Advances in Applied Mathematics 111 (2019) 101937

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

The Orlicz version of the L_p Minkowski problem for $-n < p < 0$ ☆

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A R T I C L E I N F O

Article history:
Received 28 June 2019
Accepted 9 August 2019
Available online 30 August 2019

MSC:
primary 52A38, 35J96

Keywords:
L_p Minkowski problem
Orlicz Minkowski problem
Monge-Ampère equation

A B S T R A C T

Given a function f on the unit sphere S^{n-1}, the L_p Minkowski problem asks for a convex body K whose L_p surface area measure has density f with respect to the standard $(n-1)$-Hausdorff measure on S^{n-1}. In this paper we deal with the generalization of this problem which arises in the Orlicz-Brunn-Minkowski theory when an Orlicz function φ substitutes the L_p norm and p is in the range $(-n, 0)$. This problem is equivalent to solve the Monge-Ampere equation

$$\varphi(h)\det(\nabla^2 h + hI) = f \quad \text{on } S^{n-1},$$

where h is the support function of the convex body K.

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☆ First and third authors are supported in part by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). Second author is supported in part by NKFIH grants 116451, 121649 and 129630.

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https://doi.org/10.1016/j.aam.2019.101937
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1. Introduction

We work in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$. A *convex body* K in $\mathbb{R}^n$ is a compact convex set that has non-empty interior. Given a convex body K, for $x \in \partial K$ we denote by $\nu_K(x) \subset S^{n-1}$ the family of all unit exterior normal vectors to K at x (the *Gauß map*). We can then define the *surface area measure* S_K of K, which is a Borel measure on the unit sphere S^{n-1} of $\mathbb{R}^n$, as follows: for a Borel set $\omega \subset S^{n-1}$ we set

$$S_K(\omega) = \mathcal{H}^{n-1}\left(\nu_K^{-1}(\omega)\right) = \mathcal{H}^{n-1}\left(\{x \in \partial K : \nu_K(x) \cap \omega \neq \emptyset\}\right)$$

(see, *e.g.*, Schneider [38]).

The classical Minkowski problem can be formulated as follows: *given a Borel measure μ on S^{n-1}, find a convex body K such that $\mu = S_K$*. The reader is referred to [38, Chapter 8] for an exhaustive presentation of this problem and its solution.

Throughout this paper we will consider (either for the classical Minkowski problem or for its variants) the case in which μ has a density f with respect to the $(n-1)$-dimensional Hausdorff measure on S^{n-1}. Under this assumption the Minkowski problem is equivalent to solve (in the classic or in the weak sense) a differential equation on the sphere. Namely:

$$\det(\nabla^2 h + hI) = f, \tag{1}$$

where: h is the support function of K, $\nabla^2 h$ is the matrix formed by the second covariant derivatives of h with respect to a local orthonormal frame on S^{n-1} and I is the identity matrix of order $(n-1)$.

Many different types of variations of the Minkowski problem have been considered (we refer for instance to [38, Chapters 8 and 9]). Of particular interest for our purposes is the so called L_p version of the problem (see [1–11,14–34,36,37,39–50]). At the origin of this new problem there is the replacement of the usual Minkowski addition of convex bodies by the p-addition. As an effect, the corresponding differential equation takes the form

$$h^{1-p} \det(\nabla^2 h + hI) = f, \tag{2}$$

(see [38, Section 9.2]). The study of the L_p Minkowski problems developed in a significant way in the last decades, as a part of the so called L_p Brunn-Minkowski theory, which represents now a substantial area of Convex Geometry. One of the most interesting aspects of this problem is that several threshold values of the parameter p can be identified, *e.g.* $p = 1$, $p = 0$, $p = -n$, across which the nature of the problem changes drastically. For an account on the literature and on the state of the art of the L_p Minkowski problem (especially for the values $p < 1$) we refer the reader to [1] and [2].

Of particular interest here is the range $-n < p < 0$. In this case Chou and Wang (see [10]) solved the corresponding problem when the measure μ has a density f, and

f is bounded and bounded away from zero. This result was slightly generalised by the authors in collaboration with Yang in [1], where f is allowed to be in $L_{\frac{n}{n+p}}$.

Theorem 1.1 *(Chou and Wang; Bianchi, Böröczky, Colesanti and Yang). For $n \geq 1$ and $-n < p < 0$, if the non-negative and non-trivial function f is in $L_{\frac{n}{n+p}}(S^{n-1})$ then (2) has a solution in the Alexandrov sense; namely, $f\, d\mathcal{H}^{n-1} = dS_{K,p}$ for a convex body $K \in \mathcal{K}_0^n$. In addition, if f is invariant under a closed subgroup G of $O(n)$, then K can be chosen to be invariant under G.*

As a further extension of the L_p Minkowski problem, one may consider its Orlicz version. Formally, this problem arises in the context of the Orlicz-Brunn-Minkowski theory of convex bodies (see [38, Chapter 9]). In practice, the relevant differential equation is

$$\varphi(h) \det(\nabla^2 h + hI) = f,$$

where φ is a suitable *Orlicz function*. The L_p Minkowski problem is obtained when $\varphi(t) = t^{1-p}$, for $t \geq 0$.

When $\varphi\colon (0, \infty) \to (0, \infty)$ is continuous and monotone decreasing, this problem (under a symmetry assumption) has been considered by Haberl, Lutwak, Yang, Zhang in [17]. Comparing the previous assumptions on φ with the L_p case, we see that this corresponds to the values $p \geq 1$.

We are interested in the case in which the monotonicity assumption is reversed, corresponding to the values $p < 1$. Hence we assume that $\varphi\colon [0, \infty) \to \mathbb{R}$ is continuous and monotone increasing, having the example $\varphi(t) = t^{1-p}$, $p < 1$, as a prototype. To control in a more precise form the behaviour of φ with that of a power function, we assume that there exists $p < 1$ such that

$$\liminf_{t \to 0^+} \frac{\varphi(t)}{t^{1-p}} > 0. \tag{3}$$

Concerning the behaviour of φ at ∞ we impose the condition:

$$\int_1^\infty \frac{1}{\varphi(t)}\, dt < \infty. \tag{4}$$

The corresponding Minkowski problem in this setting can be called the Orlicz L_p Minkowski problem. The solution of this problem in the range $p \in (0, 1)$ is due to Jian, Lu [25]. We also note that Orlicz versions of the so called L_p dual Minkowski problem have been considered recently by Gardner, Hug, Weil, Xing, Ye [13], Gardner, Hug, Xing, Ye [14], Xing, Ye, Zhu [43] and Xing, Ye [44].

In this paper we focus on the range of values $p \in (-n, 0)$. As an extension of the results contained in [1], we establish the following existence theorem (note that, as usual in the case of Orlicz versions of Minkowski type problems, we can only provide a solution up to a constant factor).

Theorem 1.2. *For $n \geq 2$, $-n < p < 0$ and monotone increasing continuous function $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\varphi(0) = 0$, and conditions (3) and (4), if the non-negative non-trivial function f is in $L_{\frac{n}{n+p}}(S^{n-1})$, then there exists $\lambda > 0$ and a convex body $K \in \mathcal{K}_0^n$ with $V(K) = 1$ such that*

$$\lambda\varphi(h)\det(\nabla^2 h + hI) = f$$

holds for $h = h_K$ in the Alexandrov sense; namely, $\lambda\varphi(h_K)\, dS_K = f\, d\mathcal{H}^{n-1}$. In addition, if f is invariant under a closed subgroup G of $O(n)$, then K can be chosen to be invariant under G.

We note that the origin may lie on ∂K for the solution K in Theorem 1.2.

We observe that Theorem 1.2 readily yields Theorem 1.1. Indeed if $-n < p < 0$, $f \in L_{\frac{n}{n+p}}(S^{n-1})$, $f \geq 0$, $f \not\equiv 0$ and $\lambda h_K^{1-p}\, dS_K = f\, d\mathcal{H}^{n-1}$ for $K \in \mathcal{K}_0^n$ and $\lambda > 0$, then $h_{\widetilde{K}}^{1-p}\, dS_{\widetilde{K}} = f\, d\mathcal{H}^{n-1}$ for $\widetilde{K} = \lambda^{\frac{1}{n-p}} K$.

In Section 3 we sketch the proof of Theorem 1.2 and describe the structure of the paper.

2. Notation

The scalar product on $\mathbb{R}^n$ is denoted by $\langle\cdot,\cdot\rangle$, and the corresponding Euclidean norm is denoted by $\|\cdot\|$. The k-dimensional Hausdorff measure normalized in such a way that it coincides with the Lebesgue measure on $\mathbb{R}^k$ is denoted by $\mathcal{H}^k$. The angle (spherical distance) of $u, v \in S^{n-1}$ is denoted by $\angle(u, v)$.

We write $\mathcal{K}_0^n$ ($\mathcal{K}_{(0)}^n$) to denote the family of convex bodies with $o \in K$ ($o \in \operatorname{int} K$). Given a convex body K, for a Borel set $\omega \subset S^{n-1}$, $\nu_K^{-1}(\omega)$ is the Borel set of $x \in \partial K$ with $\nu_K(x) \cap \omega \neq \emptyset$. A point $x \in \partial K$ is called smooth if $\nu_K(x)$ consists of a unique vector, and in this case, we use $\nu_K(x)$ to denote this unique vector, as well. It is well-known that $\mathcal{H}^{n-1}$-almost every $x \in \partial K$ is smooth (see, *e.g.*, Schneider [38]); let $\partial' K$ denote the family of smooth points of ∂K.

For a convex compact set K in $\mathbb{R}^n$, let h_K be its support function:

$$h_K(u) = \max\{\langle x, u\rangle : x \in K\} \quad \text{for } u \in \mathbb{R}^n.$$

Note that if $K \in \mathcal{K}_0^n$, then $h_K \geq 0$. If $p \in \mathbb{R}$ and $K \in \mathcal{K}_0^n$, then the L_p-surface area measure is defined by

$$dS_{K,p} = h_K^{1-p}\, dS_K$$

where for $p > 1$ the right-hand side is assumed to be a finite measure. In particular, if $p = 1$, then $S_{K,p} = S_K$, and if $p < 1$ and $\omega \subset S^{n-1}$ Borel, then

$$S_{K,p}(\omega) = \int_{\nu_K^{-1}(\omega)} \langle x, \nu_K(x)\rangle^{1-p} d\mathcal{H}^{n-1}(x).$$

3. Sketch of the proof of Theorem 1.2

To sketch the argument leading to Theorem 1.2, first we consider the case when $-n < p \leq -(n-1)$ and $\varphi(t) = t^{1-p}$, and $\tau_1 \leq f \leq \tau_2$ for some constants $\tau_2 > \tau_1 > 0$. We set $\psi(t) = 1/\varphi(t) = t^{p-1}$ for $t > 0$, and define $\Psi : (0, \infty) \to (0, \infty)$ by

$$\Psi(t) = \int_t^\infty \psi(s)\, ds = -\frac{1}{p}\, t^p,$$

which is a strictly convex function.

