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Introduction

The way in which a subgroup can be embedded in a group is always a question of particular interest for clearing up the structure of the group. One of the most important subgroup embedding properties is *subnormality*, the transitive closure of the relation of normality. This property was extensively studied by H. Wielandt, who developed much of the theory of subnormal subgroups between 1939 and 1981. It seems that P. Hall considered the subnormal subgroups the "bare bones" of a group, which provide the framework for all the other structures. For an exhaustive treatment of this subject, the reader is referred to the book ([**21**]) of J. C. Lennox and S. E. Stonehewer.
One of the deepest results of H. Wielandt shows that the collection of all subnormal subgroups of an arbitrary finite group forms a sublattice of the full subgroup lattice (Corollary 1.3.7 in [**21**]). Subsequently, an original important generalization of this remarkable property was found by O. H. Kegel in 1978 ([**17**]). For a given class $\mathfrak{F}$ of groups, a subgroup H of a finite group G is called $\mathfrak{F}$-subnormal in G, if there exists a chain of subgroups $H = H_0 \leq \ldots \leq H_n = G$ such that either $H_i \trianglelefteq H_{i+1}$ or $H_{i+1}/(H_i)_{H_{i+1}} \in \mathfrak{F}$ (where $(H_i)_{H_{i+1}}$ denotes the core of H_i in H_{i+1}, that is the largest normal subgroup of H_{i+1} contained in H_i). Kegel proved that if the class $\mathfrak{F}$ is closed under subgroups, epimorphic images and extensions, then the set of all $\mathfrak{F}$-subnormal subgroups of G is also a sublattice of the subgroup lattice. The works of Kegel had a significant effect on much further research in this area (see also Chapter 6 in [**4**]).

Recent developments of these ideas of Kegel can be found, for instance, in the works of A. N. Skiba ([**1**] or [**26**]) and of A. Ballester-Bolinches ([**3**]). Let σ be a partition of the set $\mathbb{P}$ of all positive prime numbers. A group G is said σ-primary if all the primes dividing its order belong to the same member of σ. In ([**26**]), Skiba introduces the concept of σ-subnormality, which is obtained by considering $\mathfrak{F}$ to be the class of all groups which are direct products of σ-primary groups. In particular, in [**5**] Ballester-Bolinches et al. extend to σ-subnormal subgroups a classical criterion of subnormality for factorisable groups (which is due to R. Maier, S. Sidki ([**23**]) and C. Casolo ([**9**])).

In this work, we propose the following generalization of subnormality. A subgroup H of G is said to be $\mathbb{P}$-*subnormal* in G, if there exists a chain of subgroups

$$H = H_0 \leq \ldots \leq H_n = G \tag{1}$$

such that, for every $i = 0, \ldots, n-1$ either $H_i \trianglelefteq H_{i+1}$ or the index $[H_{i+1} : H_i]$ is a prime power number.

It is essentially a discovery of E. Galois that every maximal subgroup of a finite soluble group has index the power of a prime number (see for instance Theorem 5.3 Chapter IV in [**29**]). Thus, it is clear that for every subgroup H of a finite

soluble group G, one can find a sequence of subgroups as in (1) for which each integer $[H_{i+1} : H_i]$ is the power of a prime number. In general, we shall say that H is a $\mathbb{P}$-*subgroup* of G if H admits a chain of this type. Therefore, if $\mathfrak{F}$ is the class of soluble groups, then every $\mathfrak{F}$-subnormal subgroup of a finite group (in the sense of Kegel) is also a $\mathbb{P}$-subnormal subgroup. Of course, the converse implication does not hold in general, for which reason a study of the $\mathbb{P}$-subnormal subgroups makes sense.
The idea for a condition as in (1) comes from the attempt to solve a problem posed by A. F. Vasil'ev, T. I. Vasil'eva and V. N. Tyutyanov in the Kourovka Notebook (Problem 18.30 in [**18**]). In fact, our notion of $\mathbb{P}$-subnormality defined above represents a natural generalization of a similar concept which had been considered by the same authors in [**30**]. The quoted Problem 18.30 was also solved by us in ([**22**]).

This thesis has the following structure. In the first section of Chapter One a series of basic properties of $\mathbb{P}$-subnormality is described. After that, we analyze the condition of $\mathbb{P}$-subnormality in finite simple groups. The main result of the first chapter is Theorem 1.7, that is a complete classification of all $\mathbb{P}$-subnormal subgroups in finite simple groups. In this regard, a key ingredient has been a well-known theorem of R. Guralnick ([**14**]) characterizing the set $\mathcal{G}$ of all the non abelian simple groups admitting maximal subgroups of prime power index (Theorem 1.5). This theorem relies on the classification of finite non abelian simple groups, and therefore most of our results.
As will become immediately clear, some examples in Section 1.5 will caution the reader that many of the classical properties related to subnormality are false in the context of $\mathbb{P}$-subnormal subgroups of an arbitrary finite group. As one might expect, in general neither the intersection nor the join of two $\mathbb{P}$-subnormal subgroups is necessarily $\mathbb{P}$-subnormal; thus the collection of these subgroups does not form a lattice. Further, by considering H and K two $\mathbb{P}$-subnormal subgroups of a group G with $H \leq K$, then H may fail to be a $\mathbb{P}$-subnormal subgroup of K (Example 1.16). However, under the more restrictive assumption that both H and K are $\mathbb{P}$-subgroups of G, then one can show that H is a $\mathbb{P}$-subgroup of K. This is done in Chapter Two (Theorem 2.9), where we also give a description of finite groups G all of whose $\mathbb{P}$-subnormal subgroups are $\mathbb{P}$-subgroups (Theorem 2.5), as well as a characterization of groups with all subgroups being $\mathbb{P}$-subnormal (Proposition 2.8).

In Chapter Three we consider the two classes of groups:

$$\mathcal{C} = \{\text{finite groups all of whose Sylow subgroups are } \mathbb{P}\text{-subnormal}\}$$

$$\mathcal{R} = \{\text{finite groups all of whose soluble subgroups are } \mathbb{P}\text{-subnormal}\},$$

and we prove that they coincide (Theorem 3.1).

The search of possible invariants associated to $\mathbb{P}$-subnormal subgroups is the topic of Chapter Four. For any given $\mathbb{P}$-subnormal subgroup H of a group G, there is naturally defined a family of maximal $\mathbb{P}$-subnormal chains from H to G (Lemma 1.20). We prove the existence of a set of prime numbers $\mathbb{P}_G(H)$ that is associated to H and which does not depend on the choice of the maximal $\mathbb{P}$-subnormal chain (Theorem 4.1). In Proposition 4.4, with the aim of generalizing a criterion of permutability of Wielandt, we show that $\mathbb{P}_G(H) \cap \mathbb{P}_G(K) = \emptyset$ provides a sufficient condition, for arbitrary $\mathbb{P}$-subnormal subgroups H and K of G, for $H \cap K$ and $\langle H, K\rangle$ both being $\mathbb{P}$-subnormal in G.
In the second part of Chapter Four we investigate the normal links of a maximal

$\mathbb{P}$-subnormal chain as in (1). These are those pairs of consecutive terms (H_i, H_{i+1}) such that $H_i \trianglelefteq H_{i+1}$. We adapt to our situation some ideas of R. W. Carter in ([**8**]) and A. Mann in ([**24**]) developed in the context of soluble groups. We prove that when G has no composition factors belonging to the Guralnick's list $\mathcal{G}$, to every $\mathbb{P}$-subnormal subgroup H of G there is associated another invariant, namely the product of all the indices of normal links in a maximal $\mathbb{P}$-subnormal chain having a maximal number of normal links (Theorem 4.10).

Notation. In this thesis we are exclusively concerned with finite groups. Mostly of the notation is standard and we generally follow the book [**20**], with some exceptions. For instance, in the framework of simple groups we use the notation of [**19**]. In particular, we write $L_n(q)$ instead of $PSL_n(q)$, for the n-dimensional projective special linear groups over the field of q elements. Similarly, the symbol $U_n(q)$ is used in place of $PSU_n(q)$, where in this case q is a square. The cyclic group of order n is sometimes denoted by C_n or n.
The symbol $\mathbb{P}$ stands for the set of all prime positive numbers, and for any $n \in \mathbb{N}$ we define

$$\pi(n) := \{p \in \mathbb{P} \mid p \text{ divides } n\}.$$

When G is a finite group we set $\pi(G) = \pi(|G|)$.

CHAPTER 1

Basic properties and $\mathbb{P}$-subnormality in finite simple groups

In this chapter we introduce the notion of $\mathbb{P}$-subnormal subgroups and of $\mathbb{P}$-subnormal chains and we present basics facts concerning these. Furthermore, we analyze the condition of $\mathbb{P}$-subnormality in finite simple groups: as a main result of this chapter, we furnish a complete classification of $\mathbb{P}$-subnormal subgroups in finite simple groups (Theorem 1.7). More specifically, detailed contents of Chapter 1 are the following.

Basic definitions and some elementary properties of $\mathbb{P}$-subnormal subgroups and of $\mathbb{P}$-subnormal chains are given in the first section.

Our account of $\mathbb{P}$-subnormal subgroups in finite simple groups starts with Section 1.2. We first recall a theorem of R. Guralnick (Theorem 1.5) which becomes of fundamental importance for our aims. This result, which relies on the classification of finite simple groups, provides a list $\mathcal{G}$ of all the non abelian simple groups admitting maximal subgroups of prime power index. After that, we state Theorem 1.7, that is a classification of $\mathbb{P}$-subnormal subgroups in finite simple groups. We have collected in Section 1.3 all the auxiliary result needed to prove Theorem 1.7, whose proof is in Section 1.4.

In Section 1.5 we present some examples, in order to show that many of the classical properties of subnormality are not satisfied by $\mathbb{P}$-subnormal subgroups (see, for instance, Example 1.16). In general, neither the intersection nor the join of two $\mathbb{P}$-subnormal subgroups is a $\mathbb{P}$-subnormal subgroup. As the reader will immediately note, these situations occur when the group under consideration is simple and non abelian. For this reason we have decided to postpone all of them nearly at the end of the chapter.

The chapter ends with Section 1.6. Here, we first fix some further terminology related to $\mathbb{P}$-subnormal chains, and then we make use of Theorem 1.5 to prove a refinement lemma for $\mathbb{P}$-subnormal chains (Lemma 1.20). Finally, we give a technical lemma that will be frequently used in the sequel (Lemma 1.24).

We remind the reader that the groups considered in this thesis are always finite.

DEFINITION 1.1. Let G be a group and H a subgroup of G. We say that:

- H is a $\mathbb{P}$-*subgroup* of G if there exists a chain of subgroups

(2) $$\alpha:\ H = H_0 \leq \ldots \leq H_n = G$$

  such that for every $i = 0, \ldots, n-1$ the index $[H_{i+1} : H_i]$ is a power of a prime number. In this case, we write $H \leq_{\mathbb{P}} G$ and any chain like α is called by us a $\mathbb{P}$-*chain from H to G*.

- H is a *$\mathbb{P}$-subnormal subgroup* of G if there exists a chain of subgroups

$$\beta: \ H = K_0 \leq \ldots \leq K_m = G \tag{3}$$

such that for every $i = 0, \ldots, m-1$ either $K_i \trianglelefteq K_{i+1}$ or $[K_{i+1} : K_i]$ is a power of a prime number. In this case, we write $H \trianglelefteq\trianglelefteq_{\mathbb{P}} G$ and any chain like β is called by us a *$\mathbb{P}$-subnormal chain from H to G*.

Note that every $\mathbb{P}$-subgroup of G is a $\mathbb{P}$-subnormal subgroup of G, and that the concept of $\mathbb{P}$-subnormality extends the one of subnormality.
It is a well known fact that when G is soluble then every maximal subgroup of G has prime power index. Therefore, if H is any given subgroup of a soluble group G then there exists a chain of subgroups from H to G which is a $\mathbb{P}$-chain, forcing H to be a $\mathbb{P}$-subgroup of G. Note that the same happens for the simple group $L_2(7)$ (see Remark 1.6).

Definition 1.2. Consider H a $\mathbb{P}$-subnormal subgroup of a group G. If α and β are two $\mathbb{P}$-subnormal chains from H to G, we call β *a refinement of* α if every term of α is also a term of β. If there is at least one term of β which is not a term of α, then β is a *proper refinement of* α. The chain α is a *maximal $\mathbb{P}$-subnormal chain* if α is a $\mathbb{P}$-subnormal chain that has no proper $\mathbb{P}$-subnormal refinements.

The relation of refinement is a partial ordering on the set of all $\mathbb{P}$-subnormal chains from H to G. For any group G and $\mathbb{P}$-subnormal subgroup H of G, we denote by

$$\mathscr{M}(H, G)$$

the set of all maximal $\mathbb{P}$-subnormal chains from H to G.

Note also that if H and K be subgroups of a group G such that $H \leq K$, and if $\alpha: \ H = H_0 \leq \ldots \leq H_n = K$ is a $\mathbb{P}$-subnormal chain from H to K and $\beta: K = K_0 \leq \ldots \leq K_m = G$ is a $\mathbb{P}$-subnormal chain from K to G, then $\alpha\beta$ denotes the $\mathbb{P}$-subnormal chain from H to G obtained by juxtaposing the two chains, that is, $\alpha\beta: \ H = H_0 \leq \ldots \leq H_n = K = K_0 \leq \ldots \leq K_m = G$.

1.1. First properties of $\mathbb{P}$-subnormality

We first list some elementary properties of $\mathbb{P}$-subnormality which follow directly from the definition.

