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Just infinite profinite structures

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Dottorando
Dario Villanis Ziani

Tutore
Prof. Orazio Puglisi

Coordinatore
Prof. Paolo Salani

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Contents

Symbols

$\mathbb{N}$	natural numbers (0 included)
$\mathbb{N}^+$	positive natural numbers
$\mathbb{F}_n$	finite field with n elements
$\mathbb{F}_n^*$	multiplicative subgroup of the finite field $\mathbb{F}_n$
$\prod$	direct product
Car	Cartesian product
$\mathrm{Sym}(X)$	symmetric group on a set X
S_n	symmetric group of order n
C_n	(abstract) cyclic group of order n
$N \rtimes K$	semidirect product
$H \wr K$	wreath product
$B \leqslant_c A$	closed subgroup or subring
$B \leqslant_o A$	open subgroup or subring
$B \leqslant_f A$	finite-index subgroup or subring

Introduction

An infinite group is called just infinite if each of its non-trivial normal subgroup has finite index or, equivalently, if it has only finite proper images. Beyond the just infinite property, we can require a further condition, namely that each finite-index subgroup of the group is just infinite; a group which satisfies this additional condition is called hereditarily just infinite.

The class of just infinite groups was first introduced by Wilson in [20], where the author analyzes basic properties of such groups, especially with regard to their normal and subnormal structure. From that time, several authors focused on this class of groups, and some well-known groups were found to be just infinite, such as projective linear groups over the ring of the integers, or the Nottingham group, which is, by the way, also a pro-p group. Later, Wilson himself brought the attention in particular on just infinite profinite groups, extending the previous work he developed about abstract just infinite groups and giving in [22] a simultaneous treatment of both the abstract and the profinite case.

The importance of just infinite groups is due to many factors. Amongst them, one of the most relevant is the similarity between the role of just infinite groups in the context of profinite group theory and the role played by simple groups in finite group theory. For instance, since each finitely generated pro-p group admits a just infinite quotient, every question about pro-p groups can be analyzed, in the first place, as a question about just infinite groups. Despite this crucial role, only few properties of this class of groups are known.

Our aim is to make this thesis, as much as possible, self-contained. For this reason, the first preliminar chapter is entirely dedicated to the resuming of some general facts and definitions which will be useful in the subsequent chapters, as well as to fix notations and conventions that are assumed troughout the entire work.

In the second chapter we will analyze several constructions of just infinite profinite groups which are not pro-p groups. This work is motivated by the fact that for a long time only pro-p just infinite groups were known, leaving unsolved the problem of the existence of just infinite profinite groups which are not pro-p. Only recently some examples of such groups were shown and recognized. The first contribution along these lines is given by Wilson in [23], where the author looks for a hereditarily just infinite group in which all countably based profinite groups can be embedded; furthermore, Reid proves in [14] that an early example exhibited, for other purposes, by Lucchini in [10], is actually a hereditarily just infinite group that contains every countably based profinite groups, thus foreshadowing Wilson's research. Anyhow, the main goal of these works was slightly different than that of our treatment, which has as its objective a more comprehensive exhibition of examples of just infinite profinite groups which are not pro-p groups. Thus, we will first construct a just infinite profinite group which is not pro-p, which is finitely generated and with an infinite prime spectrum. Later, we will look for a group that is not finitely generated, making appropriate changes to the previous construction. Lastly, we will show an example of a just infinite profinite group which is not a pro-p group and which has a finite prime spectrum.

The third chapter is devoted to the analysis of different just infinite profinite structures. In [15] Reid provides a characterization of just infinite profinite groups in terms of the finite groups occurring in the related inverse system, thus a natural question is if similar results are true for other just infinite profinite algebraic structures. In this regard, in this chapter we will focus on just infinite profinite Lie rings. We remark that also in this case the profinite limit arises as the limit of an inverse system of finite objects, in order to preserve the important properties that characterize profinite groups, above all the compactness of the inverse limit.
In the first sections of this chapter we will prove some technical results that are necessary to work towards the subsequent results. Despite of certain similarities, some differences arise with respect to the case of profinite groups, since some properties seem to be no longer verified: for instance, the intersection of the maximal closed ideals of an ideal of a Lie ring could be not an ideal itself. Therefore a complete characterization, similar to Reid's theorem, for just infinite profinite Lie rings can not be provided. We will be able to prove only some partial statements, in particular giving some sufficient conditions to make sure that a profinite Lie ring which is not virtually abelian is just infinite. The stronger assumption of hereditarily just infiniteness allows us to prove the converse relations, making a complete characterization not possible in full generality. Due to these obstacles we will look at a particular subclass of Lie rings, showing a whole characterization for just infinite profinite Lie rings in this family and which are not virtually abelian. Thus, we will show two instances of Lie algebras actually belonging to the class we have defined. The first one is given by Lie FC-algebras, namely Lie algebras in which each element has a centralizer of finite codimension; the choice of this class of Lie algebras is justified by the fact that it can be defined in parallel with the important class of FC-groups, that are groups in which each element has a centralizer of finite index. Later, we will prove that also for just infinite profinite residually solvable Lie algebras a full characterization

in the Reid's fashion is possible.

Lastly, in the fourth and final chapter we will display some results concerning profinite Noetherian groups, that are groups in which every ascending chain of closed subgroups stabilizes in finitely many steps. The basic motivation for focusing the attention on this class of groups is given by the study of p-adic analytic groups, that are defined as pro-p groups which are Lie groups over the field of p-adic numbers or, equivalently, as pro-p groups with finite profinite rank. This second definition gives the chance to formulate several different characterizations of p-adic analytic groups, for example in terms of virtual properties (that is, properties which are satisfied by a finite-index subgroup), in terms of the sections of the group or in terms of subgroup growth. The work of Lubotzky and Mann [8] brings out a question about this class of groups, asking whether a Noetherian pro-p group should be analytic; on the other hand, we can easily convince that a pro-p group with finite rank is Noetherian, then a positive answer to the question posed by Lubotzky and Mann would provide a further characterization of p-adic analytic groups. Despite the Noetherianity constitutes a strong condition, very little is known about the class of profinite Noetherian groups. Thus, in this chapter we will make some comments on the structure of this kind of groups, pointing out some consequences of results which have been established more generally, when they are applied to this class of groups. In particular, we will show that such a group is a finite extension of a direct product of a finite number of copies of a hereditarily just infinite group, as a corollary of the important result proved by Grigorchuk which goes by the name of Wilson's dichotomy. Furthermore, we will prove that a profinite Noetherian group has finite prime spectrum and that a just infinite profinite Noetherian group is virtually pro-p for some prime p.

All the results and all the notions which are already known from previous works are accompanied by the related reference. Every statement for which no reference is given is intended as original or, at least, we are not aware of already known proofs by other authors. Furthermore, for the sake of completeness, we provide a complete proof for some results that might be already known but for which we did not find any mention.

Chapter 1

Basic notions

In this preliminary chapter we make a brief review of some basic concepts and results that will be useful throughout the thesis.

1.1 Group actions

In this section we assume the finiteness of the groups we are dealing with. Here, we quickly revise some definitions and results about finite group actions that will be useful in the following. For a further discussion see for example [6] or [17].

Given a group G and a set X, a *right action* of G on X is a function ρ defined on $X \times G$ with values in X such that for all g_1, $g_2 \in G$ and for all $x \in X$ the following conditions hold:

1. $\rho\left((x, g_1 g_2)\right) = \rho\left(\rho(x, g_1), g_2\right)$;
2. $\rho(x, 1) = x$.

The image of a pair (x, g) under ρ is commonly denoted by $x \cdot g$.
A group action on a set X can be identified with a permutation representation of G on X, namely a homomorphism $G \to \mathrm{Sym}(X)$.
A group action is *faithful* if $x \cdot g = x$ if and only if $g = 1$; it is *transitive* if for each pair of distinct elements x and y in X there exists $g \in G$ such that $x \cdot g = y$.

If the set X on which G acts possesses also a group structure, we also require, beyond the properties defining an action on a set, the following additional condition for all x_1, x_2 in X and for all $g \in G$:

3. $\rho\left((x_1 x_2), g\right) = \rho(x_1, g)\rho(x_2, g)$.

For all $S \subseteq G$ and $Y \subseteq X$ we can define some typical subgroups, such as

$$N_S(Y) = \{g \in S \mid Y \cdot g = Y\}, \quad C_S(Y) = \{g \in S \mid y \cdot g = y \quad \forall y \in Y\}$$
$$C_Y(g) = \{y \in Y \mid y \cdot g = y\}, \quad C_Y(S) = \bigcap_{g \in S} C_Y(g)$$

In particular $C_X(G)$ is the subgroup of fixed points of X under the action of G, while $C_G(X)$ is the kernel of the action of G on X.
Moreover, also the notation for the commutators can be extended when we deal with actions on groups:

$$[x, g] = x^{-1}(x \cdot g), \qquad [Y, S] = \langle [y, g] \mid y \in Y,\ g \in S \rangle$$

The action of a group G on a group X is *coprime* if and only if their orders are coprime. Note that, by Feit-Thompson theorem, which ensures that every odd-order group is solvable, the assumption of coprime orders automatically implies that at least one among the two groups G and X is solvable.

If we have an action of a group G on an abelian group V, we say that V is a *G-module*. Such a module is *irreducible* or *simple* if it does not admit proper non-trivial G-invariant subgroups, also called *G-submodules*. The action is *completely reducible* or *semisimple* if and only if each G-invariant subgroup V admits a G-invariant complement in V. Furthemore, an action of G on V is called *indecomposable* if V can not be decomposed as a direct sum of non-trivial proper G-invariant subgroups. Clearly the property of being irreducible is stronger than the one of being indecomposable, while in general an abelian group on which G acts could have a proper non-trivial G-invariant subgroup which does not admit G-invariant complements, namely such a group could be indecomposable but not irreducible.

Theorem 1.1.1 (Maschke's Theorem)**.** *Let G be a group acting on an elementary abelian p-group V and suppose that this action is coprime. Then, the action of G on V is completely reducible.*

Another very basic result is the Schur's Lemma. We remind the reader that, if V and W are two G-modules, a homomorphism of abelian groups $\varphi : V \longrightarrow W$ is a *G-homomorphism* if $\varphi(v \cdot g) = \varphi(v) \cdot g$, that is, if the G-action and φ commute.

Lemma 1.1.2 (Schur's Lemma)**.** *Let V and W be two simple G-modules. Then every G-homomorphism between V and W is invertible or zero. In particular, the endomorphism ring of a simple module is a division ring.*

The following theorem says that there is no difference between finite division rings and finite fields.

Theorem 1.1.3 (Wedderburn's Theorem)**.** *Every finite division ring is a field.*

1.2 Profinite groups

In this section we quickly recall the main concepts about profinite groups. Proofs for the statements here listed can be found in several books, such as [16] or [21].

Inverse limits

A *directed set* is a partially ordered set $(I, \preceq)$ such that for all $j, k \in I$ there exists $i \in I$ such that $j, k \preceq i$.
An *inverse system* of topological spaces indexed by a directed set I consists of a pair of families of topological spaces $(X_i)_{i \in I}$ and of continuous maps

$$(\varphi_{ij} : X_j \to X_i)_{\substack{i,j \in I \\ i \preceq j}}$$

such that

- $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;
- $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ for all $i \preceq j \preceq k$ in I.

If a different topology is not specified, we usually regard the spaces X_i as endowed with the discrete topology. Moreover, an inverse system is *surjective* if all the maps φ_{ij} are surjective.

Consider now an inverse system of topological spaces as outlined above. An *inverse limit* for such an inverse system is given by:

- a topological space X;
- a family of continuous maps $\varphi_i : X \longrightarrow X_i$ indexed by a directed set I such that $\varphi_{ij}\varphi_j = \varphi_i$ for all $i \preceq j$;

such that the following universal property is satisfied: for each topological space Y and for each family of continuous maps $(\psi_i : Y \longrightarrow X_i)_{i \in I}$ such that $\varphi_{ij}\psi_j = \psi_i$ for all $i \preceq j$, there exists a unique continuous map $\psi : Y \to X$ satisfying $\varphi_i \psi = \psi_i$ for all $i \in I$. Such an inverse limit is denoted by $X = \varprojlim_{i \in I} X_i$.

Given an inverse system of topological spaces, its inverse limit exists and it is unique (up to isomorphism). It is sometimes useful to work with the explicit construction of the inverse limit as a subspace of the Cartesian product of the spaces X_i:

$$X = \{(a_i)_{i \in I} \in \mathrm{Car}_{i \in I} X_i \mid \varphi_{ij}(a_j) = a_i \quad \forall i, j \in I,\ i \preceq j\}$$

where, for all $i \in I$, the map φ_i is the restriction to the space X of the projection map π_i onto the i-th component of the Cartesian product.

The following proposition resumes some well-known topological facts about inverse limits.

Proposition 1.2.1. *Let $(X_i, \varphi_{ij})_{i,j \in I}$ be an inverse system indexed by a directed set I, let $X = \varprojlim_{i \in I} X_i$ be its inverse limit. Then*

1. *if each X_i is Hausdorff, so too is X;*

2. *if each X_i is totally disconnected, so too is X;*

3. *if each X_i is compact and Hausdorff, so too is X;*

4. *if each X_i is a non-empty compact Hausdorff space, then X is non-empty.*

The next result focuses on inverse system of non-empty compact Hausdorff topological spaces.

Proposition 1.2.2. *Let $(X_i, \varphi_{ij})_{i,j\in I}$ be an inverse system of non-empty Hausdorff spaces indexed by a directed set I, let X be its inverse limit. Then*

1. $\varphi_i(X) = \bigcap_{j\succeq i} \varphi_{ij}(X_j)$ *for all* $i \in I$;

2. *the family of sets* $\{\varphi_i^{-1}(U) \mid U \subseteq_o X_i,\ i \in I\}$ *is a basis for the topology on* X;

3. *if* $Y \subseteq X$ *satisfies* $\varphi_i(Y) = X_i$ *for all* $i \in I$, *then* Y *is dense in* X.

If in addition each X_i is compact, then

4. *a map* $\alpha : Y \longrightarrow X$ *is continuous if and only if for each* $i \in I$ *the map* $\varphi_i\alpha$ *is continuous.*

The last result that we recall about inverse limits of topological spaces is the following.

Proposition 1.2.3. *A compact Hausdorff totally disconnected topological space is the inverse limit of its discrete quotient spaces.*

Profinite groups

A *profinite group* is the inverse limit of an inverse system of finite groups. More generally, given a class $\mathcal{C}$ of finite groups (closed with respect to taking homomorphic images), a group is *pro-$\mathcal{C}$* if it is the inverse limit of an inverse system of $\mathcal{C}$-groups. So we can talk about pro-p groups, prosolvable groups, pronilpotent groups and so on.
When we deal with the subgroup structure of profinite groups, we always refer to *closed* subgroups, since we have to ascertain that the subgroups themselves are profinite groups and not only abstract subgroups. In this preliminary chapter we always specify that the subgroups are closed, but in the following, although it is not explicitely indicated, we mean any subgroup of a profinite group as a closed subgroup.

