On the number of anisotropic simple submodules in modules with a form

ON THE NUMBER OF ANISOTROPIC SIMPLE SUBMODULES IN MODULES WITH A FORM

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Abstract. Let G be a finite group, and V a finite-dimensional semisimple G-module over a finite field. Assume that V is endowed with a nonsingular bilinear form which is symmetric or symplectic, and which is invariant under the action of G. In this setting, we compute the number of anisotropic simple submodules of V.

Introduction

Let G be a finite group, $\mathbb{F}$ a finite field, and V a finite-dimensional semisimple $\mathbb{F}G$-module which is *homogeneous* (that is, the simple submodules of V are pairwise isomorphic); also, let W be a simple submodule of V. In this setting, as it is well known, the number of all the simple submodules of V is given by $\sum_{j=0}^{m-1} q^j$, where q denotes the order of the field $\mathbb{E} := \mathrm{End}_{\mathbb{F}G}(W)$, and m is the composition length of V as an $\mathbb{F}G$-module (see [**1**, VII, proof of 9.19]).

Assume now in addition that V is endowed with a nonsingular bilinear $\mathbb{F}$-form which is symmetric or symplectic, and which is G*-invariant* (in the sense that G acts on V as a group of isometries with respect to the given form). In this richer context, the set of simple submodules of V is partitioned into two (not necessarily proper) subsets, namely the set of *totally isotropic* simple submodules, and the set of *anisotropic* simple submodules (recall that a submodule Y of V is called totally isotropic if the form vanishes on it, whereas it is called anisotropic if the zero space is the unique totally isotropic submodule of Y).

In this paper we determine the cardinality of the two subsets mentioned above; in order to outline our conclusions, the following notation is needed. If W is a simple submodule of V, and $\mathbb{E}$ is the endomorphism ring $\mathrm{End}_{\mathbb{F}G}(W)$, we set $W_{\mathbb{E}}$ to be the simple $\mathbb{E}G$-module which arises regarding W as a vector space over $\mathbb{E}$ (here we consider the 'natural' action of $\mathbb{E}$ on W). It turns out that the cardinalities of the sets under consideration are strongly related to the module structure of $W_{\mathbb{E}}$; in particular, they depend on whether $W_{\mathbb{E}}$ is $\mathbb{E}G$-isomorphic to its contragredient module $W_{\mathbb{E}}^*$ or not. Denoting by q the order of $\mathbb{E}$, and by m the composition length of V as an $\mathbb{F}G$-module, we obtain the following.

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If the set of anisotropic simple submodules of V is not empty, then the cardinality of this set is given by

$$q^{(m-1)/2} \cdot \frac{q^{m/2} + (-1)^{m+1}}{q^{1/2} + 1} \quad \text{if} \quad W_{\mathbb{E}}^* \not\simeq W_{\mathbb{E}}, \tag{1}$$

$$q^{m-1} + \lambda q^{(m/2)-1} \quad \text{if} \quad W_{\mathbb{E}}^* \simeq W_{\mathbb{E}}, \tag{2}$$

where the value of λ in (2) has to be chosen in the set $\{-1, 0, 1\}$, depending on q, m, and another structural feature of V; here we just mention that λ is 0 when m is odd, or also when m is even provided q is also even, whereas λ is ± 1 when m is even and q is odd. Note that, as shown in [**3**, 9.9], the set of anisotropic simple submodules of V is certainly not empty when m is an odd number, but it can be empty in general.

It may be worth mentioning the question which originally motivated this kind of analysis. Assume that V is a simple $\mathbb{F}G$-module which carries a G-invariant nonsingular symplectic $\mathbb{F}$-form; assume also that H is a proper subgroup of G such that V is *induced* by an anisotropic submodule W of $V{\downarrow_H}$. One can ask whether, in such a situation, V admits a *form induction* from H; this means that there exists an anisotropic submodule Z of $V{\downarrow_H}$ (not necessarily isomorphic to W) such that V is induced by Z, and *distinct translates of Z by means of elements of G are pairwise orthogonal* (in other words, the distinct translates of Z provide an orthogonal direct decomposition of the vector space V). This question on symplectic modules, which plays a key role in the analysis of a problem concerning tensor induction for complex representations, is discussed in [**3**]; in that paper examples are given to show that the answer can be negative when the index of H in G is 2, but positive results are achieved with the assumption that the index of H in G is odd, and an alternative proof for the case when H is a *normal* subgroup (of odd index) can be obtained exploiting the methods developed in this paper.

