutions of (??) is, in general, a hard problem. Some results in this direction can be obtained via a comparison criterion in [?]. Another powerful tool for obtaining a precise asymptotic analysis of positive decaying solutions is based on the framework of regular

variation in the sense of Karamata, see the book [?]. In particular, we refer to [?, ?] and [?, Theorem 3.3.], in which the generalized Thomas-Fermi equation is considered.

Open problem. A closer examination of the argument in the proof of Theorem ?? shows that a necessary and sufficient characterization of the principal solution to the half-linear equation (??) plays a crucial role. When (??) holds, this property is given by Lemma ??- i_3). Moreover, when $b_+(t) = 0$ for large t , the additional condition (??) is unnecessary, because in this case a necessary and sufficient characterization of the principal solution is given in Lemma ?. If $b_+ \not\equiv 0$ in any neighborhood of infinity, then (??) can have positive solutions with changing-sign derivatives and the characterizations of principal solutions given in Lemma ?? can fail. Thus the following open problem arises: when b does not have fixed sign, is Theorem ?? valid without the assumption (??)?

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Authors’ addresses:

Zuzana Došlá, Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, CZ-61137 Brno, Czech Republic. E-mail: `dosla@math.muni.cz`

Mauro Marini, Department of Mathematics and Informatics “Ulisse Dini”, University of Florence, I-50139 Florence, Italy. E-mail: `mauro.marini@unifi.it`

Serena Matucci, Department of Mathematics and Informatics “Ulisse Dini”, University of Florence, I-50139 Florence, Italy. E-mail: serena.matucci@unifi.it