Lemma 1.3. *Let H and N be two subgroups of G.*

1. *If $N \trianglelefteq\trianglelefteq G$ and H is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of G, then $H \cap N$ is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of N.*
2. *If $N \trianglelefteq G$ and H is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of G, then H is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of HN and HN/N is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of G/N.*
3. *Assume that $N \trianglelefteq G$ and $N \leq H$. If H/N is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of G/N, then H is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of G.*
4. *Assume that $H \cap N \trianglelefteq G$ and $G = HN$. If S is a $\mathbb{P}$-subnormal subgroup of G and $S \leq H$, then S is a $\mathbb{P}$-subnormal subgroup of H.*

Proof. (1) Assume that H is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of G and let $N = N_0 \lhd \ldots \lhd N_m = G$ be a subnormal chain from N to G. We argue by induction on $m \geq 1$. Consider $\alpha :\ H = H_0 < \ldots < H_{n-1} < H_n = G$, a $\mathbb{P}$-chain (resp. a $\mathbb{P}$-subnormal chain) from H to G.

If $m = 1$ then N is a normal subgroup of G. If $H_i \trianglelefteq H_{i+1}$ then $H_i \cap N \trianglelefteq H_{i+1} \cap N$. Also, it is an easy calculation to check that the following relation

$$(4) \qquad [H_{i+1} : H_i] = [H_{i+1}N : H_iN]\,[H_{i+1} \cap N : H_i \cap N]\,,$$

is true. Thus, if $[H_{i+1} : H_i] = p^a$, for some $p \in \mathbb{P}$ and $a \geq 1$, from (4) we deduce that $[H_{i+1} \cap N : H_i \cap N]$ is still a power of p. This proves that the chain

$$\alpha_N :\ H \cap N = H_0 \cap N \leq H_1 \cap N \leq \ldots < H_m \cap N = N$$

is a $\mathbb{P}$-subnormal chain. In particular, the previous argument shows that if α is a $\mathbb{P}$-chain then α_N is a $\mathbb{P}$-chain.

When $m \geq 2$ then $H \cap N_{m-1}$ is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of N_{m-1}, for what we have already proved above for $m = 1$ ($N_{m-1} \lhd G$). Since N is subnormal in N_{m-1}, it follows that $H \cap N$ is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of N, by induction. This proves statement (1).

(2) Assume that H is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of G and let N be a normal subgroup of G. Consider $\alpha :\ H = H_0 < \ldots < H_{n-1} < H_n = G$ a $\mathbb{P}$-chain (resp. a $\mathbb{P}$-subnormal chain) from H to G.
We first show that H is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of HN. Note that the chain α admits a $\mathbb{P}$-refinement (resp. a $\mathbb{P}$-subnormal refinement) $\beta :\ H = K_0 \leq \ldots \leq K_m = G$ with the property that K_i is a maximal subgroup of K_{i+1} whenever $K_i \ntrianglelefteq K_{i+1}$ and $i = 0, \ldots, m-1$. By induction on m, we now prove that the chain

$$\alpha^N :\ H = H\,(K_0 \cap N) \leq H\,(K_1 \cap N) \leq \ldots \leq H\,(K_m \cap N) = HN$$

is a $\mathbb{P}$-chain (resp. a $\mathbb{P}$-subnormal chain) from H to HN.
For $m = 0$ there is nothing to do, so we can suppose $m \geq 1$. By the inductive hypothesis we have that H is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of $W = H\,(K_{m-1} \cap N) = HN \cap K_{m-1}$. Now two possibilities arise: either K_{m-1} is maximal in G of prime power index in G or not.
Assume that $[G : K_{m-1}] = p^a$ for some prime number p and $a \geq 1$. Then K_{m-1} is maximal in G. Clearly, if $N \leq K_{m-1} = W$ then H is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of W. Otherwise, N is not contained in K_{m-1} and then $K_{m-1}N = G$, by the maximality of K_{m-1}. Whence,

$$\begin{aligned}[HN : W] &= [HN : H\,(K_{m-1} \cap N)] = [N : N \cap K_{m-1}] \\ &= [K_{m-1}N : K_{m-1}] = [G : K_{m-1}] = p^a.\end{aligned}$$

Thus, W is a $\mathbb{P}$-subgroup of HN and therefore H is a $\mathbb{P}$-subgroup (resp. a $\mathbb{P}$-subnormal subgroup) of HN.
Assume that K_{m-1} is not of prime power index in G. Thus, $K_{m-1} \trianglelefteq G$ and, in this case, α^N is a $\mathbb{P}$-subnormal chain and not a $\mathbb{P}$-chain. It follows that $K_{m-1} \cap N \trianglelefteq G$ and then $W = H\,(K_{m-1} \cap N) = K_{m-1} \cap HN \trianglelefteq HN$. Since H is $\mathbb{P}$-subnormal in W, we have that H is a $\mathbb{P}$-subnormal subgroup of HN.
Finally, we show that $HN/N = H_0N/N \leq \ldots \leq H_mN/N = G/N$ is a $\mathbb{P}$-chain (resp. a $\mathbb{P}$-subnormal chain) from HN/N to G/N. For every $i \in \{0, \ldots, n-1\}$, the condition $H_i \trianglelefteq H_{i+1}$ clearly implies that $H_iN/N \trianglelefteq H_{i+1}N/N$. Also, when $[H_{i+1} : H_i] = p^a$ for some p prime number and $a \geq 1$, the relation (4) yields that $[H_{i+1}N/N : H_iN/N] = [H_{i+1}N : H_iN]$ is a power of p.

(3) By assuming that $H/N = H_0/N \leq \ldots \leq H_m/N = G/N$ is a $\mathbb{P}$-chain (resp. a $\mathbb{P}$-subnormal chain) from H/N to G/N, the same argument just used in the previous case shows that $H = H_0 \leq \ldots \leq H_n = G$ is a $\mathbb{P}$-chain (resp. a $\mathbb{P}$-subnormal chain) from H to G.

(4) Let S be a $\mathbb{P}$-subnormal subgroup of G such that $S \leq H \leq G$. Since $H \cap N \trianglelefteq G$, we obtain that S is $\mathbb{P}$-subnormal in $S\,(H \cap N) = H \cap SN$ and SN/N is $\mathbb{P}$-subnormal in G/N, as a result of (2). Hence,

$$(H \cap SN)\,/\,(H \cap N) = S\,(H \cap N)\,/\,(H \cap N) \trianglelefteq\trianglelefteq_{\mathbb{P}} H/\,(H \cap N)\,,$$

because $G/N = HN/N$ is isomorphic to $H/\,(H \cap N)$. Finally, by using the property in (3) with $H \cap N \vartriangleleft H$ one gets that $S\,(H \cap N)$ is $\mathbb{P}$-subnormal in H. By the transitivity, $S \trianglelefteq\trianglelefteq_{\mathbb{P}} H$. □

We recall that two subgroups H and K of a group G are said to be *permutable* if $HK = KH$. This is equivalent to say that HK is a subgroup of G.

LEMMA 1.4. *Let G be a group and let H, K be two permutable subgroups of G with $K \trianglelefteq\trianglelefteq_{\mathbb{P}} G$.*

(1) *If $HK = G$ then $H \cap K$ is $\mathbb{P}$-subnormal in H. Moreover, when $K \leq_{\mathbb{P}} G$ then $H \cap K \leq_{\mathbb{P}} H$.*
(2) *If H is subnormal in G then K is $\mathbb{P}$-subnormal in HK and HK is $\mathbb{P}$-subnormal in G.*

PROOF. (1) Let $\alpha: K = K_0 < K_1 < \ldots < K_m = G$ be a $\mathbb{P}$-subnormal chain from K to G. In order to prove item (1) it is enough to check that the chain $\alpha_K : H \cap K = H \cap K_0 \leq H \cap K_1 \leq \ldots \leq H \cap K_m = H$ is a $\mathbb{P}$-subnormal chain from $H \cap K$ to H. Now, when $K_i \trianglelefteq K_{i+1}$ then $H \cap K_i \trianglelefteq H \cap K_{i+1}$. Assume that $[K_{i+1} : K_i] = p^a$ for some prime number p and $a \geq 1$. Then $K_i\,(H \cap K_{i+1}) = HK_i \cap K_{i+1} = G \cap K_{i+1} = K_{i+1}$. It follows that $[H \cap K_{i+1} : H \cap K_i] = [K_{i+1} : K_i]\,/\,[HK_{i+1} : HK_i] = [K_{i+1} : K_i] = p^a$. In particular, the previous argument shows that if α is a $\mathbb{P}$-chain then α_K is also a $\mathbb{P}$-chain.

(2) In a minimal counterexample G, choose $K \trianglelefteq\trianglelefteq_{\mathbb{P}} G$ with $[G : K]$ minimal. Clearly, $H \neq 1$ and $HK < G$. Consider $1 \neq S \trianglelefteq\trianglelefteq H$ such that S has minimal order. Then $N = \langle S^g |\ g \in G\rangle$ is a minimal normal subgroup of G.

If $N \leq K$ then K/N is a $\mathbb{P}$-subnormal subgroup of G/N, by Lemma 1.3(2). Since HN/N is subnormal in G/N, the minimal choice of G implies $K/N \trianglelefteq\trianglelefteq_{\mathbb{P}} HK/N \trianglelefteq\trianglelefteq_{\mathbb{P}} G/N$. But this is impossible, by Lemma 1.3(3). Thus $N \not\leq K$. We now have that $[G : KN] < [G : K]$ and $KN \trianglelefteq\trianglelefteq_{\mathbb{P}} G$ (use Lemma 1.3(2) again). By assumption, we have that

$$KN \trianglelefteq\trianglelefteq_{\mathbb{P}} H\,(KN) = (KN)\,H \trianglelefteq\trianglelefteq_{\mathbb{P}} G.$$

Clearly, H is subnormal in $H\,(KN)$ and furthermore K is $\mathbb{P}$-subnormal in $H\,(KN)$, as a consequence of Lemma 1.3(2). For if $H\,(KN) = (KN)\,H < G$ then we would get the contradiction $K \trianglelefteq\trianglelefteq_{\mathbb{P}} HK \trianglelefteq\trianglelefteq_{\mathbb{P}} H\,(KN) \trianglelefteq\trianglelefteq_{\mathbb{P}} G$, by minimality of G.

Hence, we are reduced to the case $G = (HK)\,(KN)$. As a consequence of (1), it follows that $KN \cap HK$ is a $\mathbb{P}$-subnormal subgroup of HK.
Suppose that S is abelian, that is $N \leq O_p\,(G)$ for some prime p. In this case, we have that $[K\,(N \cap HK) : K] = [N \cap HK : N \cap K] = p^a$ and also $[G : HK] = [(HK)\,N : HK] = [N : N \cap HK] = p^b$, for some $a, b \geq 0$. Therefore,

$$K \trianglelefteq\trianglelefteq_{\mathbb{P}} K\,(H \cap KN) = KN \cap HK \trianglelefteq\trianglelefteq_{\mathbb{P}} HK \trianglelefteq\trianglelefteq_{\mathbb{P}} G$$

and this is impossible. Thus, N is a direct product of isomorphic copies of a non abelian simple group S. In particular, the subgroup $M = \left\langle S^{hk} \middle| \; h \in H, \, k \in K \right\rangle$ is normalized both by N and HK. By the condition $G = HKN$, it follows that $M \trianglelefteq G$. But $N \geq M$ and then $N = M$, since N is minimal normal in G. Hence, $N \leq HK$ and then $G = HK$, which is a contradiction. $\square$

1.2. $\mathbb{P}$-subnormal subgroups in finite simple groups

If G is a finite arbitrary group then the set of all $\mathbb{P}$-subnormal chains of G is non empty, since it clearly contains at least the series $1 \lhd G$. We remind the reader that a chain $1 = H_0 \trianglelefteq H_1 \lhd \ldots \lhd H_n = G$ in the group G is said to be a composition series if each H_i is maximal among the proper normal subgroups of H_{i+1}. A composition series can be found by going downward (starting with G) and choosing H_i among the maximal normal subgroups of H_{i+1}. In this way, one can refine any subnormal series to obtain a composition series. The composition factors H_{i+1}/H_i of any given composition series are all simple groups and they constitute a set of invariants of the group, by the Jordan-Hölder Theorem (see for instance 1.8.1 in [**20**]).

Of course, however, if S is any (finite) non abelian simple group, then $1 = H_0 \lhd H_1 = S$ may not be the unique $\mathbb{P}$-subnormal chain from 1 to S, since the group S may admit maximal subgroups of prime power index. In this regard, we recall the reader the following well known result of R. Guralnick below, which relies on the classification of finite non abelian simple groups.

Theorem 1.5 (R. Guralnick, [**14**]). *Let S be a finite non abelian simple group with $H < S$ and $[S : H] = p^a$, for p a prime number. Then one of the following holds.*

(1) *$S = A_n$ and $H \simeq A_{n-1}$ with $n = p^a$.*
(2) *$S = M_{23}$ and $H \simeq M_{22}$ or $S = M_{11}$ and $H \simeq M_{10}$.*
(3) *$S = U_4(2)$ and H is a parabolic subgroup of index 27.*
(4) *$S = L_2(11)$ and $H \simeq A_5$.*
(5) *$S = L_n(q)$ and H is the stabilizer of a line (or of a hyperplane), where $[S : H] = (q^n - 1) / (q - 1) = p^a$ (note that this implies n to be a prime number).*

Moreover, in cases (1), (2), (3) and (5) for $n = 2$, there is a unique S-conjugacy class of such subgroups $H \leq S$, while in cases (4) and (5) with $n > 2$, there are two S-conjugacy classes which are fused in $Aut(S)$. The subgroups H are Hall p'-subgroups of S except in case (1) with $n = p^a > p$ and in case (3).

Remark 1.6. The simple group $L_2(7) \simeq L_3(2)$ has order $168 = 2^3 \cdot 3 \cdot 7$. Its maximal subgroups lie in three different conjugacy classes, two classes of subgroups isomorphic to S_4, having index 7 and the other one of subgroups of type $7 : 3$, having index 8. In particular, it can be easily checked that $L_2(7)$ is the unique simple group having maximal subgroups of two different prime power indices. Since all its maximal subgroups are soluble, we deduce that every subgroup of $L_2(7)$ is a $\mathbb{P}$-subgroup.

