A weak version of the strong exponential closure

A WEAK VERSION OF THE STRONG EXPONENTIAL CLOSURE

P. D'AQUINO, A. FORNASIERO, AND G. TERZO

Abstract. Assuming Schanuel's Conjecture we prove that for any variety $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ over the algebraic closure of $\mathbb{Q}$, of dimension n, and with dominant projections on both the first n coordinates and the last n coordinates, there exists a generic point $(a, e^a) \in V$. We obtain in this way many instances of the Strong Exponential Closure axiom introduced by Zilber in 2004.

1. Introduction

In [28] Zilber conjectured that the complex exponential field is quasi-minimal, i.e. every subset of $\mathbb{C}$ definable in the language of rings expanded by an exponential function is either countable or co-countable. If the conjecture is true the complex exponential field should have good geometric properties.

He introduced and studied a class of new exponential fields now known as Zilber fields via axioms of algebraic and geometrical nature. There are many novelties in his analysis, including a reinterpretation of Schanuel's Conjecture in terms of Hrushovski's very general theory of predimension and strong extensions.

Zilber fields satisfy Schanuel's Conjecture and a version of Nullstellensatz for exponential equations. Moreover, in each uncountable cardinality there is a privileged such field, satisfying a countable closure condition and a strengthened Nullstellensatz. Privileged means that the structure in each uncountable cardinality is unique up to isomorphism. Zilber conjectured that the one in cardinal $2^{\aleph_0}$ is $\mathbb{C}$ as exponential field. This would, of course, imply that $\mathbb{C}$ satisfies Schanuel's Conjecture (SC), and Zilber's Nullstellensatz. Comparing the complex exponential field and Zilber's fields has been object of study in [25], [7], [8], [24], [9].

In this paper we will analyze one of the axioms introduced by Zilber, the **Strong Exponential Closure** (SEC), in the complex exponential field. Modulo Schanuel's Conjecture, (SEC) is the only axiom still unknown for $(\mathbb{C}, exp)$. Some instances of (SEC) have been proved in

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[6], [24], [25]. Here we obtain a more general result which includes those in [6].

We recall that a variety V is *rotund* if for every nonzero matrix $M \in \mathcal{M}_{n\times n}(\mathbb{Z})$ then $\dim(M \cdot V) \geq rank(M)$, i.e. all the images of V under suitable homomorphisms are of large dimension.

A variety V is *free* if V does not lie inside any subvariety of the form either $\{(\bar{x}, \bar{y}) : r_1x_1 + \ldots + r_nx_n = b, r_i \in \mathbb{Z}, b \in \mathbb{C}\}$ or $\{(\bar{x}, \bar{y}) : y_1^{r_1} \cdot \ldots \cdot y_n^{r_n} = b, r_i \in \mathbb{Z}, b \in \mathbb{C}^*\}$.

Strong Exponential Closure. If $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ is a rotund and free algebraic variety of dimension n, and $\bar{a}$ is a finite tuple of elements of $\mathbb{C}$ then there is $\bar{z} \in \mathbb{C}^n$ such that $(\bar{z}, e^{\bar{z}}) \in V$, and is generic in V over $\bar{a}$, i.e. $\mathrm{t.d.}_{\mathbb{Q}(\bar{a})}(\bar{z}, e^{\bar{z}}) = \dim(V)$.

The hypothesis of rotundity and freness on the variety V guarantee that there are no hidden relations among the coordinates of points in V except those coming from V itself and the laws of exponentiation.

We recall

Schanuel Conjecture (SC) Let $z_1, \ldots, z_n \in \mathbb{C}$. Then

$$t.d._{\mathbb{Q}}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq l.d.(z_1, \ldots, z_n).$$

In this paper assuming Schanuel's Conjecture we prove the Strong Exponential Closure for $(\mathbb{C}, exp)$ for certain varieties defined over $\mathbb{Q}^{alg}$. We denote the projections on the first n coordinates and on the last n coordinates by $\pi_1 : V \to \mathbb{C}^n$ and $\pi_2 : V \to (\mathbb{C}^*)^n$, respectively.

Main Result. (SC) Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ be a variety defined over the algebraic closure of $\mathbb{Q}$, such that $\dim V = n$, and both projections π_1 and π_2 are dominant. Then there is a Zariski dense set of generic points $(\bar{z}, e^{\bar{z}})$ in V.