Given a convex body K in $\mathbb{R}^n$, we set

$$\Phi(K, \xi) = \int_{S^{n-1}} \Psi(h_{K-\xi}) f \, d\mathcal{H}^{n-1};$$

this is a strictly convex function of $\xi \in \operatorname{int} K$. As $f > \tau_1$ and $p \leq -(n-1)$, there is a (unique) $\xi(K) \in \operatorname{int} K$ such that

$$\Phi(K, \xi(K)) = \min_{\xi \in \operatorname{int} K} \Phi(K, \xi).$$

This statement is proved in Proposition 5.2, but the conditions $f > \tau_1$ and $p \leq -(n-1)$ are actually used in the preparatory statement Lemma 5.1.

Using $p > -n$ and the Blaschke-Santaló inequality (see Lemma 5.4 and the preparatory statement Lemma 4.3), one verifies that there exists a convex body K_0 in $\mathbb{R}^n$ with $V(K_0) = 1$ maximizing $\Phi(K, \xi(K))$ over all convex bodies K in $\mathbb{R}^n$ with $V(K) = 1$.

Finally a variational argument proves that there exists $\lambda_0 > 0$ such that $f\, d\mathcal{H}^{n-1} = \lambda_0 \varphi(h_{K_0})\, dS_{K_0}$. A crucial ingredient (see Lemma 6.2) is that, as ψ is C^1 and $\psi' < 0$, $\Phi(K_t, \xi(K_t))$ is a differentiable function of K_t for a suitable variation K_t of K_0.

In the general case, when still keeping the condition $\tau_1 \leq f \leq \tau_2$ but allowing any φ which satisfies the assumptions of Theorem 1.2, we meet two main obstacles. On the one hand, even if $\varphi(t) = t^{1-p}$ but $0 < t < -(n-1)$, it may happen that for a convex body K in $\mathbb{R}^n$, the infimum of $\Phi(K, \xi)$ for $\xi \in \operatorname{int} K$ is attained when ξ tends to the boundary of K. On the other hand, the possible lack of differentiability of φ (or equivalently of ψ) destroys the variational argument.

Therefore, we approximate ψ by smooth functions, and also make sure that the approximating functions are large enough near zero to ensure that the minimum of the analogues of $\Phi(K,\xi)$ as a function of $\xi \in \operatorname{int} K$ exists for any convex body K.

Section 4 proves some preparatory statements, Section 5 introduces the suitable analogue of the energy function $\Phi(K,\xi(K))$, and Section 6 provides the variational formula for an extremal body for the energy function. We prove Theorem 1.2 if f is bounded and bounded away from zero in Section 7, and finally in full strength in Section 8.

4. Some preliminary estimates

In this section, we prove the simple but technical estimates Lemmas 4.1 and 4.3 that will be used in various settings during the main argument.

Lemma 4.1. *For $\delta \in (0,1)$, $A, \tilde{\aleph} > 0$ and $q \in (-n,0)$, let $\widetilde{\psi} : (0,\infty) \to (0,\infty)$ satisfy that $\widetilde{\psi}(t) \le \tilde{\aleph} t^{q-1}$ for $t \in (0,\delta]$ and $\int_\delta^\infty \widetilde{\psi} \le A$. If $t \in (0,\delta)$ and $\tilde{\aleph}_0 = \max\{\frac{\tilde{\aleph}}{|q|}, \frac{A}{\delta^q}\}$, then $\widetilde{\Psi}(t) = \int_t^\infty \widetilde{\psi}$ satisfies*

$$\widetilde{\Psi}(t) \le \tilde{\aleph}_0 t^q.$$

Proof. We observe that if $t \in (0,\delta)$, then

$$\widetilde{\Psi}(t) \le \int_t^\delta \tilde{\psi}(s)\,ds + A \le \tilde{\aleph} \int_t^\delta s^{q-1}\,ds + A = \frac{\tilde{\aleph}}{|q|}(t^q - \delta^q) + A \le t^q \max\left\{\frac{\tilde{\aleph}}{|q|}, \frac{A}{\delta^q}\right\}. \quad \square$$

We write B^n to denote the Euclidean unit ball in $\mathbb{R}^n$, and set $\kappa_n = \mathcal{H}^n(B^n)$. For a convex body K in $\mathbb{R}^n$, let $\sigma(K)$ denote its centroid, which satisfies (see Schneider [38])

$$-(K - \sigma(K)) \subset n(K - \sigma(K)). \tag{5}$$

Next, if $o \in \operatorname{int} K$ then the polar of K is

$$K = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \ \forall y \in K\} = \{tu : u \in S^{n-1} \text{ and } 0 \le t \le h_K(u)^{-1}\}.$$

In particular, the Blaschke-Santaló inequality $V(K)V((K - \sigma(K))^*) \le V(B^n)^2$ (see Schneider [38]) yields that

$$\int_{S^{n-1}} h_{K-\sigma(K)}^{-n}\, d\mathcal{H}^{n-1} \le \frac{nV(B^n)^2}{V(K)}. \tag{6}$$

As a preparation for the proof of Lemma 4.3, we need the following statement about absolutely continuous measures. For $t \in (0,1)$ and $v \in S^{n-1}$, we consider the spherical strip

$$\Xi(v,t) = \{u \in S^{n-1} : |\langle u, v\rangle| \le t\}.$$

Lemma 4.2. *If $f \in L_1(S^{n-1})$ and*

$$\varrho_f(t) = \sup_{v\in S^{n-1}} \int\limits_{\Xi(v,t)} |f|\, d\mathcal{H}^{n-1}$$

for $t \in (0,1)$, then we have $\lim_{t\to 0^+} \varrho_f(t) = 0$.

Proof. We observe that $\varrho_f(t)$ is decreasing, therefore the limit $\lim_{t\to 0^+} \varrho_f(t) = \delta \ge 0$ exists. We suppose that $\delta > 0$, and seek a contradiction.

Let μ be the absolutely continuous measure $d\mu = |f|\, d\mathcal{H}^{n-1}$ on S^{n-1}. According to the definition of ϱ_f, for any $k \ge 2$, there exists some $v_k \in S^{n-1}$ such that $\mu(\Xi(v_k, \frac{1}{k})) \ge \delta/2$. Let $v \in S^{n-1}$ be an accumulation point of the sequence $\{v_k\}$. For any $m \ge 2$, there exists $\alpha_m > 0$ such that $\Xi(u, \frac{1}{2m}) \subset \Xi(v, \frac{1}{m})$ if $u \in S^{n-1}$ and $\angle(u,v) \le \alpha_m$. Since for any $m \ge 2$, there exists some $k \ge 2m$ such that $\angle(v_k, v) \le \alpha_m$, we have $\mu(\Xi(v, \frac{1}{m})) \ge \mu(\Xi(v_k, \frac{1}{k})) \ge \delta/2$. We deduce that $\mu(v^\perp \cap S^{n-1}) = \mu\left(\cap_{m\ge 2}\Xi(v, \frac{1}{m})\right) \ge \delta/2$, which contradicts $\mu(v^\perp \cap S^{n-1}) = 0$. $\square$

Lemma 4.3. *For $\delta \in (0,1)$, $\tilde{\aleph} > 0$ and $q \in (-n, 0)$, let $\widetilde{\Psi} : (0,\infty) \to (0,\infty)$ be a monotone decreasing continuous function such that $\widetilde{\Psi}(t) \le \tilde{\aleph} t^q$ for $t \in (0, \delta]$ and $\lim_{t\to\infty} \widetilde{\Psi}(t) = 0$, and let $\tilde{f}$ be a non-negative function in $L_{\frac{n}{n+p}}(S^{n-1})$. Then for any $\zeta > 0$, there exists a D_ζ depending on ζ, $\widetilde{\Psi}$, δ, $\tilde{\aleph}$, q and $\tilde{f}$ such that if K is a convex body in $\mathbb{R}^n$ with $V(K) = 1$ and $\operatorname{diam} K \ge D_\zeta$ then*

$$\int\limits_{S^{n-1}} (\widetilde{\Psi} \circ h_{K-\sigma(K)})\, \tilde{f}\, d\mathcal{H}^{n-1} \le \zeta.$$

Proof. We may assume that $\sigma(K) = o$. Let $R = \max_{x\in K} \|x\|$, and let $v \in S^{n-1}$ such that $Rv \in K$. It follows from (5) that $-\frac{R}{n}\, v \in K$.

Since $\lim_{t\to\infty} \widetilde{\Psi}(t) = 0$ and $\tilde{f}$ is in $L_1(S^{n-1})$ by the Hölder inequality, we can choose $r \ge 1$ such that

$$\widetilde{\Psi}(r) \int\limits_{S^{n-1}} \tilde{f}\, d\mathcal{H}^{n-1} < \frac{\zeta}{2}. \tag{7}$$

We partition S^{n-1} into the two measurable parts

$$\Xi_0 = \{u \in S^{n-1} : h_K(u) \ge r\}$$
$$\Xi_1 = \{u \in S^{n-1} : h_K(u) < r\}.$$

Let us estimate the integrals over Ξ_0 and Ξ_1. We deduce from (7) that

$$\int_{\Xi_0} (\widetilde{\Psi} \circ h_K)\, \tilde{f}\, d\mathcal{H}^{n-1} \leq \frac{\zeta}{2}. \tag{8}$$

Next we claim that

$$\Xi_1 \subset \Xi\left(v, \frac{nr}{R}\right). \tag{9}$$

For any $u \in \Xi_1$, we choose $\eta \in \{-1, 1\}$ such that $\langle u, \eta v\rangle \geq 0$, thus $\frac{\eta R}{n}\, v \in K$ yields that $r > h_K(u) \geq \langle u, \frac{\eta R}{n}\, v\rangle$. In turn, we conclude (9). It follows from (9) and Lemma 4.2 that for the L_1 function $f = \tilde{f}^{\frac{n}{n+q}}$, we have

$$\int_{\Xi_1} \tilde{f}^{\frac{n}{n+q}} \leq \varrho_f\left(\frac{nr}{R}\right). \tag{10}$$

To estimate the decreasing function $\widetilde{\Psi}$ on $(0, r)$, we claim that if $t \in (0, r)$ then

$$\widetilde{\Psi}(t) \leq \frac{\tilde{\aleph}\delta^q}{r^q}\, t^q. \tag{11}$$

We recall that $r \geq 1 > \delta$. In particular, if $t \leq \delta$, then $\widetilde{\Psi}(t) \leq \tilde{\aleph} t^q$ yields (11). If $t \in (\delta, r)$, then using that $\widetilde{\Psi}$ is decreasing, (11) follows from

$$\widetilde{\Psi}(t) \leq \widetilde{\Psi}(\delta) \leq \frac{\tilde{\aleph}\delta^q}{t^q}\, t^q \leq \frac{\tilde{\aleph}\delta^q}{r^q}\, t^q.$$

Applying first (11), then the Hölder inequality, after that the Blaschke-Santaló inequality (6) with $V(K) = 1$ and finally (10), we deduce that

$$\begin{aligned}
\int_{\Xi_1} (\widetilde{\Psi} \circ h_K)\, \tilde{f}\, d\mathcal{H}^{n-1} &\leq \frac{\tilde{\aleph}\delta^q}{r^q} \int_{\Xi_1} h_K^{-|q|}\, \tilde{f}\, d\mathcal{H}^{n-1} \\
&\leq \frac{\tilde{\aleph}\delta^q}{r^q} \left(\int_{\Xi_1} h_K^{-n}\, d\mathcal{H}^{n-1}\right)^{\frac{|q|}{n}} \left(\int_{\Xi_1} \tilde{f}^{\frac{n}{n-|q|}}\, d\mathcal{H}^{n-1}\right)^{\frac{n-|q|}{n}} \\
&\leq \frac{\tilde{\aleph}\delta^q}{r^q} \left(nV(B^n)^2\right)^{\frac{|q|}{n}} \varrho_f\left(\frac{nr}{R}\right)^{\frac{n+q}{n}}.
\end{aligned}$$