We will denote with the symbol

$$\mathcal{G}$$

the class of non abelian simple groups that are isomorphic to one of the groups mentioned in Theorem 1.5.
Also, we reserve the symbols $\mathfrak{F}$ and $\mathfrak{M}$ to denote, respectively, the set of Fermat primes and Mersenne primes , namely:

$$\mathfrak{F} = \{p \in \mathbb{P} \mid p = 2^m + 1,\ m \in \mathbb{N}\},$$

$$\mathfrak{M} = \{p \in \mathbb{P} \mid p = 2^m - 1,\ m \in \mathbb{N}\}.$$

Finally, we remind to the reader, for any group G, then $S(G)$ is the soluble radical of G .
We now state the main result of this chapter, which is a classification of $\mathbb{P}$-subnormal subgroups of S, being S any finite simple group. It is worth remarking that, if H is soluble and H is a maximal subgroup among the soluble $\mathbb{P}$-subnormal subgroups of S, then every proper subgroup of H is $\mathbb{P}$-subnormal in S. Therefore, in the list of Theorem 1.7, we only specify the structure of maximal soluble $\mathbb{P}$-subnormal subgroups of S.

THEOREM 1.7. *Let S be a finite simple group and let H be a proper $\mathbb{P}$-subnormal subgroup of S.*

(1) *If S is abelian, then $H = 1$.*
(2) *If S is non abelian and $S \notin \mathcal{G}$, then $H = 1$.*
(3) *If $S = L_n(q) \in \mathcal{G}$ then one of the following holds.*
  (a) *$n = 2$ and either $q \in \mathfrak{M} \setminus \{7\}$ or $q + 1 \in \mathfrak{F} \cup \{9\}$. Then $H \leq B$, where B is a soluble Frobenius group with elementary abelian kernel of order q and cyclic complement of order $c = (q-1)/gcd(2, q-1)$.*
  (b) *$(n, q) = (2, 7)$ and either $H \leq 7 : 3$ or $H \leq S_4$.*
  (c) *$(n, q) = (2, 11)$ and either $H \leq A_4$ or $H = A_5$.*
  (d) *$(n, q) = (3, 3)$ and $H \leq AGL_2(3)$.*
  (e) *$(n, q) = (3, 8)$ and either $H \leq 2^9 : 7^2$ or $H/S(H) \simeq L_2(8)$.*
  (f) *$n \geq 3$ and $(n, q) \notin \{(3, 2), (3, 3), (3, 8)\}$. Then, either $H \leq F$ where F is a soluble Frobenius group with elementary abelian kernel of order q^{n-1} and cyclic complement of order $c = (q-1)/gcd(n, q-1)$, or $1 \leq S(H) < H$ where $L_{n-1}(q) \leq H/S(H) \leq PGL_{n-1}(q)$ (where $S(H)$ denotes the soluble radical of H).*
(4) *If $S = U_4(2)$ then either $H \leq 2^{1+4} : 6$ or $H/O_2(H) \simeq A_5$.*
(5) *If $S = A_{p^n} \in \mathcal{G}$ then one of the following holds.*
  (a) *$p^n = 7$ and $H \in \{1, A_6\}$.*
  (b) *$p^n = 8$ and $H \in \{1, A_6, A_7\}$.*
  (c) *$p^n = 9$ and $H \in \{1, A_6, A_7, A_8\}$.*
  (d) *$p^n = p \in \mathfrak{F} \setminus \{5\}$ and $H \in \{1, A_{p-1}\}$.*
  (e) *$p^n = 2^n$ with $2^n - 1 \in \mathfrak{M} \setminus \{7\}$ and $H \in \{1, A_{2^n-1}\}$.*
  (f) *$p^n \geq 11$ with $p^n - 1 \notin \mathbb{P}$ and $H \in \{1, A_{p^n-1}\}$.*
(6) *If $S = M_{11}$ then $H \in \{1, A_6, M_{10}\}$.*
(7) *If $S = M_{23}$ then $H \in \{1, M_{22}\}$.*

In order to prove this result we need to carefully analyze the structure of maximal subgroups described in Theorem 1.5. Therefore, we have collected in the next section some crucial preliminary results needed for the proof of Theorem 1.7.

1.3. Some properties of maximal subgroups of simple groups in the class $\mathcal{G}$

A glance at the statement of Theorem 1.5 shows that the classical simple groups in $\mathcal{G}$ are mostly linear groups. In other words, we have to deal with the case of n-dimensional projective special linear groups $S = L_n(q)$, where $n \geq 2$. Here, the last terms of the $\mathbb{P}$-subnormal chains from a given $\mathbb{P}$-subnormal subgroup H of S, are precisely the maximal parabolic subgroups that are stabilizers of lines (or of hyperplanes).

Let $G = SL_n(q)$ be the special linear group of dimension $n \geq 2$ on the field $\mathbb{F}_q$ of order $q = r^f$, where r is a prime number and $f \geq 1$. If we write V for the finite n-dimensional vector space over $\mathbb{F}_q$, a parabolic subgroup P of G is defined as the stabilizer of a flag, that is, a chain of subspaces

$$0 = V_0 < V_1 < \ldots < V_k = V,$$

ordered by inclusion. If such a chain has a subspace of each possible dimension, which means that $dim\,(V_i) = i$ for every $i = 0, \ldots, n$, then it is called a maximal flag. On the other hand, for every proper k-dimensional subspace $W < V$, then $1 < W < V$ is a minimal flag. The maximal parabolic subgroups of G are the stabilizers of minimal flags, while the stabilizers of maximal flags are the Borel subgroups of G.
If W is a k-dimensional subspace of V then we can choose a basis $\{e_1, \ldots, e_k\}$ of W and extend it to a basis $\{e_1, \ldots, e_n\}$ for V. In terms of matrices, the elements of these stabilizers have shape $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, where A and D are non singular $k \times k$ and $(n-k) \times (n-k)$ matrices respectively, such that $det\,(A) \cdot det\,(D) = 1$, and C is an arbitrary $k \times (n-k)$ matrix. The subset U of matrices of shape $\begin{pmatrix} I_k & 0 \\ C & I_{n-k} \end{pmatrix}$ is checked to be an elementary abelian normal subgroup of P of order $q^{k(n-k)}$. The subset L of matrices of shape $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, for which $det\,(A) \cdot det\,(D) = 1$, is also a subgroup of P. Moreover, $U \cap L = 1$ and $U \rtimes L$ is the full stabilizer of W. In the language of Lie theory, the factorization $P = UL$ is a *Levi decomposition* of the parabolic subgroup P, where U is the *unipotent radical* and L is a *Levi complement* of U in P.
As we have said before, in our situations we are mostly concerned with stabilizers of lines or of hyperplanes (i.e. $k \in \{1, n-1\}$) in the quotient group

$$L_n(q) = G/Z, \quad \text{where} \quad Z = Z(G).$$

The fact that every element of Z fixes all the subspaces of V, allows to define the parabolic subgroups of $L_n(q)$ at the same way as stabilizers of flags. It is immediate to see that, P is a parabolic subgroup of G if and only if P/Z is a parabolic subgroup of G/Z, and the map $H \mapsto H/Z$ is a bijection between the set parabolic subgroups P of G and the set of parabolic subgroups P/Z of G/Z. Moreover, if $P = UL$ is a Levi decomposition of the parabolic subgroup P of G, then $UZ/Z \simeq U/U \cap Z \simeq U$ (since r does not divide the order of Z) and L/Z is a Levi complement for UZ/Z in P/Z.

We collect in the next lemma some results concerning the structure of maximal parabolic subgroups of $L_n(q)$.

LEMMA 1.8. *Let P be a maximal parabolic subgroup of $S = L_n(q)$ which is the stabilizer of a line (or of a hyperplane), where $n \geq 2$ and q is the power of a prime number r. Let $P = UL$ be a Levi decomposition of P, where U is the unipotent radical of P and L is a Levi complement.*

(1) *$U = O_r(P) = F^*(P)$ is elementary abelian of order q^{n-1} and $L \simeq GL_{n-1}(q)/Z$, where Z is a central subgroup of order $gcd(n, q-1)$.*
(2) *$S(P)/U = Z(P/U)$ and $S(P) = UZ(L)$ is a Frobenius group with cyclic complement of order $(q-1)/gcd(n, q-1)$.*
(3) *$U = C_S(U)$ and U is irreducible as an L-module. Furthermore, the action of L' on U remains irreducible.*

PROOF. (1) We refer the reader to Lemma 4.1.13 and Proposition 4.1.17 in [**19**].

(2) See Proposition 1.19 in [**10**].

(3) Since P is a maximal parabolic subgroup of S, then $U = O_r(P) \neq 1$ and P is maximal in S. Thus, $C_S(U) \leq N_S(U) = P$, since P is maximal and S is simple. The fact that U coincides with $F^*(P)$ is a consequence of Theorem 1.4.2 in [**12**]. Now, $U \leq C_S(U) = C_P(U)$ because U is abelian (see (1)). Also, $C_P(U) \leq U$ by the Bender-Fitting Theorem (Theorem 9.8 in [**16**]). It follows that $U = C_S(U)$. In particular, U is irreducible as L-module and the action of L' on U remains irreducible (see the remark after Lemma 4.1.13 in [**19**]). □

REMARK 1.9. The normal structure (and, in fact, the subnormal structure) of the general linear group of degree n over a finite field $\mathbb{F}_q$ is well known. Assume that either $n \geq 3$ or $q > 4$. Then, H is a subnormal subgroup of $GL_n(q)$, if and only if, either H contains the special linear group $SL_n(q)$, or H is contained in the center of $GL_n(q)$(see for instance Theorem 9.9, Chapter I in [**28**]).

LEMMA 1.10. *Assume that $G = GL_m(q)/Z_m$, where $Z_m \leq Z(GL_m(q))$ and $L_m(q)$ is a simple group not in $\mathcal{G}$. If $H \leq G$ then every $\mathbb{P}$-subnormal chain from H to G is a subnormal chain.*

PROOF. By way of contradiction, assume there exists a subgroup $H \trianglelefteq\trianglelefteq_{\mathbb{P}} G$ admitting a maximal $\mathbb{P}$-subnormal chain $H = H_0 < H_1 \leq \ldots \leq H_n = G$ which is not a subnormal chain. Consider H of maximal order satisfying these requirements. Hence, $H_i \trianglelefteq H_{i+1}$ for every $i \geq 1$, while H is a maximal subgroup of H_1 with $[H_1 : H] = p^a$ for some $p \in \mathbb{P}$, $a \geq 1$, and H is not normal in H_1.

Since H_1 is a subnormal subgroup of G then necessarily either $H_1 \leq Z(G)$ or $G' \leq H_1$ (see Remark 1.9). Clearly, if H_1 is central in G then $H \lhd H_1$, which is not the case. Thus, $G' \leq H_1$ and of course, $G' \not\leq H$. Since H is maximal in H_1, the condition $H < HG' \leq H_1$ forces $HG' = H_1$. In particular, there exists a maximal subgroup $K < G'$ such that $H \cap G' \leq K < G'$. Since G' is perfect, then $Z(G') \leq K$. For if $Z(G') \not\leq K$ then $KZ(G') = G'$ (by maximality of K) and so

$$G' = [G', G'] = [KZ(G'), KZ(G')] = [K, K] = K' \leq K < G',$$

which is a contradiction. But now $[G' : K]$ divides $[G' : H \cap G'] = [H_1 : H] = p^a$. Thus, we have shown that $G'/Z(G') \simeq L_m(q) \in \mathcal{G}$, which is not. □

We recall the well-known Zsigmondy's result, which we present in a convenient way for our aims.

LEMMA 1.11. *Let p, t be two primes and a, b two positive integers. The equation $p^a + 1 = t^b$ is satisfied if and only if one of the following holds:*

(1) $\left(p^a, t^b\right) = (8, 9)$, *or*
(2) $t = 2$ *and* $p^a = p \in \mathfrak{M}$ *(hence b is a prime number too), or*
(3) $p = 2$ *and* $t^b = t \in \mathfrak{F}$ *(hence $b = 1$ and $a = 2^d$ for some $d \geq 0$).*

PROOF. See the Zsigmondy's result ([**25**]). □

We now prove a number-theoretic auxiliary lemma.

LEMMA 1.12. *Let p, s, t be prime numbers and a, b, c positive integers.*

(1) *If r is a prime divisor of $p^{2a} + p^a + 1$ then either $r = 3$ or $r \equiv 1 \pmod 6$.*
(2) $\left(p^a, t^b, s^c\right)$ *is a solution of the system*

$$\begin{cases} p^a + 1 = t^b \\ p^{2a} + p^a + 1 = s^c \end{cases} \tag{5}$$

if and only if $\left(p^a, t^b, s^c\right) \in \{(2, 3, 7), (3, 4, 13), (8, 9, 73)\}$.

PROOF. (1) Set $q = p^a$ and let $x = \gcd\left(q^2 + q + 1, q - 1\right)$. Since

$$q^2 + q + 1 = 3q + (q - 1)^2 \equiv 3q \pmod x,$$

we conclude that x divides $\gcd\left(3q^2, q - 1\right) = \gcd(3, q - 1) \in \{1, 3\}$. Let r be a prime divisor of $q^2 + q + 1$ and denote by h the multiplicative order of $q + r\mathbb{Z}$ in the field of r elements $\mathbb{Z}/r\mathbb{Z}$. Since r is a divisor of $q^3 - 1$ then h is a divisor of $\gcd(3, q - 1)$. If $h = 1$ the argument above yields $r = 3$ and $q \equiv 1 \pmod 3$. Otherwise, $h = 3$, which implies $r = 3k + 1$, for $k \in \mathbb{N}$. But then $r \equiv 1 \pmod 6$, because r is an odd prime number.