The results

We start with the precise definition of invariant form.

Definition 1. Let G be a group, $\mathbb{F}$ a field, V an $\mathbb{F}G$-module, and $\langle\ ,\ \rangle$ a bilinear $\mathbb{F}$-form on (the underlying vector space of) V; if $\langle v_1^g, v_2^g\rangle = \langle v_1, v_2\rangle$ holds for all v_1, v_2 in V and g in G, then the form $\langle\ ,\ \rangle$ is called G-invariant.

In what follows every abstract group, field or module is tacitly assumed to be finite. By an '$\mathbb{F}G$-module with a form' we mean an $\mathbb{F}G$-module endowed with a distinguished G-invariant nonsingular bilinear $\mathbb{F}$-form which is symmetric or symplectic; this form will be always denoted by $\langle\ ,\ \rangle$. Moreover, if V is such a module, and $W_1, \ldots, W_k$ are pairwise orthogonal submodules of V (in the sense that W_r lies in $W_s^{\perp}$ for all $s \neq r$) which provide a direct decomposition of V, then we write $V = W_1 \perp \ldots \perp W_k$.

The next lemma introduces an equivalence relation on the set of simple submodules of a module with a form (parts (a)–(d) are essentially stated and proved in [**2**, proof of Theorem 4.2]).

Lemma 2. *Let G be a group, $\mathbb{F}$ a field, V a semisimple and homogeneous $\mathbb{F}G$-module with a form, and W an anisotropic simple submodule of V; also, let $\mathbb{E}$ be the endomorphism ring $\mathrm{End}_{\mathbb{F}G}(W)$, and let q denote the order of $\mathbb{E}$.*

(a) *There exists a field automorphism τ of $\mathbb{E}$, whose order is at most 2, such that $\langle w_1\varepsilon, w_2\rangle = \langle w_1, w_2\varepsilon^\tau\rangle$ holds for all w_1, w_2 in W and ε in $\mathbb{E}$.*

(b) *If X and Y are simple submodules of V, we say that X* is equivalent to *Y (and we write $X \sim Y$) if there exists $\varphi \neq 0$ in $\mathrm{Hom}_{\mathbb{F}G}(X, Y)$ such that $\langle x_1\varphi, x_2\varphi\rangle = \langle x_1, x_2\rangle$ for all x_1, x_2 in X. This yields an equivalence relation on the set of simple submodules of V, and the totally isotropic simple submodules form a single $\sim$-equivalence class.*

(c) *If X is a simple submodule of V, and ϑ is an $\mathbb{F}G$-isomorphism from W to X, then there exists a unique δ in $\mathbb{E}$ such that $\delta^\tau = \delta$ and $\langle w_1\vartheta, w_2\vartheta\rangle = \langle w_1, w_2\delta\rangle$ holds for all w_1, w_2 in W. Moreover, we have $X \sim W$ if and only if $\delta \neq 0$ and $\delta = \varepsilon\varepsilon^\tau$ for some ε in $\mathbb{E}$.*

(d) *If τ is not the identity on $\mathbb{E}$, or also if τ is the identity on $\mathbb{E}$ and the characteristic of $\mathbb{F}$ is 2, then the anisotropic simple submodules of V form a single $\sim$-equivalence class. In any case, the set of anisotropic simple submodules of V is partitioned into at most two $\sim$-equivalence classes.*