Therefore after fixing $r \geq 1$ satisfying (7), we may choose $R_0 > r$ such that

$$\frac{\tilde{\aleph}\delta^q}{r^q}\, n^{\frac{|q|}{n}} V(B^n)^{\frac{2|q|}{n}} \varrho_f\left(\frac{nr}{R_0}\right)^{\frac{n+q}{n}} < \frac{\zeta}{2}$$

by Lemma 4.3. In particular, if $R \geq R_0$, then

$$\int_{\Xi_1} (\widetilde{\Psi} \circ h_K)\, \tilde{f}\, d\mathcal{H}^{n-1} \leq \frac{\zeta}{2}.$$

Combining this estimate with (8) shows that setting $D_\zeta = 2R_0$, if $\operatorname{diam} K \geq D_\zeta$, then $R \geq R_0$, and hence $\int_{S^{n-1}} (\widetilde{\Psi} \circ h_K)\, \tilde{f}\, d\mathcal{H}^{n-1} \leq \zeta$. $\square$

5. The extremal problem related to Theorem 1.2 when f is bounded and bounded away from zero

For $0 < \tau_1 < \tau_2$, let the real function f on S^{n-1} satisfy

$$\tau_1 < f(u) < \tau_2 \quad \text{for } u \in S^{n-1}. \tag{12}$$

In addition, let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous monotone increasing function satisfying $\varphi(0) = 0$,

$$\liminf_{t \to 0^+} \frac{\varphi(t)}{t^{1-p}} > 0 \quad \text{and} \quad \int_1^\infty \frac{1}{\varphi(t)}\, dt < \infty.$$

It will be more convenient to work with the decreasing function $\psi = 1/\varphi : (0, \infty) \to (0, \infty)$, which has the properties

$$\limsup_{t \to 0^+} \frac{\psi(t)}{t^{p-1}}, < \infty \tag{13}$$

$$\int_1^\infty \psi(t)\, dt < \infty. \tag{14}$$

We consider the function $\Psi : (0, \infty) \to (0, \infty)$ defined by

$$\Psi(t) = \int_t^\infty \psi(s)\, ds,$$

which readily satisfies

$$\Psi' = -\psi, \text{ and hence } \Psi \text{ is convex and strictly monotone decreasing,} \tag{15}$$

$$\lim_{t \to \infty} \Psi(t) = 0. \tag{16}$$

According to (13), there exist some $\delta \in (0, 1)$ and $\aleph > 1$ such that

$$\psi(t) < \aleph t^{p-1} \quad \text{for } t \in (0,\delta). \tag{17}$$

As we pointed out in Section 3, we smoothen ψ using convolution. Let $\eta : \mathbb{R} \to [0,\infty)$ be a non-negative C^∞ "approximation of identity" with $\operatorname{supp}\eta \subset [-1,0]$ and $\int_{\mathbb{R}} \eta = 1$. For any $\varepsilon \in (0,1)$, we consider the non-negative $\eta_\varepsilon(t) = \frac{1}{\varepsilon}\eta(\frac{t}{\varepsilon})$ satisfying that $\int_{\mathbb{R}} \eta_\varepsilon = 1$, $\operatorname{supp}\eta_\varepsilon \subset [-\varepsilon,0]$, and define $\theta_\varepsilon : (0,\infty) \to (0,\infty)$ by

$$\theta_\varepsilon(t) = \int\limits_{\mathbb{R}} \psi(t-\tau)\eta_\varepsilon(\tau)\, d\tau = \int\limits_{-\varepsilon}^{0} \psi(t-\tau)\eta_\varepsilon(\tau)\, d\tau.$$

As ψ is monotone decreasing and continuous on $(0,\infty)$, the properties of η_ε yield

$$\begin{aligned} \theta_\varepsilon(t) &\leq \psi(t) \quad \text{for } t > 0 \text{ and } \varepsilon \in (0,1) \\ \theta_\varepsilon(t_1) &\geq \theta_\varepsilon(t_2) \quad \text{for } t_2 > t_1 > 0 \text{ and } \varepsilon \in (0,1) \end{aligned}$$

θ_ε tends uniformly to ψ on any interval with positive endpoints as ε tends to zero.

Next, for any $t_0 > 0$, the function l_{t_0} on $\mathbb{R}$ defined by

$$l_{t_0}(t) = \begin{cases} \psi(t) & \text{if} \quad t \geq t_0 \\ 0 & \text{if} \quad t < t_0 \end{cases}$$

is bounded, and hence locally integrable. For the convolution $l_{t_0} * \eta_\varepsilon$, we have that $(l_{t_0} * \eta_\varepsilon)(t) = \theta_\varepsilon(t)$ for $t > t_0$ and $\varepsilon \in (0,1)$, thus

$$\theta_\varepsilon \text{ is } C^1 \text{ for each } \varepsilon \in (0,1).$$

As it is explained in Section 3, we need to modify ψ in a way such that the new function is of order at least $t^{-(n-1)}$ if $t > 0$ is small. We set

$$q = \min\{p, -(n-1)\},$$

and hence (17) and $\delta \in (0,1)$ yields that

$$\theta_\varepsilon(t) \leq \psi(t) < \aleph t^{q-1} \quad \text{for } t \in (0,\delta) \text{ and } \varepsilon \in (0,\delta). \tag{18}$$

Next we construct $\tilde{\theta}_\varepsilon : (0,\infty) \to (0,\infty)$ satisfying

$$\begin{aligned} \tilde{\theta}_\varepsilon(t) &= \theta_\varepsilon(t) \leq \psi(t) \quad \text{for } t \geq \varepsilon \text{ and } \varepsilon \in (0,\delta) \\ \tilde{\theta}_\varepsilon(t) &\leq \aleph t^{q-1} \quad \text{for } t \in (0,\delta) \text{ and } \varepsilon \in (0,\delta) \\ \tilde{\theta}_\varepsilon(t) &= \aleph t^{q-1} \quad \text{for } t \in (0,\tfrac{\varepsilon}{2}] \text{ and } \varepsilon \in (0,\delta) \end{aligned}$$

$$\tilde{\theta}_\varepsilon \text{ is } C^1 \quad \text{and is monotone decreasing.}$$

It follows that

$\tilde{\theta}_\varepsilon$ tends uniformly to ψ on any interval with positive endpoints as ε tends to zero.

To construct suitable $\tilde{\theta}_\varepsilon$, first we observe that the conditions above determine $\tilde{\theta}_\varepsilon$ outside the interval $(\frac{\varepsilon}{2}, \varepsilon)$, and $\tilde{\theta}_\varepsilon(\varepsilon) < \aleph\varepsilon^{q-1}$. Writing Δ to denote the degree one polynomial whose graph is the tangent to the graph of $t \mapsto \aleph t^{q-1}$ at $t = \varepsilon/2$, we have $\Delta(t) < \aleph t^{q-1}$ for $t > \varepsilon/2$ and $\Delta(\varepsilon) < 0$. Therefore we can choose $t_0 \in (\frac{\varepsilon}{2}, \varepsilon)$ such that $\tilde{\theta}_\varepsilon(\varepsilon) < \Delta(t_0) < \aleph\varepsilon^{q-1}$. We define $\tilde{\theta}_\varepsilon(t) = \Delta(t)$ for $t \in (\frac{\varepsilon}{2}, t_0)$, and construct $\tilde{\theta}_\varepsilon$ on (t_0, ε) in a way that $\tilde{\theta}_\varepsilon$ stays C^1 on $(0, \infty)$. It follows from the way $\tilde{\theta}_\varepsilon$ is constructed that $\tilde{\theta}_\varepsilon(t) \le \aleph t^{q-1}$ also for $t \in [\frac{\varepsilon}{2}, \varepsilon]$.

In order to ensure a negative derivative, we consider $\psi_\varepsilon : (0, \infty) \to (0, \infty)$ defined by

$$\psi_\varepsilon(t) = \tilde{\theta}_\varepsilon(t) + \frac{\varepsilon}{1 + t^2} \tag{19}$$

for $\varepsilon \in (0, \delta)$ and $t > 0$. This C^1 function ψ_ε has the following properties:

$$\begin{array}{llll} \psi_\varepsilon(t) & \le & \psi(t) + \frac{1}{1+t^2} & \text{for } t \ge \varepsilon \text{ and } \varepsilon \in (0,\delta) \\ \psi'_\varepsilon(t) & < & 0 & \text{for } t > 0 \text{ and } \varepsilon \in (0,\delta) \\ \psi_\varepsilon(t) & < & 2\aleph t^{q-1} & \text{for } t \in (0,\delta) \text{ and } \varepsilon \in (0,\delta) \\ \psi_\varepsilon(t) & > & \aleph t^{q-1} & \text{for } t \in (0,\frac{\varepsilon}{2}) \text{ and } \varepsilon \in (0,\delta) \end{array}$$
$$\psi_\varepsilon \text{ tends uniformly to } \psi \text{ on any interval with positive endpoints as } \varepsilon \text{ tends to zero.} \tag{20}$$

For $\varepsilon \in (0, \delta)$, we also consider the C^2 function $\Psi_\varepsilon : (0, \infty) \to (0, \infty)$ defined by

$$\Psi_\varepsilon(t) = \int_t^\infty \psi_\varepsilon(s)\, ds,$$

and hence (20) yields

$$\lim_{t \to \infty} \Psi_\varepsilon(t) = 0 \tag{21}$$

$$\Psi'_\varepsilon = -\psi_\varepsilon, \quad \text{thus } \Psi_\varepsilon \text{ is strictly decreasing and strictly convex.} \tag{22}$$

For $\varepsilon \in (0, \delta)$, Lemma 4.1 and (20) imply that setting

$$A = \int_\delta^\infty \psi(t) + \frac{1}{1 + t^2}\, dt,$$

we have

$$\Psi_\varepsilon(t) \le \aleph_0 t^q \quad \text{for } \aleph_0 = \max\{\frac{2\aleph}{|q|}, \frac{A}{\delta^q}\} \text{ and } t \in (0,\delta). \tag{23}$$

On the other hand, if $\varepsilon \in (0,\delta)$ and $t \in (0, \frac{\varepsilon}{4})$, then

$$\Psi_\varepsilon(t) \ge \int_t^{\varepsilon/2} \aleph s^{q-1}\, ds = \frac{\aleph}{|q|}(t^q - (\varepsilon/2)^q) \ge \frac{\aleph}{|q|}(t^q - (2t)^q) = \aleph_1 t^q$$

$$\text{for } \aleph_1 = \frac{(1-2^q)\aleph}{|q|} > 0. \tag{24}$$

According to (20), we have $\lim_{\varepsilon\to 0^+} \psi_\varepsilon(t) = \psi(t)$ and $\psi_\varepsilon(t) \le \psi(t) + \frac{1}{1+t^2}$ for any $t > 0$, therefore Lebesgue's Dominated Convergence Theorem implies

$$\lim_{\varepsilon\to 0^+} \Psi_\varepsilon(t) = \Psi(t) \quad \text{for any } t > 0. \tag{25}$$

It also follows from (20) that if $t \ge \varepsilon$, then

$$\Psi_\varepsilon(t) = \int_t^\infty \psi_\varepsilon \le \int_t^\infty \psi(s) + \frac{1}{1+s^2}\, ds \le \Psi(t) + \frac{\pi}{2}. \tag{26}$$

For any convex body K and $\xi \in \operatorname{int} K$, we consider

$$\Phi_\varepsilon(K,\xi) = \int_{S^{n-1}} (\Psi_\varepsilon \circ h_{K-\xi}) f \, d\mathcal{H}^{n-1} = \int_{S^{n-1}} \Psi_\varepsilon(h_K(u) - \langle \xi, u\rangle) f(u)\, d\mathcal{H}^{n-1}(u).$$

Naturally, $\Phi_\varepsilon(K)$ depends on ψ and f, as well, but we do not signal these dependences.