(2) It is straightforward to check that $\left(p^a, t^b, s^c\right) \in \{(2, 3, 7), (3, 4, 13), (8, 9, 73)\}$ satisfies (5). By way of contradiction, let $\left(p^a, t^b, s^c\right)$ be another solution of (5) different from these three. Clearly, $p^a > 8$. As a consequence of Lemma 1.11, we have either $t = 2$ and $p^a = p = 2^k - 1 \in \mathfrak{M} \setminus \{3\}$ or $p = 2$ and $t^b = t = 2^b + 1 \in \mathfrak{F} \setminus \{5\}$. From part (1), one of the following situations occurs: either (a) $s = 3$ or (b) $s \equiv 1 \pmod 6$.

(a) If $s = 3$ then $c \geq 2$. If $p^a = p = 2^k - 1 \in \mathfrak{M} \setminus \{3\}$ then $k = 2l + 1$ and $l \geq 0$ is an odd number. It follows that $3^c = p^2 + p + 1 = p(p + 1) + 1 = 2^{2l+1}\left(2^{2l+1} - 1\right) - 1 = 4 \cdot 4^l\left(4 \cdot 4^l - 1\right) - 1$.
Note that

$$4^l \equiv \begin{cases} 1 \pmod 9 & \text{if } l \equiv 0 \pmod 3 \\ \\ -1 \pmod 9 & \text{if } l \equiv 1 \pmod 3 \\ \\ -2 \pmod 9 & \text{if } l \equiv 2 \pmod 3 \end{cases}.$$

Hence, the condition $p^2 + p + 1 \equiv 0 \pmod 9$ is never satisfied.
If $t^b = t = 2^b + 1 \in \mathfrak{F} \setminus \{5\}$ then $b = 2^h \geq 4$. In particular, either $2^a \equiv -2 \pmod 9$ if h is even or $2^a \equiv 4 \pmod 9$, otherwise. However,

$$3^c = 2^{2a} + 2^a + 1 \equiv \begin{cases} 1 \pmod 9 & \text{if } h \equiv 0 \pmod 2 \\ \\ 3 \pmod 9 & \text{if } h \equiv 1 \pmod 2, \end{cases}$$

which is again a contradiction.

(*b*) If $s \equiv 1 \pmod 6$ then $p^a(p^a+1) \equiv 0 \pmod 6$. By assuming $p^a = p = 2^k - 1 \in \mathfrak{M} \setminus \{3\}$, we get $p^a(p^a+1) = 2^k \cdot p$. Thus $p = 3$, a contradiction. By assuming $p = 2$ then 6 is not a divisor of $p^a(p^a+1) = 2^a \cdot (2^a+1) = 2^a \cdot t$, since t is a prime number and $t \neq 3$. □

Now consider the following subset of $\mathcal{G}$

$$\mathcal{G}_0 := \{L_2(q), L_3(3), L_3(8), U_4(2)\} \cap \mathcal{G}.$$

Recall that, by the above result, $L_2(q) \in \mathcal{G}$ if and only if either $q \in \mathfrak{M} \cup \{8\}$ or $q+1 \in \mathfrak{F}$. Also, recall that $L_2(4) \simeq L_2(5) \simeq A_5$ and $L_2(7) \simeq L_3(2)$.

If S is a 2-dimensional projective special linear group, then the maximal parabolic subgroups of S are exactly the Borel subgroups of S, thus in particular they are soluble. We now use the above description and Lemma 1.10 to characterize the $\mathbb{P}$-subnormal subgroups in linear groups $S \in \mathcal{G} \setminus \mathcal{G}_0$, which means $S = L_n(q)$, with $n \geq 3$ and $(n,q) \notin \{(3,2), (3,3), (3,8)\}$.

Lemma 1.13. *Let $S = L_n(q) \in \mathcal{G}$, where $n \geq 3$ and $(n,q) \notin \{(3,2), (3,3), (3,8)\}$. Let H be a $\mathbb{P}$-subnormal subgroup of S and let $P < S$ the stabilizer of a line (or of a hyperplane) having prime power index in S, such that $H \trianglelefteq\trianglelefteq_{\mathbb{P}} P$. Denote by U the unipotent radical of P. One of the following situations arises.*

1. *$HU \leq S(P)$: a Frobenius group with elementary abelian kernel of order q^{n-1} and cyclic complement of order c, where $c = (q-1)/gcd(n, q-1)$.*
2. *$P' \leq HU$ and either $P' \leq H$ or $H \cap U = 1$. In both cases, we have that $L_{n-1}(q) \leq H/S(H) \leq PGL_{n-1}(q)$.*

Proof. Let $q = r^k$ for $r \in \mathbb{P}$ and $k \geq 1$. By Theorem 1.5(5), n is a prime number. Let H be a $\mathbb{P}$-subnormal subgroup of S. The last proper term of any maximal $\mathbb{P}$-subnormal chain from H to S is a maximal parabolic subgroup P of S (which is the stabilizer of a line or of a hyperplane), and $[S:P] = q^{n-1} + q^{n-2} + \ldots + 1 = s^u$, for some $s \in \mathbb{P}$ and $u \geq 1$, by Theorem 1.5(5). Set $U = O_r(P)$, the unipotent radical of P, and let M be a Levi complement of U in P. By Lemma 1.8(1), it follows that U is elementary abelian and $P/U \simeq M$ is isomorphic to a quotient group of $GL_{n-1}(q)$ module a central subgroup of order $z = gcd(n, q-1)$.

Since $S \in \mathcal{G}$, if $n > 3$ then $n-1$ is not a prime number. Also, if $n = 3$ then $L_2(q)$ is a simple group not in $\mathcal{G}$. Otherwise, there would exist a prime number t and $b \geq 1$ such that $q+1 = t^b$. Moreover, we have $q^2+q+1 = s^u$. The result in Lemma 1.12(2) yields $(r^k, t^b, s^u) \in \{(2,3,7), (3,4,13), (8,9,73)\}$, which gives respectively $S \in \{L_3(2), L_3(3), L_3(8)\}$, contrary to our assumptions. In any case, the previous argument shows that $L_{n-1}(q) \notin \mathcal{G}$ and the assumptions of Lemma 1.10, are satisfied.

Hence, since HU/U is a $\mathbb{P}$-subnormal subgroup of P/U (see Lemma 1.3(2)), then HU/U is subnormal in P/U. Set $Z/U = Z(P/U)$, we conclude that either $HU/U \leq Z/U$ or $P'/U \leq HU/U$, depending on whether H is soluble or not (see Remark 1.9). If $HU \leq Z$ then H is contained in $Z = S(P)$, which is a soluble Frobenius group with elementary abelian kernel U and cyclic complement $Z(M)$ of order $c = (q-1)/gcd(n, q-1)$, by part (2) of Lemma 1.8.

Assume that $P' \leq HU$. Thus, $P'Z/Z \leq HZ/Z$ and then

$$L_{n-1}(q) \leq HZ/Z \leq PGL_{n-1}(q), \tag{6}$$

since we have $P'Z/Z \simeq L_{n-1}(q)$ and $P/Z = PGL_{n-1}(q)$, by the assumptions on P. In particular, note that the soluble radical of HZ/Z is trivial, since HZ/Z is almost simple and, further $H \cap Z \leq S(H)$. By the isomorphism $HZ/Z \simeq H/H \cap Z$, it follows that $S(H) \leq Z$. Whence, $S(H) \leq Z$ and then the equality $H \cap Z = S(H)$ holds. Thus, $H/S(H) = H/H \cap Z \simeq HZ/Z$ and we conclude that $L_{n-1}(q) \leq H/S(H) \leq PGL_{n-1}(q)$, again by (6).
Finally, we have that $[H \cap U, P'] \leq [H \cap U, HU] \leq H \cap U$, because U is abelian and $H \cap U \lhd H$. Since the action of P' on U is irreducible then either $H \cap U = 1$ or $U \leq H$ (see Lemma 1.8(3)). $\square$

For all the other simple groups in $\mathcal{G}$, the non classical ones, the set of $\mathbb{P}$-subnormal subgroups is completely determined by numerical constraints.

LEMMA 1.14. *Let $S \in \mathcal{G}$ be a non classical group. If σ is any maximal $\mathbb{P}$-subnormal chain from 1 to S then one of the following situations holds.*

(1) *$S = A_7$ and $\sigma : 1 \lhd A_6 < S$.*
(2) *$S = A_8$ and $\sigma : 1 \lhd A_6 < A_7 < S$.*
(3) *$S = A_9$ and $\sigma : 1 \lhd A_6 < A_7 < A_8 < S$.*
(4) *$S = A_{p^n}$, where either $p^n - 1 \in \mathfrak{M} \setminus \{7\}$ or $p^n = p \in \mathfrak{F} \setminus \{5\}$, and $\sigma : 1 \lhd A_{p^n-2} < A_{p^n-1} < S$.*
(5) *$S = A_{p^n}$, with $p^n \geq 11$ and $p^n - 1 \notin \mathbb{P}$, and $\sigma : 1 \lhd A_{p^n-1} < S$.*
(6) *$S = M_{11}$ and $\sigma : 1 \lhd A_6 \lhd M_{10} < S$.*
(7) *$S = M_{23}$ and $\sigma : 1 \lhd M_{22} < S$.*

PROOF. Write $\sigma : 1 = H_0 < \ldots < H_m = S$, where $m \geq 1$. By Theorem 1.5, we have that H_{m-1} is a maximal subgroup of S having prime power index. One of the following situations occurs.

Case $S \simeq A_{p^n}$ with $p^n \geq 7$ (recall that $A_5 \simeq L_2(4) \in \mathcal{G}_0$).
By Theorem 1.5, it is $H_{m-1} \simeq A_{p^n-1}$. It is straightforward to check that the values $p^n \in \{7, 8, 9\}$ give rise to a chain σ as in (1), (2) and (3) respectively. Thus, suppose $p^n \geq 11$.
Let $m \geq 3$. Then $H_{m-2} \simeq A_{p^n-2} \in \mathcal{G}$. This is equivalent to say that there exist r, s prime numbers and integers $u, v \geq 1$ such that both conditions $p^n - 1 = r^u$ and $p^n - 2 = s^v$ hold. As a consequence of Lemma 1.12(1), we deduce that either $p^n - 1 \in \mathfrak{M} \setminus \{7\}$ is a Mersenne prime or $p^n = p \in \mathfrak{F} \setminus \{5\}$ is a Fermat prime. In the first case, we get that $p^n - 2 = p^n - 1 - 1 = 2^n - 2 = 2\left(2^{n-1} - 1\right)$, with $n \geq 5$ a prime number. This clearly implies that $A_{p^a-2} \notin \mathcal{G}$. In the latter case, one has $p^n - 2 = p - 2 = 2^k + 1 - 2 = 2^k - 1$, where $k = 2^h$ for some $h \geq 1$. Arguing in the same way as before, it follows that $A_{p^n-2} \notin \mathcal{G}$, since $p^n - 2 = 2^k - 1$ is neither a Mersenne prime nor $p^n - 1 = 2^k$ is a Fermat prime. We conclude that $m = 3$ and σ is the chain in (4).
Let $m = 2$, that is $H_{m-1} \notin \mathcal{G}$. As a consequence of the previous argument, it is necessary that $p^n - 1 \notin \mathfrak{M}$ and if $n = 1$ then $p = p^n \notin \mathfrak{F}$. Since we are assuming that $p^n \geq 11$ then σ is the chain in (5).

Case $S = M_{11}$.
Then $H_{m-1} \simeq M_{10}$. Now, the Mathieu group M_{10} has a unique normal subgroup isomorphic to $A_6 \notin \mathcal{G}$ which is its only maximal subgroup with prime power index. Thus, $H_{m-2} \simeq A_6$ and $H_{m-3} = 1$, since $A_6 \notin \mathcal{G}$. We conclude that σ is the chain in (6).

Case $S = M_{23}$.
Then $H_{m-1} \simeq M_{22}$ is a non abelian simple group which does not belong to $\mathcal{G}$. Thus $m = 2$ and σ is the chain in (7). □

1.4. Proof of Theorem 1.7

We can apply all the results of the previous section to prove the classification in Theorem 1.7, which we have stated in Section 1.2. We report its statement for convenience of the reader.

THEOREM. 1.7 *Let S be a finite simple group and let H be a proper $\mathbb{P}$-subnormal subgroup of S.*

(1) *If S is abelian, then $H = 1$.*
(2) *If S is non abelian and $S \notin \mathcal{G}$, then $H = 1$.*
(3) *If $S = L_n(q) \in \mathcal{G}$ then one of the following holds.*
    (a) *$n = 2$ and either $q \in \mathfrak{M} \setminus \{7\}$ or $q + 1 \in \mathfrak{F} \cup \{9\}$. Then $H \leq B$, where B is a soluble Frobenius group with elementary abelian kernel of order q and cyclic complement of order $c = (q-1)/gcd(2, q-1)$.*
    (b) *$(n, q) = (2, 7)$ and either $H \leq 7 : 3$ or $H \leq S_4$.*
    (c) *$(n, q) = (2, 11)$ and either $H \leq A_4$ or $H = A_5$.*
    (d) *$(n, q) = (3, 3)$ and $H \leq AGL_2(3)$.*
    (e) *$(n, q) = (3, 8)$ and either $H \leq 2^9 : 7^2$ or $H/S(H) \simeq L_2(8)$.*
    (f) *$n \geq 3$ and $(n, q) \notin \{(3, 2), (3, 3), (3, 8)\}$. Then, either $H \leq F$ where F is a soluble Frobenius group with elementary abelian kernel of order q^{n-1} and cyclic complement of order $c = (q-1)/gcd(n, q-1)$, or $1 \leq S(H) < H$ where $L_{n-1}(q) \leq H/S(H) \leq PGL_{n-1}(q)$ (where $S(H)$ denotes the soluble radical of H).*
(4) *If $S = U_4(2)$ then either $H \leq 2^{1+4} : 6$ or $H/O_2(H) \simeq A_5$.*
(5) *If $S = A_{p^n} \in \mathcal{G}$ then one of the following holds.*
    (a) *$p^n = 7$ and $H \in \{1, A_6\}$.*
    (b) *$p^n = 8$ and $H \in \{1, A_6, A_7\}$.*
    (c) *$p^n = 9$ and $H \in \{1, A_6, A_7, A_8\}$.*
    (d) *$p^n = p \in \mathfrak{F} \setminus \{5\}$ and $H \in \{1, A_{p-1}\}$.*
    (e) *$p^n = 2^n$ with $2^n - 1 \in \mathfrak{M} \setminus \{7\}$ and $H \in \{1, A_{2^n-1}\}$.*
    (f) *$p^n \geq 11$ with $p^n - 1 \notin \mathbb{P}$ and $H \in \{1, A_{p^n-1}\}$.*
(6) *If $S = M_{11}$ then $H \in \{1, A_6, M_{10}\}$.*
(7) *If $S = M_{23}$ then $H \in \{1, M_{22}\}$.*

PROOF. We may assume that S is non abelian, otherwise the result is trivial. If $S \notin \mathcal{G}$ then, by definition, 1 is the only proper normal subgroup of S and it is $\mathbb{P}$-subnormal in S. Thus, assume that S is one of the groups listed in Theorem 1.5. If S is not a classical group then H is one of the groups in items (5)-(7), by Lemma 1.14. We are left with the case of linear groups in $\mathcal{G}$.