We recall that π_1 and π_2 being dominant means that $\pi_1(V)$ and $\pi_2(V)$ are Zariski dense in $\mathbb{C}^n$. As observed in [2], π_1 being dominant implies that V is rotund and both projections being dominant imply that V is free. It is easy to construct a free and normal variety in $\mathbb{C}^n \times (\mathbb{C}^*)^n$ where both projections are not dominant.

In Lemma 2.10 in [6] (see also [5]) the existence of a Zariski dense set of solutions of V is proved under the hypothesis that π_1 is dominant. No appeal to Schanuel's conjecture is necessary, and moreover there is no restriction on the set of parameters.

Bays and Kirby in [2] proved the quasi-minimality of $(\mathbb{C}, exp)$ assuming a weaker condition than the strong exponential closure, requiring only the existence of a point $(\bar{z}, e^{\bar{z}})$ in V under the same hypothesis on the variety.

For the new result on the existence of generic solutions Schanuel's conjecture is crucial and there are restrictions on the set of paramenters defining the variety V.

2. Preliminaries

By "definable subset of $\mathbb{C}^n$" we mean definable in the sense of model theory in the language of rings. We will always allow a finite or a countable set of parameters P. We recall some basic facts about the notion of dimension associated to a definable set in $\mathbb{C}^n$ which will be used in the proof of the main theorem, for details see [10] and [13].

Every definable set has a dimension equal to the dimension of its Zariski closure. Moreover, for algebraically closed fields the model-theoretic algebraic closure (acl) coincides with the usual field-theoretic algebraic closure.

If X is definable over P, the dimension of X is

$$\dim(X) = \max\{d : \exists \bar{x} \in X \text{ t.d.}_P(\bar{x}) = d\}.$$

We will often use the fact that $\mathbb{C}$ is saturated.

Fact. 1. $\dim(X)$ is well-defined, i.e. it does not depend on the choice of the set P of parameters in the definition of X.

Fact. 2. Let X be a definable set in $\mathbb{C}^n$. The dimension of X is 0 iff X is finite and nonempty. We use the convention that the empty set has dimension -1.

Fact. 3. Let $(X_i)_{i\in I}$ be a definable family over P of subsets of $\mathbb{C}^n$. The set $\{i \in I : X_i \text{ is finite}\}(= \{i \in I : \dim(X_i) = 0\})$ is definable with the same parameters as $(X_i)_{i\in I}$. More generally, for every $d \in \mathbb{N}$, the set $\{i \in I : \dim(X_i) = d\}$ is definable, with the same parameters as $(X_i)_{i\in I}$.

Sketch of proof. The set $Z := \{i \in I : X_i \text{ is finite}\}$ is definable since ACF_0 eliminates the quantifier $\exists^\infty$. Since Z is also invariant over automorphisms of $\mathbb{C}$ fixing P, Z is definable over P.

By induction on d, we can also prove that the set $\{i \in I : \dim(X_i) = d\}$ is also definable (see [10]). $\square$

Notation. Let $Y \subseteq \mathbb{C}^{n+m}$. For every $\bar{x} \in \mathbb{C}^n$ we denote the fiber of Y at $\bar{x}$ by $Y_{\bar{x}} = \{\bar{y} \in \mathbb{C}^m : (\bar{x}, \bar{y}) \in Y\}$.

Let $\pi_1 : Y \to \mathbb{C}^n$ and $\pi_2 : Y \to \mathbb{C}^m$ be the projections on the first n and the last m coordinates, respectively.

Lemma 2.1. *Let $Y \subseteq \mathbb{C}^{n+m}$ be definable over P, and $X := \pi_1(Y)$. Assume that, for every $\bar{x} \in X$, $\dim(Y_{\bar{x}}) = d$. Then, $\dim(Y) = \dim(X)+d$. In particular, if $Y_{\bar{x}}$ is infinite for every $\bar{x} \in X$, then $\dim(Y) > \dim(X)$.*

Notice that an equivalent result holds in the case of $X := \pi_2(Y)$, relativizing the arguments to the last m-coordinates. For the proof of the above lemma see [10].

Remark 2.2. Let $X \subseteq \mathbb{C}^n$ be definable with parameters in P.