Theorem 1.2.4. *Let G be a topological group. The following conditions are equivalent:*

1. *G is profinite;*

2. *G is isomorphic as a topological group to a closed subgroup of a Cartesian product of finite groups;*

3. *G is compact and the set of open normal subgroups forms a residual system, that is*
$$\bigcap_{N \lhd_o G} N = 1$$

4. *G is compact and totally disconnected.*

Given a profinite group G, we can define its order as follows. First, we define the *index* of a closed subgroup H of G as the least common multiple of the indices of the open subgroups of G containing H. Then, we define the *order* of G as the index of the trivial subgroup in G. Thus, we can consider the prime spectrum of a profinite group G, whose elements are the prime numbers which divide the index of some open normal subgroup.
Also a profinite version for the Lagrange's theorem holds, and moreover there is a Sylow theory for profinite groups. A *p-Sylow subgroup* of a profinite group G is a closed subgroup P such that its order is a (possibly infinite) power of p and its index in G is coprime to p. Thus, p-Sylow subgroups are maximal pro-p subgroups, extending in the more natural way the role that p-Sylow subgroups play in the theory of finite groups.

Proposition 1.2.5. *Let G be a profinite group, let p be a prime. Then, the following conditions hold:*

1. *G has p-Sylow subgroups.*

2. *Every pro-p subgroup is contained in a p-Sylow.*

3. *Any two p-Sylows of G are conjugate.*

The following result generalizes to the class of profinite groups the Frattini argument.

Proposition 1.2.6 (Frattini argument, profinite case)**.** *Let G be a profinite group, let p be a prime in the prime spectrum of G. Let H be a closed normal subgroup of G and let P be a p-Sylow of H.*
Then, the normalizer of P in G is a closed subgroup of G and moreover $G = HN_G(P)$.

As in the finite case, when we deal with a set of primes π, we can refer to pro-π groups. A profinite group is a *pro-π group* if every prime divisor of its order is a prime occurring in the set π. Equivalently, a pro-π group is the inverse limit of an inverse system of finite π-groups.
A *π-Hall subgroup* of a profinite group G is a closed subgroup H such that its order is divisible only by primes in π, while its index is divisible only by

primes that are not in π. Clearly a π-Hall subgroup is a maximal pro-π subgroup, while the converse does not hold, similarly to the finite case. Thus, if $\pi = \{p\}$ is a singleton, a π-Hall subgroup is a p-Sylow subgroup; dually, if $\{p\}'$ denotes the complement, in the set of all primes, of the singleton $\{p\}$, a $\{p\}'$-Hall subgroup is called a *p-complement.*

Let now G be a prosolvable group. A *Sylow basis* $\{S_p \mid p \text{ prime}\}$ for a group G is a collection of p-Sylow subgroups, one for each prime number p, such that its members are permutable, namely $S_pS_q = S_qS_p$ for each pair of primes p and q. If we look at finite groups, each finite solvable group admits a Sylow basis, and moreover any two such bases are conjugate, as proved by Hall. The following theorem generalizes to prosolvable groups this property.

Proposition 1.2.7. *Let G be a prosolvable group. For each prime number p, let $S_{p'}$ be a p'-Hall subgroup of G. Then*

1. *For each prime q,*
$$S_q = \bigcap_{p\neq q} S_{p'}$$
is a q-Sylow subgroup of G and moreover the topological closure of the product $\prod_q S_q$ coincides with the entire G.

2. *The collection of closed subgroups $\{S_q\}$ defined above is a Sylow basis of G.*

3. *Any two Sylow bases $\{S_q\}$ and $\{R_q\}$ of G are conjugate.*

The following result is the profinite version of a theorem proved by Hall for finite solvable groups.

Proposition 1.2.8. *A profinite group is prosolvable if and only if it has p-complements for all primes p.*

The following statement generalizes to π-Hall subgroups the analogue result which holds for p-Sylow subgroups.

Proposition 1.2.9. *Let G be a prosolvable group, let π be a set of prime numbers. Then the following conditions hold.*

1. *G has π-Hall subgroups.*

2. *Every pro-π subgroup is contained in a π-Hall subgroup.*

3. *Any two π-Hall subgroups are conjugate.*

We now focalise on finitely generated profinite groups. We say that a subset X of a group G *generates* G if the abstract subgroup $\langle X \rangle$ is dense in G. Clearly, a profinite group is *finitely generated* if there exists a finite subset such that the closure of the abstract subgroup generated by it coincides with the entire group G.

Proposition 1.2.10. *Let G be a finitely generated profinite group. Then the identity element has a fundamental system of neighbourhoods consisting of a countable chain of open characteristic subgroups.*

1.3 Lie rings

A *Lie ring* is an abelian group $(L, +)$ endowed with a bilinear product, denoted by $[\cdot, \cdot]$, satisfying the following properties:

- $[x, x] = 0$ for all $x \in L$;
- the *Jacobi identity*: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$

The first property is equivalent to anticommutativity, say $[x, y] = -[y, x]$ for all x, $y \in L$.
A *Lie subring* is an abelian subgroup $(K, +) \leqslant (L, +)$ closed under Lie bracket. An *ideal* I of a Lie ring L is a Lie subring such that $[a, x] \in I$ for all $a \in I$ and $x \in L$.
A Lie ring is *abelian* if the Lie product is trivial, namely $[x, y] = 0$ for all x, $y \in L$.
The concept of Lie ring is a generalization of Lie algebra, since every Lie algebra over a generic ring is an example of Lie ring. Morever, any associative ring R can be made into a Lie ring by defining a bracket operator as $[x, y] := xy - yx$ for all $x, y \in R$.

Chapter 2

Just infinite profinite groups

For a long time the only known examples of just infinite profinite groups were pro-p groups, leaving unsolved the problem of the existence of just infinite profinite groups which are not pro-p. In recent years the first examples of just infinite profinite groups which are not pro-p were investigated. The first contribution along these lines is given by Wilson in [23], where the author looks for a hereditarily just infinite group in which all countably based profinite groups can be embedded; furthermore, Reid proves in [14] that an early example exhibited by Lucchini for other purposes in [10] is actually a group of the same type. Anyhow, the aim of the authors who investigated such examples was slightly different than the analysis of just infinite profinite groups which are not pro-p groups by themselves, that is the main goal of this chapter.

Here, in fact, we would like to make a more exhaustive discussion of this kind of groups, by proposing various examples with different properties. To do this, we will use a result proved by Reid in [15], that characterizes just infinite profinite groups through the finite groups occurring in the associated inverse system.

First we will analyze the simplest case, which gives rise to a finitely generated group with infinite prime spectrum. The finiteness of the minimal number of generators is a consequence of the construction, since the finite groups occurring in the inverse system associated to the profinite group turn out to have a stable number of generators. Then, in order to find an infinitely generated just infinite profinite group, we will suitably modify the first construction.

Later, in the subsequent example, we will work under different hypothesis, with the aim of building a just infinite profinite group with finite prime spectrum.

Unless otherwise specified, when we deal with subgroups we always mean *closed* subgroups.

2.1 Just infinite groups

An infinite group G is called *just infinite* if every of its non-trivial normal subgroup has finite index. Equivalently, a just infinite group is a group which has only finite proper images since, certainly, the kernel of a group homomorphism is a normal subgroup of the group. An important subclass of the family of just infinite groups is given by the *hereditarily just infinite groups*, that are groups in which every open subgroup is just infinite.
Clearly every simple group is just infinite. The easiest non-trivial examples of just infinite groups are given by the group of integers $\mathbb{Z}$ and the infinite dihedral group D_∞. A more interesting early example was given by Mennicke in [12], where the author shows that, for $n \geqslant 3$, the projective special linear group $\mathrm{PSL}(n, \mathbb{Z})$ is a just infinite group. Many other arithmetic groups, such as the symplectic group $\mathrm{Sp}_{2n}(\mathbb{Z})$ for $n \geqslant 2$, are just infinite groups; furthermore, this property is also satisfied if we replace $\mathbb{Z}$ by the ring of integers of an algebraic number field.
Also the Nottingham group over $\mathbb{F}_p$ is a just infinite group; recall that the *Nottingham group* over the finite field $\mathbb{F}_p$ is the group of formal series $t + t^2\,\mathbb{F}_p[[t]]$; by the way, it is a pro-p group which, besides, contains a copy of each finitely generated pro-p group, and this fact makes it a pro-p group with infinite (profinite) rank[1].
McLain gives in [11] an example of an infinite, locally finite and locally solvable just infinite group, where the only non-trivial normal subgroups are the terms of its derived series. Lastly, in [4], is discussed an infinite just infinite group all of whose proper quotients are p-groups.

From now on we will work with just infinite profinite groups; for these groups we obviously require that all non-trivial closed normal subgroups have finite index in the entire group or, equivalently, that every non-trivial closed normal subgroup is open.

In his work published in 2011, Reid focuses both on just infinite and on hereditarily just infinite profinite groups, giving characterizations of such groups throughout properties regarding the finite groups occurring in the associated inverse system. Here we state the theorem about the just infinite case, that is the result on which we rely to construct our examples.

Theorem 2.1.1 (Reid, [15])**.** *Let G be a just infinite profinite group. Let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence of classes of finite groups such that G has infinitely many chief factors in each $\mathcal{C}_n$.*
Then G is the limit of an inverse system of finite groups $(G_n)_{n\in\mathbb{N}}$ and surjective group morphisms $(\rho_n : G_{n+1} \to G_n)_{n\in\mathbb{N}}$ where each G_n has a normal subgroup A_n such that, letting $P_n = \rho_n(A_{n+1})$, the following properties are satisfied for all $n \in \mathbb{N}^+$:

1. $A_n > P_n > 1$;

[1] The *rank* of a profinite group G is defined as $\mathrm{rk}(G) = \sup\{d(H) \mid H \leqslant_c G\}$

2. *there exists a unique maximal G_n-invariant subgroup in A_n;*

3. *each normal subgroup of G_n either contains P_n or is contained in A_n;*

4. *P_n is a minimal normal subgroup of G_n;*

5. *$P_n \in \mathcal{C}_n$.*

Conversely, every surjective inverse system satisfying, for some choice of A_n and for all but finitely many n, the first three conditions above, has a just infinite inverse limit.

In particular we will use the second part of the statement, looking for a suitable sequence of finite groups having a subgroup structure with the listed properties.

2.2 Just infinite profinite groups with infinite prime spectrum

In this section we propose two examples of just infinite profinite groups which are not pro-p groups and which have infinite prime spectrum. The first example gives rise to a finitely generated group, while in the second case we look for a non-finitely generated group.
In both cases, the main step of the construction appears in a preliminary lemma that allows us to build a new finite solvable monolithic group, with some specified properties, starting from a solvable monolithic transitive permutation group. The iteration of the construction exhibited in the proof of this lemma will lead us to the desired example of just infinite profinite group.
The two constructions are based on the product action of the permutational wreath product, that we briefly recall here; for further details, see for example section 2.1 in [1]. Let H be a permutation group on a set with n elements, namely $H \leqslant S_n$, let G be a group acting by permutations on a set X. The permutational wreath product $G \wr H = G^n \rtimes H$ acts on X^n as follows: for every $(x_1, \dots, x_n) \in X^n$, $(g_1, \dots, g_n) \in G^n$ and $\sigma \in H$ we have

$$(x_1, \dots, x_n) \cdot ((g_1, \dots, g_n), \sigma) = \left(x_{\sigma^{-1}(1)} \cdot g_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)} \cdot g_{\sigma^{-1}(n)}\right)$$

where, recall, the actions are meant as right actions.

First of all, we prove a technical lemma that will be useful in what follows. We remind the reader that a *monolithic group* is a group which admits a unique minimal non-trivial normal subgroup, called the *monolith* of the group.

Lemma 2.2.1. *Let G be a group acting on an irreducibile faithful module U. Then $U \rtimes G$ is a monolithic group with monolith U.*

Proof. Let N be a minimal normal subgroup in $U \rtimes G$; we will prove that $N = U$. If N does not contain U, then $N \cap U$ is properly contained in U and, by irreducibility of U, we have that $N \cap U$ is trivial. Now,

$$[N, U] \leqslant N \cap U = 1$$

therefore N is contained in the centralizer of U in $U \rtimes G$, which equals U itself by the faithfulness of the action of G on U; but, as $N \cap U = 1$, this leads to $N = 1$, which is a contradiction.
Thus, N must contain U, and so $N = U$ by minimality. □

Finitely generated case

As we have previously explained, the first example shows a finitely generated just infinite profinite group with infinite prime spectrum.
The following lemma constitutes the general step on which the construction of the just infinite profinite group we are looking for is based.

Lemma 2.2.2. *Let p and q be two distinct prime numbers such that $q|p-1$, let G be a finite solvable monolithic group whose order is coprime to p and q. Let moreover A and P be two non-trivial normal subgroups satisfying the following properties:*

1. *there exists a unique maximal G-invariant subgroup of A;*
2. *$P < A$;*
3. *each normal subgroup of G is either contained in A or it contains P.*

Then there exists a finite solvable monolithic group $\widetilde{G}$ which has two non-trivial normal subgroups $\widetilde{A}$ and $\widetilde{P}$ satisfying the same properties (1)-(3); furthermore there exists a surjective morphism from $\widetilde{G}$ onto G which maps $\widetilde{A}$ onto P.
Lastly, the minimal number of generators of $\widetilde{G}$ equals that of G.

Proof. By Cayley's theorem, we can pick a natural number m such that G embeds in S_m as a transitive permutation subgroup. Thus, G acts faithfully on C_q^m by permuting coordinates: if $\sigma \in G$ and $(a_1, \ldots, a_m) \in C_q^m$, we have

$$(a_1, \ldots, a_m) \cdot \sigma = \left(a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(m)}\right)$$

This action is completely reducible by Maschke's theorem, under our assumptions on $|G|$ and q. So, by monolithicity of G, we can find an irreducible faithful G-submodule $U \leqslant_G C_q^m$: if not, the kernel of the action of G on each

irreducible component would contain the monolith, so also the kernel of the entire representation would contain it, contradicting the faithfulness of the G-action on the entire C_q^m.
Furthermore, since by hypothesis $q|p-1$, the cyclic group C_q embeds in $\mathrm{Aut}(C_p)$, so C_q acts faithfully on C_p; besides, there is also a natural faithful action of G on C_p^m, by permuting coordinates, in the same way as described before.
Throughout these two actions we build the product action of the permutational wreath product $C_q \wr G = C_q^m \rtimes G$ on C_p^m, as explained previously: for all $(b_1, \dots, b_m) \in C_p^m$, $(a_1, \dots, a_m) \in C_q^m$ and $\sigma \in G$ we have

$$(b_1, \dots, b_m) \cdot ((a_1, \dots, a_m), \sigma) = \left(b_{\sigma^{-1}(1)} \cdot a_{\sigma^{-1}(1)}, \dots, b_{\sigma^{-1}(m)} \cdot a_{\sigma^{-1}(m)}\right)$$

Consider now the subgroup $U \rtimes G \leqslant C_q \wr G$, which acts in a completely reducible way and faithfully on C_p^m. UG is monolithic by Lemma 2.2.1, so there exists a UG-submodule $W \leqslant_{UG} C_p^m$ which is irreducible and faithful, by the argument already used for the submodule U.

