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Geometric properties of matrices induced by pattern avoidance

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ABSTRACT

The notion of submatrix avoidance in polyominoes has recently been introduced in [2] with the aim of extending most of the concepts and properties concerning pattern avoiding permutations to the setting of polyominoes. In this paper we use submatrix avoidance to describe families of polyominoes which, in the literature, are usually defined by means of the geometric constraints of *convexity*, *k-convexity*, and *directedness*. To reach this goal, we provide an extension of the notion of pattern in a polyomino, by introducing generalized polyomino patterns. In the second part of the paper, we tackle the same problem in the context of discrete sets, which can be naturally regarded as binary matrices. In this case, we consider two types of geometric constraints: *convexity* and *directedness*, and we study how these constraints can be imposed on matrices by using submatrix avoidance.

1. Introduction

Among the recently developed approaches to the study of combinatorial structures, one consists in describing them by means of the presence/absence of *patterns*. The notion of pattern has been considered in various human activities since ancient times, and research on patterns in combinatorial structures started with the study of pattern in words that dates back at least at the beginning of the 20th century with the works of Axel Thue [24,25]. Later, the research on patterns in combinatorial structures has concerned patterns in permutations [20], while in the last few years the notion of pattern has been defined and studied in other combinatorial objects such as set partitions [19,23], trees [21].

The work [2] fits into this research line with the introduction and the study of the concept of *pattern* in *polyominoes*.

Let us recall that a cell in the plane $\mathbb{Z} \times \mathbb{Z}$ is a unit square, and a *polyomino* is a finite union of cells that is connected and has no cut point (*i.e.* the set of cells has to be connected according to the edge adjacency). Polyominoes are defined up to translation (see Fig. 1). In this paper we use a quite common representation of discrete sets and polyominoes as binary matrices, where a 1 (resp. 0) entry stands for the presence (resp. absence) of a cell.

Polyominoes are popular combinatorial objects, introduced by S. Golomb [16], related to problems arising from different areas of mathematics, such as for instance: tiling problems [4,17], recreational mathematics and games [15].

The main objective of [2] is to extend most of the concepts and properties on pattern avoiding permutations to the setting of polyominoes. In particular, algebraic tools are employed in order to provide a unified framework to describe and to handle some known families of polyominoes, by the avoidance of patterns. Therefore, in order to fruitfully present our paper in Section 2 we recall some definitions and the main results from [2].

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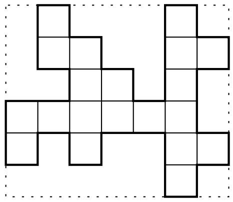

(a)

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(b)

Fig. 1. A polyomino and its representation as a binary picture (or matrix).

This paper originates from [1], presented at the *18th International Conference on Discrete Geometry for Computer Imagery* (2014). The principal purpose of this paper is to extend the studies of [1] and use the notion of submatrix avoidance in order to describe families of polyominoes defined by means of geometric constraints or combinatorial properties. More precisely we investigate the problem of representing the geometric constraints of *convexity*, *k-convexity*, and *directedness* in polyominoes by combining the avoidance of five matrices, denoted by H, V, D, L_1, L_2. To reach this goal, we have run across the following related problems:

i) given a set of patterns $\mathcal{M}$, to study the class of polyominoes avoiding the patterns of $\mathcal{M}$ as submatrices, and to give a characterization of this class in terms of the geometric properties of its elements.
ii) to extend the notion of pattern in a polyomino, by introducing *generalized polyomino patterns*, in a way that is possible to describe further families of polyominoes known in the literature. Such a generalization resembles what was done for pattern avoiding permutations by the introduction of *vincular, bivincular* patterns [5].

In the second part of the paper we study the same kind of problems in the context of discrete sets, which can be naturally regarded as binary matrices. In this case, we consider two types of geometric constraints: *convexity* and *directedness*, and we study how these constraints can be imposed on matrices by using the avoidance of some of the patterns H, V, D, L_1, L_2 used for polyominoes. In the authors' opinion this research guideline gives a new insight into the study of discrete sets, in particular it lets us tackle the longstanding problem of describing sets connectedness by using this new notion of pattern avoidance.

2. Polyomino classes

Throughout all the paper we use the representation of discrete sets (resp. polyominoes) as binary matrices, understanding that the matrix has a 1 entry if there is a cell of the set (resp. polyomino) in the corresponding position, 0 otherwise. Notice that in a binary matrix representing a polyomino both the first and the last rows (resp. columns) should contain at least a 1. In this section we recall some basic definitions and results from [2], which is useful in the rest of the paper.

Definition 1. Let $\mathfrak{M}$ be the class of *matrices*. We denote by $\preccurlyeq$ the usual *submatrix* order on $\mathfrak{M}$, *i.e.* $M' \preccurlyeq M$ if M' may be obtained from M by deleting any collection of rows and/or columns.

Definition 2. Let $\preccurlyeq_P$ be the restriction of the submatrix order $\preccurlyeq$ on the set of polyominoes $\mathfrak{P}$.

This defines the poset $(\mathfrak{P}, \preccurlyeq_P)$ and the pattern order on polyominoes: a polyomino P is a *pattern* of a polyomino Q (which we denote $P \preccurlyeq_P Q$) if the binary picture representing P is a submatrix of the one representing Q. We point out that the order $\preccurlyeq_P$ has already been studied by the name of *subpicture order* in [8] in which the authors proved that $(\mathfrak{P}, \preccurlyeq_P)$ contains infinite antichains, and it is a graded poset (the rank function being the semi-perimeter of the bounding box of the polyominoes).

This allows to introduce a natural analogue of permutation classes for polyominoes:

Definition 3. A *polyomino class* is a set of polyominoes $\mathcal{C}$ that is downward closed for $\preccurlyeq_P$: for all polyominoes P and Q, if $P \in \mathcal{C}$ and $Q \preccurlyeq_P P$, then $Q \in \mathcal{C}$.