The following are equivalent:

(1) $\dim(X) = n$;
(2) X is Zariski dense in $\mathbb{C}^n$;
(3) X has non-empty interior (in the Zariski topology);
(4) $\dim(\mathbb{C}^n \setminus X) < n$;
(5) X contains a point which is generic in $\mathbb{C}^n$ over P;
(6) X contains all points which are generic in $\mathbb{C}^n$ over P;
(7) the points of X which are generic in X over P are exactly the points of $\mathbb{C}^n$ which are generic in $\mathbb{C}^n$ over P.

Lemma 2.3. *Let $Y \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be a definable set over P, such that* $\dim(Y) \leq n$. *Let $\bar{c} \in \mathbb{C}^n$ be generic over P, i.e. $t.d._P(\bar{c}) = n$. Then, the fiber $Y_{\bar{c}} := \{\bar{z} \in \mathbb{C}^m : (\bar{c}, \bar{z}) \in Y\}$ is finite.*

Proof. Let

$$X := \{\bar{a} \in \mathbb{C}^n : Y_{\bar{a}} \text{ infinite}\}.$$

Notice that X is definable over P by Fact 3. We claim that $\dim(X) < n$, and therefore a generic point $\bar{c}$ cannot be in X.

Assume, by contradiction, that $\dim(X) = n$, and let

$$W := \pi_1^{-1}(X) = \{\bar{y} \in Y : \text{ for infinitely many } \bar{y}' \in Y, \pi_1(\bar{y}') = \pi_1(\bar{y})\}.$$

Notice that for every $\bar{x} \in X$, $W_{\bar{x}}$ is infinite, and hence $\dim(W_{\bar{x}}) \geq 1$. Therefore by Lemma 2.1,

$$n \geq \dim(Y) \geq \dim(W) \geq \dim(X) + 1 = n + 1,$$

a contradiction.

□

3. Exponential Closure and quasiminimality

Recently in [4] Boxall proved that certain definable sets in $(\mathbb{C}, exp)$ are either countable or co-countable, i.e. he obtained a special case of quasi-minimality for $(\mathbb{C}, exp)$. It is known that if X is a subset of $\mathbb{C}$ defined by either quantifier-free formulas or by $\forall \overline{y}(P(x, \overline{y}) = 0)$ where P is a term in the language $\{+, \cdot, 0, 1, exp\}$ then X is either countable or co-countable. Boxall extends this result to existential definable sets.

Theorem 3.1. *Let X a subset of $\mathbb{C}$ defined by $\exists \overline{y}(P(x, \overline{y}) = 0)$, where P is a term in the language $\{+, \cdot, 0, 1, exp\}$ together with parameters from $\mathbb{C}$. Then either X or $\mathbb{C} \setminus X$ is countable.*

In [28] Zilber proved that if $(\mathbb{C}, exp)$ satisfies Schanuel's Conjecture and the Strong Exponential Closure axiom (SEC) then $(\mathbb{C}, exp)$ is quasi-minimal. In [2], Bays and Kirby considered a weaker condition, the Exponential-Algebraic Closedness which simply requires that V intersects the graph of exponentiation under the same hypothesis in (SEC). They prove:

Theorem 3.2. *If $(\mathbb{C}, exp)$ is exponential-algebraically closed then $(\mathbb{C}, exp)$ is quasi-minimal.*

No appeal to Schanuel's Conjecture is made. In [6], [24], [25] the proof of the existence of generic points of the form $(\overline{a}, e^{\overline{a}})$ in the given variety depends on Schanuel's Conjecture.

In the remaining part of this section we review a result already announced in [6] which we will use in the proof of the main result. We recall a very powerful criterium for solvability of systems of exponential equations due to Brownawell and Masser (see [5]).

Theorem 3.3. *Let $P_1(\overline{x}), \ldots, P_n(\overline{x}) \in \mathbb{C}[\overline{x}]$, where $\overline{x} = (x_1, \ldots, x_n)$, and $P_i(\overline{x})$ are non zero polynomials in $\mathbb{C}[\overline{x}]$. Then there exist $z_1, \ldots, z_n \in \mathbb{C}$ such that*

$$(1) \qquad \begin{cases} e^{z_1} = P_1(z_1, \ldots, z_n) \\ e^{z_2} = P_2(z_1, \ldots, z_n) \\ \vdots \\ e^{z_n} = P_n(z_1, \ldots, z_n) \end{cases}$$

The proof of Theorem 3.3 can be generalized to algebraic functions, see [6]. This will be crucial in the proof of the main result of this paper, and for completeness we prefer to recall the basic notions that we will use.