We are now ready to build the desired group. Consider the semidirect product $\widetilde{G} := W \rtimes (U \rtimes G)$ and its normal subgroup $\widetilde{A} := WUP$.
The group $\widetilde{G}$ is monolithic with monolith W, once again thanks to Lemma 2.2.1. This unique minimal normal subgroup is exactly the subgroup $\widetilde{P}$ we were looking for.
We claim now that the unique maximal $\widetilde{G}$-invariant subgroup which is properly contained in $\widetilde{A}$ is WU. For, if there exists another $M \lhd \widetilde{G}$ which is a maximal $\widetilde{G}$-invariant subgroup of $\widetilde{A}$, then $M \cap WU$ is a proper subgroup of $\widetilde{A}$ which is normal in $\widetilde{G}$; thus, $M \cap WU$ must contain W, according to the fact that $\widetilde{G}$ is monolithic. Since $WU/W \cong U$ is irreducible, we can have only two distinct cases: either $M \cap WU = WU$, which implies $M = WU$ by maximality; or $M \cap WU = W$, which implies $M \cap U = 1$ or $M \cap U = W$ by the Dedekind modular law (remembering that $W \leqslant M$); this second case actually can not hold, as $U \cap W = 1$, hence we certainly have $M \cap U = 1$.
Thus, as WU is normal in $\widetilde{G}$, we have $[M, U] \leqslant M \cap UW = W$, hence MW/W centralizes WU/W and, as G acts faithfully on U, we deduce that $MW \leqslant WU$, and thus $M \leqslant WU$.
Lastly, also the third requirement is satisfied, since the surjective map $\pi : \widetilde{G} \twoheadrightarrow G$ has kernel UW and so maps $\widetilde{A}$ onto P.

It remains to prove the statement about the minimal number of generators of the new group $\widetilde{G}$. In [9] the authors proved, using the classification of finite simple groups, that for every non-cyclic finite monolithic group H, the number of generators of H is the maximum among 2 and $d(H/N)$, where N is the unique minimal normal subgroup of H. In our case we see that, as G is non-cyclic,

$$d\left(\widetilde{G}\right) = \max\left\{2, d\left({}^{\widetilde{G}}\!/_{W}\right)\right\} = d\left({}^{\widetilde{G}}\!/_{W}\right) = d(UG)$$

and, in turn, $d(UG) = d(G)$, as also UG is a monolithic group with monolith U. □

Iterating the construction provided by the previous lemma we are able to prove the following theorem.

Theorem 2.2.3. *There exists a finitely generated just infinite profinite group which is not a pro-p group.*

Proof. Let X be a solvable monolithic group which is a transitive permutation group on a set with m_1 elements. Consider two prime numbers p_1 and q_1 such that $q_1 | p_1 - 1$ and $(p_1, |X|) = (q_1, |X|) = 1$, as in the hypothesis of Lemma 2.2.2.
As we have seen in Lemma 2.2.2, we have a permutational action of X on $C_{q_1}^{m_1}$, so we can find a faithful irreducible X-submodule $U_1 \leqslant_X C_{q_1}^{m_1}$. Furthermore, there is also an action of the wreath product $C_{q_1} \wr X = C_{q_1}^{m_1} \rtimes X$ on $C_{p_1}^{m_1}$, with respect to which we can find a faithful irreducible U_1X-submodule $W_1 \leqslant_{U_1X} C_{p_1}^{m_1}$. We then define the semidirect product $G_1 := W_1 \rtimes (U_1 \rtimes X)$ and we consider its subgroup $A_1 := W_1U_1$.
The group G_1 is monolithic, with monolith W_1, by Lemma 2.2.1, so any normal subgroup N of G_1 must contain W_1. Moreover, if N is also properly contained in A_1, we have $W_1 \leqslant N < W_1U_1$ which implies $N = W_1$ by irreducibility of $(U_1W_1)/W_1$. So $N = W_1$ is the unique maximal G_1-invariant subgroup of A_1.

Once the first step is done, we can use straightly Lemma 2.2.2. Note that we can consider two sequences of (different) prime numbers $(p_k)_{k\in\mathbb{N}^+}$ and $(q_k)_{k\in\mathbb{N}^+}$ such that for every positive natural numbers i and j we have $p_i \neq q_j$, $\mathrm{GCD}\,(p_{i+1}, |X|) = \mathrm{GCD}\,(q_{i+1}, |X|) = 1$ and $q_i | p_i - 1$. For, suppose that we have already built the first n finite groups $G_1, \ldots, G_n$ and that we have already chosen $p_1, \ldots, p_n, q_1, \ldots, q_n$ with the desired properties. Then, pick q_{n+1} different from all the p_i and the q_j and such that it is coprime to $|X|$ for all $i \leqslant n$. By Dirichlet's theorem on arithmetic progressions[2], we can find a prime number p_{n+1} which is congruent to 1 modulo q_{n+1} and which is coprime to $|G_n|$.
Applying now Lemma 2.2.2 we can build the group G_{n+1} which has two subgroups A_{n+1} and P_{n+1} satisfying the desired properties and with a suitable surjective morphism π_n onto G_n.
So, the inverse system of groups $(G_i, \pi_i)_{i\in\mathbb{N}}$ gives rise to a profinite group $G = \varprojlim_{i\in\mathbb{N}} G_i$ that, by Reid's theorem, comes out to be just infinite. Clearly, by construction such a group is not a pro-p group.

Now we look at the minimal numbers of generators of the groups occurring in the inverse system associated to the profinite group we have built.

[2]Dirichlet's theorem on arithmetic progressions claims that, for every two positive coprime integers a and b, there are infinitely many $n \in \mathbb{N}^+$ such that $a + nb$ is a prime number.

Thanks to the last claim stated in Lemma 2.2.2, in this construction the minimal number of generators of the components of the inverse system does not increase, thus

$$d\left(\varprojlim_{i\in\mathbb{N}} G_i\right) = d\left(G_i\right) \quad \forall i \in \mathbb{N}$$

and the profinite group comes out to be finitely generated. □

In what follows, we will try to modify the last example, in order to obtain an inverse system in which the minimal number of generators of each term increases, and that, therefore, will give rise to a non-finitely generated profinite group.

Infinitely generated case

The goal of this section is the construction of a new inverse system of finite groups, giving rise to a non-finitely generated just infinite profinite group. To do this, we need that the number of generators of the terms in the inverse system increases in a finite number of steps.
As in the previous case, we first prove a lemma that will be the cornerstone for our main result.

Lemma 2.2.4. *Let p and q be two distinct prime numbers such that $q|p-1$, let G be a finite solvable group whose order is coprime to p and q and such that its abelianization has order not coprime to q. Let moreover A and P be two non-trivial normal subgroups satisfying the following properties:*

1. *there exists a unique maximal normal subgroup of G contained in A;*
2. *$P < A$;*
3. *each normal subgroup of G is either contained in A or it contains P.*

Then there exists a finite monolithic solvable group $\widetilde{G}$ which has two non-trivial normal subgroups $\widetilde{A}$ and $\widetilde{P}$ satisfying the same properties (1)-(3) and such that $d(\widetilde{G}) > d(G)$; moreover there exists a surjective morphism from $\widetilde{G}$ onto G which maps $\widetilde{A}$ onto P.

Proof. Let G be as in the hypothesis. Consider the group $W \rtimes (U \rtimes G)$ as built in the proof of Lemma 2.2.2: U is an irreducible faithful G-submodule of C_q^m, and W is an irreducible faithful UG-submodule of C_p^m.
The ring of the UG-invariant endomorphisms of W is a division ring by Schur's lemma and so, by finiteness, it is a field, by Wedderburn's theorem; moreover, the finite field $\mathbb{F}_p$ is contained in $\mathrm{End}_{UG}(W)$.
Pick $x \in \mathrm{End}_{UG}(W)$ such that its order is not coprime to the order of UG; such an element exists: for instance, we can pick x as an element of order q in the multiplicative group of the field $\mathbb{F}_p$.

Of course W is still irreducible for the action of $UG \times \langle x \rangle$, since it was already irreducible as a UG-module; furthermore, W is also still faithful for the action of $UG \times \langle x \rangle$, since the new element x can not act as an element of UG, by its choice in the multiplicative group of the field $\mathbb{F}_p$.
So we can consider the new semidirect product $G_1 = W \rtimes (UG \times \langle x \rangle)$. Defining the subgroups $\widetilde{P} = W$ and $\widetilde{A} = WUP$, we can prove similarly to the previous case that they satisfy the required properties (1)-(3).
First of all, G_1 is monolithic, with monolith W, by Lemma 2.2.1.
Then, we have to prove that WU is the unique maximal subgroup of $\widetilde{A}$ that is G_1-invariant. For, let M be another non-trivial normal subgroup which is a maximal subgroup of $\widetilde{A}$; then $M \cap WU$ must contain W and, by irreducibility of WU/W, we conclude that $M = WU$ or $M \cap WU = W$, which implies $M \cap U = 1$. Thus, as WU is normal in $\widetilde{G}$, we have $[M, U] \leqslant M \cap UW = W$, hence MW/W centralizes WU/W and, as G acts faithfully on U, we have $MW \leqslant WU$, and thus $M \leqslant WU$.

Now, we look at the minimal number of generators for G_1. First of all, given a group product NH, where N is normal, through the usual commutator relations we can easily convince that its derived subgroup is

$$N'H'[N, H]$$

Thus, in our case we have

$$\begin{aligned} G_1' &= W' \,(UG \times \langle x \rangle)' \,[W, UG \times \langle x \rangle] = \\ &= (UG)' \,[W, UG \times \langle x \rangle] = \\ &= U'G'[U, G] \,[W, UG \times \langle x \rangle] = \\ &= G'[U, G] \,[W, UG \times \langle x \rangle] = \\ &= G'UW \end{aligned}$$

since $U' = W' = \langle x \rangle' = 1$ (being abelian), $[UG, \langle x \rangle] = 1$ as both UG and $\langle x \rangle$ are factors of a direct product, and since the actions on U and W are irreducible.
Then, we note that the minimal number of generators of the abelianization of G_1 is strictly greater than $d(G^{\mathrm{ab}})$, due to the choice of the element x. Now, if also $d(G_1) > d(G)$ holds, then we have found a group with the desired property and we redefine $\widetilde{G} = G_1$. Otherwise, we can repeat the argument, finding a sequence of groups $(G_n)_{n \in \mathbb{N}^+}$ such that the properties (1)-(3) are satisfied and such that, for each n, the minimal number of generators of the abelianization of G_{n+1} is strictly greater than the corresponding number for G_n and $d(G_{n+1}) \geqslant d(G_n)$; since obviously $d(G_n)$ is not smaller than $d\left(G_n^{\mathrm{ab}}\right)$, there surely exists $k \in \mathbb{N}^+$ such that $d(G_{n+k}) > d(G_n)$; choosing k minimal with this property, we can define $\widetilde{G} = G_{n+k}$. □

Iterating the construction provided by the previous lemma, we are able to state the main theorem of this section, which proves the existence of a just infinite profinite group that is not a pro-p group and that is not finitely generated.

Theorem 2.2.5. *There exists a just infinite profinite group which is not a pro-p group and which is not finitely generated.*

Proof. The first step is completely analogous to the construction made in the previous case: given a transitive solvable monolithic permutation group $X \leqslant S_{m_1}$ and given two prime numbers p_1 and q_1 such that $q_1|p_1 - 1$ and $(p_1, |X|) = (q_1, |X|) = 1$, first we pick an irreducible faithful X-submodule $U_1 \leqslant_X C_{q_1}^{m_1}$ and then an irreducible faithful U_1X-submodule $W_1 \leqslant_{U_1X} C_{p_1}^{m_1}$; therefore we define $G_1 = W_1 \rtimes (U_1 \rtimes X)$.
Applying Lemma 2.2.4 to G_1 we can build a group G_2 that satisfies the conditions of the Reid's theorem and such that $d(G_2) > d(G_1)$.
Suppose now that we have already built a group G_i, for some $i > 1$, such that $d(G_i) \geqslant d(G_1) + i$ and, of course, which satisfies the hypothesis of Lemma 2.2.4. Now, using the same lemma, we can construct a new group G_{i+1} which has one more generator than G_i and also satisfying the conditions given by Reid.
Applying Reid's theorem to the sequence of groups which we have found, we conclude that the profinite group $G = \varprojlim_{i\in\mathbb{N}} G_i$ is just infinite. Furthermore, since $d(G) \geqslant d(G_i)$ for every $i \in \mathbb{N}^+$ and as the sequence $\{d(G_i)\}_{i\in\mathbb{N}}$ is unbounded, the just infinite group G is not finitely-generated. □

2.3 Just infinite profinite groups with finite prime spectrum

The previous examples were made through two sequences of suitable prime numbers, chosen in order to deal with coprime actions only, allowing us to make use of Maschke's theorem.
In what follows, we will work with finitely many primes; certainly we will have a price to pay, due to the lack of coprimality of the actions we work with.
As usual, the main step for the construction is enclosed in a lemma, which, with respect to the prime spectrum of G, is stated "dually" to the previous ones: in fact, here the primes p and q are prime divisors of the order of the finite group G.
First of all, we prove an auxiliary lemma that will be useful in the following.

Lemma 2.3.1. *Let q be a prime number, let G be a finite monolithic group, whose monolith is not a q-group.*
Then, G admits a faitfhul simple module which is a q-group.

Proof. Consider the regular representation V of G over the ring of integers modulo q. The action of G on V may be not coprime, since we did not assume that the order of G must be coprime to q, so such an action could be not completely reducible. Therefore, we consider a composition series for V (with respect to the G-module structure), say

$$1 = V_0 \lhd V_1 \lhd \ldots \lhd V_n = V$$

If the monolith acts trivially on each factor of this composition series, then the monolith would have a unipotent action, and it would be a q-group, contradicting the hypothesis. Hence, there is a simple G-module which is a q-group on which G acts faithfully. □

We are now ready for the main lemma. The idea is similar to the previous examples. In this case, the construction of the iterated wreath product is based on the previous lemma, which is applied to a monolithic group whose monolith is alternatively not a q-group or not a p-group.