(e) *Let X be an anisotropic simple submodule of V which is orthogonal to W, and suppose that τ is the identity on $\mathbb{E}$; suppose also that $\mathbb{F}$ has odd characteristic. If -1 is a square in $\mathbb{E}^\times$, then $W \perp X$ contains: two totally isotropic simple submodules and two $\sim$-classes, containing $(q-1)/2$ elements each, of anisotropic simple submodules* in the case *$W \sim X$; no totally isotropic simple submodules and two $\sim$-classes, containing $(q+1)/2$ elements each, of anisotropic simple submodules* in the case *$W \not\sim X$. If -1 is not a square in $\mathbb{E}^\times$, then the two cases above are switched.*

(f) *Let X be an anisotropic simple submodule of V which is orthogonal to W, and suppose that τ is the identity on $\mathbb{E}$; if the characteristic of $\mathbb{F}$ is 2, then $W \perp X$ contains a unique totally isotropic simple submodule, and q anisotropic simple submodules (constituting a single $\sim$-equivalence class).*

Proof of (a). The restriction of the form $\langle\ ,\ \rangle$ to $W \times W$ is a G-invariant nonsingular $\mathbb{F}$-form on W, hence the claim follows from [**3**, 6.1, 6.3].

Proof of (b). This is clear.

Proof of (c). If the submodule X is anisotropic, then the map

$$f : W \times W \to \mathbb{F}, \ \ (w_1, w_2) \mapsto \langle w_1\vartheta, w_2\vartheta\rangle$$

is a G-invariant nonsingular bilinear $\mathbb{F}$-form on W of the same type as $\langle\ ,\ \rangle$ (that is, symmetric or symplectic), and the first claim of (c) follows from [**3**, 6.1, 6.2]. If X is totally isotropic, then of course the zero endomorphism of W (and only that one) satisfies the required conditions.

Observe that, if ϑ' is any $\mathbb{F}G$-isomorphism from W to X, then there exists ε in $\mathbb{E}^\times$ such that $\vartheta' = \varepsilon\vartheta$, and the element of $\mathbb{E}$ attached to ϑ' as above is $\varepsilon\delta\varepsilon^\tau$; now the second claim of (c) can be easily proved.

Proof of (d). If τ is not the identity, it is clear that we have a single $\sim$-equivalence class of anisotropic simple submodules of V, as any element of $\mathbb{E}^\times$ which is fixed by τ is of the kind $\varepsilon\varepsilon^\tau$ for some ε in $\mathbb{E}^\times$ (this follows from the fact that, denoting by p the characteristic of $\mathbb{F}$, the order of $\mathbb{E}$ must be in this case an even power of p, say $p^{2\beta}$, and τ is precisely the p^β-th powering map). Assume now that τ is the identity, and let W, Y, Y' be anisotropic simple submodules of V such that Y and Y' are both not equivalent to W. If ϑ is an $\mathbb{F}G$-isomorphism from W to Y, and ϑ' an $\mathbb{F}G$-isomorphism from W to Y', then the elements δ and δ', attached (as in (c)) to the pairs (Y,ϑ) and (Y',ϑ') respectively, are both non-squares in $\mathbb{E}^\times$ (in particular, the characteristic of $\mathbb{F}$ is odd); hence there exists ε in $\mathbb{E}^\times$ such that $\delta = \varepsilon^2\delta'$. Now, consider the $\mathbb{F}G$-isomorphism φ from Y to Y' defined as the composite map $\vartheta^{-1}\varepsilon\vartheta'$; we get

$$\langle (w_1\vartheta)\varphi, (w_2\vartheta)\varphi\rangle = \langle w_1\varepsilon\vartheta', w_2\varepsilon\vartheta'\rangle = \langle w_1, w_2\varepsilon^2\delta'\rangle = \langle w_1, w_2\delta\rangle = \langle w_1\vartheta, w_2\vartheta\rangle$$

for all w_1, w_2 in W. We conclude that Y and Y' are equivalent, and the claim is proved.