We point out that a polyomino class is just an *ideal* in the poset $(\mathfrak{P}, \preccurlyeq_P)$.

In [2] the authors established that some known families of polyominoes, including *convex*, *column-convex*, *L-convex*, *directed-convex* polyominoes, are polyomino classes. On the other side, there are several families of polyominoes, which are not polyomino classes, such as the family of polyominoes having a square shape or the family of polyominoes having exactly $k > 1$ columns.

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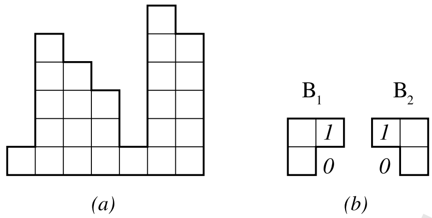

Fig. 2. (a) A bargraph; (b) The p-basis of the class of bargraphs.

Similarly to the case of permutations, for any set $\mathcal{B}$ of polyominoes, let us denote by $Av_P(\mathcal{B})$ the set of all polyominoes that do not contain any element of $\mathcal{B}$ as a pattern. Every such set $Av_P(\mathcal{B})$ of polyominoes defined by pattern avoidance is a polyomino class. Conversely, as for permutation classes, every polyomino class may be characterized in this way [2].

Proposition 4. *For every polyomino class $\mathcal{C}$, there is a unique antichain $\mathcal{B}$ of polyominoes such that $\mathcal{C} = Av_P(\mathcal{B})$. The set $\mathcal{B}$ consists of all minimal polyominoes, wrt $\preccurlyeq_P$, that do not belong to $\mathcal{C}$.*

We call $\mathcal{B}$ the *polyomino-basis* (or *p-basis* for short), to distinguish from other kinds of bases. We observe that, denoting $Av_M(\mathcal{M})$ the set of binary matrices that do not have any submatrix in $\mathcal{M}$, we have $Av_P(\mathcal{M}) = Av_M(\mathcal{M}) \cap \mathfrak{P}$.

On the other side, it is quite natural to describe classes of polyominoes by the avoidance of submatrices, then we introduce the notion of *matrix-basis* (or *m-basis*) of a polyomino class $\mathcal{C}$, which is an antichain $\mathcal{M}$ of matrices such that $\mathcal{C} = Av_P(\mathcal{M})$. Differently from the p-basis, the m-basis needs not be unique.

We recall from [2] that the p-basis and an m-basis of a polyomino class are related by the following.

Proposition 5. *Let $\mathcal{C}$ be a polyomino class, and let $\mathcal{M}$ be an m-basis of $\mathcal{C}$. Then the p-basis of $\mathcal{C}$ consists of all polyominoes that contain a submatrix in $\mathcal{M}$, and that are minimal (w.r.t. $\preccurlyeq_P$) for this property.*

Example 6 *(Bargraphs).* A *bargraph* is a polyomino obtained by juxtaposing a sequence of connected columns of nonzero height, whose bottom cells lie on the x-axis, see Fig. 2(a). It is clear that bargraphs form a polyomino class. So, by Proposition 4 they are described by the avoidance of the two polyominoes B_1, B_2 in Fig. 2(b), which are their p-basis.

Observe that bargraphs can also be described by the avoidance of the matrix $M = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which is a m-basis. By Proposition 5 it turns out that B_1 and B_2 are the minimal polyominoes containing M.

3. Polyomino constraints determined by submatrix avoidance

Recall from our previous work [1] that "if a family of polyominoes is defined by imposing some kind of geometric constraints on its elements, then these constraints can be often represented in a natural way by the avoidance of submatrix patterns". In this section, we aim at giving a formalization of this statement, by providing a unified setting to describe some kind of constraints by means of pattern avoidance. More precisely, we are concerned with the following geometric constraints on polyominoes:

i) *convexity*: a polyomino is said to be *convex* if its rows and columns are connected (see Fig. 3(a)). Convex polyominoes have been extensively investigated in the literature [6,11].

ii) *k-convexity*: an *(internal) path* of a polyomino is a sequence of distinct cells $(c_1, \ldots, c_n)$ of the polyomino such that every two consecutive cells in this sequence are edge-connected; according to the respective positions of the cells c_i and c_{i+1}, we say that the pair (c_i, c_{i+1}) forms a *north, south, east or west step* in the path. A path is monotone if it uses steps of only two types. A convex polyomino is *k-convex* if every pair of its cells can be connected by a monotone path, with at most k changes of direction [8]. For $k = 1$ we have L-convex polyominoes, studied by Castiglione and Restivo [8], where each pair of cells can be connected by means of a path using at most one change of direction (see Fig. 3(d)). Recently, L-convex polyominoes have been considered from several points of view: in [9,10] the authors solve the main enumeration problems for L-convex polyominoes. We underline that the characterization of a path in the discrete plane by means of a sequence of directed steps is equivalent to a restriction of the standard contour encoding of an arbitrary geometric configuration provided by Freeman in one of his capital works [13].

iii) *directedness*: a polyomino P is *directed* when every cell of P can be reached from a distinguished cell (called *source*) by a path that uses only north and east steps, called *ne*-path. A polyomino is *directed-convex* if it is both directed and convex. These families of polyominoes were studied and enumerated in [12], and Fig. 3(b) depicts an example of a directed non-convex polyomino.