Lemma 2.3.2. *Let q be a prime number, let G be a finite solvable monolithic group whose monolith is not a q-group. Let moreover A and P be two non-trivial normal subgroups satisfying the following properties:*

1. *there exists a unique maximal G-invariant subgroup of A;*
2. *$P < A$;*
3. *each normal subgroup of G is either contained in A or it contains P.*

Then there exists a finite solvable monolithic group $\widetilde{G}$ which has two non-trivial normal subgroups $\widetilde{A}$ and $\widetilde{P}$ satisfying the same properties (1)-(3) and whose monolith is not a q-group; furthermore there exists a surjective map from $\widetilde{G}$ onto G which maps $\widetilde{A}$ onto P.
In addition, the prime spectrum of $\widetilde{G}$ equals that of G.

Proof. Let m be a natural number such that G embeds in S_m as a transitive permutation subgroup.
By Lemma 2.3.1, there exist a simple module, say S, which is a q-group and on which the action of G is faithful.

Now, note that also $S \rtimes G$ is a monolithic group, with monolith S, by Lemma 2.2.1. As before, the action of SG on C_p^m could be not coprime, but, again by Lemma 2.3.1, as the monolith S is not a p-group, we can find an irreducible module, say T, which is a p-group and on which the SG-action is faithful.
We then consider the semidirect product $\widetilde{G} := T \rtimes (S \rtimes G)$ and its normal subgroup $\widetilde{A} := TSP$. The checking of the required properties goes ahead similarly to what we have done in Lemma 2.2.2.
First of all, $\widetilde{G}$ is certainly monolithic, with monolith T, by Lemma 2.2.1. Rename the unique minimal normal subgroup as $\widetilde{P} := T$.
Moreover, we have to prove that TS is the unique maximal $\widetilde{G}$-invariant

subgroup of $\widetilde{A}$: if M is another $\widetilde{G}$-invariant subgroup of $\widetilde{A}$, then $M \cap TS$ is normal in G and it must contain T, by monolithicity. As $TS/T \cong S$ is irreducible, we can have either $M \cap TS = TS$ or $M \cap TS = T$. In the first case we have $M = TS$ by maximality; while in the second case we have either $M \cap S = 1$ or $M \cap S = T$ by the Dedekind modular law; this second case actually can not hold, as $T \cap S = 1$.
Thus, as TS is normal in $\widetilde{G}$, we have $[M, S] \leqslant M \cap TS = T$, hence MT/T centralizes TS/T and we have $MT \leqslant TS$, hence $M \leqslant TS$.
Lastly, the projection morphism $\pi : \widetilde{G} \to G$ has kernel TS and maps $\widetilde{A}$ onto P, as required. □

We are now able to prove the existence of the group we are looking for.

Theorem 2.3.3. *There exists a just infinite profinite group that is not a pro-p group and which has finite prime spectrum.*

Proof. Let p and q be two distinct prime numbers, let X be a monolithic group such that p and q are in the prime spectrum of X and such that its monolith is not a q-group. Let moreover m be a natural number such that X embeds in S_m as a transitive permutation group.
The group X acts faithfully on C_q^m and, even if the action is not coprime, by Lemma 2.3.1 we can find an irreducible faithful submodule $U_1 \leqslant_X C_q^m$.
Now, the group $U_1 \rtimes X$ is monolithic, by Lemma 2.2.1, with monolith U_1, which is a q-group. Thus, again by Lemma 2.3.1, we can find an irreducible faithful submodule $W_1 \leqslant_{U_1 X} C_p^m$.
We then define the group $G_1 := W_1 \rtimes (U_1 \rtimes X)$ and consider its subgroup $A_1 := W_1 U_1$. Once again thanks to Lemma 2.2.1, the group G_1 is monolithic, with monolith W_1. We then put $P_1 := W_1$.
Once the first step is done, we can make the inductive step via Lemma 2.3.2. Suppose that we have already built a group G_i which has two subgroups A_i and P_i and embed it transitively in S_{m_i} for a suitable $m_i \in \mathbb{N}^+$. The prime numbers p and q are both divisors of the order of G_i, and moreover G_i is a monolithic group whose monolith is not a q-group. Applying Lemma 2.3.2 to G_i we can find a new group G_{i+1} with two subgroups A_{i+1} and P_{i+1} satisfying the desired properties.

So, by Reid's theorem, the inverse limit $G = \varprojlim_{i \in \mathbb{N}^+} G_i$ is a just infinite profinite group whose prime spectrum is finite and contains $\{p, q\}$. □

Chapter 3

Profinite structures

The aforementioned theorem proved by Reid in [15] characterizes just infinite profinite groups in terms of the finite groups occurring in the associated inverse system. This result is the endpoint of a discussion about just infinite profinite groups, during which other results about this kind of groups are established. It is quite natural to wonder if analogous results hold for other algebraic structures, as long as a suitable definition of the just infinite property for these structures is provided. Thus, in this chapter we will discuss this topic, in particular looking at Lie rings.
Unfortunately, we will not be able to prove a complete characterization for just infinite profinite Lie rings, since it seems that some properties which hold in the case of groups are no longer true for profinite Lie rings. In particular, the intersection of all the maximal ideals of an ideal of a Lie ring could be not, in general, an ideal itself (although we are not able to provide an explicit counterexample). By contrary, in the case of groups, the intersection of all the maximal normal subgroups of a normal subgroup is normal itself. Thus, in general, we will only be able to provide partial results.
The lack of a satisfactory comprehensive result for Lie rings brought us to examine more specific structures, whose properties could help us to achieve more interesting conclusions. For this reason, we will focus on an abstract subclass of profinite Lie rings, for which we are able to provide a full characterization.
Therefore, we will show that two specific kinds of Lie algebras belong to the class we have defined. The first family of such Lie algebras is given by profinite Lie FC-algebras, that are Lie algebras whose elements have centralizers of finite codimension. The definition of this class of Lie algebras is made in complete analogy to the concept of FC-groups, which, in fact, are infinite groups whose elements have finite-index centralizers. This condition is one of the possible finiteness conditions on an infinite group, so such groups are being studied by several authors. Thus, the analysis of an analogue class of Lie algebras could lead to some interesting results.

Subsequently, we will establish that also residually solvable profinite Lie algebras belong to the class we have defined.

3.1 Profinite Lie rings

In this section we make a brief overview on profinite Lie rings. Many of the statements here presented are similar to the corresponding results for profinite groups.
A *profinite Lie ring* R is the inverse limit of a surjective inverse system (indexed over a directed set $(\Omega, \preceq)$) consisting of a pair of families of discrete finite Lie rings $(R_i)_{i\in\Omega}$ and of continuous maps

$$(\varphi_{i,j} : R_j \longrightarrow R_i)_{\substack{i,j\in\Omega \\ i\preceq j}}$$

We remark that also in this case the profinite limit arises as the limit of an inverse system of finite objects, in order to preserve some important properties that characterize profinite groups, above all the compactness of the inverse limit.
As usual, we endow such an inverse limit with the subspace topology inherited by the product topology on $\mathrm{Car}_{i\in I} R_i$.
As a topological space, a profinite Lie ring is compact, Hausdorff and totally disconnected, since it is the inverse limit of an inverse system of Hausdorff totally disconnected finite topological spaces (see Proposition 1.2.1).

Lemma 3.1.1. *Let $R = \varprojlim_{i\in\Omega} R_i$ be an inverse limit of an inverse system of compact Hausdorff topological Lie rings, let I be an open ideal of R. Then, R/I is isomorphic to a quotient of some R_i.*

Proof. For every $i \in \Omega$ let $\varphi_i : R \to R_i$ be the projection map from R onto the i-th component of the inverse system. Remind that the family of sets

$$\left\{\varphi_i^{-1}(U) \mid i \in \Omega,\ U \subseteq_o R_i\right\}$$

is a basis for the topology on R, by Proposition 1.2.2. Thus, since I is an open ideal, there exists $j \in \Omega$ and an open neighbourhood V of 0 in R_j such that the open set $\varphi_j^{-1}(V)$ is contained in I; thus, $\ker(\varphi_j) \leqslant I$. Therefore

$$R/I \cong (R/\ker\varphi_j)/(I/\ker\varphi_j) \cong R_j/(I/\ker\varphi_j)$$

as desired. □

Lemma 3.1.2. *Let R be a profinite Lie ring, let I be a closed ideal. Then I is open if and only if its index in R is finite.*

Proof. The open covering $R = \bigcup_{i\in I}(x_i + I)$ admits, by compactness, a finite subcovering, then the index of I is finite.
Conversely, if the index of a closed ideal is finite, then its complementary is a finite union of closed subsets, then I is open. □

Let $\mathcal{I}$ be a family of open ideals of a topological ring R such that for any pair of ideals I_1 and I_2 in $\mathcal{I}$ also $I_1 \cap I_2 \in \mathcal{I}$. Make it a partially ordered set definining the following order

$$I_1 \preceq I_2 \iff I_2 \leqslant I_1$$

$\mathcal{I}$ is a directed set with respect to this order, since $I_1, I_2 \preceq I_1 \cap I_2$ for all I_1 and I_2 in $\mathcal{I}$.
The family of finite quotient rings with quotient maps

$$\left\{ R/I,\ \pi_{I,J} : R/J \longrightarrow R/I \mid I, J \in \mathcal{I},\ I \preceq J \right\} \tag{3.1}$$

is an inverse system.

Lemma 3.1.3. *Let R be a compact topological Lie ring. Let $\widehat{R}$ be the inverse limit of the inverse system (3.1), with maps $\varphi_I : \widehat{R} \longrightarrow R/I$ for all $I \in \mathcal{I}$. If $\bigcap_{I\in\mathcal{I}} I = 0$, then R and $\widehat{R}$ are isomorphic as topological Lie rings.*

Proof. Consider the map

$$\psi : R \longrightarrow \widehat{R} \leqslant \underset{I\in\mathcal{I}}{\mathrm{Car}}\ R/I$$

defined by $a \mapsto (a + I)_{I\in\mathcal{I}}$. It is a continuous homomorphism of rings with kernel given by $\bigcap_{I\in\mathcal{I}} I$, that is trivial by hypothesis.
Since $\varphi_I(\psi(R)) = R/I$ for all $I \in \mathcal{I}$, by Proposition 1.2.2 (3) the image of R under ψ is dense in $\widehat{R}$ and, by compactness, we have the thesis. □

Theorem 3.1.4. *Let R be a topological Lie ring. The following conditions are equivalent:*

1. *R is a profinite ring.*
2. *R is compact and the intersection of all open ideals of R is trivial.*
3. *R is compact and totally disconnected.*

Proof. [(2) ⇒ (1)] By Lemma 3.1.3 we immediately have that R is isomorphic as topological Lie ring to the inverse limit $\widehat{R} = \varprojlim_{I\in\mathcal{I}} R/I$, where $\mathcal{I}$ is the family of all open ideals, thus R is profinite.
[(1) ⇒ (3)] By hypothesis R is the inverse limit of an inverse system of discrete finite Lie rings, which are obviously compact, totally disconnected and Hausdorff, hence R is compact and totally disconnected by Proposition 1.2.1.

[(3) $\Rightarrow$ (2)] Let $y \in R$ be a non-zero element. Then, there exists an open subset U of R containing y such that $0 \notin U$. Since every open set of R is a union of cosets of open ideals, there exists an open ideal I such that $y + I \subseteq U$, hence $0 \notin y + I$ and $y \notin I$. Thus, the intersection of all open ideals must be trivial. □

Lemma 3.1.5. *Let R be a profinite Lie ring, let K be an open ideal of R. Let $(S_i)_{i\in\Omega}$ be a descending chain of closed subrings such that each one is not contained in K. Let S be the intersection of these subrings. Then also S is not contained in K.*

Proof. Consider the closed subset $C_i = S_i \cap K^c$ (where K^c is the complementary of K as a subset in R, namely $K^c = R \setminus K$). Since the intersection of finitely many C_i is not empty (as the chain $(C_i)_{i\in\Omega}$ is descending), then also the intersection of all the C_i is not empty, by the finite intersection property. □

3.2 Subcartesian products

This section is an interlude in which we prove a technical result, that we will need in the sequel, about subcartesian products.
Remind that a set X is a *subcartesian product* of a family of sets $(X_i)_{i\in I}$ if $X \subseteq \mathrm{Car}_{i\in I} X_i$ and if, for each $i \in I$, the projection map on the i-th component $\pi_i : X \to X_i$ is surjective. This condition does not ensure on its own that X is itself a Cartesian product (and this is not even true for direct products, think, for instance, about diagonal maps), but we can investigate additional conditions to make this true. We state the following lemma in the case of finite Lie rings, but similar results can be proved for other algebraic structures by referring to their simple objects.

Lemma 3.2.1. *Let S be a closed subcartesian product of pairwise non-isomorphic finite simple Lie rings $\mathrm{Car}_{i\in I} S_i$. Then S coincides with the entire Cartesian product.*

Proof. For each finite subset $J = \{j_1, \ldots, j_n\}$ contained in I we denote by $\pi_J : S \to \prod_{h=1}^{n} S_{j_h}$ the map defined by

$$s \mapsto (\pi_{j_1}(s), \ldots, \pi_{j_n}(s))$$

First, we prove the surjectivity of each π_J. The composition map

$$\pi_{j_k} \circ \pi_J : S \longrightarrow \prod_{h=1}^{n} S_{j_h} \longrightarrow S_{j_k}$$

is surjective for each $k = 1, \ldots, n$, by definition of subcartesian product, so for all k there exists a composition factor of $\pi_J(S)$ isomorphic to S_{j_k}; in other words, each S_{j_k} occurs, up to isomorphism, among the composition factors of $\pi_J(S)$. Hence, all the simple Lie rings $S_{j_1}, \ldots, S_{j_n}$ must occur in the composition series of $\pi_J(S)$, since they are pairwise non-isomorphic by hypothesis. Therefore, as the cardinality of $\pi_J(S)$ equals to the product of the cardinalities of its composition factors, we have

$$|\pi_J(S)| \geqslant \prod_{h=1}^{n} |S_{j_h}|$$

But $\pi_J(S)$ is also the image of S in $\prod_{h=1}^{n} S_{j_h}$, thus also the opposite inequality holds, so $\pi_J(S) = \prod_{h=1}^{n} S_{j_h}$ and each map π_J, with J finite, is surjective.

We now focus on the general case. Suppose that there exists $(s_i)_{i\in I}$ belonging to the Cartesian product but not to S. As S is closed, $(s_i)_{i\in I}$ has an open neighbourhood disjoint from S, say, by definition of product topology,

$$\pi_{j_1}^{-1}(U_1) \cap \cdots \cap \pi_{j_1}^{-1}(U_r) \cap S = \emptyset$$

for some positive natural number r and some $j_1, \ldots, j_r \in I$, where $s_{j_h} \in U_h$ for every $h = 1, \ldots, r$. So, the element $(s_{j_1}, \ldots, s_{j_r})$ belongs to $\prod_{h=1}^{r} S_{j_h}$ but it is not in $\pi_{\{j_1,\ldots,j_r\}}(S)$; this contradicts the surjectivity of the maps π_J with J finite. Hence $(s_i)_{i\in I} \in S$ and $S = \mathrm{Car}_{i\in I} S_i$. □

A more general property holds, as we prove in the following lemma.