We equip $\mathcal{K}_0^n$ with the Hausdorff metric, which is the L_∞ metric on the space of the restrictions of support functions to S^{n-1}. For $v \in S^{n-1}$ and $\alpha \in [0, \frac{\pi}{2}]$, we consider the spherical cap

$$\Omega(v,\alpha) = \{u \in S^{n-1}\, \langle u, v\rangle \ge \cos\alpha\}.$$

We write $\pi : \mathbb{R}^n \backslash \{o\} \to S^{n-1}$ the radial projection:

$$\pi(x) = \frac{x}{\|x\|}.$$

In particular, if π is restricted to the boundary of a $K \in \mathcal{K}_{(0)}^n$, then this map is Lipschitz. Another typical application of the radial projection is to consider, for $v \in S^{n-1}$, the composition $x \mapsto \pi(x+v)$ as a map $v^\perp \to S^{n-1}$ where

$$\text{the Jacobian of } x \mapsto \pi(x+v) \text{ at } x \in v^\perp \text{ is } (1+\|x\|^2)^{-n/2}. \tag{27}$$

The following Lemma 5.1 is the statement where we apply directly that ψ is modified to be essentially t^q if t is very small.

Lemma 5.1. *Let $\varepsilon \in (0, \delta)$, and let $\{K_i\}$ be a sequence of convex bodies tending to a convex body K in $\mathbb{R}^n$, and let $\xi_i \in \operatorname{int} K_i$ such that $\lim_{i\to\infty} \xi_i = x_0 \in \partial K$. Then*

$$\lim_{i\to\infty} \Phi_\varepsilon(K_i, \xi_i) = \infty.$$

Proof. We may assume that $\lim_{i\to\infty} \xi_i = x_0 = o$. Let $v \in S^{n-1}$ be an exterior normal to ∂K at o, and choose some $R > 0$ such that $K \subset RB^n$. Therefore we may assume that $K_i - \xi_i \subset (R+1)B^n$, $h_{K_i}(v) < \varepsilon/8$ and $\|\xi_i\| < \varepsilon/8$ for all ξ_i, thus $h_{K_i-\xi_i}(v) < \varepsilon/4$ for all i.

For any $\zeta \in (0, \frac{\varepsilon}{8})$, there exists I_ζ such that if $i \geq I_\zeta$, then $\|\xi_i\| \leq \zeta/2$ and $\langle y, v\rangle \leq \zeta/2$ for all $y \in K_i$, and hence $\langle y, v\rangle \leq \zeta$ for all $y \in K_i - \xi_i$. For $i \geq I_\zeta$, any $y \in K_i - \xi_i$ can be written in the form $y = sv + z$ where $s \leq \zeta$ and $z \in v^\perp \cap (R+1)B^n$, thus if $\angle(v, u) = \alpha \in [\zeta, \frac{\pi}{2})$ for $u \in S^{n-1}$, then we have

$$h_{K_i-\xi_i}(u) \leq (R+1)\sin\alpha + \zeta\cos\alpha \leq (R+2)\alpha. \tag{28}$$

We set $\beta = \frac{\varepsilon}{4(R+2)}$, and for $\zeta \in (0, \beta)$, we define

$$\Omega_\zeta = \Omega(v, \beta)\backslash\Omega(v, \zeta).$$

In particular, as $\Psi_\varepsilon(t) \geq \aleph_1 t^q$ for $t \in (0, \frac{\varepsilon}{4})$ according to (24), if $u \in \Omega_\zeta$, then (28) implies

$$\Psi_\varepsilon(h_{K_i-\xi_i}(u)) \geq \gamma(\angle(v, u))^q$$

for $\gamma = \aleph_1(R+2)^q$.

The function $x \mapsto \pi(x+v)$ maps $B_\zeta = v^\perp \cap \big((\tan\beta)B^n\backslash(\tan\zeta)B^n\big)$ bijectively onto Ω_ζ, while $\beta < \frac{1}{8}$ and (27) yield that the Jacobian of this map is at least 2^{-n} on B_ζ.

Since $f > \tau_1$ and $\angle(v, \pi(x+v)) \leq 2x$ for $x \in B_\zeta$, if $i \geq I_\zeta$, then

$$\begin{aligned}\Phi_\varepsilon(K_i, \xi_i) &= \int_{S^{n-1}} \Psi_\varepsilon(h_{K_i-\xi_i}(u)) f(u)\, d\mathcal{H}^{n-1}(u) \geq \int_{\Omega_\zeta} \tau_1\gamma(\angle(v, u))^q\, d\mathcal{H}^{n-1}(u)\\ &\geq \frac{\tau_1\gamma}{2^{n+|q|}} \int_{B_\zeta} \|x\|^q\, d\mathcal{H}^{n-1}(x) = \frac{(n-1)\kappa_{n-1}\tau_1\gamma}{2^{n+|q|}} \int_{\tan\zeta}^{\tan\beta} t^{q+n-2}\, dt.\end{aligned}$$

As $\zeta > 0$ is arbitrarily small and $q \leq 1-n$, we conclude that $\lim_{i\to\infty} \Phi_\varepsilon(K_i, \xi_i) = \infty$. □

Now we single out the optimal $\xi \in \operatorname{int} K$.

Proposition 5.2. *For $\varepsilon \in (0, \delta)$ and a convex body K in $\mathbb{R}^n$, there exists a unique $\xi(K) \in \operatorname{int} K$ such that*

$$\Phi_\varepsilon(K, \xi(K)) = \min_{\xi \in \operatorname{int} K} \Phi_\varepsilon(K, \xi).$$

In addition, $\xi(K)$ and $\Phi_\varepsilon(K, \xi(K))$ are continuous functions of K, and $\Phi_\varepsilon(K, \xi(K))$ is translation invariant.

Proof. The first part of this proof, the one regarding the existence of $\xi(K) \in \operatorname{int} K$ and its uniqueness, is very similar to the proof of [1, Proposition 3.2] given by the authors and Yang for the L_p Minkowski problem. It is very short and we rewrite it here for completeness.

Let $\xi_1, \xi_2 \in \operatorname{int} K$, $\xi_1 \neq \xi_2$, and let $\lambda \in (0, 1)$. If $u \in S^{n-1} \setminus (\xi_1 - \xi_2)^\perp$, then $\langle u, \xi_1 \rangle \neq \langle u, \xi_2 \rangle$, and hence the strict convexity of Ψ_ε (see (22)) yields that

$$\Psi_\varepsilon(h_K(u) - \langle u, \lambda\xi_1 + (1-\lambda)\xi_2 \rangle) > \lambda \Psi_\varepsilon(h_K(u) - \langle u, \xi_1 \rangle) + (1-\lambda)\Psi_\varepsilon(h_K(u) - \langle u, \xi_2 \rangle),$$

thus $\Phi_\varepsilon(K, \xi)$ is a strictly convex function of $\xi \in \operatorname{int} K$ by $f > \tau_1$.

Let $\xi_i \in \operatorname{int} K$ such that

$$\lim_{i \to \infty} \Phi_\varepsilon(K, \xi_i) = \inf_{\xi \in \operatorname{int} K} \Phi_\varepsilon(K, \xi).$$

We may assume that $\lim_{i \to \infty} \xi_i = x_0 \in K$, and Lemma 5.1 yields $x_0 \in \operatorname{int} K$. Since $\Phi_\varepsilon(K, \xi)$ is a strictly convex and continuous function of $\xi \in \operatorname{int} K$, x_0 is the unique minimum point of $\xi \mapsto \Phi_\varepsilon(K, \xi)$, which we denote by $\xi(K)$ (not signalling the dependence on ε, ψ and f).

Readily $\xi(K)$ is translation equivariant, and $\Phi_\varepsilon(K, \xi(K))$ is translation invariant.

For the continuity of $\xi(K)$ and $\Phi_\varepsilon(K, \xi(K))$, let us consider a sequence $\{K_i\}$ of convex bodies tending to a convex body K in $\mathbb{R}^n$. We may assume that $\xi(K_i)$ tends to a $x_0 \in K$.

For any $y \in \operatorname{int} K$, there exists an I such that $y \in \operatorname{int} K_i$ for $i \geq I$. Since h_{K_i} tends uniformly to h_K on S^{n-1}, we have that

$$\limsup_{i \to \infty} \Phi_\varepsilon(K_i, \xi(K_i)) \leq \lim_{\substack{i \to \infty \\ i \geq I}} \Phi_\varepsilon(K_i, y) = \Phi_\varepsilon(K, y).$$

Again Lemma 5.1 implies that $x_0 \in \operatorname{int} K$. It follows that $h_{K_i - \xi_i(K_i)}$ tends uniformly to $h_{K - x_0}$, thus

$$\Phi_\varepsilon(K, x_0) = \lim_{i \to \infty} \Phi_\varepsilon(K_i, \xi(K_i)) \leq \lim_{\substack{i \to \infty \\ i \geq I}} \Phi_\varepsilon(K_i, y) = \Phi_\varepsilon(K, y).$$

In particular, $\Phi_\varepsilon(K, x_0) \leq \Phi_\varepsilon(K, y)$ for any $y \in \operatorname{int} K$, thus $x_0 = \xi(K)$. In turn, we deduce $\xi(K_i)$ tends to $\xi(K)$, and $\Phi_\varepsilon(K_i, \xi(K_i))$ tends to $\Phi_\varepsilon(K, \xi(K))$. □

Since $\xi \mapsto \Phi_\varepsilon(K,\xi) = \int_{S^{n-1}} \Psi_\varepsilon(h_K(u) - \langle u, \xi\rangle) f(u)\, d\mathcal{H}^{n-1}(u)$ is maximal at $\xi(K) \in \operatorname{int} K$ and $\Psi'_\varepsilon = -\psi_\varepsilon$, we deduce

Corollary 5.3. *For $\varepsilon \in (0,\delta)$ and a convex body K in $\mathbb{R}^n$, we have*

$$\int_{S^{n-1}} u\ \psi_\varepsilon\Big(h_K(u) - \langle u, \xi(K)\rangle\Big) f(u)\, d\mathcal{H}^{n-1}(u) = o.$$

For a closed subgroup G of $O(n)$, we write $\mathcal{K}^{n,G}_{(0)}$ to denote the family of $K \in \mathcal{K}^n_{(0)}$ invariant under G.