Proof of (e). Suppose that -1 is a square in $\mathbb{E}^\times$; if we fix $\vartheta \neq 0$ in $\mathrm{Hom}_{\mathbb{F}G}(W,X)$, then every simple submodule $Y \neq X$ of $W\perp X$ can be written as $\{w+w\varepsilon\vartheta\ ,\ w \in W\}$ for some ε in $\mathbb{E}$. Consider the map $\gamma : W \to Y$ defined as $w\gamma := w + w\varepsilon\vartheta$; γ is clearly an $\mathbb{F}G$-isomorphism, and we have

$$\langle w_1\gamma, w_2\gamma\rangle = \langle w_1 + w_1\varepsilon\vartheta, w_2 + w_2\varepsilon\vartheta\rangle = \langle w_1, w_2\rangle + \langle w_1, w_2\varepsilon^2\delta\rangle$$

where δ is the element of $\mathbb{E}$ defined in (c). Now, Y is totally isotropic if and only if $\varepsilon^2\delta = -1$, and this equation has two solutions provided we have $W \sim X$ (in this case δ is a square in $\mathbb{E}^\times$ as well), otherwise it has no solutions. Moreover, we have $Y \sim W$ if and only if $1 + \varepsilon^2\delta$ is a square in $\mathbb{E}^\times$; if δ is a square (that is, $W \sim X$), we find $(q-3)/2$ elements ε in $\mathbb{E}$ such that $1+\varepsilon^2\delta$ is a square different from 0 (see [**2**, 4.4]) and, if we add the contribution of X, we get $(q-1)/2$ anisotropic simple submodules in $W\perp X$ which are equivalent to W. It is now clear that the other $\sim$-equivalence class of anisotropic simple submodules contains $(q-1)/2$ elements as well. If δ is *not* a square, then clearly we have $Y \not\sim Y^\perp$ for each anisotropic simple submodule Y of $W\perp X$ (here $Y^\perp$ is meant to be the orthogonal of Y in $W\perp X$); otherwise $W\perp X = Y\perp Y^\perp$ would contain two totally isotropic simple submodules, which is not the case. Therefore, the map $Y \mapsto Y^\perp$ provides a bijection between the two $\sim$-equivalence classes of anisotropic simple submodules, whence they contain $(q+1)/2$ elements each.

If -1 is *not* a square in $\mathbb{E}^\times$, the result can be obtained in a similar way.

Proof of (f). This is easily achieved repeating the first few lines of the previous point. ■

We note that the statement (d) of the previous lemma might be strengthened by claiming that, if τ is the identity on $\mathbb{E}$ and the characteristic of $\mathbb{F}$ is odd, then the set of anisotropic simple submodules of V is partitioned into *exactly* two $\sim$-equivalence classes (except of course when V is simple). This is a consequence of Lemma 2(e) and Lemma 4.

Remark 3. As it is shown in [**3**, 6.5, 6.6], the nature of the automorphism τ defined in Lemma 2 depends only on the isomorphism type of W as an $\mathbb{F}G$-module (hence, on the isomorphism type of V). Indeed, we have that *τ is the identity on $\mathbb{E}$ if and only if the $\mathbb{E}G$-module $W_{\mathbb{E}}$ is self-contragredient.*

The next step shall be to establish, in the relevant setting, the existence of orthogonal direct decompositions with simple summands.

Lemma 4. *Let G be a group, $\mathbb{F}$ a field, and V a semisimple and homogeneous $\mathbb{F}G$-module with a form. If V has an anisotropic simple submodule W, then V admits an orthogonal direct decomposition in which the summands are (anisotropic) simple submodules.*

Proof. We proceed by induction on the composition length m of V. For $m = 1$ the claim is clearly true. Suppose then $m > 1$. We have $V = W \perp W^{\perp}$ and, if $W^{\perp}$ has an anisotropic simple submodule, then the result is achieved by means of the inductive hypothesis. Therefore we may assume that all the simple submodules of $W^{\perp}$ are totally isotropic. In this case it is easy to check that, if X is a simple submodule of V which is not contained in $W^{\perp}$, then X is anisotropic: indeed, X can be written as $\{w + w\alpha\ ,\ w \in W\}$ for some α in $\mathrm{Hom}_{\mathbb{F}G}(W, W^{\perp})$ and hence, for all x_1, x_2 in X, we get

$$\langle x_1, x_2\rangle = \langle w_1 + w_1\alpha, w_2 + w_2\alpha\rangle = \langle w_1, w_2\rangle + \langle w_1\alpha, w_2\alpha\rangle.$$