Case $S = L_2(q)$ and $q \notin \{7, 11\}$.
Then P is a Borel subgroup of S. Further, since $q + 1$ is the power of a prime number t, then either $q \in \mathfrak{M} \setminus \{7\}$ or $q + 1 \in \mathfrak{F} \cup \{9\}$, by Lemma 1.11. We conclude that H is contained in a soluble Frobenius group $P \simeq q : c$ with elementary abelian kernel of order q and cyclic complement of order $c = (q-1)/gcd(2, q-1)$ (see Lemma 1.8(1)). Thus, item $(3a)$ holds.

Case $S = L_2(7)$.
From the isomorphism $L_2(7) \simeq L_3(2)$, it follows that S admits two conjugacy classes of maximal subgroups of different prime power index. Thus, either $P \simeq 7:3$ or $P \simeq S_4$ (see Remark 1.6), as described in $(3b)$.

Case $S = L_2(11)$.
Then $P \simeq A_5 \simeq L_2(4)$. If $H \neq P$ then H is isomorphic to a subgroup of $A_4 \simeq 2^2 : 3$, by the result in $(3a)$. Otherwise, $H = P = A_5$. We have proved that $(3c)$ holds.

Case $S = L_n(q)$ with $n \geq 3$ and $(n,q) \notin \{(3,2), (3,3), (3,8)\}$.
Then P is a maximal parabolic subgroup of S which is the stabilizer of a line or of a hyperplane and Lemma 1.13 holds. If H is soluble, then H is contained in a soluble Frobenius group F, with elementary abelian kernel of order q^{n-1} and cyclic complement of order $c = (q-1)/(n, q-1)$. Otherwise, H is a non soluble group and $L_{n-1}(q) \leq H/S(H) \leq PGL_{n-1}(q)$.

Case $S = L_3(3)$.
Then $P \simeq 3^2 : GL_2(3)$ and $H \leq P$, which is the case in item $(3d)$.

Case $S = L_3(8)$.
Then $P \simeq 2^6 : GL_2(8)$ is a maximal parabolic subgroup of S. Set $N = S(P)$. We claim that $HN < P$ if and only if H is soluble. Indeed, note that H is soluble if and only if HN/N is soluble. Furthermore, HN/N is a $\mathbb{P}$-subnormal subgroup of P/N, by Lemma 1.3(2). Since $P/N \simeq L_2(8)$, we may use the result in $(3a)$ to get that HN/N is soluble if and only if $HN/N \leq M/N$, where M/N is a soluble $\{2,7\}$-Hall subgroup of P/N. Whence, H is soluble if and only if $HN < P$ and, in particular $H \leq M$, where $M \simeq 2^9 : 7^2$.
Now suppose that H is non soluble. By the argument above, then $HN = P$ and, furthermore, one has $L_2(8) \simeq P/N = HN/N \simeq H/H \cap N$. Hence, $H \cap N \leq S(H)$ since $H \cap N \trianglelefteq H$ and N is soluble. It follows that $H \cap N = S(H)$, because the quotient group $H/H \cap N$ is simple and non abelian. We conclude that $H/Sol(H) \simeq L_2(8)$. This proves the result in item $(3e)$.

Case $S = U_4(2)$.
Let P be the last proper term of a $\mathbb{P}$-subnormal chain from H to S. Then P is a maximal parabolic subgroup of S and $P \simeq 2^4 : A_5$, by Lemma 1.8(1). Set $N = O_2(P)$ the unipotent radical of P, then HN/N is a $\mathbb{P}$-subnormal subgroup of P/N, by Lemma 1.3(2).
Let $HN < P$. Since $P/N \simeq A_5$, by the result in $(3a)$ it follows that $HN/N \leq M/N$, where $M/N \simeq A_4$. Hence $H \leq M$ and M is soluble because N is soluble. In particular, we get that H is contained in a soluble subgroup $M \simeq 2^{1+4} : 6$.
Finally, let $HN = P$. Then $H/H \cap N \simeq HN/N \simeq P/N \simeq A_5$ and, furthermore, $H \cap N = H \cap O_2(P) \leq O_2(H)$. But $H/H \cap N$ is simple and non abelian, thus $O_2(H) = H \cap N$. All these situations are described in item (4). $\square$

1.5. Bad behavior of $\mathbb{P}$-subnormality

In this section we exploit our knowledge of $\mathbb{P}$-subnormal subgroups in non abelian simple groups coming from the results of Section 1.3, in order to make some examples. Through the description of these situations our aim is to caution the reader that many interesting properties about subnormality fail for $\mathbb{P}$-subnormal subgroups.

In general the intersection of two arbitrary $\mathbb{P}$-subgroups of G may fail to be a $\mathbb{P}$-subgroup of G, so that the assumption of $N \trianglelefteq\trianglelefteq G$ in Lemma 1.3(1) can not be dropped.

EXAMPLE 1.15. Let $G = A_7$ be the alternating group on seven objects. For every $i \in \{1, 2, \ldots, 7\}$ the corresponding one point stabilizer $M_i = Stab_G(i) \simeq A_6$ is a $\mathbb{P}$-subgroup of G, because it has index 7. Let $H = M_i \cap M_j$ for $i \neq j$. Then $H \simeq A_5$ and the only subgroups of G of prime power index containing H are precisely M_i and M_j, each of which contains H as a maximal subgroup, by Lemma 1.14(1). This shows that H is not a $\mathbb{P}$-subgroup of G, since $[M_i : H] = 6$.

When dealing with subnormal subgroups the following property is basic and crucial.

E: Let H and K be two subnormal subgroups of a group G. If $H \leq K$ then H is subnormal in K.

The next example shows that the property **E** is not true in general in the realm of $\mathbb{P}$-subnormality.

EXAMPLE 1.16. Consider $G = SL_5(13) = L_5(13)$. Let $\{e_1, \ldots, e_5\}$ be a basis for a 5-dimensional vector space on the field $\mathbb{F}_{13}$, and let λ be a generator for the multiplicative group $\mathbb{F}_{13}^*$ and $g = diag\left(\lambda^8, \lambda, \lambda, \lambda, \lambda\right) \in G$. The maximal parabolic subgroups $P = Stab_G(\langle e_1, e_2, e_3, e_4\rangle)$ and $Q = Stab_G(\langle e_2, e_3, e_4, e_5\rangle)$ are $\mathbb{P}$-subnormal in G, both having prime index 30941. The element g belongs to the center of Q, therefore $\langle g\rangle$ is $\mathbb{P}$-subnormal in G. However, $g \in L$ where L is a Levi factor of P. Note that $L \simeq GL_4(13)$ and g is not central in L. Let $N = UZ(L)$, where U is the unipotent radical of P and use the "bar notation" for the elements and the subgroups of $P/N = \overline{P}$. By Lemma 1.3(2), if $\langle g\rangle$ were $\mathbb{P}$-subnormal in P then $\overline{\langle g\rangle}$ would be $\mathbb{P}$-subnormal in $\overline{P} \simeq PGL_4(13)$. This contradicts Lemma 1.10, since $L_4(13) \notin \mathcal{G}$ (see Theorem 1.5).

It is a result of H. Wielandt that the subgroup generated by two subnormal subgroups is still a subnormal subgroup (see for instance Theorem 6.7.1 in [**20**]). With regard to this issue and with respect to the $\mathbb{P}$-subnormality condition, we remind the following.

EXAMPLE 1.17. Let $s = 2^b + 1$ be a Fermat prime number greater that 5 and let $G = L_2\left(2^b\right)$. Consider G as a permutation group on the points of the projective line over the field $\mathbb{F}_q$, with $q = 2^b$ elements. Then G is doubly transitive of prime degree s and the identity is the only element that fixes three distinct points (Zassenhaus considered this class of groups and classified those groups which are triply transitive on a set of finite symbols, see for instance Section 1 of [**27**]).

If B is a Borel subgroup, namely the stabilizer in G of one point, then $[G : B] = s$ and B is a Frobenius group with Frobenius kernel U, an elementary abelian Sylow 2-subgroup of G, and complement L, cyclic of order $2^b - 1$ (see Theorem 1.7($3a$)). In particular, $U = C_G(u)$, whenever $1 \neq u \in U$. On the other hand, let v be an involution of G which is not contained in B. Then $L = B \cap B^v$ consists of elements inverted by v and $N_G(L) = L \rtimes \langle v\rangle$.

Thus, if $w \in L$ is an element of prime order p, then $w^v = w^{-1}$ and $T = \langle v, w\rangle$ is a dihedral group of order $2p$. Since B is soluble we have that both U and $\langle w\rangle$ are $\mathbb{P}$-subgroups of B and, since B is a $\mathbb{P}$-subgroup of G, we see that U and $\langle w\rangle$ are $\mathbb{P}$-subgroups of G. In particular, $\langle v\rangle$ is a $\mathbb{P}$-subgroup of G, because $v \in G$ is conjugated to an involution u which lies in B. However, T is not a $\mathbb{P}$-subnormal subgroup of G, since every $\mathbb{P}$-subnormal subgroup of G lies in a conjugate of B (see Theorem 1.7(3a)), which is impossible for T by the structure of the Borel subgroups.

We conclude this section with a remark.

REMARK 1.18. The assumption of $H \trianglelefteq\trianglelefteq G$ in Lemma 1.4(2) can not be dropped. It suffices to consider the subgroups $H = \langle v \rangle$ and $K = \langle w \rangle$ as defined in Example 1.17(1). Indeed, H and K are $\mathbb{P}$-subnormal subgroups of G such that $HK = \langle v, w \rangle = KH$ is not $\mathbb{P}$-subnormal in G.

1.6. $\mathbb{P}$-subnormal refinements

In this section we introduce some further terminology associated to $\mathbb{P}$-subnormal chains.

DEFINITION 1.19. Let G be a group and let be a H be $\mathbb{P}$-subnormal subgroup of G with $\mathbb{P}$-subnormal chain

$$\alpha: \; H = H_0 < \ldots < H_m = G.$$

We say that (H_i, H_{i+1}) is a *p-link* of α if $[H_{i+1} : H_i] = p^a$, for some prime p and $a \geq 1$. We call (H_i, H_{i+1}) a *normal link* of α if $H_i \trianglelefteq H_{i+1}$. If (H_i, H_{i+1}) is a normal link which is also a p-link then we sometimes refer to it as an *abelian normal link* of α.

Guralnick's Theorem 1.5 allows us to consider a natural family of $\mathbb{P}$-subnormal refinements for any given $\mathbb{P}$-subnormal chain.

LEMMA 1.20. *Let G be a group with $H \trianglelefteq\trianglelefteq_{\mathbb{P}} G$, and let α be a $\mathbb{P}$-subnormal chain from H to G. Then α has a maximal $\mathbb{P}$-subnormal refinement $\beta: \; H = T_0 < T_1 < \ldots < T_m = G$, such that, for every $i = 0, \ldots, m-1$*

- *T_i is a maximal subgroup of T_{i+1}, if (T_i, T_{i+1}) is a p-link of β (for some p prime number);*
- *T_{i+1}/T_i is a non abelian simple group not in $\mathcal{G}$, if (T_i, T_{i+1}) is a non abelian normal link of β.*

PROOF. By induction on $[G : H]$.

If $H = G$ there is nothing to prove, so we may assume that $\alpha: \; H = H_0 < H_1 \leq \ldots \leq H_m = G$ is a $\mathbb{P}$-subnormal chain with $m \geq 1$.

If H has prime power index in H_1 then consider $H = K_0 < K_1 < \ldots < K_n = H_1$, where each K_i is maximal in the next. Since $[G : H_1] < [G : H]$, by inductive hypothesis there exists $\beta_1 : \; H_1 = H_{1,0} < H_{1,1} < \ldots < H_{1,n} = G$ maximal $\mathbb{P}$-subnormal chain which is a refinement of $H_1 \leq H_2 \leq \ldots < H_m = G$ satisfying the requirements. Now $\beta: \; H = K_0 < K_1 < \ldots < K_n = H_1 = H_{1,0} < H_{1,1} < \ldots < H_{1,n} = G$ is a desired $\mathbb{P}$-subnormal refinement of α.

Assume now that H is a normal subgroup of H_1, then two cases arise: either H is a maximal normal subgroup of H_1 or not.

If $H \lhd K_1 \lhd H_1$ then since $[G : K_1]$ and $[K_1 : H]$ are smaller that $[G : H]$, by the inductive hypothesis, one can find a maximal $\mathbb{P}$-subnormal refinement β_1 of $H \lhd K_1$ and a maximal $\mathbb{P}$-subnormal refinement β_2 of $K_1 \leq H_1 \leq \ldots \leq H_m = G$ with the required properties. Then $\beta = \beta_1\beta_2$ is a chain as described in the statement.