Moreover, we deal with the following patterns:

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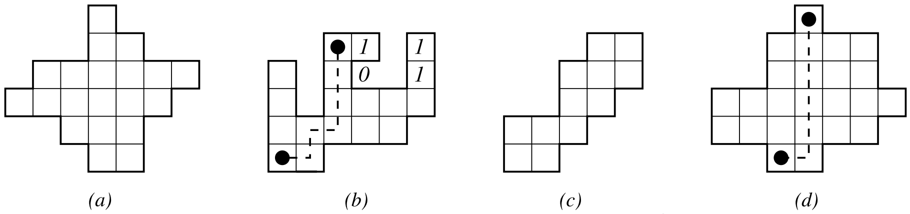

Fig. 3. (a) A convex polyomino; (b) A directed polyomino; (c) A parallelogram polyomino; (d) A L-convex polyomino.

$$H = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \; V = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \; D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \; L_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Throughout all the paper, these patterns are referred by using the above notation. Our aim is to study how it is possible to represent the convexity, k-convexity, and directedness constraints on polyominoes by imposing the avoidance of (some of) the patterns above. The case of convex polyominoes is perhaps the simplest one, and it was considered in [2].

Proposition 7 *(Convex polyominoes). The class of convex polyominoes is described by the avoidance of H and V.*

3.1. The family of directed polyominoes

The reader can easily check that the family of directed polyominoes is not a polyomino class as shown in Fig. 3(b). But if we impose the convexity constraint, then we have a polyomino class:

Proposition 8 *(Directed-convex polyominoes). The class $\mathcal{DC}$ of directed-convex polyominoes is characterized by the avoidance of the submatrices H, V and D.*

By Proposition 4, directed polyominoes cannot be expressed in terms of submatrix avoidance. In order to overcome this problem we extend the notion of pattern avoidance introducing the notion of *generalized pattern avoidance*.

Our extension consists in imposing the adjacency of two columns or rows by introducing special symbols, i.e. vertical/horizontal lines. Let M be a matrix, a vertical line between two columns of M, c_i and c_{i+1} (resp. a horizontal line between two rows r_i and r_{i+1}) means that c_i and c_{i+1} (resp. r_i and r_{i+1}) must be adjacent. If the vertical (resp. horizontal) line is external, the column (resp. row) of the pattern must touch the minimal bounding rectangle of the polyomino. Moreover, we use the $*$ symbol to denote 0 or 1, indifferently.

In the following we prove that directed polyominoes can be described using the notion of pattern avoidance. In the proofs the notation $M(i, j)$ refers to the entry in row i and column j of the matrix M.

Proposition 9. *The class of directed polyominoes is represented as the class of polyominoes avoiding the following patterns:*

$$\mathcal{D} = \left\{ P_1 = \left[\begin{array}{c|c} 0 & 1 \\ \hline * & 0 \end{array}\right], \; P_2 = \left[\begin{array}{cc} 0 & 1 \\ \hline \end{array}\right], \; P_3 = \left[\begin{array}{c|c} & 1 \\ & 0 \end{array}\right] \right\}$$

Proof. ($\Rightarrow$) If P is a directed polyomino, then P avoids $\mathcal{D}$. Let us proceed by contradiction assuming that P is a directed polyomino containing one of the patterns of $\mathcal{D}$. If P contains P_1, then every path running from the source to the entry 1 of the pattern ends with a s or w step. Indeed, by definition of generalized pattern, the entry 1 in P_1 needs be adjacent to the entries 0 (below and on the left), against the assumption. A similar reasoning holds for the patterns P_2 and P_3 considering the first column and the last row of P.

($\Leftarrow$) If P is a polyomino avoiding $\mathcal{D}$, then P is directed. Let us assume, by contradiction, that P avoids $\mathcal{D}$ without being directed. Let m be the number of rows of P. Since P avoids P_2 and P_3, the entry $P(m, 1) = 1$, i.e. there is a cell of P in position $(m, 1)$. Moreover, since P is not directed, there exists a cell $P(i_1, j_1)$ of P which cannot be reached by a ne-path starting from $P(m, 1)$.

If $i_1 = m$, then we move westwards from $P(i_1, j_1)$, up to reach $P(i_1, j_2)$ on the boundary of P. Since there does not exist a ne-path from $P(m, 1)$ to $P(i_1, j_1)$, then $P(i_1, j_2) \neq P(m, 1)$, so P contains P_2 against the assumption, see Fig. 4 (a). A similar reasoning holds for $j_1 = 1$ and the pattern P_3.

So, let us assume that $i_1 \neq m$ and $j_1 \neq 1$, i.e. $P(i_1, j_1)$ does not lie in the first column or in the last row of P. Let us consider a sw-path (i.e., using south and west steps) defined as follows: it moves from $P(i_1, j_1)$ starting with a west step (if possible) and changes direction only if it hits the boundary of P, as shown in Fig. 4(b). Let $P(i_2, j_2)$ be the ending cell of such path. If $P(i_2, j_2)$ lies in the last row (resp. first column) of P, then P contains P_2 (resp. P_3), against the assumption.

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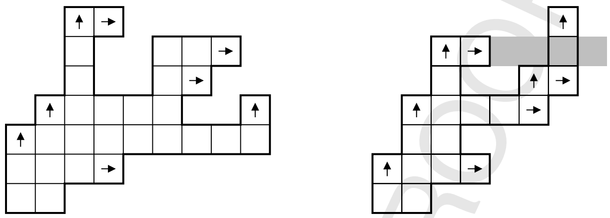
(a) (b)

Fig. 4. (a) A non-directed polyomino containing P_2; (b) A non-directed polyomino containing P_1.



(a) (b)

Fig. 5. (a) A prudent non-convex polyomino; (b) A directed non-prudent polyomino.

Otherwise, since $P(i_2, j_2)$ is the ending cell of the path, then there are 0 entries below and on the left of $P(i_2, j_2)$. Hence P contains one of the two configurations of P_1, against the assumption. □

An important polyomino class between directed convex and directed polyominoes is that of *prudent* polyominoes. Before defining these objects, let us give some definitions.