Lemma 3.2.2. *Let S be a closed subcartesian product of finite simple non-abelian Lie rings $\mathrm{Car}_{i\in I} S_i$. Then S is itself a Cartesian product.*

Proof. If the simple rings are pairwise non-isomorphic, we can apply Lemma 3.2.1.
Otherwise, we partition the set I through the following equivalence relation: two indices i and j are equivalent if and only if $\ker \pi_i = \ker \pi_j$. Consider the quotient set of indices $I/\sim$ corresponding to this equivalence relation and let T be a transversal set for this relation, namely a set consisting of one class representative $t \in I$ for each class in $I/\sim$. Then, we denote by

$$\pi_T : S \longrightarrow \underset{t\in T}{\mathrm{Car}}\ S_t$$

the map defined as $s \mapsto (\pi_t(s))_{t\in T}$.
Such a function is clearly injective: for, if $x \in \ker \pi_T$, then $x \in \ker \pi_t$ for every $t \in T$, thus $x \in \ker \pi_i$ for every $i \in I$ by definition of our equivalence relation; hence $x = 0$ since $S \leqslant \mathrm{Car}_{i\in I} S_i$.

We now look at surjectivity. We first assume that the transversal set is finite. For all $s \in S$ consider the support of $\pi_T(s)$:

$$\mathrm{supp}\,(\pi_T(s)) = \{t \in T \mid s_t \neq 0\}$$

where s_t denotes the t-th component of $\pi_T(s)$. Clearly the support is non-empty for every $s \in S$, since otherwise the map π_T would be not injective.
By finiteness of T we can pick $x \in S$ such that $\pi_T(x)$ has minimal support; namely, such that there are no elements with fewer non-trivial components in the projection on $\mathrm{Car}_{t\in T} S_t$; we want to prove that the support of such x is a singleton.
Suppose by contradiction that there exist two distinct indices t and u in T such that both x_t and x_u are non-trivial. Since S_t is simple non-abelian, there exists $a \in S_t$ such that a does not commute with $\pi_t(x)$, namely such that $[a, \pi_t(x)] \neq 0$. Certainly, by surjectivity of π_t, there exists $y \in S$ such that $\pi_t(y) = a$; furthermore, such a y can be chosen in $\ker \pi_u$, since, by maximality of the kernels, $S = \ker \pi_u + \ker \pi_t$. Thus, $\pi_t([x, y]) \neq 0$ while $\pi_u([x, y]) = 0$, as $\pi_u(y) = 0$.
Moreover, if v does not belong to the support of $\pi_T(x)$, then v does not either belong to the support of $\pi_T([x, y])$, since $\pi_v(x) = 0$.
Therefore, we can conclude that the support of $[x, y]$ is strictly contained in the support of x, contradicting the minimality of x. Thus, the support of a minimal element x is a singleton.
Now, we prove that π_T is surjective by induction on $|T|$. If T is a singleton, the thesis holds by injectivity of π_T.
If $|T| > 1$ we proceed as follows: for all $t \in T$ we say that an element $x \in S$ is t-minimal if it has minimal support and if its unique non-trivial component is the t-th component. If for all $t \in T$ there exists a t-minimal element, then each S_t is contained in the image of S under π_T, hence π_T is surjective. Otherwise, we consider the proper subset $T' \subsetneq T$ whose elements are the indices $t \in T$ such that there is not a t-minimal element. We then consider the map

$$\pi_{T'} : S \longrightarrow \operatorname*{Car}_{t\in T'} S_t$$

Such a map is not injective, since $\pi_{T'}(y) = 0$ for every y which is t-minimal for some $t \in T \smallsetminus T'$, thus we quotient S by $\ker(\pi_{T'})$ in order to have injectivity. We can prove as above that, if the support of an element in $S/\ker(\pi_{T'})$ is minimal, then it is a singleton. If for all $t \in T'$ the simple Lie ring S_t is contained in the image of $S/\ker(\pi_{T'})$, then the map

$$S/\ker(\pi_{T'}) \longrightarrow \operatorname*{Car}_{t\in T'} S_t$$

is surjective, hence also the map $\pi_{T'}$ is surjective. Otherwise, we repeat the same argument, through a new proper subset $T'' \subsetneq T' \subsetneq T$. Since $|T|$ is finite, this argument must ends in a finite number of steps, showing that each S_t is contained in the image of S. Hence, we have proved the surjectivity of the map π_T if T is finite.
We have now to prove the surjectivity of π_T in the case T is not finite, and we proceed similarly to Lemma 3.2.1. Suppose by contradiction that there

exists $(s_t)_{t\in T}$ belonging to the Cartesian product $\mathrm{Car}_{t\in T}S_t$ but not to S. As S is closed, we can find an open neighbourhood of $(s_t)_{t\in T}$ disjoint from S, say

$$\pi_{t_1}^{-1}(U_1)\cap\cdots\cap\pi_{t_r}^{-1}(U_r)\cap S=\emptyset$$

for some $r\in\mathbb{N}^+$, where each U_i is an open subset of S_{t_i}. Thus, the element $(s_{t_1},\ldots,s_{t_r})$ belongs to $S_{t_1}\times\cdots\times S_{t_r}$ while it does not belong to $\pi_{\{t_1,\ldots,t_r\}}(S)$. Note that the support of $\pi_{\{t_1,\ldots,t_r\}}(s)$ is not empty, since, otherwise, $s\in\ker\pi_{t_j}$ for each $j=1,\ldots,r$, hence $\pi_{t_1}^{-1}(U_1)\cap\cdots\cap\pi_{t_r}^{-1}(U_r)\cap S$ would contains 0 and would be not empty. Therefore, we can repeat the argument used in the case with T finite to prove that $\pi_{\{t_1,\ldots,t_r\}}$ is surjective, contradicting the fact that $(s_{t_1},\ldots,s_{t_r})$ is not in $\pi_{\{t_1,\ldots,t_r\}}(s)$. Hence, $S=\mathrm{Car}_{t\in T}S_t$. □

Lemma 3.2.3. *Let R be an abstract Lie ring, let S be an open ideal. Suppose that S decomposes as a Cartesian product of finite simple non-abelian Lie rings, say $S=Car_{\lambda\in\Lambda}T_\lambda$. Then, each T_λ is an ideal in R.*

Proof. Let $x\in R$ and $\alpha\in\Lambda$. Consider the subset $[T_\alpha,x]+T_\alpha$ in S. We claim that it is an ideal of S. Clearly, it is an additive subgroup.
If $b\in S$ we have

$$[[T_\alpha,x]+T_\alpha,b]=[[T_\alpha,x],b]+[T_\alpha,b]$$

where the second summand is contained in T_α since it is an ideal of S; on the other hand, for the first summand we have, by Jacobi identity,

$$[[T_\alpha,x],b]\subseteq[[b,T_\alpha],x]+[[x,b],T_\alpha]\subseteq[T_\alpha,x]+T_\alpha$$

Thus, for all $b\in S$ we have

$$[[T_\alpha,x]+T_\alpha,b]\subseteq[T_\alpha,x]+T_\alpha$$

as desired.
Let now $C:=C_S(T_\alpha)$ be the centralizer of T_α in S; because of the decomposition of S, C certainly contains $\mathrm{Car}_{\lambda\neq\alpha}T_\lambda$; besides, if $C=S$ we would have T_α abelian, contradicting the hypothesis; thus, $C=\mathrm{Car}_{\lambda\neq\alpha}T_\lambda$.
Now, if $y\in C$, we have for every $a\in T_\alpha$ and $x\in R$

$$[[a,x],y]=-[[x,y],a]-[[y,a],x]\in T_\alpha$$

hence $[[T_\alpha,x],C]\leqslant T_\alpha$, from which $[[T_\alpha,x],C,C]=0$ follows by definition of C.
Now, if $[T_\alpha,x]\nleqslant T_\alpha$, then there exists $\lambda\neq\alpha$ such that $[T_\alpha,x]\cap T_\lambda\neq 0$, so we can pick $z\in T_\lambda$ which is not centralized by $C_S(T_\alpha)$ since $C_S(C_S(T_\alpha))=T_\alpha$. Thus, the condition $[[T_\alpha,x],C,C]=0$ implies $[T_\alpha,x]\leqslant T_\alpha$ (these conditions are indeed equivalent), and our claim is proved. □

3.3 Narrow ideals

Now we introduce the notion of narrow ideal of a Lie ring. Such ideals will be useful in the following sections.

Definition 3.3.1. Let R be a Lie ring, let A and B be two ideals of R such that B is properly contained in A. The factor ring A/B is called a *chief factor* of the Lie ring R if A/B is a simple ring.

Definition 3.3.2. Two chief factors I/J and A/B are *associated* if and only if $I + B = A + J$ and $J + B < I + A$.

Definition 3.3.3. Let R be a Lie ring, let K be a non-trivial ideal of R. Let $\mathcal{I}_R(K)$ be the family of proper subrings of finite index of K which are ideals in R. We denote by $M_R(K)$ the intersection of the maximal elements in $\mathcal{I}_R(K)$.
The ideal K of R is *narrow* if $M_R(K)$ is the unique maximal element in $\mathcal{I}_R(K)$.
A narrow ideal K of R is *associated* to a chief factor I/J if $K/M_R(K)$ is associated to I/J.

Lemma 3.3.4. *Let I and J be closed ideals of a profinite Lie ring R. Then $I + M_R(J) \geqslant J$ if and only if $I \geqslant J$.*

Proof. Suppose that $I + M_R(J) \geqslant J$ and assume, by contradiction, that $I \ngeqslant J$. Then $I \cap J$ is an ideal in R which is strictly contained in J, thus there exists a maximal element K in $\mathcal{I}_R(J)$ such that $I \cap J \leqslant K$ (possibly $I \cap J = K$). Now, since $M_R(J) < J$, we have

$$(I + M_R(J)) \cap J = I \cap J + M_R(J) \leqslant K + M_R(J) \leqslant K < J$$

where the first relation holds by Dedekind's law. Therefore, $I + M_R(J) < J$, contradicting the hypothesis.
The converse is clear. □

In the following lemma we prove the existence of narrow ideals associated to chief factors and we characterize them.

Lemma 3.3.5. *Let I/J be a chief factor of a profinite Lie ring R. Then, there exists a narrow ideal K of R associated to such a chief factor.*
The narrow ideals associated to I/J are exactly the narrow ideals of R contained in I but not in J; moreover, in this case $M_R(K) = K \cap J$.
In particular, every non-trivial ideal of R contains a narrow ideal of R.

Proof. First, we prove that there exists a narrow ideal of R associated to a chief factor I/J. Let

$$\mathcal{K}_R(I) = \{H \lhd_c R \mid H \leqslant I\}$$

and let

$$\mathcal{D}_R(I,J) = \mathcal{K}_R(I) \setminus \mathcal{K}_R(J)$$

be the set of closed ideals of R which are contained in I but not in J.
Let $(D_i)_{i\in I}$ be a descending chain in $\mathcal{D}_R(I,J)$; since, by Lemma 3.1.5, the intersection $\bigcap_{i\in I} D_i$ is not contained in J, we have $\bigcap_{i\in I} D_i \in \mathcal{D}_R(I,J)$, hence $\mathcal{D}_R(I,J)$ admits, by Zorn's lemma, a minimal element, say K. Now, let $H \lhd_c R$ properly contained in K; by minimality of K, we have $H \leqslant J$, hence $K \cap J$ is the unique maximal element in $\mathcal{I}_R(K)$. This proves that K is a narrow ideal of R.

Now we prove the characterization of narrow ideals associated to I/J. Let K such an ideal; then, by definition, we have

$$I + M_R(K) = K + J \tag{3.2}$$

By this equation it follows that $I + M_R(K) \geqslant K$, hence, by Lemma 3.3.4, we have $I \geqslant K$. Moreover, again by relation (3.2), we deduce $K \nleqslant J$, since otherwise $J \geqslant I$.
Conversely, let K be a narrow ideal of R that is contained in I but not in J. Certainly, $K + J = I$ since I/J is a chief factor and $K + J > J$. Hence, since

$$K/_{K \cap J} \cong K + J/_J = I/_J$$

we deduce that $K/(K\cap J)$ is a chief factor, which implies $K\cap J = M_R(K)$. It remains to prove that $K/(K \cap J)$ and I/J are associated: for, as $K \cap J \leqslant I$, we have certainly $K + J = I + K \cap J$; moreover $K \cap J + J < K + I$, hence the thesis. □

3.4 Just infinite profinite Lie rings

A profinite Lie ring is *just infinite* if each non-trivial closed ideal is open or, equivalently, if it has finite index.

In this section we analyze some properties of just infinite profinite Lie rings which are not virtually abelian, similarly to what Reid has done for just infinite profinite groups. For some results we need to look at the subclass of profinite *hereditarily just infinite* Lie rings, that are profinite Lie rings in which every open Lie subring is just infinite.

First we state a technical lemma. Given a profinite Lie ring R, its *Jacobson radical* $J(R)$ is the intersection of all of its maximal closed (indeed, open) ideals.

Lemma 3.4.1. *Let R be a profinite Lie ring, let I be a closed ideal of R and let S be a closed subring of R containing I.*
Then, the Jacobson radical of S contains the Jacobson radical of I.