Assume now that H is a maximal normal subgroup of H_1. Then H_1/H is simple.

By applying Theorem 1.5 and Lemma 1.3(3), the chain $H \lhd H_1$ has a proper $\mathbb{P}$-subnormal refinement if and only if H_1/H is non abelian simple group which lies in $\mathcal{G}$. In this case, there exists a maximal subgroup M of H_1 containing H such that the index $[H_1 : M]$ is a prime power. Now both $[M : H]$ and $[G : M]$ are smaller that $[G : H]$, and we can conclude the argument as before by induction. Finally, if H_1/H is isomorphic to a simple group not in $\mathcal{G}$, then we are done if $H_1 = G$, otherwise we use induction to refine the $\mathbb{P}$-subnormal chain $H_1 \leq \ldots \leq H_m = G$ and conclude the proof in the same way. □

We apply the Lemma 1.20 in the following example.

EXAMPLE 1.21. Let $G = GL_2(8) = Z(G) \times G' \simeq C_7 \times L_2(8)$. Then G is the group of the linear maps of a vector space V of dimension 2 over the field $\mathbb{F}_8$ with 8 elements. If $\{v, w\}$ is a basis for V then

$$B = B(v) = \left\{ \left(\begin{array}{cc} \delta_1 & 0 \\ \lambda & \delta_2 \end{array} \right) \middle| \ \delta_1, \delta_2 \in \mathbb{F}_8^*, \ \lambda \in \mathbb{F}_8 \right\}$$

is the stabilizer of the 1-dimensional subspace $\langle v \rangle = \{ kv | \ k \in \mathbb{F}_8 \}$ generated by v, and $[G : B] = 9$. We may decompose B as a semidirect product $B = U \rtimes L$, where $U = \left\{ \left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right) \middle| \ \lambda \in \mathbb{F}_8 \right\}$ is a Sylow 2-subgroup of G and $L = B(v) \cap B(w) = \left\{ \left(\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array} \right) \middle| \ \delta_1, \delta_2 \in \mathbb{F}_8^* \right\}$ is elementary abelian of order 49. Note that L is maximal in B, whereas $U < T$, where

$$T = \left\{ \left(\begin{array}{cc} 1 & 0 \\ \lambda & \delta_2 \end{array} \right) \middle| \ \delta_2 \in \mathbb{F}_8^*, \ \lambda \in \mathbb{F}_8 \right\} \lhd B$$

is the kernel of the action of B on $\langle v \rangle$.

Let $H \trianglelefteq\trianglelefteq_{\mathbb{P}} G$. Consider a maximal $\mathbb{P}$-subnormal chain $\mu : \ H = H_0 < H_1 < \ldots < H_m = G$ from H to G and set $M = H_{m-1}$. By Lemma 1.20, one of the following situations arises.

(1) M is not normal in G. Then (M, G) is a p-link of μ (for some prime p) and M is a maximal subgroup of G. Thus, $M \neq G'$ and $G'M = G$. One can check that there exists a suitable basis $\{v, w\}$ of V such that $M = B(v)$ and $[G : B] = 9$, so $p = 3$.

(2) M is normal in G. If $M = Z(G)$ then $G/Z(G)$ would be isomorphic to $L_2(8) \in \mathcal{G}$, which contradicts our assumption on μ. Therefore, it is necessarily true that $M = G'$ and (M, G) is an abelian normal link of μ. Moreover, if $H < G'$ then $H \leq B(v) \cap G'$ for some $0 \neq v \in V$ (Theorem 1.5(5)). We deduce that either $H = G'$ or H is a soluble group; in particular, μ is always a $\mathbb{P}$-chain from H to G.

DEFINITION 1.22. Let G be a group and H a $\mathbb{P}$-subnormal subgroup of G with $\mathbb{P}$-subnormal chain $\alpha : \ H = H_0 < \ldots < H_m = G$. For every $i \in \{0, \ldots, m-1\}$ we denote $[H_{i+1} : H_i]$ by $\alpha(i)$. Set

$$\Lambda_G(H, \alpha) = \{\alpha(i)\}_{i=0,\ldots,m-1},$$

and

$$\mathbb{P}_G(H, \alpha) = \{p \in \mathbb{P} \mid \exists \, i \in \{0, \ldots, m-1\} \text{ such that } \alpha(i) = p^a\}.$$

REMARK 1.23. It is clear from the definition above that $[G : H] = \prod \alpha(i)$, the product being taken over all $\alpha(i) \in \Lambda_G(H, \alpha)$. Moreover, up to removing the repeated terms, $|\Lambda_G(H, \alpha)|$ is the length of α.

The following Lemma describes the behavior of the list $\Lambda_G(H,\alpha)$ when G has a normal subgroup, for every maximal $\mathbb{P}$-subnormal chain α from a given $\mathbb{P}$-subnormal subgroup H of G.

LEMMA 1.24. *Let G be a group, let $H \trianglelefteq\trianglelefteq_{\mathbb{P}} G$ and let $N \trianglelefteq G$. Assume that $\alpha :\ H = H_0 < \ldots < H_m = G$ is a maximal $\mathbb{P}$-subnormal chain from H to G. Set*

$$\alpha_N :\ H \cap N = H_0 \cap N \leq \ldots \leq H_m \cap N = N,$$

and

$$\alpha^{G/N} :\ HN/N = H_0N/N \leq \ldots \leq H_mN/N = G/N.$$

Then the following hold.

(1) *α_N is a $\mathbb{P}$-subnormal chain such that $\mathbb{P}_N(H\cap N, \alpha_N) = \mathbb{P}_N(H \cap N, \beta)$, whenever β is a maximal $\mathbb{P}$-subnormal refinement of α_N.*
(2) *$\alpha^{G/N}$ is a maximal $\mathbb{P}$-subnormal chain from HN/N to G/N.*

Moreover,

$$\Lambda_G(H,\alpha) = \Lambda_N(H\cap N, \alpha_N) \cup \Lambda_{G/N}(HN/N, \alpha^{G/N}) \tag{7}$$

and consequently,

$$\mathbb{P}_G(H,\alpha) = \mathbb{P}_N(H\cap N, \alpha_N) \cup \mathbb{P}_{G/N}(HN/N, \alpha^{G/N}). \tag{8}$$

PROOF. For every $i \in \{0, \ldots, m-1\}$ the subgroup $H_i\,(H_{i+1} \cap N) = H_{i+1} \cap H_iN$ lies between H_i and H_{i+1}. By the assumptions on α (Lemma 1.20), one of two possibilities arises: either H_i is a maximal subgroup of H_{i+1} or H_i is a normal maximal subgroup of H_{i+1}.
In both cases, the condition $H_i \leq (H_{i+1} \cap N)\, H_i \leq H_{i+1}$ implies that either $H_iN = H_{i+1}N$ or $H_i \cap N = H_{i+1} \cap N$. Since this holds for every $i = 0, \ldots, m-1$, by the identity $[H_{i+1} : H_i] = [H_{i+1}N : H_iN]\,[H_{i+1} \cap N : H_i \cap N]$, we get

$$\Lambda_G(H,\alpha) = \Lambda_N(H\cap N, \alpha_N) \cup \Lambda_{G/N}(HN/N, \alpha^{G/N}),$$

which is (7). Equality in (8) is an immediate consequence of the above relation.

(1) We have already noted in the proof of Lemma 1.3(1) that α_N is a $\mathbb{P}$-subnormal chain. Consider $\beta :\ H \cap N = K_0 < \ldots < K_n = N$ to be any maximal $\mathbb{P}$-subnormal refinement of α_N. By definition, for every $i \in \{0, \ldots, m-1\}$ such that $H_i \cap N < H_{i+1} \cap N$ there exists $j\,(i) = j \in \{0, \ldots, n-1\}$ such that $K_j = H_i \cap N < K_{j+1} \leq H_{i+1} \cap N$, where K_{j+1} is a $\mathbb{P}$-subnormal subgroup of $H_{i+1} \cap N$. Clearly, if $(H_i \cap N, H_{i+1} \cap N)$ is a p-link of α_N, for some p prime number, then (K_j, K_{j+1}) is a p-link of β. Assume that $(H_i \cap N, H_{i+1} \cap N)$ is not a p-link of α_N. Then necessarily $H_i \lhd H_{i+1}$. It follows that

$$(H_{i+1} \cap N)\,/\,(H_i \cap N) \simeq (H_{i+1} \cap N)\, H_i/H_i = H_{i+1}/H_i$$

is a non abelian simple group which does not belong to $\mathcal{G}$, because α is maximal (Lemma 1.20). Hence, we conclude that $(H_i \cap N, H_{i+1} \cap N)$ is not a p-link of α_N. Moreover, $K_{j+1}/\,(H_i \cap N)$ is a $\mathbb{P}$-subnormal subgroup of $(H_{i+1} \cap N)\,/\,(H_i \cap N)$, since Lemma 1.3(2) holds. In particular, we have that $K_{j+1} = H_{i+1} \cap N$, because $H_{i+1}/H_i \notin \mathcal{G}$.
We have proved that (K_j, K_{j+1}) is a p-link of β (for some prime p) if and only if $(H_i \cap N, H_{i+1} \cap N)$ is a p-link of α_N. This means that

$$\mathbb{P}_N\,(H \cap N, \alpha_N) = \mathbb{P}_N\,(H \cap N, \beta)\,,$$

as desired.

(2) The result in Lemma 1.3(2) shows that $\alpha^{G/N}$ is a $\mathbb{P}$-subnormal chain in G/N. In particular, by applying the correspondence theorem, the relation

$$[H_{i+1} : H_i] = [H_{i+1}N : H_iN]\,[H_{i+1} \cap N : H_i \cap N]$$

yields that $\alpha^{G/N}$ is a maximal $\mathbb{P}$-subnormal chain of $HN/N \leq G/N$. $\square$

As the reader might certainly imagine, the chains α_N as described in Lemma 1.24(1) do not need to be maximal $\mathbb{P}$-subnormal chains.

EXAMPLE 1.25. Let p be a prime number and let V be a vector space of dimension $n \geq 2$ over the field $\mathbb{F}_p$. Regard the group $K = GL_n(V)$ of linear transformations of V as a permutation group on V. If $v \in V$, the associated translation v^* is the permutation of V defined as $x \mapsto x + v$, for each $x \in V$. The mapping $v \mapsto v^*$ is an injective homomorphism from the additive group of V into $Sym(V)$, the image V^* being the translation group of V. The affine group of V is defined as $AGL_n(V) = \langle V^*, K\rangle \leq Sym(V)$ and, in fact, $AGL_n(V)$ is a split extension $V^* \rtimes K$, where the action of K on V^* is given by $k^{-1}v^*k = (vk)^*$ whenever $k \in K$ and $v \in V$.

Consider any subgroup G of $AGL_n(V)$ containing the translation group V^*. Then G is a primitive permutation group on V and $G = NL$, where $N = V^* = O_p(G)$ is a regular normal subgroup of G and L is the stabilizer of a point $v \in V$ acting irreducibly and faithfully on the $\mathbb{F}_p$-module V. The chain $\alpha :\ L < G$ is a maximal $\mathbb{P}$-chain from L to G, because $[G : L] = |N| = p^n$ and L is a maximal subgroup of G. However, $\alpha_N :\ 1 < N$ is not a maximal $\mathbb{P}$-subnormal chain in N.

CHAPTER 2

Some results on $\mathbb{P}$-subnormality in arbitrary finite groups

In the first section of this chapter we give a description of groups all of whose $\mathbb{P}$-subnormal subgroups are $\mathbb{P}$-subgroups (see Theorem 2.5). In order to obtain this result, we first investigate the non abelian simple groups S in $\mathcal{G}$ for which there exists a $\mathbb{P}$-chain from the trivial subgroup to S. Classical groups are treated separately in Lemma 2.2 and Lemma 2.4. For all the remaining groups, there are few possibilities for such $\mathbb{P}$-subnormal chains so that these can be listed explicitly, as we have already shown in Lemma 1.14 of Chapter 1. As a consequence of this analysis, it turns out that for any given group G, admitting the existence of a $\mathbb{P}$-chain from 1 to G is equivalent to asking that every $\mathbb{P}$-subnormal subgroup of G is a $\mathbb{P}$-subgroup.

Further, as a consequence of Guralnick's Theorem 1.5, we obtain a characterization of groups all of whose subgroups are $\mathbb{P}$-subnormal (Proposition 2.8). In particular, we may apply our result in Proposition 2.8 to deduce that, for this class of groups (which clearly contains the class of soluble groups), every subgroup is a $\mathbb{P}$-subgroup.

Finally, we complete our account on the relation $\leq_{\mathbb{P}}$ by showing that for any group G, if H and K are two $\mathbb{P}$-subgroups of G with $H \leq K$, then H is a $\mathbb{P}$-subgroup of K (see Theorem 2.9).

2.1. Groups where the trivial subgroup is a $\mathbb{P}$-subgroup

For any group G we denote by the symbol $\Gamma_0(G)$ the (possibly empty) set of all $\mathbb{P}$-chains from 1 to G, that is, the chains of subgroups

$$1 = H_0 < H_1 < \ldots < H_n = G$$

for which each $[H_{i+1} : H_i]$ is a prime power. For instance, when G is soluble then $\Gamma_0(G)$ contains all the possible chains of subgroups from 1 to G (since, in this case, every subgroup is a $\mathbb{P}$-subgroup).

We need first to classify the non abelian simple groups S admitting $\Gamma_0(S) \neq \emptyset$. As the reader might certainly imagine, a trivial example is given by A_5.

Example 2.1. In $S = A_5$, the sequence of subgroups

$$1 < \langle (123) \rangle < Stab_{A_5}(5) < S$$

is a $\mathbb{P}$-chain from 1 to S. It is easy to check that any maximal $\mathbb{P}$-subgroup of S is isomorphic to A_4, since these are the only maximal subgroups having prime power index in A_5 (see Theorem 1.7(3a)). Note that A_4 is a soluble group and then all its subgroups are $\mathbb{P}$-subgroups. Thus, one has that H is a proper $\mathbb{P}$-subgroup of A_5 if and only if H is contained in some subgroup isomorphic to A_4. Equivalently, this happens if and only if either H is not isomorphic to S_3 or the order of H is not divisible by 5.