But $W\alpha$ is a simple submodule of $W^{\perp}$ (therefore it is totally isotropic), so we have $\langle x_1, x_2\rangle = \langle w_1, w_2\rangle$. It is now clear that, since W is anisotropic, we can not have $\langle x_1, x_2\rangle = 0$ for all x_1, x_2 in X. Choose now a simple submodule of V, say Y, such that $Y \neq W$ and $Y \not\leq W^{\perp}$ (such a submodule certainly exists, as $\mathrm{Hom}_{\mathbb{F}G}(W, W^{\perp})$ is not the zero space). We have $Y^{\perp} \cap W^{\perp} < Y^{\perp}$, otherwise $Y^{\perp}$ would lie in $W^{\perp}$, and consequently W in Y, which is not the case. Since $Y^{\perp}$ is semisimple, we get $Y^{\perp} = Z \oplus (Y^{\perp} \cap W^{\perp})$, where Z is a simple (because of its dimension) submodule of V. Observe now that $Y^{\perp}$ has a simple submodule, namely Z, which is anisotropic (as it is not contained in $W^{\perp}$), and the result is achieved by applying the inductive hypothesis to $Y^{\perp}$. ■

The following analysis (5–10) focuses on the case in which the module under consideration has at least one anisotropic simple submodule W, and *this W is such that $W_{\mathbb{E}}$ is self-contragredient.*

Remark 5. Let G be a group, $\mathbb{F}$ a field of odd characteristic, and V a semisimple and homogeneous $\mathbb{F}G$-module with a form. Assume that V has an anisotropic simple submodule W such that $W_{\mathbb{E}}$ is self-contragredient (where $\mathbb{E}$ denotes the endomorphism ring $\mathrm{End}_{\mathbb{F}G}(W)$), and assume also that we have $V = W_1 \perp W_2 \perp \ldots \perp W_m$, where the W_j are anisotropic simple submodules of V. Now, suppose that W_r and W_s are distinct $\sim$-equivalent summands of the given decomposition, and let X be an anisotropic simple submodule of $W_r \perp W_s$ *which lies in the other $\sim$-equivalence class* (by Lemma 2(e) such a submodule X does exist). Then we get $W_r \perp W_s = X \perp X^{\perp}$ (where $X^{\perp}$ denotes the orthogonal of X in $W_r \perp W_s$), and of course $X^{\perp}$ is $\sim$-equivalent to X. It follows that *if m is odd, then V has an orthogonal direct decomposition in which the summands are anisotropic simple submodules belonging to the same $\sim$-equivalence class; if m is even, then* either *the same holds,* or *V has an orthogonal direct decomposition in which the summands are anisotropic simple submodules,* all but one *belonging to the same $\sim$-equivalence class.*

Lemma 6. *Let G be a group, $\mathbb{F}$ a field of odd characteristic, and V a semisimple and homogeneous $\mathbb{F}G$-module with a form. Suppose that V has an anisotropic simple submodule W such that $W_{\mathbb{E}}$ is self-contragredient (where $\mathbb{E}$ denotes the endomorphism ring $\mathrm{End}_{\mathbb{F}G}(W)$), and assume the following conditions:*

(a) *-1 is a square in $\mathbb{E}^{\times}$;*

(b) *$V = W_1 \perp W_2 \perp \ldots \perp W_m$, where the W_j are anisotropic simple submodules of V which belong to the same $\sim$-equivalence class.*

Also, let q denote the order of $\mathbb{E}$. If the composition length m of V is odd, say $m = 2k+1$, then V contains q^{2k} anisotropic simple submodules, $(q^{2k}+q^{k})/2$ of them equivalent to the W_j, $(q^{2k}-q^{k})/2$ of them constituting the other $\sim$-equivalence class.