Definition 10. Let P be a directed polyomino. Let $w(P)$ be the boundary word of P, consisting of north, east, south and west unit steps, and which is obtained by following the boundary of P clockwise starting from the source. Any occurrence of the factor *ne* (resp. *se*) in $w(P)$ identifies a cell which is called a north (resp. east) corner of P. Moreover, a polyomino has no holes if the boundary is a simple loop.

Observe that north (resp. east) corners are precisely those corners c of P such that every directed path internal to P from the source to c ends with a north (resp. east) step.

Definition 11. A directed polyomino P without holes is said to be *prudent* if every north (resp. east) corner c is the uppermost (resp. rightmost) cell of P with the same abscissa (resp. ordinate) of c.

Fig. 5(a) shows a prudent polyomino. It is clear that every directed convex polyomino is prudent while, not every directed polyominoes is prudent (see Fig. 5(b)). In the graphical representation we identify north (resp. east) corners by means of an up (resp. right) arrow in the corresponding cell.

The name "prudent" is motivated by the fact that in these polyominoes, no east/north corner points the boundary. We draw attention to the fact that another family of polyominoes, called *prudent polygons* was considered in [3], but these objects are not relevant for the scope of this paper, since they do not form a polyomino class.

Proposition 12 *(Prudent directed polyominoes). The family $\mathcal{PD}$ of prudent directed polyominoes is a polyomino class described by the avoidance of D.*

Proof. ($\Leftarrow$) If $P \in Av_P(D)$, then $P \in \mathcal{PD}$. Assume that P avoids the pattern D. Then, P must contain at least a cell in the last row, so let c_1 be the leftmost cell of this row. Since the first column should contain at least a cell c_2, and $D \not\preccurlyeq_P P$, c_1 and c_2 coincide or they lie in the same column. Moreover, in order not to have the pattern D any other cell c' of P must be reached from c_1 using north and east steps. So P is directed. Furthermore, P has no holes, otherwise trivially it would contain the pattern

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

which contains D. Now we prove that there is no cell of P above every north corner. So, in the matrix representation of P, let c_N be a 1 entry corresponding to a north corner, which means there are 0 entries above and on the left of this 1, i.e.

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$\begin{bmatrix} * & 0 \\ 0 & 1 \end{bmatrix}$. Now, assuming that in P there is at least a cell above c_N in the same column, let c' be the bottommost cell among these. Since P is directed, there must be a cell on the left of c', thus we have the pattern D. The property for east corners can be proved in a completely analogous way.

($\Rightarrow$) If $P \in \mathcal{PD}$, then $P \in Av_P(D)$. Proceeding by contradiction, assume that $P \notin Av_P(D)$, P is directed and without holes. We prove that $P \notin \mathcal{PD}$. Thus, P contains the pattern D, and let $a = 0$, $b = 1$, $c = 1$, and $d = 1$ be four cells leading to the pattern D, arranged as

$$U = \begin{bmatrix} b & \dots & c \\ \vdots & \ddots & \vdots \\ a & \dots & d \end{bmatrix}.$$

First we prove the following stronger statement:

Claim 1: U contains one of the two generalized patterns:

$$D_1 = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ \hline 0 & * & 1 \end{array}\right] \quad \text{or} \quad D_2 = \left[\begin{array}{c|c} 1 & 1 \\ * & 0 \\ \hline 0 & 1 \end{array}\right]$$

Proof of Claim 1: Since P is directed, one of the two cases must hold:

1. there is a 0 entry between b and c which is connected to a by means of a sequence of 0 entries which are edge- or vertex-connected;
2. there is a 0 entry between d and c, which is connected to a by means of a sequence of 0 entries which are edge- or vertex-connected.

It is clear, that, if 1. and 2. do not hold, then a is a cell of a hole contained in P, which is not possible. So let us consider the two cases separately:

1. Since there is a sequence $\mathcal{O}$ of 0 between a and the first row of U, it means there is a submatrix $U' = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & * \end{array}\right]$ of U. Moreover, by directedness, c is connected to the source by means of a ne-path which lies on the right of $\mathcal{O}$. Due to this fact, for each pair of adjacent rows in U exists a column of 1 on the right of $\mathcal{O}$. Therefore we clearly have the pattern D_1. An example of this situation is represented in the matrix below, where the entries forming the pattern D_1 are underlined:

$$\begin{bmatrix} b & 1 & 0 & 0 & c \\ 1 & \underline{1} & \underline{0} & \underline{1} & 1 \\ 1 & \underline{0} & \underline{1} & \underline{1} & 0 \\ a & 0 & 1 & 0 & d \end{bmatrix}$$

2. Similarly to Case 1., since there is a sequence of 0 between a and the last column of U, using the same arguments we have the pattern D_2. An example of this situation is represented in the matrix below, where the entries forming the pattern D_2 are underlined:

$$\begin{bmatrix} b & 1 & 0 & 0 & c \\ 0 & 0 & \underline{1} & \underline{1} & 1 \\ 1 & 1 & \underline{1} & \underline{0} & 0 \\ a & 0 & \underline{0} & \underline{1} & d \end{bmatrix}$$

Now it is clear that if P contains D_1, then the entry $D_1(1,1)$ is an east corner of P, and there is a cell in front of it. So $P \notin \mathcal{PD}$. Similarly if P contains D_2, then the entry $D_2(3,2)$ is a north corner of P, and there is a cell in front of it. So $P \notin \mathcal{PD}$ as well. $\square$

By symmetry, there is an analogous characterization of the class of polyominoes avoiding the pattern $D' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. The avoidance of both submatrices D and D' describes a well-known class of polyominoes.

Definition 13. A parallelogram polyomino is a polyomino whose boundary can be decomposed in two paths, the upper and the lower ones, made of north and east unit steps and meeting only at their starting and final points.

Parallelogram polyominoes (see Fig. 3(d)) are proved to be a polyomino class [2].