Lemma 5.4. *For $\varepsilon \in (0,\delta)$, there exists a $K^\varepsilon \in \mathcal{K}^n_{(0)}$ with $V(K^\varepsilon) = 1$ such that*

$$\Phi_\varepsilon(K^\varepsilon, \xi(K^\varepsilon)) \geq \Phi_\varepsilon(K, \xi(K)) \quad \textit{for any } K \in \mathcal{K}^n_{(0)} \textit{ with } V(K) = 1.$$

In addition, if f is invariant under a closed subgroup G of $O(n)$, then K^ε can be chosen to be invariant under G.

Proof. We choose a sequence $K_i \in \mathcal{K}^n_{(0)}$ with $V(K_i) = 1$ for $i \geq 1$ such that

$$\lim_{i\to\infty} \Phi(K_i, \xi(K_i)) = \sup\{\Phi(K, \xi(K)) : K \in \mathcal{K}^n_{(0)} \text{ with } V(K) = 1\}.$$

Writing $B_1 = \kappa_n^{-1/n} B^n$ to denote the unit ball centred at the origin and having volume 1, we may assume that each K_i satisfies

$$\Phi_\varepsilon(K_i, \sigma(K_i)) \geq \Phi_\varepsilon(K_i, \xi(K_i)) \geq \Phi_\varepsilon(B_1, \xi(B_1)). \tag{29}$$

According to Proposition 5.2, we may also assume that $\sigma_{K_i} = o$ for each K_i.

We deduce from Lemma 4.3, (21), (23) and (29) that there exists some $R > 0$ such that $K_i \subset RB^n$ for any $i \geq 1$. According to the Blaschke selection theorem, we may assume that K_i tends to a compact convex set K^ε with $o \in K^\varepsilon$. It follows from the continuity of the volume that $V(K^\varepsilon) = 1$, and hence $\operatorname{int} K^\varepsilon \neq \emptyset$. We conclude from Lemma 5.2 that $\Phi_\varepsilon(K^\varepsilon, \xi(K^\varepsilon)) = \lim_{i\to\infty} \Phi_\varepsilon(K_i, \xi(K_i))$.

If f is invariant under a closed subgroup G of $O(n)$, then we apply the same argument to convex bodies in $\mathcal{K}^{n,G}_{(0)}$ instead of $\mathcal{K}^n_{(0)}$. $\square$

Since $\Phi(5) < \Phi(4)$, (25) yields some $\tilde{\delta} \in (0,\delta)$ such that $\Psi_\varepsilon(4) \geq \Phi(5)$ for $\varepsilon \in (0,\tilde{\delta})$. For future reference, the monotonicity of Ψ_ε, $\operatorname{diam} \kappa_n^{-1/n} B^n \leq 4$ and (29) yield that if $\varepsilon \in (0,\tilde{\delta})$, then

$$\begin{aligned} \Phi_\varepsilon(K^\varepsilon, \sigma(K^\varepsilon)) &\geq \Phi_\varepsilon\big(\kappa_n^{-1/n} B^n, \xi(\kappa_n^{-1/n} B^n)\big) \\ &\geq \int_{S^{n-1}} \Psi_\varepsilon(4) f\, d\mathcal{H}^{n-1} \geq \Psi(5) \int_{S^{n-1}} f\, d\mathcal{H}^{n-1}. \end{aligned} \tag{30}$$

6. Variational formulae and smoothness of the extremal body when f is bounded and bounded away from zero

In this section, again let $0 < \tau_1 < \tau_2$ and let the real function f on S^{n-1} satisfy $\tau_1 < f < \tau_2$. In addition, let φ be the continuous function of Theorem 1.2, and we use the notation developed in Section 5, say $\psi : (0, \infty) \to (0, \infty)$ is defined by $\psi = 1/\varphi$.

Now that we have constructed an extremal body K^ε, we want to show that it satisfies the required differential equation in the Alexandrov sense by using a variational argument. This section provides the formulae that we will need, and ensure the required smoothness of K^ε.

Concerning the variation of volume, a key tool is Alexandrov's Lemma 6.1 (see Lemma 7.5.3 in [38]). To state this, for any continuous $h : S^{n-1} \to (0, \infty)$, we define the Alexandrov body

$$[h] = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(u) \ \text{ for } u \in S^{n-1}\}$$

which is a convex body containing the origin in its interior. Obviously, if $K \in \mathcal{K}^n_{(0)}$ then $K = [h_K]$.

Lemma 6.1 *(Alexandrov). For $K \in \mathcal{K}^n_{(0)}$ and a continuous function $g : S^{n-1} \to \mathbb{R}$, $K(t) = [h_K + tg]$ satisfies*

$$\lim_{t\to 0} \frac{V(K(t)) - V(K)}{t} = \int_{S^{n-1}} g(u)\, dS_K(u).$$

To handle the variation of $\Phi_\varepsilon(K(t), \xi(K(t)))$ for a family $K(t)$ is a more subtle problem. The next lemma shows essentially that if we perturb a convex body K in a way such that the support function is differentiable as a function of the parameter t for $\mathcal{H}^{n-1}$-almost all $u \in S^{n-1}$, then $\xi(K)$ changes also in a differentiable way. Lemma 6.2 is the point of the proof where we use that ψ_ε is C^1 and $\psi'_\varepsilon < 0$.

Lemma 6.2. *For $\varepsilon \in (0, \delta)$, let $c > 0$ and $t_0 > 0$, and let $K(t)$ be a family of convex bodies with support function h_t for $t \in [0, t_0)$. Assume that*

(i) $|h_t(u) - h_0(u)| \leq ct$ *for each $u \in S^{n-1}$ and $t \in [0, t_0)$,*
(ii) $\lim_{t\to 0^+} \frac{h_t(u)-h_0(u)}{t}$ *exists for $\mathcal{H}^{n-1}$-almost all $u \in S^{n-1}$.*

Then $\lim_{t\to 0^+} \frac{\xi(K(t))-\xi(K(0))}{t}$ exists.

Proof. We set $K = K(0)$. We may assume that $\xi(K) = o$, and hence Proposition 5.2 yields that

$$\lim_{t\to 0^+} \xi(K(t)) = o.$$

There exists some $R > r > 0$ such that $r \leq h_t(u) - \langle u, \xi(K(t))\rangle = h_{K(t)-\xi(K(t))}(u) \leq R$ for $u \in S^{n-1}$ and $t \in [0, t_0)$. Since ψ_ε is C^1 on $[r, R]$, we can write

$$\psi_\varepsilon(t) - \psi_\varepsilon(s) = \psi'_\varepsilon(s)(t-s) + \eta_0(s,t)(t-s)$$

for $t, s \in [r, R]$ where $\lim_{t\to s} \eta_0(s,t) = 0$. Let $g(t,u) = h_t(u) - h_K(u)$ for $u \in S^{n-1}$ and $t \in [0, t_0)$. Since $h_{K(t)-\xi(K(t))}$ tends uniformly to h_K on S^{n-1}, we deduce that if $t \in [0, t_0)$, then

$$\begin{aligned} &\psi_\varepsilon\Big(h_t(u) - \langle u, \xi(K(t))\rangle\Big) - \psi_\varepsilon(h_K(u)) \\ &\quad = \psi'_\varepsilon(h_K(u))\big(g(t,u) - \langle u, \xi(K(t))\rangle\big) + e(t,u) \end{aligned} \tag{31}$$

where

$$|e(t,u)| \leq \eta(t)|g(t,u) - \langle u, \xi(K_t)\rangle| \quad \text{and} \quad \eta(t) = \eta_0(h_K(u), h_t(u) - \langle u, \xi(K(t))\rangle).$$

Note that $\lim_{t\to 0^+} \eta(t) = 0$ uniformly in $u \in S^{n-1}$.

In particular, (i) yields that $e(t,u) = e_1(t,u) + e_2(t,u)$ where

$$|e_1(t,u)| \leq c\eta(t)t \quad \text{and} \quad |e_2(t,u)| \leq \eta(t)\|\xi(K(t))\|. \tag{32}$$

It follows from (31) and from applying Corollary 5.3 to $K(t)$ and K that

$$\int_{S^{n-1}} u\, \Big(\psi'_\varepsilon(h_K(u))\, \big(g(t,u) - \langle u, \xi(K(t))\rangle\big) + e(t,u)\Big)\, f(u) d\mathcal{H}^{n-1}(u) = o,$$

which can be written as

$$\begin{aligned} &\int_{S^{n-1}} u\, \psi'_\varepsilon(h_K(u))\, g(t,u)\, f(u) d\mathcal{H}^{n-1}(u) \\ &\qquad + \int_{S^{n-1}} u\, e_1(t,u)\, f(u) d\mathcal{H}^{n-1}(u) = \int_{S^{n-1}} u\, \langle u, \xi(K_t)\rangle \psi'_\varepsilon(h_K(u))\, f(u) d\mathcal{H}^{n-1}(u) \\ &\qquad\qquad - \int_{S^{n-1}} u\; e_2(t,u)\, f(u) d\mathcal{H}^{n-1}(u). \end{aligned}$$

Since $\psi'_\varepsilon(s) < 0$ for all $s > 0$, the symmetric matrix

$$A = \int_{S^{n-1}} u \otimes u\, \psi'_\varepsilon(h_K(u))\, f(u) d\mathcal{H}^{n-1}(u)$$

is negative definite because for any $v \in S^{n-1}$, we have

$$v^T A v = \int_{S^{n-1}} \langle u, v\rangle^2 \, \psi'_\varepsilon(h_K(u))\, f(u)\, d\mathcal{H}^{n-1}(u) < 0.$$

In addition, A satisfies

$$\int_{S^{n-1}} u \, \langle u, \xi(K_t)\rangle \, \psi'_\varepsilon(h_K(u))\, f(u) d\mathcal{H}^{n-1}(u) = A\, \xi(K_t).$$

It follows from (32) that if $t \geq 0$ is small, then

$$A^{-1} \int_{S^{n-1}} u \, \psi'_\varepsilon(h_K(u)) \; g(t,u)\, f(u) d\mathcal{H}^{n-1}(u) + \tilde{e}_1(t) = \xi(K_t) - \tilde{e}_2(t), \tag{33}$$

where $\|\tilde{e}_1(t)\| \leq \alpha_1 \eta(t) t$ and $\|\tilde{e}_2(t)\| \leq \alpha_2 \eta(t) \|\xi(K_t)\|$ for constants $\alpha_1, \alpha_2 > 0$. Since $\eta(t)$ tends to 0 with t, if $t \geq 0$ is small, then $\|\xi(K(t)) - \tilde{e}_2(t)\| \geq \frac{1}{2} \|\xi(K_t)\|$. Adding the estimate $g(t,u) \leq ct$, we deduce that $\|\xi(K(t))\| \leq \beta\, t$ for a constant $\beta > 0$, which in turn yields that $\lim_{t\to 0^+} \frac{\|\tilde{e}_i(t)\|}{t} = 0$ and $\tilde{e}_i(0) = 0$ for $i = 1, 2$. Since there exists $\lim_{t\to 0^+} \frac{g(t,u) - g(0,u)}{t} = \partial_1 g(0,u)$ for $\mathcal{H}^{n-1}$ almost all $u \in S^{n-1}$, and $\frac{g(t,u)-g(0,u)}{t} < c$ for all $u \in S^{n-1}$ and $t > 0$, we conclude that

$$\frac{d}{dt}\, \xi(K(t))\Big|_{t=0^+} = A^{-1} \int_{S^{n-1}} u \, \psi'_\varepsilon(h_K(u)) \; \partial_1 g(0,u)\, f(u)\, d\mathcal{H}^{n-1}(u). \quad \square$$

Corollary 6.3. *Under the conditions of Lemma 6.2, and setting $K = K(0)$, we have*

$$\frac{d}{dt}\, \Phi_\varepsilon(K(t), \xi(K(t)))\Big|_{t=0^+} = - \int_{S^{n-1}} \frac{\partial}{\partial t} h_{K(t)}(u)\Big|_{t=0^+} \psi_\varepsilon\big(h_K(u) - \langle u, \xi(K)\rangle\big)\, f(u)\, d\mathcal{H}^{n-1}(u).$$

We omit the proof of this result since it is very similar to that of [1, Corollary 3.6], given by the authors and Yang for the L_p Minkowski problem, with $f(u)\, d\mathcal{H}^{n-1}(u)$, Ψ_ε, $-\psi_\varepsilon$, Lemma 6.2 and Corollary 5.3 replacing respectively $d\mu(u)$, φ_ε, φ'_ε, Lemma 3.5 and Corollary 3.3.