Proof. Consider the submodule $W_1^{\perp}$ of V; if X is a simple submodule of V which is different from W_1, denoting by π the projection on $W_1^{\perp}$, clearly $X\pi$ is a simple submodule of $W_1^{\perp}$ and X is contained in $W_1 \perp X\pi$; moreover, if Y is a simple submodule of $W_1^{\perp}$ such that X is contained in $W_1 \perp Y$, then we have $Y = X\pi$. It is now clear that, denoting by $\mathcal{S}_U$ the set of all simple submodules of an $\mathbb{F}G$-module U, we have

$$\mathcal{S}_V = \Big(\bigcup_{Y \in \mathcal{S}_{W_1^{\perp}}} (\mathcal{S}_{W_1 \perp Y} \setminus \{W_1\}) \Big) \cup \{W_1\}. \tag{1}$$

We can now prove the lemma by induction on k. If $k=0$ the claim is true. Assume then $k>0$, and set $S := W_1 \perp W_2 \perp \ldots \perp W_{2k}$, $T := W_1 \perp W_2 \perp \ldots W_{2k-1}$. We can apply the inductive hypothesis on T, hence assuming that T has $q^{2(k-1)}$ anisotropic simple submodules, among which $(q^{2(k-1)}+q^{k-1})/2$ are equivalent to the W_j, and $(q^{2(k-1)}-q^{k-1})/2$ are not. If $Z \leq T$ is a totally isotropic simple submodule, then it is easy to check that all the simple submodules of $Z \perp W_{2k}$ (except of course for Z) are anisotropic and equivalent to the W_j. By Lemma 2(e) and (1) we conclude that the number of totally isotropic simple submodules of S is given by

$$\Big(\sum_{j=0}^{2k-3} q^j \Big) + 2\,\frac{q^{2(k-1)}+q^{k-1}}{2} = \Big(\sum_{j=0}^{2k-2} q^j \Big) + q^{k-1}.$$

We compute next the number of anisotropic simple submodules of S which are equivalent to the W_j; this is

$$1 + \Big(\sum_{j=0}^{2k-3} q^j \Big)(q-1) + \frac{q^{2(k-1)}+q^{k-1}}{2}\Big(\frac{q-1}{2}-1\Big) + \\ + \frac{q^{2(k-1)}-q^{k-1}}{2}\Big(\frac{q+1}{2}-1\Big) = \frac{q^{2k-1}-q^{k-1}}{2},$$

and the other $\sim$-equivalence class contains $(q^{2k-1}-q^{k-1})/2$ elements as well.

Finally, we analyse the situation for V. The number of totally isotropic simple submodules is given by

$$\left(\sum_{j=0}^{2k-2} q^j\right)+q^{k-1}+2\,\frac{q^{2k-1}-q^{k-1}}{2}=\sum_{j=0}^{2k-1} q^j,$$

hence we have q^{2k} anisotropic simple submodules in V, as desired. The number of simple submodules equivalent to the W_j is now

$$1+\left(\left(\sum_{j=0}^{2k-2} q^j\right)+q^{k-1}\right)(q-1)+\frac{q^{2k-1}-q^{k-1}}{2}\left(\frac{q-1}{2}-1\right)+$$
$$+\frac{q^{2k-1}-q^{k-1}}{2}\left(\frac{q+1}{2}-1\right)=\frac{q^{2k}+q^k}{2}$$

whereas the number of anisotropic simple submodules in the other equivalence class is of course $(q^{2k}-q^k)/2$. ■

In a totally analogous way it is possible to treat the following case.