Proposition 14 *(Parallelogram polyominoes). Parallelogram polyominoes are characterized by the avoidance of D and D'.*

It is interesting to observe that the avoidance of D and D' imposes convexity. This is due to the fact that if a polyomino contains H or V as a pattern, then it necessarily contains D or D'.

3.2. The family of k-convex polyominoes

In this section we consider the families of k-convex polyominoes, $k \geq 1$, and their representability by means of submatrix avoidance.

L-convex polyominoes The family of L-convex polyominoes forms a polyomino class, as proved in [2].

Proposition 15. *L-convex polyominoes are described by the avoidance of the submatrices H, V, L_1, L_2.*

We would like to point out that L-convex polyominoes play an important role among polyomino classes. We start by introducing a different notion of pattern containment, obtained by strengthening the constraints of a polyomino class. In the rest of the paper, for a matrix M, $\widehat{M}$ denotes its *reduction*, i.e. the matrix obtained from M by removing possible 0 rows/columns at the extremities of M.

Definition 16. A polyomino class $\mathcal{S}$ is said to be a *strong* polyomino class if for every polyomino $P \in \mathcal{S}$, and for every pattern $M \preccurlyeq P$, we have that $\widehat{M}$ is a polyomino in $\mathcal{S}$.

On the other side, the class of bargraphs is not strong, in fact a bargraph may contain the reduced pattern H which is not a bargraph.

Clearly, every strong polyomino class is a polyomino class. But only few polyomino classes are strong classes: for instance, rectangles, Ferrers diagrams, stack polyominoes. Compared with the notion of polyomino class, we have the rather surprising result that there is a maximal strong class, and it is the class of *L-convex polyominoes*.

Proposition 17. *Let $\mathcal{S}$ be a class of polyominoes.*

i) *if $\mathcal{S}$ is a strong class then $\mathcal{S}$ is contained in the class of L-convex polyominoes;*
ii) *the class of L-convex polyominoes is a strong class.*

Proof.

i) We proceed by contradiction. Suppose that $\mathcal{S}$ is not contained in the class of L-convex polyominoes. Then, by Proposition 15 there is a polyomino $P \in \mathcal{S}$ and a pattern $M \in \{H, V, L_1, L_2\}$ such that $M \preccurlyeq P$. Since $M = \widehat{M}$, and M is not a polyomino, then $\mathcal{S}$ is not a strong class.
ii) Suppose that the family of L-convex polyominoes is not a strong class. Then, there exists a L-convex polyomino P and a pattern $M \preccurlyeq P$ such that $\widehat{M}$ is not a L-convex polyomino. So, we have that both of the following two possibilities are impossible:
   1. $\widehat{M}$ is a polyomino containing one of the four patterns H, V, L_1, L_2, which is against hypothesis of P being L-convex.
   2. $\widehat{M}$ is not a polyomino. Then, it contains at least two disconnected cells lying on the same row (resp. column), hence $\widehat{M}$ contains the pattern H (resp. V) (against our hypothesis). Otherwise, if all rows and columns are connected, the only possibility to have two disconnected cells is that $\widehat{M}$ contains L_1 or L_2 (against our hypothesis). □

2-convex polyominoes Differently from L-convex polyominoes, 2-convex polyominoes do not form a polyomino class. As a matter of fact, the 2-convex polyomino in Fig. 6(a) contains the 3-convex polyomino (b) as a pattern, so the class is not downward closed w.r.t. $\preccurlyeq_P$. Similarly, the set of k-convex polyominoes is not a polyomino class, for $k \geq 2$.

In practice, this means that 2-convex polyominoes cannot be described in terms of pattern avoidance. Let us recall some definitions and results concerning k-convex polyominoes from [7].

Definition 18 *(Bounce paths).* Let P be a convex polyomino, and b, c be two cells of P, in positions (i_b, j_b), and (i_c, j_c), respectively, with $i_b \leq i_c$ and $j_b \leq j_c$. The bounce path u_{bc} (resp. r_{bc}) is the monotone path internal to P starting at b with a north (resp. east) unit step and ending at c, where each horizontal side has maximal length up to reach i_c, and each vertical side has maximal length up to reach j_c (see Fig. 7(a)). Observe that, due to the position of b in P, one of the two paths may not exist.

If $j_b > j_c$, then a bounce path d_{bc} is defined analogously to u_{bc}, changing north with south steps.

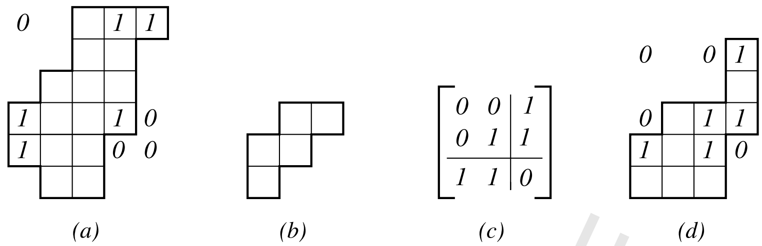

Fig. 6. (a) A 2-convex polyomino P; (b) A pattern of P that is not a 2-convex polyomino; (c) A generalized pattern, which is not contained in (a), but is contained in the 3-convex polyomino (non-2-convex) (d).

Lemma 19. *Let P be a convex polyomino and let b, c be two cells of P. The minimal number of changes of direction of the bounce paths joining b to c is less than or equal to the number of changes of direction of any path joining b to c.*

Now, we can prove that 2-convex polyominoes is expressed by the avoidance of generalized patterns.