A cone is an open subset $U \subseteq \mathbb{C}^n$ such that for every $1 \leq t \in \mathbb{R}$, if $\overline{x} \in U$ then $t\overline{x} \in U$. An algebraic function (in many variables) is an analytic function $f : U \to \mathbb{C}$, where $U \subseteq \mathbb{C}^n$ is a cone, and there exists a nonzero polynomial $p(\bar{x}, u) \in \mathbb{C}[\bar{x}, u]$ with $p(\bar{x}, f(\bar{x})) = 0$ for all $\overline{x} \in U$. If the polynomial p is monic in u, we say that f is integral algebraic.

Theorem 3.4. *Let $f_1, \ldots, f_n : U \to \mathbb{C}$ be nonzero algebraic functions, defined on some cone U. Assume that $U \cap (2\pi i\mathbb{Z}^*)^n$ is Zariski dense in $\mathbb{C}^n$. Then, the system*

$$(2) \qquad \begin{cases} e^{z_1} = f_1(\overline{z}), \\ \ldots \\ e^{z_n} = f_n(\overline{z}) \end{cases}$$

has a solution $\overline{a} \in U$.

For the proof see [6]. Using Theorem 3.4 we can prove a version of Exponential-Algebraic Closedness for $(\mathbb{C}, exp)$.

Let $G_n(\mathbb{C}) = \mathbb{C}^n \times (\mathbb{C}^*)^n$ be the algebraic group.

Lemma 3.5. *Let $p_1, \ldots, p_n \in \mathbb{C}[\overline{x}, u]$ be nonzero irreducible polynomials of degree at least 1 in u, and not of the form a constant times u. Let $V \subseteq G_n(\mathbb{C})$ be an irreducible component of the set*

$$\{(\bar{x}, \overline{y}) \in G_n(\mathbb{C}) : \bigwedge_{i=1}^{n} p_i(\bar{x}, y_i) = 0\}.$$

Assume that $\pi_1(V)$ *is Zariski dense in* $\mathbb{C}^n$. *Then, the set* $\{\overline{a} \in \mathbb{C}^n : (\overline{a}, e^{\overline{a}}) \in V\}$ *is Zariski dense in* $\mathbb{C}^n$.

Proof. It is well known that there exists a nonempty cone $U \subseteq \mathbb{C}^n$ and algebraic functions

$$f_1, \ldots, f_n : U \to \mathbb{C},$$

such that $p_i(\overline{x}, f_i(\overline{x})) = 0$ on all U. Moreover, since $(2\pi i\mathbb{Z}^*)^n$ is Zariski dense in $\mathbb{C}^n$, we can also assume that $(2\pi i\mathbb{Z}^*)^n \cap U$ is Zariski dense. Thus, in order to find a solution of the system

$$p_1(\overline{x}, e^{x_1}) = 0, \ldots, p_n(\overline{x}, e^{x_n}) = 0,$$

it suffices to find $\overline{a} \in U$ such that $e^{a_1} = f_1(\overline{a}), \ldots, e^{a_n} = f_n(\overline{a})$, and we apply Theorem 3.4 to find such $\overline{a}$.

□

Theorem 3.6. *Let* $W \subseteq G_n(\mathbb{C})$ *be an irreducible algebraic variety such that* $\pi_1(W)$ *is Zariski dense in* $\mathbb{C}^n$. *Then, the set* $\{\overline{a} \in \mathbb{C}^n : (\overline{a}, e^{\overline{a}}) \in W\}$ *is Zariski dense in* $\mathbb{C}^n$.

Proof. We describe the main steps of the proof. Let K be a finite extension of $\mathbb{Q}$ where W is defined. Let $\overline{a}$ be a generic point in $\pi_1(W)$, and $\overline{b}$ in $W_{\overline{a}}$. By model completeness of algebraically closed fields we can assume that the b_i's are all algebraic over $K(\overline{a})$. Hence, there exist polynomials $p_1(\overline{x}, u), \ldots, p_n(\overline{x}, u) \in K[\overline{x}, u]$ such that