Lemma 7. *Let G be a group, $\mathbb{F}$ a field of odd characteristic, and V a semisimple and homogeneous $\mathbb{F}G$-module with a form. Suppose that V has an anisotropic simple submodule W such that $W_{\mathbb{E}}$ is self-contragredient (where $\mathbb{E}$ denotes the endomorphism ring $\operatorname{End}_{\mathbb{F}G}(W)$), and assume the following conditions:*

(a) -1 *is* not *a square in* $\mathbb{E}^\times$;

(b) $V=W_1\perp W_2\perp\ldots\perp W_m$, *where the W_j are anisotropic simple submodules of V which belong to the same $\sim$-equivalence class.*

Also, let q denote the order of $\mathbb{E}$. If the composition length m of V is odd, say $m=2k+1$, then V contains q^{2k} anisotropic simple submodules, $(q^{2k}+(-1)^k q^k)/2$ of them $\sim$-equivalent to W, $(q^{2k}+(-1)^{k+1}q^k)/2$ of them constituting the other $\sim$-equivalence class.

At this stage, the intermediate step in the proof of Lemma 6 (adapted of course to the various situations) provides a proof for the following lemma, concerning modules of *even* composition length.

Lemma 8. *Let G be a group, $\mathbb{F}$ a field of odd characteristic, and V a semisimple and homogeneous $\mathbb{F}G$-module with a form. Suppose that V has an anisotropic simple submodule W such that $W_{\mathbb{E}}$ is self-contragredient (where $\mathbb{E}$ denotes the endomorphism ring $\operatorname{End}_{\mathbb{F}G}(W)$), and consider the following conditions:*

(a) -1 *is a square in* $\mathbb{E}^\times$;

(b) $V=W_1\perp W_2\perp\ldots\perp W_m$, *where the W_j are anisotropic simple submodules of V which belong to the same $\sim$-equivalence class;*

(a′) -1 *is* not *a square in* $\mathbb{E}^\times$;

(b′) $V=W_1\perp W_2\perp\ldots\perp W_m$, *where the W_j are anisotropic simple submodules of V,* all but one *belonging to the same $\sim$-equivalence class.*

Also, let q denote the order of $\mathbb{E}$. If the composition length m of V is even, say $m = 2k$, then the number of anisotropic simple submodules of V is as follows:

(1) $q^{2k-1} - q^{k-1}$ if (a) and (b) hold,
(2) $q^{2k-1} + q^{k-1}$ if (a) and (b$'$) hold,
(3) $q^{2k-1} + (-1)^{k-1}q^{k-1}$ if (a$'$) and (b) hold,
(4) $q^{2k-1} + (-1)^{k}q^{k-1}$ if (a$'$) and (b$'$) hold.

In any case, the set of anisotropic simple submodules of V is partitioned into two $\sim$-equivalence classes of the same cardinality.

Finally we move to characteristic 2.

Lemma 9. *Let G be a group, $\mathbb{F}$ a field of characteristic 2, and V a semisimple and homogeneous $\mathbb{F}G$-module with a form. Suppose that V has an anisotropic simple submodule W such that $W_{\mathbb{E}}$ is self-contragredient (where $\mathbb{E}$ denotes the endomorphism ring $\mathrm{End}_{\mathbb{F}G}(W)$). Then, denoting by q the order of $\mathbb{E}$ and by m the composition length of V, V contains q^{m-1} anisotropic simple submodules.*

Proof. The claim is easily proved by induction on m, recalling Lemma 2(f), Lemma 4, and (1) in the proof of Lemma 6. ∎

We are now in a position to collect the information obtained so far.

Theorem 10. *Let G be a group, $\mathbb{F}$ a field, and V a semisimple and homogeneous $\mathbb{F}G$-module with a form. Suppose that V has an anisotropic simple submodule W such that $W_{\mathbb{E}}$ is self-contragredient (where $\mathbb{E}$ denotes the endomorphism ring $\mathrm{End}_{\mathbb{F}G}(W)$). Then, denoting by q the order of $\mathbb{E}$ and by m the composition length of V, the number of anisotropic simple submodules of V is as follows:*

(1) q^{m-1} if m is odd, or if m and q are both even,
(2) $q^{m-1} \pm q^{(m/2)-1}$ if m is even and q is odd,

where the sign in (2) is determined by Lemma 8.