Proof. If each maximal ideal of S contains I, then certainly the Jacobson radical of S contains that of I. Otherwise, if we consider a closed maximal ideal M of S such that M does not contain I, we have

$$S/M = I + M/M \cong I/I \cap M$$

so $I/(I \cap M)$ is simple, hence $I \cap M$ is maximal in I. This proves the statement. □

Lemma 3.4.2. *Let R be a just infinite profinite Lie ring that is not virtually abelian, let S be an open subring. Then, the Jacobson radical $J(S)$ of S is not trivial.*

Proof. Each open subring S contains an open ideal I, therefore, by Lemma 3.4.1, the Jacobson radical of S contains that of I; hence, it is sufficient to prove the statement for an open ideal.
For our purpose, we partition the set of open maximal ideals of S as follows: let $\mathcal{A}$ be the set of open maximal ideals of S such that the corresponding quotient is simple abelian and let $\mathcal{B}$ be the set of open maximal ideals of S such that the corresponding quotient is simple non-abelian. We also denote by A, respectively B, the intersection of the elements in $\mathcal{A}$, respectively in $\mathcal{B}$.
Suppose by contradiction that the Jacobson radical is trivial. Then, at least one among A and B must have infinite index in S.
If $A = 0$, then S decomposes as a Cartesian product of simple abelian Lie rings and so R is a finite extension of an abelian Lie ring, case that we have excluded by hypothesis.
If $B = 0$, then S embeds in a Cartesian product of simple non-abelian factors, and so it is a subcartesian product since $\pi_M(S) = S/M$ for every open maximal ideal $M \lhd S$. Hence, by Lemma 3.2.2, the subring S itself is a Cartesian product of finite simple non-abelian Lie rings. By Lemma 3.2.3 each component of this Cartesian product is a finite ideal of R, hence each such component is trivial by just infiniteness of R, and so $S = 0$.
If both A and B are not trivial, then necessarily B has infinite index; in fact A contains $[S, S]$, that is the smallest ideal of S that makes the quotient abelian and that, in addition, is an ideal in R because of the Jacobi identity. Thus, A must have finite index in S and $S \cong S/(A \cap B)$ embeds in $S/A \times S/B$, that, in turn, embeds in a Cartesian product of simple factors among which only finitely many are abelian. Note that $[S, S]$, that is in turn an open subring of R, decomposes as a Cartesian product of finite simple non-abelian Lie rings, since certainly the abelian terms are killed by the Lie bracket; hence, applying Lemma 3.2.3 to $[S, S]$, each finite simple non-abelian term is a finite ideal of R, so every such term is trivial. Therefore, S embeds in S/A which is finite. Thus, also this case can not hold.
In conclusion, assuming the triviality of the Jacobson radical we deduce a contradiction, and our statement is proved. □

Lemma 3.4.3. *Let R be a just infinite profinite Lie ring. Then the Jacobson radical of each open ideal I has finite index in I if and only if the Jacobson radical of each open Lie subring S has finite index in S.*

Proof. We have only to prove that if the condition holds for open ideals, then it also holds for open subrings, as the converse implication trivially holds.
Thus, let S be an open subring of R; of course S contains an open ideal I of R. By Lemma 3.4.1, we have $J(I) \leqslant J(S)$. Therefore

$$|S : J(S)| \leqslant |S : J(I)| = |S : I||I : J(I)|$$

Thus we deduce the finiteness of $|S : J(S)|$ by the finiteness of both $|S : I|$ and $|I : J(I)|$. □

We recall now the König's Lemma, which will be useful in the subsequent result. We remind the reader that a *directed graph* is an ordered pair (V, E) where V is a set of vertices and E is a set of ordered pairs of vertices. Moreover, a *simple path* is a sequence of distinct vertices with the property that each vertex in the sequence is adjacent to the vertex next to it. A graph is *locally finite* if each vertex is adjacent to finitely many vertices.

Lemma 3.4.4 (König's lemma)**.** *Let Γ be a directed locally finite infinite graph. Then Γ contains an infinite simple path.*

For the statement of the following technical lemma, we have to assume that the Jacobson radical of every open subring S of a just infinite profinite Lie ring L has finite index in S. This condition seems essential, since there is no evidence that it holds in general, and it forces us to restrict the class of profinite Lie rings we are dealing with.

Lemma 3.4.5. *Let R be a just infinite profinite Lie ring in which the Jacobson radical of every open subring S has finite index in S. Let $\mathcal{I}$ be an infinite set of open ideals and suppose that the following property holds: for every ideal $I_1 \in \mathcal{I}$, if $I_2 \geqslant I_1$ then also $I_2 \in \mathcal{I}$.*
Then, there exists a strictly descending sequence $(I_i)_{i \in \mathbb{N}^+}$ of open ideals of R such that $I_i \in \mathcal{I}$ for every i.

Proof. We build a directed graph Γ whose vertices are elements of $\mathcal{I}$ and whose edges are couple of ideals (I_1, I_2) such that $I_2 < I_1$ and such that there is not any ideal properly contained between them. If $(I, I_1) \in E(\Gamma)$, then I_1 contains the Jacobson radical of I. By assumption there are finitely many such ideals I_1, so Γ is a locally finite directed graph and by König's lemma it contains an infinite path; thus, there is an infinite descending chain of open ideals in R. □

To formulate the following statements we have to deal with Lie rings which satisfy the condition on the Jacobson radical of open subrings stated in the hypothesis of Lemma 3.4.5. Certainly, hereditarily just infinite profinite Lie rings satisfy this condition, hence we look at this subclass of just infinite profinite Lie rings.

Lemma 3.4.6. *Let R be a hereditarily just infinite profinite Lie ring that is not virtually abelian, let S be an open subring. Then, the Jacobson radical of S has finite index in S.*

Proof. By Lemma 3.4.2, the Jacobson radical can not be trivial. Moreover, as the ring is hereditarily just infinite, also S is just infinite, thus the Jacobson radical must have finite index in S, hence also in R. □

Let R be a profinite Lie ring, let S be a closed subring of R. We denote by $\mathcal{I}_S$ the set of open ideals of R which are not contained in S. Then we define the *obliquity subring* of S as

$$\mathrm{Ob}_R(S) = S \cap \bigcap \mathcal{I}_S$$

We can now state the following results, that relate the just infiniteness of a profinite Lie ring which is not virtually abelian with the finiteness of the index of the obliquity subrings of the open subrings.

Theorem 3.4.7. *Let R be a profinite Lie ring that is not virtually abelian. If it is hereditarily just infinite, then for every open subring S the set of ideals $\mathcal{I}_S$ is finite.*
If $\mathcal{I}_S$ is finite for every open subring S, then R is just infinite.

Proof. Assume that R is hereditarily just infinite and suppose that the set $\mathcal{I}_S$ is infinite. By Lemma 3.4.6 the Jacobson radical of each open ideal of R has finite index, hence we can apply Lemma 3.4.5 to $\mathcal{I}_S$, which obviously satisfies the condition of the lemma. This ensures the existence of an infinite descending chain in $\mathcal{I}_S$. Now, by Lemma 3.1.5 the intersection of the elements in this family would be a non-trivial ideal having infinite index in R, contradicting the hypothesis.
For the second statement, let I be a closed nontrivial ideal of R: we have to prove that the quotient R/I is finite. Since the intersection of all the open subrings is trivial, there exists an open subring T such that $I \not\subseteq T$. Now, by definition of the profinite topology, the ideal I is the intersection of all the open ideals of R containing it; moreover, this intersection is extended only over the open ideals containing I and not contained in T: otherwise, in fact, I would be contained in T. Hence,

$$\mathrm{Ob}_R(T) = T \cap \bigcap \mathcal{I}_T = T \cap \bigcap_{\substack{J \lhd_o R \\ J \not\subseteq T}} J \ \subseteq \ T \cap \bigcap_{\substack{I \leqslant J \lhd_o R \\ J \not\subseteq T}} J \ = \ T \cap I \ \subsetneq \ I$$

Now, $\text{Ob}_R(T)$ has finite index, since, by hypothesis, it is the intersection of an open subring and finitely many finite-index ideals, then also I has finite index, therefore the thesis holds. □

Corollary 3.4.8. *Let R be a profinite Lie ring which is not virtually abelian. If R is hereditarily just infinite then the index of $Ob_R(T)$ is finite for every open subring T.*
If the index of $Ob_R(T)$ is finite for every open subring T, then R is just infinite.

Proof. If R is a hereditarily just infinite Lie ring, then $\text{Ob}_R(T)$ is an intersection of finitely many ideals of finite index, by Theorem 3.4.7, so it also has finite index.
On the other hand, if R is not just infinite, then, again by Theorem 3.4.7, there exists a subring T such that the set $\mathcal{I}_T$ is infinite, and so $\text{Ob}_R(T)$ has infinite index, contradicting the hypothesis. □

Corollary 3.4.9. *Let R be a hereditarily just infinite profinite Lie ring which is not virtually abelian, let S be an open subring.*
Then $I \leqslant S$ for all but finitely many chief factors I/J in R.

Proof. By Theorem 3.4.7 the family of open ideals that are not contained in S is finite, hence the thesis holds. □

Now we state another result analogue to what Reid has done for just infinite profinite groups, describing the just-infiniteness of a profinite Lie ring which is not virtually abelian in terms of the finite Lie rings occurring in the associated inverse system.

Theorem 3.4.10. *Let R be a hereditarily just infinite profinite Lie ring which is not virtually abelian, let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of classes of finite Lie rings such that R has infinitely many chief factors in $\mathcal{A}_n$ for all $n \in \mathbb{N}$. Then R is the limit of an inverse system of finite Lie rings $(R_n)_{n\in\mathbb{N}}$ and surjective ring morphisms $(\rho_n : R_{n+1} \longrightarrow R_n)_{n\in\mathbb{N}}$ where each R_n has a subring A_n such that, letting $B_n = \rho_n(A_{n+1})$, we have*

1. *$A_n > B_n > 1$;*
2. *A_n is a narrow subring in R_n*
3. *every ideal contains B_n or it is contained in A_n;*
4. *B_n is a minimal ideal for R_n;*
5. *$B_n \in \mathcal{A}_n$.*

On the other hand, every surjective inverse system satisfying conditions (1)-(3) (for some choice of A_n) for all but finitely many n has a just infinite limit.

Proof. First we prove that a hereditarily just infinite profinite Lie ring as in the hypothesis satisfies properties (1)-(5). We will obtain a descending chain $(K_n)_{n\in\mathbb{N}}$ of narrow subrings of R and we will build the required inverse system through this sequence.
The first term of the sequence is $K_0 = R$. Afterwards, for the definition of the general term, suppose we have built K_n. Consider then $S := \mathrm{Ob}_R(K_n)$, which has finite index in R. Thence, pick a chief factor I/J such that $I \leqslant S$ and $I/J \in \mathcal{A}_n$: such a chief factor exists, since by hypothesis each class $\mathcal{A}_n$ contains infinitely many chief factors and since, by Corollary 3.4.9, S contains all but finitely many chief factors. We then pick K_{n+1} to be a narrow ideal associated to I/J, whose existence is justified by Lemma 3.3.5. Such an ideal is contained in S by Lemma 3.3.4.
After we have constructed the sequence of narrow subrings, if we define

$$R_n = R/M_R(K_{n+1}), \quad A_n = K_n/M_R(K_{n+1}), \quad B_n = K_{n+1}/M_R(K_{n+1})$$

all the conditions are satisfied.
Now we prove that every surjective inverse system satisfying properties (1)-(3) for some choice of A_n has a just infinite limit. Let I be a closed nontrivial ideal of R. For n sufficiently large, if $\pi_n : R \longrightarrow R/I_n$ is the surjective map associated to the inverse limit, we have that $\pi_n(I)$ is not contained in A_n, so it must contain B_n by condition (3). Since $M_R(A_{n+1})$ contains $\ker \rho_n$, by condition (1), we have that $\pi_{n+1}(I)$ contains A_{n+1} and in particular $\pi_{n+1}(I)$ contains $\ker \rho_n$.
Since this applies for all n sufficiently large, we have that N contains $\ker \pi_n$ for some n, hence it has finite index. □

3.5 A condition for a full characterization

In the previous section we have seen that, for a general profinite Lie ring, we are not able to provide a complete characterization of just infinite profinite Lie rings in terms of the associated inverse system, since we need the additional hypothesis of hereditarily just-infiniteness to formulate the statements, in order to make sure that the Jacobson radical of an open subring has finite index in the subring, as required in the statement of Lemma 3.4.5. In this section we define an abstract class of Lie rings satisfying the desired condition on the Jacobson radical of its subrings; for just infinite profinite Lie rings belonging to this family we are able to give a full characterization. Later, we will show that two families of Lie algebras, the Lie FC-algebras and the residually solvable Lie algebras, actually belong to this class.

Let $\mathcal{X}$ be the class of Lie rings R such that, for every open subring S of R, the Jacobson radical of S has finite index in S. For this class of Lie rings we can provide more precise statements for Theorems 3.4.7 and 3.4.10.

Theorem 3.5.1. *Let R be a profinite Lie ring in $\mathcal{X}$ which is not virtually abelian. The following conditions are equivalent:*

1. *R is just infinite;*
2. *for every open subring S the set of ideals*
$$\mathcal{I}_S = \{I \lhd_o R \mid I \nleqslant S\}$$
is finite;
3. *there exists a family $\mathcal{F}$ of open subrings in R with trivial intersection such that $\mathcal{I}_S$ is finite for every $S \in \mathcal{F}$.*

Proof. [(1) ⇒ (2)] Suppose that the set $\mathcal{I}_S$ is infinite. Then, by Lemma 3.4.5 we could find an infinite descending chain in $\mathcal{I}_S$. Now, by Lemma 3.1.5 the intersection of the elements in this family would be a non-trivial closed ideal having infinite index in R, contradicting the hypothesis.
[(2) ⇒ (3)] This implication is clear.
[(3) ⇒ (1)] This implication holds for every profinite Lie ring which is not virtually abelian, see proof of Theorem 3.4.7. □

Corollary 3.5.2. *Let R be a profinite Lie ring in $\mathcal{X}$ which is not virtually abelian. R is just infinite if and only if the index of $Ob_R(T)$ is finite for every open subring T.*

Proof. If R is a just infinite Lie ring, then $\mathrm{Ob}_R(T)$ is an intersection of finitely many ideals of finite index, so it also has finite index.
Conversely, if R is not just infinite, then, by Theorem 3.5.1, there exists a subring T such that the set $\mathcal{I}_T$ is infinite, and so $\mathrm{Ob}_R(T)$ has infinite index. □

Corollary 3.5.3. *Let R be a just infinite profinite Lie ring in $\mathcal{X}$ which is not virtually abelian, let S be an open subring. Then $I \leqslant S$ for all but finitely many chief factors I/J in R.*

Proof. By Theorem 3.5.1 the family of open ideals that are not contained in S is finite, hence the thesis holds. □

We are now ready for the main result, which characterizes just infinite profinite Lie rings in $\mathcal{X}$ which are not virtually abelian in terms of the finite Lie rings occuring in the related inverse system.

Theorem 3.5.4. *Let R be a just infinite profinite Lie ring in $\mathcal{X}$ which is not virtually abelian, let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of classes of finite Lie rings such that R has infinitely many chief factors in $\mathcal{A}_n$ for all $n \in \mathbb{N}$.*
Then R is the limit of an inverse system of finite Lie rings $(R_n)_{n\in\mathbb{N}}$ and surjective ring morphisms $(\rho_n : R_{n+1} \longrightarrow R_n)_{n\in\mathbb{N}}$ where each R_n has a subring A_n such that, letting $B_n = \rho_n(A_{n+1})$, we have

1. *$A_n > B_n > 1$;*

2. *A_n is a narrow subring in R_n;*

3. *every ideal contains B_n or it is contained in A_n;*

4. *B_n is a minimal ideal for R_n;*

5. *$B_n \in \mathcal{A}_n$.*

Conversely, every surjective inverse system satisfying conditions (1)-(3) (for some choice of A_n) for all but finitely many n has a just infinite limit.

Proof. The proof is completely similar to the general case. To prove that a just infinite profinite Lie ring as in the hypothesis is the inverse limit of an inverse system of finite Lie rings with the listed properties, we build the inverse system starting from a descending chain of narrow subrings and applying Corollary 3.5.3 in place of Corollary 3.4.9.
On the other hand, the proof of the converse implication is valid for a general profinite Lie ring which is not virtually abelian. □

In the following, we show two examples of Lie algebras satisfying the condition on the Jacobson radical of open subrings.