$$p_1(\overline{a}, b_1) = \ldots = p_n(\overline{a}, b_n) = 0.$$

Let V be an irreducible component of $\{(\overline{x}, \overline{y}) \in G_n(\mathbb{C}) : \bigwedge_{i=1}^n p_i(\overline{x}, y_i) = 0\}$ which contains $(\overline{a}, \overline{b})$, and let $U = \{(\overline{x}, \overline{y}) \in V : V_{\pi_1(\overline{x}, \overline{y})} \text{ is finite}\}$. Clearly, $(\overline{a}, \overline{b}) \in U$, and hence $(\overline{a}, \overline{b}) \in \overline{U}^{Zar}$, the Zariski closure of U. Hence, $\dim(\overline{U}) = n$ since $t.d._K(\overline{a}, \overline{b}) = t.d._K(\overline{a}) = n$. Let U_0 be an irreducible component of $\overline{U}^{Zar}$ which contains $(\overline{a}, \overline{b})$. We have $\dim(U_0) = n$, and so $\pi_1(U_0)$ is Zariski dense in $\mathbb{C}^n$. A generic point $(\overline{a}, \overline{b})$ of U_0 is in W, and so $U_0 \subseteq W$. Hence, we work with U_0 instead than W and the result follows from Lemma 3.5.

□

The hypothesis that $\pi_1(W)$ is Zariski dense is a non-trivial condition, and it implies that the variety is rotund. Theorem 3.6 states the Exponential-Algebraic Closedness of $(\mathbb{C}, exp)$ for irreducible variety W with π_1 dominant. Indeed, there is a Zariski-dense sets of points $(\overline{a}, e^{\overline{a}})$ in W. A major problem is to replace the hypothesis that π_1 is dominant with much weaker ones like rotundity and freeness while still retaining the conclusion of the theorem.

Notice that no restriction is made on the coefficients of the polynomials defining W, and the result is independent from Schanuel's Conjecture.

4. Strong Exponential Closure

We now go back to analyze Zilber original axiom (SEC), i.e. we want to prove the existence of a point in the variety V of the form $(\overline{a}, e^{\overline{a}})$ which is generic in V. Assuming Schanuel's Conjecture we can prove (SEC) for algebraic varieties satisfying certain conditions.

Theorem 4.1. *(SC) Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ be an irreducible variety over the algebraic closure of $\mathbb{Q}$ with $\dim V = n$. Assume that both projections π_1 and π_2 are dominant. Then there is $\overline{a} \in \mathbb{C}^n$ such that $(\overline{a}, e^{\overline{a}}) \in V$ and $t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = n$.*

As observed in [2] π_1 being dominant implies that V is a rotund variety, and both projections being dominant imply that V is free.

In the proof of Theorem 4.1 we will use the following known result. For completeness we sketch the proof.

Fact. Let $M \in \mathbb{Z}^m \times \mathbb{Z}^n$, $L_M = \{\overline{x} \in \mathbb{C}^n : M \cdot \overline{x} = 0\}$, and $T_M = \{\overline{y} \in (\mathbb{C}^*)^n : \overline{y}^M = 1\}$. The hyperspace L_M and the algebraic subgroup T_M have the same dimension.

Proof. Let $Z = \exp^{-1}(T_M) = \{\overline{x} \in \mathbb{C}^n : M \cdot \overline{x} = 2\pi i\overline{k}$, where $\overline{k} \in \mathbb{Z}^m\}$, Z is the union of the translates of L_M by the vectors $2\pi i\overline{k}$. The algebraic subgroup T_M is a closed differential submanifold in $(\mathbb{C}^*)^n$, and since exp is a diffeomorphism Z is a differential submanifold of $\mathbb{C}^n$. Notice that L_M is the tangent space of Z at 0. Let L be the tangent space of T_M at 1. Clearly, L_M and L are isomorphic, hence they have the same dimension, and so $\dim(T_M) = \dim(L_M)$.