Given a family $K(t)$ of convex bodies for $t \in [0, t_0)$, $t_0 > 0$, to handle the variation of $\Phi_\varepsilon(K(t), \xi(K(t)))$ at $K(0) = K$ via applying Corollary 6.3, we need the property (see Lemma 6.2) that there exists $c > 0$ such that

$$|h_{K(t)}(u) - h_K(u)| \leq c\,|t| \quad \text{for any } u \in S^{n-1} \text{ and } t \in [0, t_0) \tag{34}$$

$$\lim_{t\to 0^+} \frac{h_{K(t)}(u) - h_K(u)}{t} \quad \text{exists for } \mathcal{H}^{n-1} \text{ almost all } u \in S^{n-1}. \tag{35}$$

However, even if $K(t) = [h_K + th_C]$ for $K, C \in \mathcal{K}^n_{(0)}$ and for $t \in (-t_0, t_0)$, K must satisfy some smoothness assumption in order to ensure that (35) holds also for the two sided limits (problems occur say if K is a polytope and C is smooth).

We recall that $\partial' K$ denotes the set of smooth points of ∂K. We say that K is *quasi-smooth* if $\mathcal{H}^{n-1}(S^{n-1}\backslash\nu_K(\partial' K)) = 0$; namely, the set of $u \in S^{n-1}$ that are exterior normals only at singular points has $\mathcal{H}^{n-1}$-measure zero. The following Lemma 6.4, taken from Bianchi, Böröczky, Colesanti, Yang [1], shows that (34) and (35) are satisfied even if $t \in (-t_0, t_0)$ at least for $K(t) = [h_K + th_C]$ with arbitrary $C \in \mathcal{K}^n_{(0)}$ if K is quasi-smooth.

Lemma 6.4. *Let $K, C \in \mathcal{K}^n_{(0)}$ be such that $rC \subset K$ for some $r > 0$. For $t \in (-r, r)$ and $K(t) = [h_K + th_C]$,*

(i) *if $K \subset RC$ for $R > 0$, then $|h_{K(t)}(u) - h_K(u)| \leq \frac{R}{r}|t|$ for any $u \in S^{n-1}$ and $t \in (-r, r)$;*

(ii) *if $u \in S^{n-1}$ is the exterior normal at some smooth point $z \in \partial K$, then*

$$\lim_{t\to 0} \frac{h_{K(t)}(u) - h_K(u)}{t} = h_C(u).$$

We will need the condition (35) in the following rather special setting taken from Bianchi, Böröczky, Colesanti, Yang [1].

Lemma 6.5. *Let K be a convex body with $rB^n \subset \operatorname{int} K$ for $r > 0$, let $\omega \subset S^{n-1}$ be closed, and if $t \in [0, r)$, then let*

$$K(t) = [h_K - \mathbf{1}_\omega] = \{x \in K : \langle x, u\rangle \leq h_K(u) - t \quad \textit{for every } u \in \omega\}.$$

(i) *We have $\lim_{t\to 0^+} \frac{h_{K(t)}(u)-h_K(u)}{t}$ exists and is non-positive for all $u \in S^{n-1}$, and if $u \in \omega$, then even $\lim_{t\to 0^+} \frac{h_{K(t)}(u)-h_K(u)}{t} \leq -1$.*

(ii) *If $S_K(\omega) = 0$, then $\lim_{t\to 0^+} \frac{V(K(t))-V(K)}{t} = 0$.*

Proposition 6.6. *For $\varepsilon \in (0, \delta)$, K^ε is quasi-smooth.*

Proof. We suppose that K^ε is not quasi-smooth, and seek a contradiction. It follows that $\mathcal{H}^{n-1}(X) > 0$ for $X = S^{n-1}\backslash\nu_{K^\varepsilon}(\partial' K^\varepsilon)$, therefore there exists a closed $\omega \subset X$ such that $\mathcal{H}^{n-1}(\omega) > 0$. Since $\nu^{-1}_{K^\varepsilon}(\omega) \subset \partial K^\varepsilon\backslash\partial' K^\varepsilon$, we deduce that $S_{K^\varepsilon}(\omega) = 0$.

We may assume that $\xi(K^\varepsilon) = o$ and $rB^n \subset K^\varepsilon \subset RB^n$ for $R > r > 0$. As in Lemma 6.5, if $t \in [0, r)$, then we define

$$K(t) = [h_{K^\varepsilon} - \mathbf{1}_\omega] = \{x \in K^\varepsilon : \langle x, u\rangle \leq h_K(u) - t \quad \text{for every } u \in \omega\}.$$

Clearly, $K(0)$ equals K^ε. We define $\alpha(t) = V(K(t))^{-1/n}$, and hence $\alpha(0) = 1$, and Lemma 6.5 (ii) yields that $\alpha'(0) = 0$.

We set $\widetilde{K}(t) = \alpha(t)K(t)$, and hence $\widetilde{K}(0) = K^\varepsilon$ and $V(\widetilde{K}(t)) = 1$ for all $t \in [0, r)$. In addition, we consider $h(t, u) = h_{K(t)}(u)$ and $\tilde{h}(t, u) = h_{\widetilde{K}(t)}(u) = \alpha(t)h(t, u)$ for $u \in S^{n-1}$ and $t \in [0, r)$. Since $[h_{K^\varepsilon} - th_{B^n}] \subset K(t)$, Lemma 6.4 (i) yields that $|h(t, u) - h(0, u)| \le \frac{R}{r} t$ for $u \in S^{n-1}$ and $t \in [0, r)$. Hence $\alpha'(0) = 0$ implies that there exist $c > 0$ and $t_0 \in (0, r)$ such that $|\tilde{h}(t, u) - \tilde{h}(0, u)| \le ct$ for $u \in S^{n-1}$ and $t \in [0, t_0)$. Applying $\alpha(0) = 1$, $\alpha'(0) = 0$ and Lemma 6.5 (i), we deduce that

$$\partial_1 \tilde{h}(0, u) = \lim_{t \to 0^+} \frac{\tilde{h}(t, u) - \tilde{h}(0, u)}{t} = \lim_{t \to 0^+} \frac{h(t, u) - h(0, u)}{t} \le 0 \quad \text{exists for all } u \in S^{n-1},$$

$$\partial_1 \tilde{h}(0, u) \le -1 \quad \text{for all } u \in \omega.$$

As ψ_ε is positive and monotone decreasing, $f > \tau_1$ and $\mathcal{H}^{n-1}(\omega) > 0$, Corollary 6.3 implies that

$$\begin{aligned} \frac{d}{dt} \Phi_\varepsilon(\widetilde{K}(t), \xi(\widetilde{K}(t)))\Big|_{t=0^+} &= - \int_{S^{n-1}} \partial_1 \tilde{h}(0, u) \cdot \psi_\varepsilon(h_K(u))\, f(u)\, d\mathcal{H}^{n-1}(u) \\ &\ge - \int_\omega (-1)\psi_\varepsilon(R)\tau_1\, d\mathcal{H}^{n-1}(u) > 0. \end{aligned}$$

Therefore $\Phi_\varepsilon(\widetilde{K}(t), \xi(\widetilde{K}(t))) > \Phi_\varepsilon(K^\varepsilon, \xi(K^\varepsilon))$ for small $t > 0$. This contradicts the definition of K^ε and concludes the proof. □

For $\varepsilon \in (0, \delta)$, we define

$$\lambda_\varepsilon = \frac{1}{n} \int_{S^{n-1}} h_{K^\varepsilon - \xi(K^\varepsilon)} \cdot \psi_\varepsilon(h_{K^\varepsilon - \xi(K^\varepsilon)}) \cdot f\, d\mathcal{H}^{n-1}. \tag{36}$$

Proposition 6.7. *For $\varepsilon \in (0, \delta)$, $\psi_\varepsilon(h_{K^\varepsilon - \xi(K^\varepsilon)}) \cdot f\, d\mathcal{H}^{n-1} = \lambda_\varepsilon\, dS_{K^\varepsilon}$ as measures on S^{n-1}.*

We omit the proof of this result since it is very similar to that of [1, Proposition 6.1], given by the authors and Yang for the L_p Minkowski problem, with $-\lambda_\varepsilon$, $-\psi_\varepsilon$, Lemma 6.1, Lemma 6.4, Corollary 6.3, and [38] replacing respectively λ_ε, φ'_ε, Lemma 5.2, Lemma 2.3, Corollary 3.6 and [35].

7. The proof of Theorem 1.2 when f is bounded and bounded away from zero

In this section, again let $0 < \tau_1 < \tau_2$, let the real function f on S^{n-1} satisfy $\tau_1 < f < \tau_2$, and let φ be the continuous function on $[0, \infty)$ of Theorem 1.2. We use the notation developed in Section 5, and hence $\psi : (0, \infty) \to (0, \infty)$ and $\psi = 1/\varphi$.