Lie FC-algebras

Here we consider just infinite profinite Lie algebras which are FC-algebras. An *FC-algebra* is an infinite-dimensional algebra in which the centralizer of each element has finite codimension. Clearly also the centralizer of each finite subset has finite codimension, since it is the intersection of finitely many finite-codimensional subalgebras. Moreover, for every vector subspace V and for every finite set F, also $[V, F]$ is finite.

We prove that Lie FC-algebras are actually algebras for which the Jacobson radical of every open subalgebra has finite codimension in the subalgebra. This fact allows us to apply the previous result of this section to Lie FC-algebras, providing a full characterization of just infinite profinite Lie FC-algebras in terms of the associated inverse system. Furthermore, we emphasize that the limitation to Lie algebras which are not virtually abelian is not really restrictive in this case, since a profinite just infinite virtually abelian Lie FC-algebra is finite; hence for such Lie FC-algebras the result is trivial.

Thus, we look at Jacobson radical of open subalgebras in just infinite profinite Lie FC-algebras. Let L be a profinite Lie FC-algebra, let S be an open subalgebra of L. Then we can find a finite vector subspace V of L such that $L = S + V$ (where the sum is intended as sum of vector spaces).

As usual, we assume the following notation for iterated Lie brackets:

$$[J(S),_n V] := \begin{cases} J(S), & n = 0 \\ [[J(S),_{n-1} V], V], & n > 0 \end{cases}$$

We also set $I_n := \sum_{k=0}^{n} [J(S),_n V]$.

Lemma 3.5.5. *If S is an ideal in L, then I_n is an ideal of S for all $n \in \mathbb{N}$.*

Proof. First we prove, by induction on $n > 0$, that $[[J(S),_n V], S] \subseteq I_n$.
If $n = 1$, using Jacobi identity

$$[[J(S), V], S] \subseteq [[S, J(S)], V] + [[V, S], J(S)] \subseteq [J(S), V] + J(S)$$

where the second inclusion holds since $J(S) \lhd S$ and $[V, S] \subseteq S$ as S is an ideal in L.
For $n > 1$, using again Jacobi identity and the inductive hypothesis,

$$\begin{aligned} [[J(S),_{n+1} V], S] &= [[[J(S),_n V], V], S] \subseteq \\ &\subseteq [[S, [J(S),_n V]], V] + [[V, S], [J(S),_n V]] \subseteq \\ &\subseteq [I_n, V] + [S, [J(S),_n V]] = \\ &= \sum_{k=1}^{n+1} [J(S),_n V] + [S, [J(S),_n V]] \subseteq \\ &\subseteq I_{n+1} + I_n = I_{n+1} \end{aligned}$$

Now we prove the statement of the lemma, by induction on $n \geqslant 0$.
If $n = 0$ the conclusion is immediate. If $n > 0$

$$[I_{n+1}, S] = [I_n + [J(S),_{n+1} V], S] \subseteq I_n + I_{n+1} = I_{n+1}$$

where the second inclusion holds by inductive hypothesis and by the fact $[[J(S),_n V], S] \subseteq I_n$, previously proved. □

Lemma 3.5.6. *Let L be a just infinite Lie FC-algebra that is not virtually abelian, let S be an ideal of L with finite codimension. Then $J(S)$ is either trivial or it has finite codimension in L.*

Proof. If there are infinitely many maximal abelian ideals, then their intersection is trivial, and even more so $J(S)$ is trivial. Thus, suppose that there are finitely many maximal abelian ideals and let J be the intersection of all the non-abelian maximal ideals. If such intersection is trivial, also the Jacobson radical is; on the other hand, if such intersection has finite codimension in L, then also the Jacobson radical has finite codimension in L. Hence, we will prove the statement for the ideal J.
Remembering the decomposition as vector spaces $L = S + V$, it is sufficient to prove that $[J, x] \leqslant J$ for every $x \in V$, since we have to prove that J is

an ideal in L and we already know that it is an ideal in S. Furthermore, since the Lie algebra is FC, we have that $[J, V]$ is finite; more than that, for each $n \in \mathbb{N}^+$, also $[J,_n V] = [J, V, \ldots, V]$ is finite. So, if we prove that I_n has finite codimension, we deduce that also J has finite codimension.
First, we note that the increasing sequence $(I_n)_{n\in\mathbb{N}^+}$ stabilizes in finitely many steps, since each summand is contained in $[L, V]$, which is finite, as L is an FC-algebra and V finite. Hence there exists $m \in \mathbb{N}^+$ such that $I_n = I_m$ for all $n \geqslant m$.
Now, suppose by contradiction that I_m, that is an ideal of S by Lemma 3.5.5, has infinite codimension in L. Thus, in the quotient

$$S/I_m \cong \mathrm{Car}_{i\in I} T_i$$

there occur infinitely many factors among the finite simple factors of the decomposition of S/J.
Let moreover $\widetilde{T}_i$ be an ideal of S such that its quotient by I_m is the ideal T_i. Now, for each $x \in V$, we claim that $\widetilde{T}_i + \left[\widetilde{T}_i, x\right]$ is an ideal in S: in fact, we have

$$\left[\widetilde{T}_i + \left[\widetilde{T}_i, x\right], S\right] \subseteq \left[\widetilde{T}_i, S\right] + \left[\left[S, \widetilde{T}_i\right], x\right] + \left[[x, S], \widetilde{T}_i\right] \subseteq \widetilde{T}_i + \left[\widetilde{T}_i, x\right]$$

since $\widetilde{T}_i$ is an ideal of S and S is an ideal of L. Hence, the image of $\widetilde{T}_i + \left[\widetilde{T}_i, x\right]$ in the quotient S/I_m is an ideal, that is, besides, non-trivial, since $\widetilde{T}_i \subseteq \widetilde{T}_i + \left[\widetilde{T}_i, x\right]$.
Let now C_i be the centralizer of T_i in S/I_m; we obviously have $C_i = \mathrm{Car}_{j\neq i} T_j$. Then, define as

$$\widetilde{C}_i = \left\{ y \in S \mid \left[\widetilde{T}_i, y\right] \leqslant I_m \right\}$$

the inverse image of C_i in S.
Now, we claim that the image in S/I_m of $\left[\widetilde{T}_i + \left[\widetilde{T}_i, x\right], \widetilde{C}_i, \widetilde{C}_i\right]$ is trivial. In fact

$$\begin{aligned}\left[\widetilde{T}_i + \left[\widetilde{T}_i, x\right], \widetilde{C}_i\right] \subseteq \left[\widetilde{T}_i, \widetilde{C}_i\right] + \left[\left[\widetilde{C}_i, \widetilde{T}_i\right], x\right] + \left[\left[x, \widetilde{C}_i\right], \widetilde{T}_i\right] &\subseteq \\ \subseteq I_m + [I_m, x] + \widetilde{T}_i &\subseteq \\ \subseteq I_m + [I_m, V] + \widetilde{T}_i &= \\ = I_m + \widetilde{T}_i &\end{aligned}$$

where the second inclusion holds since $\left[\widetilde{T}_i, \widetilde{C}_i\right] \subseteq I_m$ by the characterization of $\widetilde{C}_i$ and since $\left[x, \widetilde{C}_i\right] \leqslant S$, as $\widetilde{C}_i \leqslant S \lhd L$; the last equality holds since $I_m + [I_m, V] \subseteq I_m + I_{m+1} = I_m$ by the choice of m. Therefore

$$\begin{aligned}\left[\widetilde{T}_i + \left[\widetilde{T}_i, x\right], \widetilde{C}_i, \widetilde{C}_i\right] \subseteq \left[I_m + \widetilde{T}_i, \widetilde{C}_i\right] &\subseteq \\ \subseteq I_m + I_m = I_m &\end{aligned}$$

so we have

$$\pi_{S/I_m}\left(\left[\widetilde{T_i}+\left[\widetilde{T_i},x\right],\widetilde{C_i},\widetilde{C_i}\right]\right)=0$$

as desired.
It follows that $\widetilde{T_i}+\left[\widetilde{T_i},x\right]=\widetilde{T_i}$ and $\left[\widetilde{T_i},x\right]\subseteq\widetilde{T_i}$ for all $x\in V$, hence $\widetilde{T_i}$ is an ideal in L.
Besides, $\widetilde{T_i}$ has infinite codimension in L, since T_i has infinite codimension in S/I_m. This leads to a contradiction, as the algebra is just infinite. So I_m has finite codimension in S. Hence, since $\sum_{k=1}^{n}[J,_k V]$ is finite, we have that also J has finite codimension in S. □

Lemma 3.5.7. *Let L be a just infinite profinite Lie FC-algebra that is not virtually abelian, let S be an open subalgebra. The Jacobson radical $J(S)$ of S has finite codimension in S.*

Proof. By Lemma 3.4.3 it is sufficient to prove the statement for S an ideal. Since by Lemma 3.4.2, the Jacobson radical of S is not trivial, then its codimension in S is finite by Lemma 3.5.6. □

Then, thanks to Lemma 3.5.7, we can apply to Lie FC-algebras Theorems 3.5.1 and 3.5.4, providing two characterizations of just infinite profinite Lie FC-algebras which are not virtually abelian; on the other hand, as already mentioned, the virtually abelian case is not relevant, since a profinite just infinite virtually abelian Lie FC-algebra is finite.

Residually solvable profinite Lie algebras

Another family of Lie algebras which satisfy the required condition on the Jacobson radical is the class of residually solvable Lie algebras. A Lie algebra L is *residually solvable* if the family of ideals such that the corresponding quotient is solvable has trivial intersection. In other words, L is residually solvable if the derived series has trivial intersection, that is if

$$L^{(\omega)}=\bigcap_{n\in\mathbb{N}^+}L^{(n)}$$

is trivial.
The following lemma is what we need to make sure that for a just infinite profinite residually solvable Lie algebra which is not virtually abelian a full characterization in the Reid's fashion is possible.

Lemma 3.5.8. *Let L be a just infinite profinite residually solvable Lie algebra which is not virtually abelian, let S be an open subalgebra. Then, the Jacobson radical $J(S)$ has finite codimension in S.*

Proof. By Lemma 3.4.3 we can assume without loss of generality that S is an ideal.

The derived ideal $[S, S]$ is an ideal in L, by Jacobi identity. Moreover, since L is residually solvable, S is residually solvable too, hence the ideal $[S, S]$ is properly contained in S; otherwise, the derived series of S would not have trivial intersection. Furthermore, since the Lie algebra is not virtually abelian, the abelian quotient $S/[S, S]$ must be finite, or, equivalently, $[S, S]$ must have finite codimension in S. □

Thus, also in the case of residually solvable Lie algebras we can apply the two characterization theorems 3.5.1 and 3.5.4.

Chapter 4

Profinite Noetherian groups

Since the concept of p-adic analytic pro-p group was introduced by Lazard [7] in 1960s, several important characterizations have been given. The most important result in this sense is the equivalence between p-adic analytic groups and pro-p groups of finite rank, see for instance [5]. This first result leads to other remarkable characterizations of p-adic analytic groups, for example in terms of virtual properties (that is, properties which are satisfied by a finite-index subgroup), in terms of the sections of the group (see [19]) and in terms of subgroup growth, which was proved to be polynomial, giving also a quantitative result (see [18]).

During the development of the theory of p-adic analytic pro-p groups, also several questions have been posed; one of these, asked by Lubotzky and Mann in [8], is about Noetherian pro-p groups. It is easy to prove that each pro-p group of finite rank is Noetherian, so they asked if also the converse should be true. A positive answer to this question would provide a further characterization of p-adic analytic groups. Despite the Noetherianity constitutes a strong condition, very little is known about the class of profinite Noetherian groups. Here, we will make some comments on the structure of profinite Noetherian groups, pointing out some consequences of results which have been established more in general, when they are applied to this class of groups.

In particular, we will show that a profinite Noetherian just infinite group is a finite extension of a direct product of a finite number of copies of a hereditarily just infinite group, as a corollary of an important result proved by Grigorchuk which goes by the name of Wilson's dichotomy.

Furthermore, we will prove that a profinite Noetherian group has finite prime spectrum and that a just infinite profinite Noetherian group is virtually pro-p for some prime p.

4.1 Non-Noetherianity of branch groups

A *profinite Noetherian group* is a profinite group in which every ascending chain of closed subgroups stabilizes in finitely many steps. For instance, the *group of p-adic integers* $\mathbb{Z}_p$, namely

$$\mathbb{Z}_p = \varprojlim_{n \in \mathbb{N}^+} \mathbb{Z}/p^n\mathbb{Z}$$

is a pro-p Noetherian group.

In order to analyze the behaviour of profinite Noetherian just infinite groups, a first step is represented by a direct consequence of a result proved by Grigorchuk in [2] which goes by the name of Wilson's dichotomy. Here we present this statement as a specialization to just infinite profinite groups of a result which holds more in general for abstract just infinite groups. Recall that a profinite group G is *hereditarily just infinite* if every its open subgroup is just infinite.

Theorem 4.1.1 (Wilson's dichotomy, [2])**.** *Just infinite profinite groups are either branch groups or finite extensions of a direct product of a finite number of copies of a hereditarily just infinite group.*

In this section we will prove that branch groups do not satisfy the Noetherian property. To do this we will show an increasing sequence of closed subgroups which does not stabilize. So, by Wilson's dichotomy, we will have that a just infinite profinite Noetherian group must be a finite extension of a direct product $L \times \cdots \times L$, where L is a hereditarily just infinite group.

Definition 4.1.2. A group G is a *branch group* if there exist two decreasing sequences of subgroups $(H_i)_{i\in\mathbb{N}}$ and $(K_i)_{i\in\mathbb{N}}$ and a strictly increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$ such that $H_0 = K_0 = G$, $n_0 = 1$ and, for all $i \in \mathbb{N}$, the following conditions hold:

1. $\bigcap_{j\in\mathbb{N}} H_j$ is trivial;
2. $H_i \lhd G$ and $|G : H_i|$ is finite;
3. H_i is a direct product of n_i copies of K_i; that is,

   $$H_i = K_i^{(1)} \times \cdots \times K_i^{(n_i)}$$

   where $K_i^{(j)} \cong K_i$ for all $1 \leqslant j \leqslant n_i$;
4. n_i is a proper divisor of n_{i+1} and, given $m_{i+1} := \frac{n_{i+1}}{n_i} \geqslant 2$, each direct factor $K_i^{(j)}$ of H_i contains m_{i+1} among the factors of H_{i+1}; more precisely, $K_i^{(j)}$ contains all the factors $K_{i+1}^{(h)}$ such that

   $$(j-1)m_{i+1} + 1 \leqslant h \leqslant jm_{i+1}$$

5. conjugations by the elements in G transitively permute the factors in the product decomposition of H_i.

The sequence of such pairs $(H_i, K_i)_{i\in\mathbb{N}}$ is called a *branch structure* on G.