Proposition 20. *The class of 2-convex polyominoes is described by the avoidance of the set $\mathcal{M}$ of generalized patterns:*

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \left[\begin{array}{c|cc} 0 & * & 1 \\ \hline * & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 1 & * \\ \hline 1 & * & 0 \end{array}\right] \left[\begin{array}{cc|c} 1 & * & 0 \\ \hline 0 & 1 & * \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c|cc} 1 & 0 & 0 \\ * & 1 & 0 \\ \hline 0 & * & 1 \end{array}\right]$$
$$\left[\begin{array}{cc|cc} 1 & * & 0 & 0 \\ * & 1 & * & 0 \\ \hline 0 & * & 1 & * \\ 0 & 0 & * & 1 \end{array}\right] \left[\begin{array}{cc|cc} 0 & 0 & * & 1 \\ 0 & * & 1 & * \\ \hline * & 1 & * & 0 \\ 1 & * & 0 & 0 \end{array}\right].$$

Proof. ($\Rightarrow$) If P is a 2-convex polyomino then P avoids $\mathcal{M}$.
Let us proceed by contradiction assuming that P is a 2-convex polyomino containing one of the patterns of $\mathcal{M}$. Clearly, P cannot contain H and V, by convexity. So, let consider the two patterns:

$$Z_1 = \left[\begin{array}{c|cc} 0 & * & 1 \\ \hline * & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \text{ and } Z_2 = \left[\begin{array}{cc|cc} 0 & 0 & * & 1 \\ 0 & * & 1 & * \\ \hline * & 1 & * & 0 \\ 1 & * & 0 & 0 \end{array}\right].$$

Let us assume that P contains Z_1 (resp. Z_2). We remark the other cases can be obtained by symmetry. It follows that P contains a submatrix P' (resp. P'') of the form:

$$P' = \begin{array}{c|cccccc} 0 & * & . & . & . & . & \mathbf{1} \\ \hline * & 1 & . & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \mathbf{1} & 0 & . & . & . & . & 0 \end{array} \qquad P'' = \begin{array}{ccccc|ccccc} 0 & . & . & . & 0 & * & . & . & . & \mathbf{1} \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & * & 1 & . & . & . & * \\ \hline * & . & . & . & 1 & * & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ \mathbf{1} & . & . & . & * & 0 & . & . & . & 0 \end{array},$$

where 0, 1, and $*$ entries correspond to those of Z_1 (resp. Z_2), while dots represent a 0 or 1 entry indifferently, provided that convexity and connectedness constraints are preserved.

Among all polyominoes that contain P' (resp. P''), those where each $*$ entry is replaced by 1 clearly have minimal convexity degree, and it is easy to verify that, for each of them, the bounce paths coincide, and the minimal number of changes of direction in every path connecting the boldface 1 entries of P' (resp. P'') is three, against our assumption.

($\Leftarrow$) If P avoids $\mathcal{M}$, then P is a 2-convex polyomino.
Again we proceed by contradiction assuming that P avoids $\mathcal{M}$ and it is a 3-convex polyomino, i.e. there exists a couple of cells of P, indicated by $P(i_1, j_1)$ and $P(i_2, j_2)$, such that one of the bounce paths connecting them has three changes of direction. Let us assume w.l.g. that such bounce path uses only n and e steps, and it starts with a n step, so it holds $i_1 > i_2$ and $j_1 < j_2$ (the other cases can be treated symmetrically). Furthermore, let the three changes of direction lie in positions $P(i_3, j_1)$, $P(i_3, j_3)$, and $P(i_2, j_3)$, such that $i_1 > i_3 > i_2$, and $j_1 < j_3 < j_2$, as depicted in Fig. 7(a).

We consider the leftmost cell $P(i_2, j_4)$ of P in row i_2: by definition of bounce path, it holds $j_4 \le j_3$. Since, by assumption, there are no internal paths running from $P(i_1, j_1)$ to $P(i_2, j_2)$ with less than three changes of directions, then it holds $P(i_2, j_4 - 1) = 0$ and $P(i_1, j_4) = 0$. A similar reasoning allows us to determine a row i_4 such that $i_4 < i_3$, $P(i_4, j_2) = 1$,

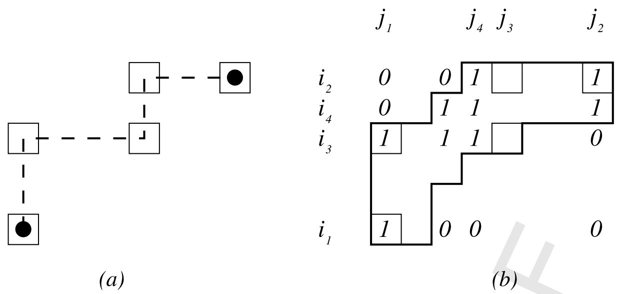

Fig. 7. (a) The bounce path $u_{b,c}$ connecting two cells b and c of P; (b) A submatrix of P containing the bounce path, where the forbidden pattern Z_2 is highlighted.

$P(i_4 + 1, j_2) = 0$ and $P(i_4, j_1) = 0$. The situation is depicted in Fig. 7(b); we observe that the highlighted entries, i.e. those cells in the intersections between rows i_1, i_3, and i_4, and columns j_4, j_3, and j_2, as defined, form one of the patterns of Z_2, against our hypothesis.

We point out that the pattern Z_1, and its rotations, is obtained from the pattern Z_2 (or its rotation) replacing appropriately $*$ entries. However we have added Z_1, and its rotations to $\mathcal{M}$ in order to avoid the 3-convex polyominoes having three rows or columns. □

Let us just observe, referring to Fig. 6, that the pattern (c) is not contained in the 2-convex polyomino (a), but it is contained in the 3-convex polyomino (d). It is possible to generalize the previous result and to give a characterization of the class of k-convex polyominoes, with $k > 2$, using generalized patterns.