Proof. The claim follows at once from 4–9. ∎

To conclude, we focus on the case in which the module under consideration has at least one anisotropic simple submodule W, and *this W is such that $W_{\mathbb{E}}$ is not self-contragredient.*

Theorem 11. *Let G be a group, $\mathbb{F}$ a field of characteristic p, and V a semisimple and homogeneous $\mathbb{F}G$-module with a form. Suppose that V has an anisotropic simple submodule W such that $W_{\mathbb{E}}$ is* not *self-contragredient (where $\mathbb{E}$ denotes the endomorphism ring $\mathrm{End}_{\mathbb{F}G}(W)$). Then, denoting by q the order of $\mathbb{E}$ and by m the composition length of V, we have $q = p^{2k}$ for some k in $\mathbb{N}$, and the number of anisotropic simple submodules of V is given by*

$$f(m) = q^{(m-1)/2} \cdot \frac{q^{m/2} + (-1)^{m+1}}{q^{1/2} + 1}.$$

Proof. Let r be an integer such that $q = p^r$; we know that the group of field automorphisms of $\mathbb{E}$, $\mathrm{Aut}(\mathbb{E})$, is a cyclic group of order r, and it contains a (nontrivial) involution τ. This forces r to be even, whence $r = 2k$ for some positive integer k.

Consider now any two anisotropic simple submodules of V, say X and Y; by Lemma 2(d) we have $X \sim Y$, therefore we can choose ϑ in $\mathrm{Hom}_{\mathbb{F}G}(X, Y)$ such that $\langle x_1\vartheta, x_2\vartheta\rangle = \langle x_1, x_2\rangle$ for all x_1, x_2 in X. If X and Y are orthogonal, we can determine the number of totally isotropic simple submodules contained in $X \perp Y$ as follows. Every simple submodule Z of $X \perp Y$, except for Y, can be written as $\{x + x\varepsilon\vartheta \ , \ x \in X\}$ where ε is a suitable element of $\mathbb{E}_X := \mathrm{End}_{\mathbb{F}G}(X)$; therefore, denoting by τ_X the involution of $\mathrm{Aut}(\mathbb{E}_X)$,

$$\langle z_1, z_2\rangle = \langle x_1 + x_1\varepsilon\vartheta, x_2 + x_2\varepsilon\vartheta\rangle = \langle x_1, x_2\rangle + \langle x_1, x_2\varepsilon\varepsilon^{\tau_X}\rangle$$

for all z_1, z_2 in Z, and it is now clear that Z is totally isotropic if and only if $\varepsilon\varepsilon^{\tau_X} = -1$. Now, setting $h := p^k$, we get $\varepsilon^{\tau_X} = \varepsilon^h$, so that the equation becomes $\varepsilon^{h+1} = -1$ and it has $h+1$ solutions in $\mathbb{E}_X$. We conclude that $X \perp Y$ contains $h+1$ totally isotropic simple submodules, so that the number of anisotropic simple submodules of it is given by $h^2 + 1 - h - 1 = h^2 - h$.

We proceed by induction on m. The discussion above yields the result for $m = 2$, so that we may assume $m > 2$. Observe that, by Lemma 4, we can also assume $V = W \perp W_2 \perp \ldots \perp W_m$, where the W_j are anisotropic simple submodules of V; moreover, the $\mathbb{F}G$-module $W^{\perp} = W_2 \perp W_3 \perp \ldots \perp W_m$ satisfies our hypotheses. We already know that, if X and Y are simple submodules of V which are orthogonal, then $X \perp Y$ has $f(2) = h^2 - h$ anisotropic simple submodules in the case that X and Y are both anisotropic, whereas it contains h^2 anisotropic simple submodules if X (or Y) is anisotropic and the other one is totally isotropic. Recalling (1) in Lemma 6, we can compute the number of anisotropic simple submodules of V, which is given by

$$f(m) = 1 + f(m-1)\cdot(h^2 - h - 1) + \left(\frac{h^{2(m-1)} - 1}{h^2 - 1} - f(m-1)\right)\cdot(h^2 - 1) =$$
$$= f(m-1)\cdot(-h) + h^{2(m-1)}.$$

Now we are in a position to apply the inductive hypothesis, achieving the desired conclusion. ■

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