As we will see, the non abelian simple groups S admitting $\mathbb{P}$-chains from 1 to S are mainly two-dimensional projective special linear groups, in which the Borel subgroups have prime power index in S. All the other cases are due to numerical coincidences.

LEMMA 2.2. *Let S be a non abelian simple group in $\mathcal{G}_0$. If H is a $\mathbb{P}$-subnormal subgroup of S then H is a $\mathbb{P}$-subgroup of S. In particular, $\Gamma_0(S) \neq \emptyset$.*

PROOF. Let $S \in \mathcal{G}_0$ and let H be a $\mathbb{P}$-subnormal subgroup of S. By Theorem 1.5, the last proper term of any maximal $\mathbb{P}$-subnormal chain from H to S is a maximal subgroup P of S having prime power index and one of the following situations occurs.

Case $S = L_2(q)$, where either $q \in \mathfrak{M} \cup \{7\}$ or $q+1 \in \mathfrak{F} \cup \{9\}$ (see Theorem 1.7(3a),(3b)).
Then P is a soluble Frobenius group of order $q\,(q-1)\,/d$, where $d = gcd\,(2, q-1)$. Clearly, H is a $\mathbb{P}$-subgroup of P and therefore of S.

Case $S = L_2\,(11)$.
Then $P \simeq A_5$. We have already noted in Example 2.1 that every $\mathbb{P}$-subnormal subgroup of A_5 is a $\mathbb{P}$-subgroup of A_5. Therefore, every $\mathbb{P}$-subnormal subgroup of S is a $\mathbb{P}$-subgroup of S.

Case $S = U_4\,(2)$.
Then $P \simeq 2^4 : A_5$ is a maximal parabolic subgroup of S. Set $N = O_2\,(P) \simeq 2^4$, the unipotent radical of P. Then $HN/N \trianglelefteq\trianglelefteq_{\mathbb{P}} P/N$ by Lemma 1.3(2). Since $P/N \simeq A_5$, then $HN/N \leq_{\mathbb{P}} P/N$. Thus $HN \leq_{\mathbb{P}} P$ by Lemma 1.3(3). Clearly, $H \leq_{\mathbb{P}} HN$ (since $[HN : H] = [N : H \cap N]$ is a power of 2). We deduce that H is a $\mathbb{P}$-subgroup of P as required.

Case $S = L_3\,(3)$.
Then $P \simeq 3^2 : GL_2\,(3)$. In particular, P is soluble and so H is a $\mathbb{P}$-subgroup of P, and of S as well.

Case $S = L_3\,(8)$.
Then $P \simeq 2^6 : GL_2\,(8)$ is a maximal parabolic subgroup of S. Set $N = O_2\,(P) \simeq 2^6$, the unipotent radical of P. The result in Lemma 1.3(2) yields $HN/N \trianglelefteq\trianglelefteq_{\mathbb{P}} P/N$. Since $P/N \simeq GL_2\,(8)$, Example 1.21 shows that $HN/N \leq_{\mathbb{P}} P/N$ thus $HN \leq_{\mathbb{P}} P$ (by Lemma 1.3(3) again). Since $[HN : H] = [N : H \cap N] = 2^a$ for some $a \geq 0$, we conclude that $H \leq_{\mathbb{P}} HN$. It follows that H is a $\mathbb{P}$-subgroup of S. The proof is now complete. □

We may use the characterization of $\mathbb{P}$-subnormal subgroups in simple linear groups $S \in \mathcal{G} \setminus \mathcal{G}_0$ given in Lemma 1.13 to identify those $\mathbb{P}$-subnormal subgroups of S which are not $\mathbb{P}$-subgroups of S. In particular, for such a subgroups H we determine the set

$$\mathbb{P}_S(H, \alpha) = \{p \in \mathbb{P} \mid \exists\, i \in \{0, \ldots, m-1\} \text{ such that } [H_{i+1} : H_i] = p^a\},$$

whenever $\alpha : H = H_0 < H_1 < \ldots < H_m = S$ is a maximal $\mathbb{P}$-subnormal chain from H to S (see Definition 1.22 in Chapter 1).
In Chapter 4, we will consider the set of prime numbers $\mathbb{P}_G(H, \alpha)$ which is associated to any maximal $\mathbb{P}$-subnormal chain α for a given $H \trianglelefteq\trianglelefteq_{\mathbb{P}} G$, where G is arbitrary finite group. More specifically, in Theorem 4.1, we will show that the set $\mathbb{P}_G(H, \alpha)$

is independent on the choice of the chain $\alpha \in \mathscr{M}(H, G)$; in order to establish this result, the formula given in Lemma 2.4 will be fundamental.

REMARK 2.3. Let G be a group and let H be a $\mathbb{P}$-subgroup of G. Clearly, if α is a $\mathbb{P}$-chain from H to G then, by definition, we have simply $\mathbb{P}_G(H, \alpha) = \pi([G : H])$ (see also Remark 1.23).
Also, note that if $H \leq K \leq G$ then we may write

$$\pi([G : H]) = \pi([G : K]) \cup \pi([K : H]).$$

LEMMA 2.4. *Let $S = L_n(q) \in \mathcal{G} \setminus \mathcal{G}_0$. Let H be a $\mathbb{P}$-subnormal subgroup of S and let $P < S$ be the stabilizer of a line (or of a hyperplane) having prime power index in S, such that $H \trianglelefteq\trianglelefteq_{\mathbb{P}} P$. The following conditions are equivalent.*

(1) *H is soluble.*
(2) *$H \leq S(P)$.*
(3) *H is not a $\mathbb{P}$-subgroup of P (and then of S).*

Moreover, if H is soluble and $\alpha \in \mathscr{M}(H, P)$, we have

$$\mathbb{P}_P(H, \alpha) = \pi([P : P'S(P)]) \cup \pi([S(P) : H]).$$

PROOF. (1) $\Leftrightarrow$ (2) See Lemma 1.13.

(2) $\Leftrightarrow$ (3) Since the last proper term of any $\mathbb{P}$-subnormal chain from H to S is always a maximal parabolic subgroup P, which is the stabilizer of a line (or of a hyperplane) by Theorem 1.5(5), and P is $\mathbb{P}$-subgroup of S, then it is clear that H is a $\mathbb{P}$-subgroup of S if and only if H is a $\mathbb{P}$-subgroup of P. Thus, we only need to prove that $H \leq S(P)$ is equivalent to the fact that H is not a $\mathbb{P}$-subgroup of P.
Set $U = O_r(P)$ the unipotent radical of P. Hence, the relation in (7) of Lemma 1.24 implies that H is a $\mathbb{P}$-subgroup of P if and only if PU/U is a $\mathbb{P}$-subgroup of P/U. We may apply the same number-theoretic argument as in Lemma 1.13 to get that P/U satisfies the assumptions of Lemma 1.10 and then HU/U is a subnormal subgroup of P/U. Since $Z(P/U) = S(P)/U$ and $P/S(P)$ has a unique non abelian factor isomorphic to $L_{n-1}(q)$ (see items (1) and (2) in Lemma 1.8), by the argument above we conclude that $H \leq S(P)$ if and only if HU/U is not a $\mathbb{P}$-subgroup of P/U (see also Remark 1.9). This proves the equivalence of the three statements.

We now prove the formula in the last assertion. If we consider a maximal $\mathbb{P}$-subnormal chain α from H to P, then $\alpha^{P/U} : HU/U = K_0 \leq K_1 \leq \ldots \leq K_v = P/U$ is a maximal $\mathbb{P}$-subnormal chain from HU/U to P/U, by Lemma 1.24(2). Note that $\alpha^{P/U}$ is a composition series from HU/U in P/U, by the maximality of $\alpha^{P/U}$ (use Lemma 1.20). By the argument in the proof of (2) $\Leftrightarrow$ (3), we get that $H \leq S(P)$ if and only if $\alpha^{P/U}$ has a unique non abelian composition factor, say K_{j+1}/K_j, which is isomorphic to $L_{n-1}(q)$; in particular, we get that $L_{n-1}(q) \notin \mathcal{G}$ by our assumptions on S. Whence, it follows that

$$\mathbb{P}_{P/U}\left(HU/U, \alpha^{P/U}\right) = \pi\left(\prod_{j \neq i \in \{0,\ldots,v-1\}} [K_{i+1} : K_i]\right), \tag{9}$$

because (K_i, K_{i+1}) is a p-link, for every $i = 0, \ldots, v-1$ and $i \neq j$ (see also Remark 1.23).

But $K_{j+1}/K_j \simeq P'S(P)/S(P)$ and $HU \le S(P)$, thus

$$\begin{aligned}
\pi\left(\prod_{j\neq i\in\{0,\ldots,v-1\}}[K_{i+1}:K_i]\right) &= \pi\left(\tfrac{[P/U:HU/U]}{[P'S(P)/U:S(P)/U]}\right)\\
&= \pi\left([P/U:P'S(P)/U]\,[S(P)/U:HU/U]\right)\\
&= \pi\left([P:P'S(P)]\,[S(P):HU]\right)
\end{aligned}$$

which implies

$$\mathbb{P}_{P/U}\left(HU/U,\alpha^{P/U}\right) = \pi\left([P:P'S(P)]\right)\cup\pi\left([S(P):HU]\right), \tag{10}$$

by (9). Therefore, we have proved that, for any maximal ℙ-subnormal chain α from H to P, we have $H \le S(P)$ if and only if $\alpha^{P/U}$ is a subnormal chain from HU/U to P/U (which is not a ℙ-chain) and $\alpha^{P/U}$ satisfies (10).

Now, by using the relation (8) in Lemma 1.24, we get

$$\mathbb{P}_P(H,\alpha) = \mathbb{P}_{P/U}(HU/U,\alpha^{P/U}) \cup \mathbb{P}_U(U,\alpha_U).$$

Note that

$$\mathbb{P}_U(U,\alpha_U) = \pi\left([U:H\cap U]\right) = \pi\left([HU:H]\right) \subseteq \{r\},$$

since U is an r-group. As a consequence,

$$\begin{aligned}
\mathbb{P}_P(H,\alpha) = {} & \pi\left([P:P'S(P)]\right)\cup\pi\left([S(P):HU]\right)\cup\pi\left([HU:H]\right)\\
= {} & \pi\left([P:P'S(P)]\right)\cup\pi\left([S(P):H]\right),
\end{aligned}$$

since (10) holds. □

THEOREM 2.5. *Let G be a group. The following three conditions are equivalent.*

1. $\Gamma_0(G) \neq \emptyset$.
2. *Every non abelian composition factor of G belongs to $\mathcal{G}_0$.*
3. *The set of ℙ-subgroups of G coincides with the set of ℙ-subnormal subgroups of G.*

PROOF. (1) ⇒ (2) Use induction on $|G|$. Of course, we can assume that G is non soluble. By the assumption, there exists a ℙ-chain $\alpha:\ 1 = H_0 \le H_1 \le \ldots \le H_n = G$. If the group G is non simple, let N be a non trivial proper normal subgroup of G. Hence, α_N is a ℙ-chain in N and similarly, $\alpha^{G/N}$ is a ℙ-chain in G/N (use items (1) and (2) of Lemma 1.24). The inductive hypothesis applied to the groups G/N and N implies the result for G. Thus, suppose that G is a non abelian simple group in $\mathcal{G}$. If $S \in \mathcal{G}$ is a non classical group, that is one of the groups in Lemma 1.14, then it is straightforward to check that $\Gamma_0(S) = \emptyset$. Assume that $G \notin \mathcal{G}_0$ and let $i \ge 1$ be the maximal integer such that H_i is a soluble group. By Lemma 2.4 we know that H_i is not a ℙ-subgroup of G and this is a contradiction.

(2) ⇒ (3) If H is any ℙ-subnormal subgroup of G, we need to show that H is a ℙ-subgroup of G. We proceed by induction on $|G|$, and clearly we can assume that G is non soluble (otherwise there is nothing to prove). Consider α any ℙ-subnormal chain from H to G. If N is a proper normal subgroup of G, then by items (1) and (2) of Lemma 1.24, we know that α_N and $\alpha^{G/N}$ are ℙ-subnormal chains in N and in G/N respectively. Thus, α_N and $\alpha^{G/N}$ are ℙ-chains, by the inductive hypothesis.

Now equality (7) in Lemma 1.24 implies that α is a $\mathbb{P}$-chain. When G is a non abelian simple group then $G \in \mathcal{G}_0$ and the result follows by Lemma 2.2.

(3) $\Rightarrow$ (1) Trivial. $\square$

We have summed up in a table the values in $\mathbb{P}_S(1,\alpha)$, where S is any finite simple group and α is an arbitrary maximal $\mathbb{P}$-subnormal chain from 1 to S. In particular, we observe that this set of prime numbers does not depend on the choice of the maximal $\mathbb{P}$-subnormal chain α.