In a merely theoretical way one can observe that it should be possible to describe every family of polyominoes $\mathcal{F}$ using an infinite set of generalized patterns: since every polyomino P is uniquely represented by a pattern which has a vertical/horizontal line for every row/column, then to describe $\mathcal{F}$ it suffices to consider the polyominoes avoiding one by one *all* polyominoes not in $\mathcal{F}$. In practice, we would like to point out that there are families of polyominoes which cannot be described by means of a finite set of generalized patterns.

Proposition 21. *The family of polyominoes having square shape cannot be described by the avoidance of a finite set of generalized patterns.*

4. Matrix constraints defined by submatrix avoidance

A discrete set can be naturally regarded as a binary matrix. Thus, using the notion of submatrix avoidance, it is possible to study families of discrete sets which are characterized in terms of pattern avoidance. Let us start by giving some basic definitions (for simplicity's sake, we often use the term matrix in place of discrete set, and 1 entry in place of cell).

4.1. Matrix classes

Definition 22. A *matrix class* is a set of (binary) matrices $\mathcal{M}$ that is downward closed for $\preccurlyeq$: for all matrices M and M', if $M' \in \mathcal{M}$ and $M \preccurlyeq M'$, then $M \in \mathcal{M}$.

The reader can check that the following are examples of matrix classes: the set of matrices which contain at most a 0 entry for every row and column; the set of matrices with all 0 entries; the set of matrices with at most k 1 entries. On the other side, there are also families of matrices which are not matrix classes, such as the family of square matrices, the family of permutation matrices, the family of matrices which do not have two adjacent 1 entries.

For any set $\mathcal{M}$ of matrices, let us denote by $Av_M(M)$ the set of all matrices that do not contain any element of $\mathcal{M}$ as a pattern. Every such set $Av_M(M)$ of matrices defined by pattern avoidance is a matrix class. Conversely, like for polyomino classes, every matrix class may be characterized in terms of submatrix avoidance.

The following property resembles Proposition 4. Its proof is rather straightforward, and it follows from the fact that the poset of matrices is well-founded, i.e. every non-empty subset has minimal elements.

Proposition 23. *For every matrix class $\mathcal{M}$, there is a unique antichain $\mathcal{B}$ of matrices such that $\mathcal{M} = Av_P(\mathcal{B})$. The set $\mathcal{B}$ consists of all minimal matrices that do not belong to $\mathcal{M}$.*

Example 24 *(Bar matrices).* Let M be a binary matrix, and $r = (r(1), \ldots, r(n))$ and $r' = (r'(1), \ldots, r'(n))$ be two rows of M. We write that $r \subseteq r'$ if $r(i) = 1$ implies $r'(i) = 1$, for $i = 1, \ldots, n$. A binary matrix is a *bar matrix* if any two rows of M are comparable according to $\subseteq$ (see Fig. 8).

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$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&1\\ 0&1&0&0&0&1&0&0&1\\ 0&1&1&0&0&1&0&1&1\\ 0&1&1&0&1&1&0&1&1\\ 0&1&1&0&1&1&1&1&1 \end{bmatrix}$$

Fig. 8. A bar matrix.

It can easily be proved that the family of bar matrices is a matrix class, and precisely they can be described by the avoidance of the submatrix $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We point out that, as shown in Example 6, imposing the avoidance of B to polyominoes gives the class of bargraphs. Moreover, an explicit link between bargraphs and bar matrices is that bar matrices can be effectively decomposed as sequences of bargraphs separated by columns of 0 entries and with (possible) rows of 0 at the top.

As we did for polyominoes, in this section we study the possibility of representing geometric constraints on matrices by imposing the avoidance some of the patterns H, V, D, L_1, and L_2, used for polyominoes. In particular, we are now interested in the *connectedness* and *convexity* constraints on matrices, where a binary matrix is said to be *convex* if every sequence of 1 entries in each row and each column is connected.

On the other side, we try to give a combinatorial/geometrical characterization to classes of matrices defined by the avoidance of some combination of the same patterns. Again, the case of convex matrices is perhaps the simplest one.

Proposition 25 *(Convex matrices). Convex matrices are a matrix class, and they can be described by the avoidance of the submatrices H and V.*

4.2. Unique matrices

According to the vocabulary of Discrete Tomography [18] a discrete set (matrix) is said to be *unique* if it is uniquely determined by its horizontal and vertical projections. Ryser's Theorem [22] provides a characterization of unique matrices, reformulated as follows:

Proposition 26 *(Unique matrices). The family of unique binary matrices is a matrix class and it is described by the avoidance of L_1 and L_2.*

From the previous proposition follows directly a geometric characterization of unique matrices.

Proposition 27 *(Unique matrices). Let M be a binary matrix. The following are equivalent:*

i) *M is unique;*
ii) *all rows of M are pairwise comparable;*
iii) *all columns of M are pairwise comparable.*

Fig. 9(a) shows a unique matrix. In this section we study geometric properties of subclasses of unique matrices defined by imposing pattern avoidance. The first class we are going to consider is the class of convex unique matrices $Av_M(H, V, L_1, L_2)$, showing the following remarkable property.

Proposition 28 *(Convex unique matrices). Every convex unique matrix is connected.*

Proof. Let M be a convex unique matrix and suppose that it is not connected. So, there exist two 1 entries $M(i_1, j_1)$ and $M(i_2, j_2)$ of M, such that there are no paths connecting them. Let us assume w.l.g. that $i_1 < i_2$ and $j_1 < j_2$, and consider the sequence: $s = (M(i_1, j_1), M(i_1, j_1+1), \dots, M(i_1, j_2), M(i_1+1, j_2), \dots, M(i_2, j_2))$. By hypothesis, at least one of the entries of s is 0 and since M avoids H and V, then $M(i_1, j_2)$ has to be one of them. Symmetrically, also $M(i_2, j_1)$ has to be 0, against the assumption that M avoids L_2. $\square$

As a byproduct of Propositions 15 and 28, we have that reduced convex unique matrices are precisely L-convex polyominoes. An example of a convex unique matrix is given in Fig. 9(b). Moreover, from the proof of the previous proposition we have that, even removing the H or V pattern, we still have connectedness.