□

Proof of Theorem 4.1. Theorem 3.6 guarantees the nonempty intersection of V with the graph of exponentiation. Let $(\overline{a}, e^{\overline{a}}) \in V$, and suppose that $(\overline{a}, e^{\overline{a}})$ is not generic in V, i.e. $t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = m < n$. By Lemma 2.1 and the hypothesis that both projections are dominant without loss of generality we can assume that both fibers $V_{\overline{a}}$ and $V^{e^{\overline{a}}}$ are finite. The finite cardinality of $V_{\overline{a}}$ implies that all coordinates of the tuple $e^{\overline{a}}$ are algebraic over $\overline{a}$, since they are in $\mathrm{acl}(\overline{a})$. Exchanging $\overline{a}$ and $e^{\overline{a}}$ we have that each coordinate of the tuple $\overline{a}$ is algebraic over $e^{\overline{a}}$. Hence,

$$m = t.d._{\mathbb{Q}}(\overline{a}) = t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = t.d._{\mathbb{Q}}(e^{\overline{a}}). \tag{3}$$

Schanuel's Conjecture implies $l.d.(\overline{a}) \leq t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = m < n$. By equation (3) we can then conclude that $l.d.(\overline{a}) = m$. Hence, there exists a matrix $M \in \mathbb{Z}^{n-m} \times \mathbb{Z}^n$ of rank $n - m$ such that $M \cdot \overline{a} = 0$, which together with its multiplicative version give the following hyperspace and torus:

$$L_M = \{\overline{x} : M \cdot \overline{x} = 0\} \text{ and } T_M = \{\overline{y} : \overline{y}^M = 1\}.$$

As observed $\dim T_M = \dim L_M = m$. So, $\overline{a}$ is generic in L_M and $e^{\overline{a}}$ is generic in T_M. Then the non genericity of $(\overline{a}, e^{\overline{a}})$ in V is witnessed either by $\overline{a}$ or $e^{\overline{a}}$.

Without loss of generality we can assume that T_M is irreducible. If not, we consider the irreducible component of T_M containing $\overline{1}$ whose associated matrix we call M'. By results on pages 82-83 in [3] the associate hyperspace $L_{M'}$ coincides with L_M.

Let $N \in \mathbb{C}^{n-m} \times \mathbb{C}^n$, and

$$W_N = \{(\overline{x}, \overline{y}) \in V : \overline{x} \in L_N \wedge \mid V_{\overline{x}} \mid < \infty \wedge \mid V^{\overline{y}} \mid < \infty\}.$$

Clearly, W_N is definable, and so $(W_N)_N$ is a definable family.

If $N = M$ then $(\overline{a}, e^{\overline{a}}) \in W_M$, and so $\dim W_M \geq \dim L_M$. Moreover, (3) implies $\dim W_M = \dim \pi_1(W_M)$, and so $\dim W_M \leq \dim L_M$. Hence, $\dim W_M = \dim L_M$.

For any given definable family of definable sets in $\mathbb{C}$ the family of the Zariski closures is still a definable family, see [17].

Let W'_M be the irreducible components of the Zariski closure of W_M containing the point $(\overline{a}, e^{\overline{a}})$.

Since $(\overline{a}, e^{\overline{a}})$ is generic in W'_M, and $e^{\overline{a}} \in \pi_2(W'_M) \cap T_M$ is generic in $\pi_2(W'_M)$ we have that $\pi_2(W'_M) \subseteq T_M$. Hence, the Zariski closure of the projection, $\overline{\pi_2(W'_M)}^{Zar}$, is contained in T_M. Moreover, $e^{\overline{a}}$ is generic in T_M, and this implies that $T_M = \overline{\pi_2(W'_M)}^{Zar}$.
We denote by S_N the Zariski closure of $\pi_2(W'_N)$ where W'_N is an irreducible component of W_N and $N \in \mathbb{C}^{n-m} \times \mathbb{C}^n$. Let

$$\mathcal{U} = \{S_N : S_N \text{ is a torus }\}.$$

Since $\mathcal{U}$ is a countable definable family in $(\mathbb{C}^*)^n$, and $\mathbb{C}$ is saturated then $\mathcal{U}$ is either finite or uncountable. Then $\mathcal{U}$ is necessarily finite, i.e. $\mathcal{U} = \{H_1, \dots, H_l\}$. So, $T_M = H_i$, for some $i = 1, \dots, l$. We now construct a new Masser's system in order to exclude a non generic solutions in V by requiring that the last n coordinates do not belong to one of the finitely many tori above. Then by Theorem 3.4 the new Masser system has a solutions which is necessarily a generic solution in V. $\square$

Remark 4.2. *Notice that the set of generic solutions is Zariski dense since the non generic ones belong to a finite union of tori.*

References

[1] J. Ax: *On Schanuel's Conjecture*, Ann. of Math., (2), 93, (1971), 252-268.