To ensure that a convex body is "fat" enough in Lemma 7.2 and later, the following observation is useful:

Lemma 7.1. *If K is a convex body in $\mathbb{R}^n$ with $V(K) = 1$ and $K \subset \sigma(K) + RB^n$ for $R > 0$, then*

$$\sigma(K) + rB^n \subset K \quad \text{for } r = \frac{1}{c\kappa_{n-1}} n^{-3/2} R^{-(n-1)}.$$

Proof. Let $z_0 + r_0 B^n$ be a largest ball in K. According to the Steinhagen theorem [12, Theorem 50], there exists $v \in S^{n-1}$ such that

$$|\langle x - z_0, v\rangle| \le c\sqrt{n} r_0 \quad \text{for } x \in K,$$

where c is a positive universal constant. It follows that $1 = V(K) \le c\sqrt{n} r_0 \kappa_{n-1} R^{n-1}$, thus $r_0 \ge \frac{1}{c\kappa_{n-1}} n^{-1/2} R^{-(n-1)}$. Since $\sigma(K) + \frac{r_0}{n} B^n \subset K$ by $-(K - \sigma(K)) \subset n(K - \sigma(K))$, we may choose $r = \frac{1}{c\kappa_{n-1}} n^{-3/2} R^{-(n-1)}$. $\square$

We recall (compare (36)) that if $\varepsilon \in (0, \delta)$ and $\xi(K^\varepsilon) = o$, then λ_ε is defined by

$$\lambda_\varepsilon = \frac{1}{n} \int_{S^{n-1}} h_{K^\varepsilon} \psi_\varepsilon(h_{K^\varepsilon}) f \, d\mathcal{H}^{n-1}. \tag{37}$$

Lemma 7.2. *There exist $R_0 > 1$, $r_0 > 0$ and $\tilde{\lambda}_2 > \tilde{\lambda}_1 > 0$ depending on $f, q, \psi, \aleph$ such that if $\varepsilon \in (0, \delta_0)$ for $\delta_0 = \min\{\tilde{\delta}, \frac{r_0}{2}\}$ where $\tilde{\delta}$ comes from (30), then $\tilde{\lambda}_1 \le \lambda_\varepsilon \le \tilde{\lambda}_2$ and*

$$\sigma(K^\varepsilon) + r_0 B^n \subset K^\varepsilon \subset \sigma(K^\varepsilon) + R_0 B^n.$$

Proof. According to (23), there exists $\aleph_0 > 0$ depending on $q, \psi, \aleph$ such that if $\varepsilon \in (0, \delta)$ and $t \in (0, \delta)$, then $\Psi_\varepsilon(t) \le \aleph_0 t^q$. In addition, $\lim_{t\to\infty} \Psi_\varepsilon(t) = 0$ by (21), therefore we may apply Lemma 4.3. Since (30) provides the condition

$$\int_{S^{n-1}} \Psi_\varepsilon(h_{K^\varepsilon - \sigma(K^\varepsilon)}) f \, d\mathcal{H}^{n-1} \ge \Psi(5) \int_{S^{n-1}} f \, d\mathcal{H}^{n-1}$$

for any $\varepsilon \in (0, \tilde{\delta})$, we deduce from Lemma 4.3 the existence of $R_0 > 0$ such that $K^\varepsilon \subset \sigma(K^\varepsilon) + R_0 B^n$ for any $\varepsilon \in (0, \tilde{\delta})$. In addition, the existence of r_0 independent of ε such that $\sigma(K^\varepsilon) + r_0 B^n \subset K^\varepsilon$ follows from Lemma 7.1.

To estimate λ_ε, we assume $\xi(K^\varepsilon) = o$. Let $w_\varepsilon \in S^{n-1}$ and $\varrho_\varepsilon \ge 0$ be such that $\sigma(K^\varepsilon) = \varrho_\varepsilon w_\varepsilon$, and hence $r_0 w_\varepsilon \in K^\varepsilon$. It follows that $h_{K^\varepsilon}(u) \ge r_0/2$ holds for $u \in \Omega(w_\varepsilon, \frac{\pi}{3})$, while $K^\varepsilon \subset 2R_0 B^n$, $R_0 > 1$ and the monotonicity of ψ_ε imply that $\psi_\varepsilon(h_{K^\varepsilon}(u)) \ge \psi_\varepsilon(2R_0) = \psi(2R_0)$ for all $u \in S^{n-1}$.

We deduce from (37) that

$$\lambda_\varepsilon = \frac{1}{n} \int_{S^{n-1}} h_{K^\varepsilon} \psi_\varepsilon(h_{K^\varepsilon}) f \, d\mathcal{H}^{n-1} \ge \frac{1}{n} \cdot \frac{r_0}{2} \cdot \psi(2R_0) \cdot \tau_1 \cdot \mathcal{H}^{n-1}\left(\Omega\left(w_\varepsilon, \frac{\pi}{3}\right)\right) = \tilde{\lambda}_1.$$

To have a suitable upper bound on λ_ε, we define $\alpha \in (0, \frac{\pi}{2})$ with $\cos\alpha = \frac{r_0}{2R_0}$, and hence

$$\Omega(-w_\varepsilon, \alpha) = \left\{ u \in S^{n-1} : \langle u, w_\varepsilon \rangle \leq \frac{-r_0}{2R_0} \right\}.$$

A key observation is that if $u \in S^{n-1} \backslash \Omega(-w_\varepsilon, \alpha)$, then $\langle u, w_\varepsilon \rangle > -\frac{r_0}{2R_0}$ and $\varrho_\varepsilon \leq R_0$ imply

$$h_{K^\varepsilon}(u) \geq \langle u, \varrho w_\varepsilon + r_0 u \rangle \geq r_0 - \frac{r_0 \varrho_\varepsilon}{2R_0} \geq r_0/2,$$

therefore $\varepsilon < \frac{r_0}{2}$ yields

$$\psi_\varepsilon(h_{K^\varepsilon}(u)) \leq \psi_\varepsilon(r_0/2) = \psi(r_0/2). \tag{38}$$

Another observation is that $K^\varepsilon \subset 2R_0 B^n$ implies

$$h_{K^\varepsilon}(u) < 2R_0 \quad \text{for any } u \in S^{n-1}. \tag{39}$$

It follows directly from (38) and (39) that

$$\int_{S^{n-1} \backslash \Omega(-w_\varepsilon, \alpha)} h_{K^\varepsilon} \psi_\varepsilon(h_{K^\varepsilon}) f \, d\mathcal{H}^{n-1} \leq (2R_0) \psi(r_0/2) \tau_2 n \kappa_n. \tag{40}$$

However, if $u \in \Omega(-w_\varepsilon, \alpha)$, then $\psi_\varepsilon(h_{K^\varepsilon}(u))$ can be arbitrary large as $\xi(K^\varepsilon)$ can be arbitrary close to ∂K^ε if $\varepsilon > 0$ is small, and hence we transfer the problem to the previous case $u \in S^{n-1} \backslash \Omega(-w_\varepsilon, \alpha)$ using Corollary 5.3. First applying $\langle u, -w_\varepsilon \rangle \geq \frac{r_0}{2R_0}$ for $u \in \Omega(-w_\varepsilon, \alpha)$, then Corollary 5.3, and after that $\langle u, w_\varepsilon \rangle \leq 1$, $f \leq \tau_2$ and (38) implies

$$\begin{aligned}
\int_{\Omega(-w_\varepsilon, \alpha)} \psi_\varepsilon(h_{K^\varepsilon}(u)) f(u) \, d\mathcal{H}^{n-1}(u) &\leq \frac{2R_0}{r_0} \int_{\Omega(-w_\varepsilon, \alpha)} \langle u, -w_\varepsilon \rangle \psi_\varepsilon(h_{K^\varepsilon}(u)) f(u) \, d\mathcal{H}^{n-1}(u) \\
&= \frac{2R_0}{r_0} \int_{S^{n-1} \backslash \Omega(-w_\varepsilon, \alpha)} \langle u, w_\varepsilon \rangle \psi_\varepsilon(h_{K^\varepsilon}(u)) f(u) \, d\mathcal{H}^{n-1}(u) \\
&\leq \frac{2R_0}{r_0} \cdot \psi\left(\frac{r_0}{2}\right) \tau_2 n \kappa_n.
\end{aligned}$$

Now (39) yields

$$\int_{\Omega(-w_\varepsilon, \alpha)} h_K \psi_\varepsilon(h_K) f \, d\mathcal{H}^{n-1} \leq \frac{(2R_0)^2}{r_0} \cdot \psi\left(\frac{r_0}{2}\right) \tau_2 n \kappa_n,$$

which estimate combined with (40) leads to $\lambda_\varepsilon < \left(\frac{(2R_0)^2}{r_0} + 2R_0 \right) \psi(\frac{r_0}{2}) \tau_2 n \kappa_n$. In turn, we conclude Lemma 7.2. □

Now we prove Theorem 1.2 if f is bounded and bounded away from zero.

Theorem 7.3. *For $0 < \tau_1 < \tau_2$, let the real function f on S^{n-1} satisfy $\tau_1 < f < \tau_2$, and let $\varphi : [0,\infty) \to [0,\infty)$ be increasing and continuous satisfying $\varphi(0) = 0$, $\liminf_{t\to 0^+} \frac{\varphi(t)}{t^{1-p}} > 0$, and $\int_1^\infty \frac{1}{\varphi} < \infty$. Let $\Psi(t) = \int_t^\infty \frac{1}{\varphi}$. Then there exist $\lambda > 0$ and a $K \in \mathcal{K}_0^n$ with $V(K) = 1$ such that*

$$f\, d\mathcal{H}^{n-1} = \lambda\varphi(h_K)\, dS_K,$$

as measures on S^{n-1}, and

$$\int\limits_{S^{n-1}} \Psi(h_{K-\sigma(K)}) f\, d\mathcal{H}^{n-1} \geq \Psi(5) \int\limits_{S^{n-1}} f\, d\mathcal{H}^{n-1}. \tag{41}$$

In addition, if f is invariant under a closed subgroup G of $O(n)$, then K can be chosen to be invariant under G.

Proof. We assume that $\xi(K^\varepsilon) = o$ for all $\varepsilon \in (0, \delta_0)$ where $\delta_0 \in (0, \delta]$ comes from Lemma 7.2. Using the constant R_0 of Lemma 7.2, we have that

$$K^\varepsilon \subset 2R_0 B^n \text{ and } \ h_{K^\varepsilon}(u) < 2R_0 \ \text{ for any } u \in S^{n-1} \text{ and } \varepsilon \in (0, \delta_0). \tag{42}$$

We consider the continuous increasing function $\varphi_\varepsilon : [0,\infty)] \to [0,\infty)$ defined by $\varphi_\varepsilon(0) = 0$ and $\varphi_\varepsilon(t) = 1/\psi_\varepsilon(t)$ for $\varepsilon \in (0, \delta)$. We claim that

$$\varphi_\varepsilon \text{ tends uniformly to } \varphi \text{ on } [0, 2R_0] \text{ as } \varepsilon > 0 \text{ tends to zero.} \tag{43}$$

For any small $\zeta > 0$, there exists $\delta_\zeta > 0$ such that $\varphi(t) \leq \zeta/2$ for $t \in [0, \delta_\zeta]$. We deduce from (20) that if $\varepsilon > 0$ is small, then $|\varphi_\varepsilon(t) - \varphi(t)| \leq \zeta/2$ for $t \in [\delta_\zeta, 2R_0]$. However φ_ε is monotone increasing, therefore $\varphi_\varepsilon(t), \varphi(t) \in [0, \zeta]$ for $t \in [0, \delta_\zeta]$, completing the proof of (43).

For any $\varepsilon \in (0, \delta_0)$, it follows from Lemma 6.7 that $\psi_\varepsilon(h_{K^\varepsilon}) f\, d\mathcal{H}^{n-1} = \lambda_\varepsilon\, dS_{K^\varepsilon}$ as measures on S^{n-1}. Integrating $g\varphi_\varepsilon(h_{K^\varepsilon})$ for any continuous real function g on S^{n-1}, we deduce that

$$f\, d\mathcal{H}^{n-1} = \lambda_\varepsilon \varphi_\varepsilon(h_{K^\varepsilon})\, dS_{K^\varepsilon} \tag{44}$$

as measures on S^{n-1}.