Corollary 29. *Every matrix in $Av_M(V, L_1, L_2)$ (resp. $Av_M(H, L_1, L_2)$) is connected.*

$$\begin{bmatrix} 1&1&0&1&1&1&0&0\\ 1&0&0&1&1&1&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0&0\\ 1&0&0&1&1&1&0&1\\ 0&0&0&0&1&1&0&0\\ 1&1&0&1&1&1&1&1 \end{bmatrix} \qquad \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&1&1&1&1&1\\ 0&0&0&1&1&1&0&0\\ 0&1&1&1&1&1&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

(a) (b)

Fig. 9. (a) An example of unique matrix; (b) An example of convex unique matrix.

4.3. Other relevant matrix classes

Here we analyze how some other geometric properties are obtained by combining our set of patterns.

Definition 30. Two rows $r = (r(1), \dots, r(n))$ and $r' = (r'(1), \dots, r'(n))$ are said to be *half-comparable* if there is an index $1 \le j \le n$ such that

$$(r(1), \dots, r(j)) \subseteq (r'(1), \dots, r'(j))$$

and

$$(r(j+1), \dots, r(n)) \supseteq (r'(j+1), \dots, r'(n)).$$

If $j = n$ then $r \subseteq r'$.

As usual, let r_i denote the ith row of a matrix M.

Proposition 31. *A matrix $M \in Av_M(L_1)$ if and only if any two rows r_h and r_k of M, with $h > k$ are pairwise half-comparable.*

Proof. Let M be a binary matrix, and r_h and r_k two of its rows, with $h < k$.
($\Rightarrow$) Let us assume that $M \in Av_M(L_1)$, and j be the smallest column index such that $M(h, j) = 0$ and $M(k, j) = 1$, if it exists. Two cases arise: if j does not exist, then $r_h \subseteq r_k$; on the other hand, if j exists, then it holds that $(r_h(1), \dots, r_h(j)) \supseteq (r_k(1), \dots, r_k(j))$, and $(r_h(j+1), \dots, r_h(n)) \subseteq (r_k(j+1), \dots, r_k(n))$ by the avoidance of the pattern L_1. So, in both cases, r_h and r_k are half-comparable.

($\Leftarrow$) Now we assume that any two rows r_h and r_k of M, with $h < k$, are half-comparable. Let j be a column index such that $r_h(j) = 0$ and $r_k(j) = 1$, if it exists. Again two cases arise: if j does not exist, then $r_h \subseteq r_k$, and the two rows avoid the pattern $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and, consequently, L_1. Secondly, if j exists, then, by definition, $(r_h(1), \dots, r_h(j)) \supseteq (r_k(1), \dots, r_k(j))$, and $(r_h(j+1), \dots, r_h(n)) \subseteq (r_k(j+1), \dots, r_k(n))$, so the pattern $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ may appear only before column j, and L_1 is avoided as well. $\square$

Recall that, given two matrices A and B, their *direct sum* $A \oplus B$ produces the matrix

$$\begin{bmatrix} A & \emptyset \\ \emptyset & B \end{bmatrix}$$

Proposition 32. *Every matrix in $Av_M(H, V, L_1)$ is expressed by a direct sum of connected matrices in $Av_M(L_1)$.*

Proposition 33. *If M is a matrix in $Av_M(H, V, D)$, then M or M' (the symmetric of M according to the y-axis) is a direct sum of matrices A_i, $i \ge 1$, such that $\widehat{A_i}$ is a directed-convex polyomino.*

At the end of this section we would like to conclude that, in the framework of matrices, the avoidance of the patterns D (which imposes directedness on polyominoes) and L_1, L_2 (which impose uniqueness), when combined with convexity lead to direct sum of connected sets (see Fig. 10).

5. Further work

In this paper we use the notion of submatrix avoidance in order to describe families of polyominoes and of matrices defined by means of geometric constraints or combinatorial properties. Concerning this subject, there are several interesting problems which remain unsolved and should be better investigated. In the sequel we just mention a few of them.

$$\begin{bmatrix} 1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \qquad \begin{bmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&1\\ 0&0&1&1&1&1&1\\ 0&0&1&0&0&0&0\\ 1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0 \end{bmatrix}$$

(a) (b)

Fig. 10. (a) An example of matrix avoiding H, V and L_1. (b) An example of matrix avoiding H, V and D.

Polyominoes as pattern avoiding matrices In the second part of the paper we focused on the problem of obtaining the connectedness of binary matrices by imposing submatrix avoidance. We reached our goal only for special cases where other constraints are imposed. Therefore, one of the most challenging problems in our research line is clearly the following.

Problem 1. Is there a set $\mathcal{M}$ of generalized patterns such that the class of polyominoes can be described by the avoidance of $\mathcal{M}$?

The previous problem can be regarded as the problem of describing the connectedness of a generic set by submatrix avoidance. This is still an open problem, and up to the authors' opinion the answer is likely to be negative.

More on matrix classes There are several examples of matrix classes, defined by imposing the avoidance of some subset of the patterns $\{H, V, D, L_1, L_2\}$ that we have not been able to describe using geometric constraints. Perhaps the most intriguing is $Av_M(D)$.

Matrix classes as two-dimensional languages A matrix class can be regarded as a two-dimensional language on the alphabet $\{0, 1\}$. For basic definition of two-dimensional languages we address the reader to [14]. First, it would be interesting to study some closure properties of these languages wrt to union, intersection and complement. On the other side, one of the most important families of two-dimensional languages is that of tiling system recognizable languages. We believe it is worth investigating possible connections between these families of languages and matrix classes.

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