[2] M. Bays and J. Kirby: *Pseudo-exponential maps, variants, and quasiminimality* Algebra Number Theory, (12), 3, (2018), 493-549.

[3] E. Bombieri and W. Gluber: *Hights in Diofantine geometry*, Cambridge University Press, (2006).

[4] G. Boxall: *A special case of quasiminimality*, arXiv: 1608.06400, (2016).

[5] W.D. Brownawell and W. Masser: Zero estimates of moving targets, Journal of the London Mathematical Society (2), 95, (2017), 441-454.

[6] P. D'Aquino, A. Fornasiero and G. Terzo: *Generic solutions of equations with iterated exponentials.* Transactions of the American Mathematical society, (370), 2, (2018), 1393–1407.

[7] P. D'Aquino, A. Macintyre and G. Terzo: *From Schanuel's conjecture to Shapiro's conjecture.* Comment. Math. Helv., (89), 3, (2014), 597-616.

[8] P. D'Aquino, A. Macintyre and G. Terzo: *Schanuel Nullstellensatz for Zilber fields*, Fundamenta Mathematicae, 207, (2010), 123–143.

[9] P. D'Aquino, A. Macintyre and G. Terzo: *Comparing $\mathbb{C}$ and Zilber's exponential fields: zero sets of exponential polynomials*, Journal of the Institute of Mathematics of Jussieu, (1), 15, (2016), 71–84.

[10] L. van den Dries: *Dimension of definable sets, algebraic boundedness and Henselian fields*, in Annals of Pure and Applied Logic, 45, (1989), 189–209.

[11] L. van den Dries: *Exponential rings, exponential polynomials and exponential functions*, Pacific Journal of Mathematics, (1), 113, (1984), 51–66.

[12] J. E. Dennis, Jr., Robert B. Schnabel: *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Classics in Applied Mathematics, (1996).

[13] A. Fornasiero, *Dimensions, matroids, and dense pairs of first-order structures*, Annals of Pure and Applied Logic, 162, (7), (2011), 514-543.

[14] A. Günaydin : *Rational solutions of polynomial-exponential equations.* Int. J. Number Theory, (6), 8, (2012), 1391–1399.

[15] C. W. Henson and L. A. Rubel: *Some applications of Nevanlinna Theory to Mathematical Logic: identities of exponential functions*, in Transactions of the American Mathematical Society, (1), 282, (1984), 1-32.

[16] E. Hrushovski: *Strongly minimal expansions of algebraically closed fields*, Israel J. Math., (2-3) 79, (1992), 129-151.

[17] W. A. Johnson, Fun with Fields, Ph.D. Thesis, University of California at Berkeley, 2016.

[18] H. Katzberg: *Complex exponential terms with only finitely many zeros*, Seminarberichte, Humboldt-Univ. Berlin, Sekt. Math. 49, (1983), 68–72 .

[19] J. Kirby *Finitely presented exponential fields*, Algebra and Number Theory, (4), 7, (2013), 943–980.

[20] J. Kirby, A. Macintyre, A. Onshuus: *The algebraic numbers definable in various exponential fields.* Journal of the Institute of Mathematics of Jussieu, (4), 11 (2012), 825–834.

[21] S. Lang: *Complex Analysis*, Springer, (1993).

[22] A. Macintyre: *Schanuel's conjecture and free exponential rings.* Annals Pure Applied Logic (3), 51, (1991), 241–246.

[23] A. Macintyre and A. Wilkie: *On the decidability of the real exponential field.* Kreiseliana, A K Peters, Wellesley, MA, (1996), 441–467.

[24] V. Mantova, U. Zannier: *Polynomial-exponential equations and Zilber's conjecture*, Bulletin of the London Mathematical Society, (2), 48, (2016), 309–320.

[25] D. Marker: *A remark on Zilber's pseudoexponentiation*, The Journal of Symbolic Logic, (3) 71, (2006), 791-798.

[26] W. M. Schmidt, Equations over Finite Fields, An Elementary Approach, Lecture Notes in Mathematics 536, Springer (1976).

[27] G. Terzo: *Some Consequences of Schanuel's Conjecture in exponential rings*, Communications of Algebra, (3), 36, (2008), 1171–1189.

[28] B. Zilber: *Pseudo-exponentiation on algebraically closed fields of characteristic zero*, Annals of Pure and Applied Logic, (1) 132, (2004), 67-95.

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