Theorem 4.1.3. *A branch group does not satisfy the Noetherian property on subgroups; that is, not all ascending sequences of subgroups stabilize in finitely many steps.*

Proof. We construct explicitly a sequence of subgroups that does not stabilize. First, we define the following (strictly increasing) sequence of natural numbers:

$$\begin{cases} a_1 = 1 \\ a_{i+1} = m_{i+1} \cdot a_i + 1, \qquad i \geqslant 1 \end{cases}$$

where the numbers m_i are defined as Definition 4.1.2.
Now consider the sequences $(H_i)_{i\in\mathbb{N}}$ and $(K_i)_{i\in\mathbb{N}}$ as in the definition of branch group, and define the following sequence of subgroups

$$\begin{cases} L_1 = K_1^{(a_1)} \\ L_{i+1} = K_1^{(a_1)} \times K_2^{(a_2)} \times \cdots \times K_{i+1}^{(a_{i+1})} \end{cases}$$

which is constructed by picking one direct factor for each level of the branch structure. Note that for every $i \geqslant 2$ and for every $1 < j \leqslant i$ the subgroup $K_{j+1}^{(a_{j+1})}$ is not a factor of the decomposition of $K_j^{(a_j)}$. In fact, the direct factors of the latter one are

$$K_{j+1}^{((a_j-1)m_{i+1}+1)}, \ldots, K_{j+1}^{(a_j m_{i+1})}$$

while

$$a_{j+1} = m_{i+1} a_j + 1$$

So, explicitely, the j-th factor of L_i is the first direct factor that does not appear as a direct factor of the previous subgroups.
Moreover, the last element in each level of the branch structure is $K_i^{(n_i)}$ and we can easily prove by induction that $a_i < n_i$. The base case is trivial, as $a_1 = 1 \leqslant n_1 - 1$; for the general case we have

$$a_i = m_i a_{i-1} + 1 \leqslant m_i(n_{i-1} - 1) + 1 = n_i - m_i + 1 < n_i$$

This proves that at each level we can choose a new factor in the previous fashion. The chain of subgroups $(L_i)_{i\in\mathbb{N}^+}$ is strictly increasing and it does not stabilize. □

As a corollary of the last theorem and the Wilson's dichotomy, we can deduce the following result.

Corollary 4.1.4. *Let G be a Noetherian just infinite profinite group. Then it is a finite extension of a direct product of a finite number of copies of a hereditarily just infinite group.*

4.2 Profinite groups with self-normalizing Sylow pro-p subgroups

In this section we prove a prosolvability condition for profinite groups, deducing it by the corresponding result for finite groups, which is stated in a theorem due to Guralnick, Malle and Navarro.

Theorem 4.2.1 (Guralnick, Malle, Navarro, [3]). *Let G be a finite group, let $p > 3$ be a prime number*[1]*. If there exists a p-Sylow P such that $P = N_G(P)$, then G is solvable.*

The proof of the following corollary relies on a compactness argument which is often used: if $(X_i,\ \varphi_{i,j})_{\substack{i,j\in\Omega \\ i\leqslant j}}$ is an inverse system of non-empty compact Hausdorff spaces, then its inverse limit $X = \varprojlim_{i\in I} X_i$ is non-empty.

Corollary 4.2.2. *Let G be a profinite group, let $p > 3$ be a prime number, let P be a pro-p Sylow subgroup such that $P = N_G(P)$. Then G is prosolvable.*

Proof. Let $(G_i, \varphi_{i,j})_{\substack{i,j\in\Omega \\ i\leqslant j}}$ be an inverse system of finite groups and surjective morphisms whose inverse limit is the group G. Since a π-Hall subgroup in a profinite group is the inverse limit of a sequence of π-Hall subgroups in each $G_i = \varphi_i(G)$, by surjectivity of φ_i, we have that $P_i := \varphi_i(P)$ is a p-Sylow in G_i for all i. We want to prove that, for all $i \in I$, each P_i is self-normalizing, in order to apply Theorem 4.2.1.
Consider the set

$$\mathcal{A}_i = \{x_i \in G_i \mid x_i \in N_{G_i}(P_i),\ x_i \notin P_i\}$$

and suppose that it is not empty. We can easily convince that $\varphi_{i,j}(x_j) \in \mathcal{A}_i$ for all $x_j \in \mathcal{A}_j$ and for all indices i and j such that $i \leqslant j$: in fact, if $x_j \in N_{G_j}(P_j)$ and $x_j \notin P_j$, then certainly $\varphi_{i,j}(x_j) \in N_{G_i}(P_i)$ and, due to the order, $\varphi_{i,j}(x_j) \notin P_i$.
Thus, the inverse system with discrete finite sets $(\mathcal{A}_i)_{i\in I}$ and maps

$$(\varphi_{i,j} : \mathcal{A}_j \longrightarrow \mathcal{A}_i)_{\substack{i,j\in I \\ i\leqslant j}}$$

has non-empty inverse limit $\mathcal{A} := \varprojlim \mathcal{A}_i$, by Proposition 1.2.1 (4).
Then, pick an element $x = (x_i)_{i\in I} \in \mathcal{A}$: it belongs to $N_G(P)$ while it can not belong to P, due to the order, contradicting the hypothesis of self-normalization of P. Therefore, each $\mathcal{A}_i$ must be empty; in other words, for all i, we have $P_i = N_{G_i}(P_i)$, and, applying Theorem 4.2.1, we deduce that each G_i is solvable. This proves that G is prosolvable. □

[1] The case $p = 3$ requires further investigation and the exclusion of a specific case.

4.3 Prime spectrum of a profinite Noetherian group

In this section we will prove some results about the prime spectrum of the profinite Noetherian groups. First, we will see that the prime spectrum of a profinite prosolvable Noetherian group is finite. Subsequently, we will strengthen this result, proving the finiteness of the prime spectrum of a general profinite Noetherian group.
In view of the following result, we remind the reader that, fixed a prime p, a *p'-group* is a group whose order is coprime to p. Thus, a *pro-p' group* is a profinite group which is the inverse limit of an inverse system of finite p'-groups.

Proposition 4.3.1. *Let G be a profinite group. If each Sylow subgroup is finite, then G is virtually pro-p' for all p in the prime spectrum of G. In particular, G is virtually prosolvable.*

Proof. Let H be a finite closed subgroup of G. Since the set of all open normal subgroups forms a residual system, namely it has trivial intersection, there certainly exists an open normal subgroup N of G such that $H \cap N$ is trivial.
This applies in particular to every p-Sylows, that are finite by hypothesis; then, for all p prime, we are able to find an open normal subgroup N_p that intersects trivially each p-Sylow: in fact, if a normal subgroup intersects trivially a p-Sylow then it must intersect trivially all the p-Sylows since they are pairwise conjugate. Thus, N_p is an open p'-subgroup and G is virtually pro-p' for all p in the prime spectrum of G.
In particular, if we look at 2-Sylows, we can find an open normal subgroup N_2 that is the inverse limit of finite odd-order groups, that are solvable by Feit-Thompson theorem; therefore, N_2 is prosolvable. Thus, G is virtually prosolvable, since N has finite index in G. □

Let now G be a prosolvable group. The proof of the following result, that will be strengthened in the sequel, relies on the fact that each prosolvable group G admits a Sylow basis, as reminded in Proposition 1.2.7.

Proposition 4.3.2. *The prime spectrum of a profinite prosolvable Noetherian group G is finite.*

Proof. By Proposition 1.2.7, a prosolvable group G admits a Sylow basis $\{S_p \mid p \text{ prime}\}$. So, in G there is an ascending chain of subgroups

$$S_{p_1} \leqslant_c S_{p_1}S_{p_2} \leqslant_c S_{p_1}S_{p_2}S_{p_3} \leqslant_c \dots$$

where each term of the chain is actually a closed subgroup of G; in fact each Sylow subgroup is a compact subgroup, hence a finite product of Sylow

subgroups is compact, thus closed.
Now, as the group G is Noetherian by hypothesis, this ascending chain must stabilize in finitely many steps. Since, by Proposition 1.2.7 (1), the topological closure of the product of all the terms in the Sylow basis must coincide with G, we deduce that there are only finitely many primes in the prime spectrum of G. □

The following lemma make us able to find an ascending chain of closed subgroups in a finitely generated profinite group.

Lemma 4.3.3. *Let K be a closed subgroup of a finitely generated profinite group G. Then there exists a family of closed subgroups $(K_i)_{i\in\mathbb{N}}$ such that the sequence of their normalizers is an ascending chain.*

Proof. By Proposition 1.2.10, in each finitely generated profinite group the identity admits a fundamental system of neighbourhoods consisting of a countable descending chain of open characteristic subgroups, say $(H_i)_{i\in\mathbb{N}^+}$. Then we have

$$[H_{i+1} \cap K, N_G (H_i \cap K)] \leqslant [H_i \cap K, N_G (H_i \cap K)] \leqslant H_i \cap K$$

for all $i \in \mathbb{N}^+$, where the first relation holds since $H_{i+1} \leqslant H_i$ and the second one holds since we are commuting a subgroup with its normalizer. Similarly

$$[H_{i+1} \cap K, N_G (H_i \cap K)] \leqslant [H_{i+1}, N_G (H_i \cap K)] \leqslant H_{i+1}$$

where the second containment holds since each H_i is a characteristic subgroup. Thus,

$$[H_{i+1} \cap K, N_G (H_i \cap K)] \leqslant H_{i+1} \cap K$$

that is, $H_{i+1} \cap K$ is normalized by $N_G (H_i \cap K)$. Therefore, the normalizer of $H_{i+1} \cap K$ contains the normalizer of $H_i \cap K$. So, the family of closed subgroups $(H_i \cap K)_{i\in\mathbb{N}}$ is the one we were looking for. □

By the way, note that in the previous proof the set of subgroups $(H_i \cap K)_{i\in\mathbb{N}}$ is a family with trivial intersection, while the family of normalizers, being ascending, intersects in $N_G(K)$.

Now we prove that, if a profinite Noetherian group admits an infinite p-Sylow subgroup, then it is virtually pro-p. To do this, we will use the profinite version of the Frattini argument, stated in Proposition 1.2.6.

Lemma 4.3.4. *Let G be a profinite Noetherian group. Suppose that, for some p prime, there exists an infinite p-Sylow subgroup P. Then, G admits a normal p-subgroup which has finite index in P.*

Proof. Let p be a prime number in the prime spectrum of G such that $P \in \mathrm{Syl}_p(G)$ is infinite.

The group G is Noetherian, so it is finitely generated. Then, the identity element admits a fundamental system of neighbourhoods consisting of a countable chain of open characteristic subgroups $(H_i)_{i \in \mathbb{N}^+}$.
Now, the intersection $P \cap H_i$ is not trivial, since H_i has finite index in G and P is infinite, so $P \cap H_i$ is a p-Sylow of the subgroup H_i. By the profinite Frattini argument we have $G = N_G(H_i \cap P)H_i$. We can prove, by Lemma 4.3.3, that the sequence $(N_G(H_i \cap P))_{i \in \mathbb{N}^+}$ is ascending.
Now, because of Noetherianity, this sequence of subgroups must stabilize, so there exists a greatest term of the sequence, say N. Then, there is a natural number $n_0 \in \mathbb{N}^+$ such that $G = H_j N$ for all $j \geqslant n_0$. Furthermore, for all $j \in \mathbb{N}^+$, the subgroup $N_G(H_j \cap P)$ is the normalizer of a closed subgroup, so it is closed, then it is the intersection of all the open subgroups containing it; in particular, such normalizer equals the intersection of all the open subgroups of the form NH_i, as i varies in $\mathbb{N}^+$, since by construction $(H_i)_{i \in \mathbb{N}^+}$ is a fundamental system of neighbourhoods of the identity. In particular for all $j \geqslant n_0$ we have

$$N_G(P \cap H_j) = N = \bigcap_{i \in \mathbb{N}^+} NH_i \geqslant \bigcap_{i \in \mathbb{N}^+} N_G(P \cap H_i)H_i = \bigcap_{i \in \mathbb{N}^+} G = G$$

since $N \geqslant N_G(P \cap H_i)$ for every $i \in \mathbb{N}^+$.
So, for every $j \geqslant n_0$, the p-subgroup $P \cap H_j$ is normal in G. □

We are now ready for the following theorem, which strengthen the statement of Proposition 4.3.2, claiming that every profinite Noetherian group has finite prime spectrum.

Theorem 4.3.5. *The prime spectrum of a profinite Noetherian group is finite.*

Proof. If all the Sylow subgroups are finite, then by Proposition 4.3.1 the group is virtually prosolvable, and so its prime spectrum is finite, since it has a finite-index subgroup with finite prime spectrum by Proposition 4.3.2. Then, suppose that there exists at least a p-Sylow subgroup that is not finite and suppose, by way of contradiction, that the prime spectrum of G is infinite. By Lemma 4.3.4 we can find a normal p-subgroup $P \lhd G$ which has finite index in the infinite p-Sylow subgroup whose existence we have assumed. Now, if the quotient G/P was finite, then necessarily the prime spectrum would be finite, contradicting our hypothesis. Then, G/P must be infinite. As the prime spectrum of G is not finite, certainly there exists $q \neq p$ such that q divides the order of this quotient. If in this quotient there are only finite Sylow subgroups, then, again by Proposition 4.3.1, G/P is virtually prosolvable, so it has finite prime spectrum and also G has finite prime spectrum, contradicting the hypothesis. Then also in the quotient G/P there is an infinite q-Sylow subgroup and, applying again Lemma 4.3.4, G/P has a normal q-subgroup which has finite index in the infinite

q-Sylow subgroup, say $Q/P \lhd G/P$. So we have $P \lhd QP \lhd G$.
As, by assumption, the prime spectrum is infinite, we can repeat this argument indefinitely, building an infinite ascending chain of closed subgroups, contradicting the Noetherianity. So the prime spectrum must be finite. □

Theorem 4.3.6. *An infinite profinite Noetherian group is virtually pronilpotent.*

Proof. By Theorem 4.3.5 the prime spectrum of such a group is finite and so for at least one prime number p there exists an infinite p-Sylow subgroup; otherwise in fact the group would be finite. Say $\{p_1, \ldots, p_k\}$ the set of primes with this property. By Lemma 4.3.4 the profinite Noetherian group G admits a normal p_i-subgroup which has finite index in the infinite p_i-Sylow for each $i = 1, \ldots, k$. By construction, for the primes p other than $p_1, \ldots, p_k$, the related p-Sylow subgroups are finite, hence, the subgroup generated by all these p-Sylows is finite. Thus, G is a finite extension of a direct product of a finite number of normal p_i-Sylow subgroups, so G is virtually pronilpotent. □

Throughout the following result, we can deduce a corollary of the previous theorem.

Proposition 4.3.7 (Proposition 11, [13])**.** *Let G be a just infinite profinite group. The following are equivalent:*

1. *G is virtually pronilpotent;*
2. *G is virtually pro-p for some prime p;*
3. *G has finitely many maximal open subgroups.*

Corollary 4.3.8. *A just infinite profinite Noetherian group is virtually pro-p for some prime p.*

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