Since $\tilde\lambda_1 \leq \lambda_\varepsilon \leq \tilde\lambda_2$ for some $\tilde\lambda_2 > \tilde\lambda_1$ independent of ε according to Lemma 7.2, (42) yields the existence of $\lambda > 0$, $K \in \mathcal{K}_0^n$ with $V(K) = 1$ and sequence $\{\varepsilon(m)\}$ tending to 0 such that $\lim_{m\to\infty} \lambda_{\varepsilon(m)} = \lambda$ and $\lim_{m\to\infty} K^{\varepsilon(m)} = K$. As $h_{K^{\varepsilon(m)}}$ tends uniformly to h_K on S^{n-1}, we deduce that $\lambda_{\varepsilon(m)}\varphi_{\varepsilon(m)}(h_{K^{\varepsilon(m)}})$ tends uniformly to $\lambda\varphi(h_K)$ on S^{n-1}. In addition, $S_{K^{\varepsilon(m)}}$ tends weakly to S_K, thus (44) yields

$$f\,d\mathcal{H}^{n-1} = \lambda\varphi(h_K)\,dS_K.$$

We note that if f is invariant under a closed subgroup G of $O(n)$, then each K^ε can be chosen to be invariant under G according to Lemma 5.4, therefore K is invariant under G in this case.

To prove (41), if $\varepsilon \in (0,\delta_0)$, then (30) provides the condition

$$\int_{S^{n-1}} \Psi_\varepsilon(h_{K^\varepsilon-\sigma(K^\varepsilon)}) f\,d\mathcal{H}^{n-1} \geq \Psi(5) \int_{S^{n-1}} f\,d\mathcal{H}^{n-1}. \tag{45}$$

Now Lemma 7.2 yields that there exists $r_0 > 0$ such that if $\varepsilon \in (0,\delta_0)$, then $\sigma(K^\varepsilon) + r_0 B^n \subset K^\varepsilon$ where $0 < \delta_0 \leq \frac{r_0}{2}$. In particular, if $u \in S^{n-1}$, then $h_{K^\varepsilon-\sigma(K^\varepsilon)}(u) \geq r_0$, and hence we deduce from (26) that

$$\Psi_\varepsilon(h_{K^\varepsilon-\sigma(K^\varepsilon)}(u)) \leq \Psi(h_{K^\varepsilon-\sigma(K^\varepsilon)}(u)) + \frac{\pi}{2}. \tag{46}$$

Since $K^{\varepsilon(m)} - \sigma(K^{\varepsilon(m)})$ tends to $K - \sigma(K)$, (25) implies that if $u \in S^{n-1}$, then

$$\lim_{\varepsilon\to 0^+} \Psi_\varepsilon(h_{K^\varepsilon-\sigma(K^\varepsilon)}(u)) = \Psi(h_{K-\sigma(K)}(u)). \tag{47}$$

Combining (45), (46) and (47) with Lebesgue's Dominated Convergence Theorem, we conclude (41), and in turn Theorem 7.3. □

8. The proof of Theorem 1.2

Let $-n < p < 0$, let f be a non-negative non-trivial function in $L_{\frac{n}{n+p}}(S^{n-1})$, and let $\varphi : [0,\infty) \to [0,\infty)$ be a monotone increasing continuous function satisfying $\varphi(0) = 0$,

$$\liminf_{t\to 0^+} \frac{\varphi(t)}{t^{1-p}} > 0 \tag{48}$$

$$\int_1^\infty \frac{1}{\varphi(t)}\,dt < \infty. \tag{49}$$

We associate certain functions to f and φ. For any integer $m \geq 2$, we define f_m on $\mathbb{S}^{n-1}$ as follows:

$$f_m(u) = \begin{cases} m & \text{if } f(u) \geq m, \\ f(u) & \text{if } \frac{1}{m} < f(u) < m, \\ \frac{1}{m} & \text{if } f(u) \leq \frac{1}{m}. \end{cases}$$

In particular, $f_m \leq \tilde{f}$ where the function $\tilde{f} : S^{n-1} \to [0,\infty)$ in $L_{\frac{n}{n+p}}(S^{n-1})$, and hence in $L_1(S^{n-1})$, is defined by

$$\tilde{f}(u) = \begin{cases} f(u) & \text{if } f(u) > 1, \\ 1 & \text{if } f(u) \le 1. \end{cases}$$

As in Section 5, using (49), we define the function

$$\Psi(t) = \int_t^\infty \frac{1}{\varphi} \quad \text{for } t > 0.$$

According to (48), there exist some $\delta \in (0,1)$ and $\aleph > 1$ such that

$$\frac{1}{\varphi(t)} < \aleph t^{p-1} \quad \text{for } t \in (0,\delta). \tag{50}$$

We deduce from Lemma 4.1 that there exists $\aleph_0 > 1$ depending on φ such that

$$\Psi(t) < \aleph_0 t^p \quad \text{for } t \in (0,\delta). \tag{51}$$

For $m \ge 2$, Theorem 7.3 yields a $\lambda_m > 0$ and a convex body $K_m \in \mathcal{K}_0^n$ with $\xi(K_m) = o \in \operatorname{int} K_m$, $V(K_m) = 1$ such that

$$\lambda_m \varphi(h_{K_m})\, dS_{K_m} = f_m\, d\mathcal{H}^{n-1} \tag{52}$$

$$\int_{S^{n-1}} \Psi(h_{K_m - \sigma(K_m)}) f_m\, d\mathcal{H}^{n-1} \ge \Psi(5) \int_{S^{n-1}} f_m\, d\mathcal{H}^{n-1}. \tag{53}$$

In addition, if f is invariant under a closed subgroup G of $O(n)$, then f_m is also invariant under G, and hence K_m can be chosen to be invariant under G.

Since $f_m \le \tilde{f}$, and f_m converges pointwise to f, Lebesgue's Dominated Convergence theorem yields the existence of $m_0 > 2$ such that if $m > m_0$, then

$$\frac{1}{2} \int_{S^{n-1}} f < \int_{S^{n-1}} f_m < 2 \int_{S^{n-1}} f. \tag{54}$$

In particular, (53) implies

$$\int_{S^{n-1}} \Psi(h_{K_m - \sigma(K_m)}) \tilde{f}\, d\mathcal{H}^{n-1} \ge \frac{\Psi(5)}{2} \int_{S^{n-1}} f\, d\mathcal{H}^{n-1}. \tag{55}$$

We deduce from $V(K_m) = 1$, $\lim_{t \to \infty} \Psi(t) = 0$, (51), (55) and Lemma 4.3 that there exists $R_0 > 0$ independent of m such that

$$K_m \subset \sigma(K_m) + R_0 B^n \subset 2R_0 B^n \quad \text{for all } m > m_0. \tag{56}$$

Since $V(K_m) = 1$, Lemma 7.1 yields some $r_0 > 0$ independent of m such that

$$\sigma(K_m) + r_0 B^n \subset K_m \quad \text{for all } m > m_0. \tag{57}$$

To estimate λ_m from below, (56) implies that

$$\int_{S^{n-1}} \varphi(h_{K_m})\, dS_{K_m} \le \varphi(2R_0)\mathcal{H}^{n-1}(\partial K_m) \le \varphi(2R_0)(2R_0)^{n-1} n\kappa_n,$$

and hence it follows from (52) and (54) the existence of $\tilde{\lambda}_1 > 0$ independent of m such that

$$\lambda_m = \frac{\int_{S^{n-1}} f_m\, d\mathcal{H}^{n-1}}{\int_{S^{n-1}} \varphi(h_{K_m})\, dS_{K_m}} \ge \tilde{\lambda}_1 \quad \text{for all } m > m_0. \tag{58}$$

To have a suitable upper bound on λ_m for any $m > m_0$, we choose $w_m \in S^{n-1}$ and $\varrho_m \ge 0$ such that $\sigma(K_m) = \varrho_m w_m$. We set $B_m^{\#} = w_m^{\perp} \cap \operatorname{int} B^n$ and consider the relative open set

$$\Xi_m = (\partial K_m) \cap \Big(\varrho_m w_m + r_0 B_m^{\#} + (0,\infty) w_m\Big).$$

If u is an exterior unit normal at an $x \in \Xi_m$ for $m > m_0$, then $x = (\varrho_m + s)w_m + rv$ for $s > 0$, $r \in [0, r_0)$ and $v \in w_m^{\perp} \cap S^{n-1}$, and hence $\varrho_m w_m + rv \in K_m$ yields

$$\langle u, (\varrho_m + s)w_m + rv\rangle = h_{K_m}(u) \ge \langle u, \varrho_m w_m + rv\rangle,$$

implying that $\langle u, w_m\rangle \ge 0$; or in other words, $u \in \Omega(w_m, \frac{\pi}{2})$. Since the orthogonal projection of Ξ_m onto $w_m^{\perp}$ is $B_m^{\#}$ for $m > m_0$, we deduce that

$$S_{K_m}\left(\Omega\left(w_m, \frac{\pi}{2}\right)\right) \ge \mathcal{H}^{n-1}(\Xi_m) \ge \mathcal{H}^{n-1}(B_m^{\#}) = r_0^{n-1}\kappa_{n-1}. \tag{59}$$

On the other hand, if $u \in \Omega(w_m, \frac{\pi}{2})$ for $m > m_0$, then $\varrho_m w_m + r_0 u \in K_m$ yields

$$h_{K_m}(u) \ge \langle u, \varrho_m w_m + r_0 u\rangle \ge r_0. \tag{60}$$

Combining (54), (59) and (60) implies

$$\lambda_m = \frac{\int_{\Omega(w_m,\frac{\pi}{2})} f_m\, d\mathcal{H}^{n-1}}{\int_{\Omega(w_m,\frac{\pi}{2})} \varphi(h_{K_m})\, dS_{K_m}} \le \frac{2\int_{S^{n-1}} f\, d\mathcal{H}^{n-1}}{\varphi(r_0) r_0^{n-1}\kappa_{n-1}} = \tilde{\lambda}_2 \quad \text{for all } m > m_0. \tag{61}$$

Since $K_m \subset 2R_0 B^n$ and $\tilde{\lambda}_1 \le \lambda_m \le \tilde{\lambda}_2$ for $m > m_0$ by (56), (58) and (61), there exists subsequence $\{K_{m'}\} \subset \{K_m\}$ such that $K_{m'}$ tends to some convex compact set K and $\lambda_{m'}$ tends to some $\lambda > 0$. As $o \in K_{m'}$ and $V(K_{m'}) = 1$ for all m', we have $o \in K$ and $V(K) = 1$.

We claim that for any continuous function $g : S^{n-1} \to \mathbb{R}$, we have

$$\int_{S^{n-1}} g\lambda\varphi(h_K)\, dS_K = \int_{S^{n-1}} gf\, d\mathcal{H}^{n-1}. \tag{62}$$

As φ is uniformly continuous on $[0, 2R_0]$ and $h_{K_{m'}}$ tends uniformly to h_K on S^{n-1}, we deduce that $\lambda_{m'}\varphi(h_{K_{m'}})$ tends uniformly to $\lambda\varphi(h_K)$ on S^{n-1}. Since $S_{K_{m'}}$ tends weakly to S_K, we have

$$\lim_{m'\to\infty} \int_{S^{n-1}} g\lambda_{m'}\varphi(h_{K_{m'}})\, dS_{K_{m'}} = \int_{S^{n-1}} g\lambda\varphi(h_K)\, dS_K.$$

On the other hand, $|gf_m| \leq \tilde{f} \cdot \max_{S^{n-1}} |g|$ for all $m \geq 2$, and gf_m tends pointwise to gf as m tends to infinity. Therefore Lebesgue's Dominated Convergence Theorem implies that

$$\lim_{m\to\infty} \int_{S^{n-1}} gf_m\, d\mathcal{H}^{n-1} = \int_{S^{n-1}} gf\, d\mathcal{H}^{n-1},$$

which in turn yields (62) by (52). In turn, we conclude Theorem 1.2 by (62). □

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