TABLE 1. $\mathbb{P}_S(1,\alpha)$ for finite simple groups S

Simple group S	Values of the parameters	$\mathbb{P}_S(1,\alpha)$
C_p	$p \in \mathbb{P}$	$\{p\}$
A_7	-	$\{7\}$
A_8	-	$\{2,7\}$
A_9	-	$\{2,3,7\}$
A_{2^n}	$2^n - 1 \in \mathfrak{M} \setminus \{7\}$	$\{2, 2^n - 1\}$
A_p	$p \in \mathfrak{F} \setminus \{5\}$	$\{2,p\}$
A_{p^n}	$p \in \mathbb{P},\ p^n \geq 11,\ p^n - 1 \notin \mathbb{P}$	$\{p\}$
$L_2(7)$	-	$\pi(S)$
$L_2(11)$	-	$\pi(S)$
$L_2(q)$	$q \in \mathfrak{M} \setminus \{7\}$ or $q+1 \in \mathfrak{F} \cup \{9\}$	$\pi(S)$
$L_3(3)$	-	$\pi(S)$
$L_3(8)$	-	$\pi(S)$
$L_n(q)$	$(n,q) \neq (3,2),(3,3),(3,8)\ n \geq 3$ $p^a = (q^n - 1)/(q-1),\ q = r^b$ $n,r,p \in \mathbb{P},\ a,b \geq 1$	$\pi(q-1) \cup$ $\{p,r\}$
$U_4(2)$	-	$\pi(S)$
M_{11}	-	$\{2,11\}$
M_{23}	-	$\{23\}$
$S \notin \mathcal{G}$	-	$\emptyset$

REMARK 2.6. We refer the reader to the classification given in Theorem 1.7, for the values of the parameters in the second column of Table 1. All the entries in the third column, except for line 13, follow as a consequence of Lemma 1.14 and Theorem 2.5. Finally, if we assume that $S = L_n(q) \in \mathcal{G}$ and $(n,q) \notin \{(3,2),(3,3),(3,8)\}$ (that is $S \in \mathcal{G} \setminus \mathcal{G}_0$), and we adopt all the notations as in Lemma 2.4, then there exists a maximal parabolic subgroup P of S such that $\mathbb{P}_P(1,\alpha) = \pi([P : P'S(P)]) \cup \pi(|S(P)|)$. In particular, one checks that $\pi([P : P'S(P)]) \cup \pi(|S(P)|) = \pi(q-1) \cup \{r\}$ (see Lemma 1.8(1),(2)). Since $[S:P] = p^a = q^{n-1} + \ldots + q + 1$ for some prime number p and $a \geq 1$, we conclude that

$$\mathbb{P}_S(1,\alpha) = \pi(q-1) \cup \{p,r\},$$

as described in line 13.

2.2. Groups with all subgroups $\mathbb{P}$-subnormal

The result we present in this section is essentially little more than a corollary of Theorem 1.5, whose statement we record for the convenience of the reader.

COROLLARY (R. Guralnick [**14**], Corollary 3). *Let G be a finite group such that every maximal subgroup of G has prime power index. Then $G/S(G) \in \{1, L_2(7)\}$.*

We may use the result above to characterize groups all of whose subgroups are $\mathbb{P}$-subnormal. For this, we isolate in a separate lemma an easy observation which will be needed also in Section 2.3.

LEMMA 2.7. *Let G be a finite group and let $N = S(G)$ be the soluble radical of G. If $H \leq G$ then H is a $\mathbb{P}$-subgroup of HN.*

PROOF. Let H be a subgroup of G and set $N = S(G)$. Clearly, we may assume that $N < G$ otherwise the result is trivial. Consider $1 = N_0 \leq N_1 < \ldots < N_m = N$ a chief series of N. Then N_i char $N \lhd G$, so that each N_i is a normal subgroup of G. Since N is soluble, the integer $[HN_{i+1} : HN_i] = [N_{i+1} : N_i] / [H \cap N_{i+1} : H \cap N_i]$ is the power of a prime number, whenever $i = 0, \ldots, m-1$. Therefore,

$$H = HN_0 \leq HN_1 \leq \ldots \leq HN_m = HN$$

is a $\mathbb{P}$-chain from H to HN. □

PROPOSITION 2.8. *In any group G the following conditions are equivalent.*
(1) *Every subgroup of G is a $\mathbb{P}$-subgroup of G.*
(2) *The group G is soluble or $G/S(G) \simeq L_2(7)$.*
(3) *Every subgroup of G is $\mathbb{P}$-subnormal in G.*

PROOF. (1) $\Rightarrow$ (2) Apply Corollary 3 in [**14**].
(2) $\Rightarrow$ (1) Let H be a subgroup of G and set $N = S(G)$. Then H is a $\mathbb{P}$-subgroup of HN, by Lemma 2.7. Since $G/N \simeq L_2(7)$, it follows that HN/N is a $\mathbb{P}$-subgroup in G/N (see also Remark 1.6). Thus, HN is a $\mathbb{P}$-subgroup of G, by Lemma 1.3(3).
(3) $\Rightarrow$ (1) Trivial.
(1) $\Rightarrow$ (3) We have just proved that, for any group G, condition (1) is equivalent to having either G soluble or $G/S(G) \simeq L_2(7)$. In the first case the result follows immediately. In the latter case, the result is a consequence of Theorem 2.5, since $L_2(7) \in \mathcal{G}_0$. □

2.3. Another property of $\mathbb{P}$-subgroups

From the description we have obtained in Lemma 2.4, we know that for suitable integers n, q there exist linear groups $L_n(q)$ admitting proper $\mathbb{P}$-subnormal subgroups which are not $\mathbb{P}$-subgroups. As Example 1.16 shows, if G is one of these groups and H and K are two $\mathbb{P}$-subnormal subgroups of G with $H \leq K$, then H may fail to be a $\mathbb{P}$-subnormal subgroup of K.
In this section we establish the following result.

THEOREM 2.9. *Let G be a group and let H and K be two $\mathbb{P}$-subgroups of G. If $H \leq K$ then H is a $\mathbb{P}$-subgroup of K.*

Our proof makes a reduction to the case in which G is a non abelian simple group and K is a maximal $\mathbb{P}$-subgroup of G. Thus, we are led to analyze the situation in which G is one of the groups in Guralnick's list $\mathcal{G}$ and K is one of its maximal subgroups of prime power index. In particular, since they are transitively permuted under $Aut(G)$ (see the last assertion in Theorem 1.5 of Chapter 1), we are assuming there exists an automorphism ϕ of the simple group G such that H

is a $\mathbb{P}$-subgroup of K^{ϕ} (which is the last term of a $\mathbb{P}$-chain from H to G) and we would like to show that H is also a $\mathbb{P}$-subgroup of K. The proof of this fact relies on the following auxiliary result.

LEMMA 2.10. *Let G be a non soluble group and let $X = G/S(G)$.*

(1) *Assume that $S(G)$ is a 2-group and $X \simeq A_5$. Then H is a $\mathbb{P}$-subgroup of G if and only if 5 does not divide $|H|$, or H does not contain a subgroup isomorphic to S_3, or $H/S(H) \simeq A_5$.*
(2) *Assume that $S(G)$ is a $\{2,7\}$-group and $X \simeq L_2(8)$. Then H is a $\mathbb{P}$-subgroup of G if and only if either $3 \nmid |H|$ or $H/S(H) \simeq L_2(8)$.*
(3) *Assume that $X \simeq PGL_n(q)$, where either $n \geq 3$ or $q > 3$. Let $H \leq G$ such that H has a composition factor isomorphic to $L_n(q)$. Then H is a $\mathbb{P}$-subgroup of G.*

PROOF. Let $N = S(G)$ and let H be a subgroup of G. By Lemma 2.7, H is a $\mathbb{P}$-subgroup of HN. Since Lemma 1.3(3) holds, we know that if HN is a $\mathbb{P}$-subgroup of G then HN/N is a $\mathbb{P}$-subgroup of G/N. Conversely, if HN/N is a $\mathbb{P}$-subgroup of G/N then HN is a $\mathbb{P}$-subgroup of G (use Lemma 1.3(2)). Thus, H is a $\mathbb{P}$-subgroup of G if and only if HN/N is a $\mathbb{P}$-subgroup of G/N. This of course is true when $HN = G$. Note that if $X \simeq A_5$ or $X \simeq L_2(8)$ then $HN = G$ is equivalent to $H/S(H) \simeq X$. For, if $HN = G$ then $X = G/N = HN/N \simeq H/H \cap N$, where $H \cap N \leq S(H)$. It follows that $H \cap N = S(H)$ since $H/H \cap N \simeq X$ is simple and non abelian. Conversely, if $H/S(H) \simeq X$ then it is necessarily $HN = G$, by order considerations.

(1) Assume that N is a 2-group, $X \simeq A_5$ and $HN \neq G$. Then either 5 is not a divisor of HN/N or HN/N is not isomorphic to S_3 (by Example 2.1). Clearly, since $HN/N \simeq H/H \cap N$ and N is a 2-group, then 5 does not divide the order of H if and only if 5 is not a divisor of $[H : H \cap N]$. Now consider $K \leq H$ such that $K \simeq S_3$. Then $K \cap N \leq O_2(K) = 1$, so that $K \cap N = 1$. It follows that $S_3 \simeq KN/N \leq HN/N$.

(2) Assume that $\pi(N) = \{2,7\}$, $X \simeq L_2(8)$ and $HN \neq G$. Then HN/N is a soluble group with $3 \nmid |HN/N|$ by Example 1.21. Since $HN/N \simeq H/H \cap N$ and $3 \nmid |H \cap N|$, we deduce that H is a $\mathbb{P}$-subgroup of G if and only if $3 \nmid |H|$.

(3) Let $X \simeq PGL_n(q)$. Let $M/N = (G/N)'$. By assumption $M/N \simeq L_n(q)$ is simple and non abelian (see for instance Corollary 9.10, Chapter I in [**28**]). Let H be a subgroup of G such that H has a composition factor isomorphic to $L_n(q)$. Clearly, $1 \leq H \cap N \leq H \cap M \leq H$ is a normal series in H. Since $H/H \cap M$ and $H \cap N$ are soluble groups, then there exist subgroups U, V of G such that $H \cap N \leq V \lhd U \leq H \cap M$ and $U/V \simeq L_n(q)$. But $(H \cap M)/(H \cap N) \simeq (H \cap M)N/N$ is a section of $M/N \simeq L_n(q)$, so that $M \cap HN = (H \cap M)N = M$ by order considerations. It follows that $M \leq HN \leq G$ and then HN is a $\mathbb{P}$-subgroup of G (see Lemma 1.3(4)). Since H is a $\mathbb{P}$-subgroup of HN, we conclude that H is a $\mathbb{P}$-subgroup of G. The proof is complete. $\square$

LEMMA 2.11. *Assume that S is a simple group in the list $\mathcal{G}$. Let H be a $\mathbb{P}$-subgroup of S and let P be maximal $\mathbb{P}$-subgroup of S such that $H \leq_{\mathbb{P}} P$. If $H \leq P^{\phi}$ for some $\phi \in Aut(S)$, then H is a $\mathbb{P}$-subgroup of P^{ϕ}.*

PROOF. By Theorem 1.5, the last proper term of any $\mathbb{P}$-subnormal chain from H to S is a maximal subgroup P of S having prime power index. We are supposing that H is also contained in $Q = P^{\phi}$ for some $\phi \in Aut\,(S)$, and we need to prove that H is a $\mathbb{P}$-subgroup of Q. Clearly, we can assume that P is non soluble. As a consequence of Theorem 1.5, one of the following situations holds.

Assume that either $S = A_{p^n}$ and $p^n \geq 11$ or $S \in \{M_{11}, M_{23}\}$. By Lemma 1.14, it is straightforward to check that H is a $\mathbb{P}$-subgroup of P if and only if H is a $\mathbb{P}$-subgroup of Q.
We are left with the cases below.

Case $S = L_2\,(11)$.
Then $P \simeq A_5$. One checks either H does not contain a subgroup isomorphic to S_3 or 5 does not divide the order of H (see Example 2.1). Since $P \simeq Q$, this is equivalent to say that H is a $\mathbb{P}$-subgroup of Q.

Case $S = U_4\,(2)$.
Then $P \simeq 2^4 : A_5$. By Lemma 2.10(1), we deduce that 5 is not a divisor of $|H|$, or H does not contain any subgroup isomorphic to S_3, or $H/S\,(H) \simeq A_5$. Since P and Q are isomorphic groups, another application of Lemma 2.10(1) implies that H is a $\mathbb{P}$-subgroup of Q.

Case $S = L_3\,(8)$.
Then $P \simeq 2^6 : GL_2\,(8)$. In particular, one checks that $P/S\,(P) \simeq L_2\,(8)$ since $S\,(P) \simeq 2^6 : 7$. By Lemma 2.10(2), we get that either $3 \nmid |H|$ or $H/S\,(H) \simeq L_2\,(8)$. We argue as in the previous case to get that H is a $\mathbb{P}$-subgroup of Q.

Case $S = L_n\,(q)$, where $q = r^k$ for $r \in \mathbb{P}$, $k \geq 1$ and $n \geq 3$, with $(n, q) \notin \{(3,2)\,, (3,3)\,, (3,8)\}$.
By Lemma 1.13(2), we get that $H/S\,(H) \simeq PGL_{n-1}\,(q)$ and H has a unique non abelian composition factor isomorphic to $L_{n-1}\,(q)$. It follows that H is a $\mathbb{P}$-subgroup of Q as a consequence of Lemma 2.10(3). $\square$

Proof of Theorem 2.9. Assume that H and K are $\mathbb{P}$-subgroups of G. We proceed by induction on $|G| + [G : K]$.

Step I. We can assume that K is a maximal subgroup of G with $[G : K] = p^a$ (for some prime number p and $a \geq 1$).
If not, one may consider $K < K_1 < G$ such that K is a $\mathbb{P}$-subgroup of K_1 and K_1 is maximal in G with $[G : K_1] = p^a$ (for some prime number p and $a \geq 1$). Since $[G : K_1] < [G : K]$ it follows by induction that H is a $\mathbb{P}$-subgroup of K_1. Now H and K are $\mathbb{P}$-subgroup of K_1, where $|K_1| < |G|$. The inductive hypothesis yields $H \leq_{\mathbb{P}} K$.

Step II. It is not reductive to suppose that $K_G = 1$.
Note first that K is not normal in G, otherwise by Lemma 1.3(1) we immediately conclude that $H \leq_{\mathbb{P}} K$, as required. As a consequence, $K_G < K$ and we may consider G/K_G. By Lemma 1.3(2), we know that $HK_G/K_G \leq_{\mathbb{P}} G/K_G$. Thus, if $K_G \neq 1$ the inductive hypothesis yields $HK_G/K_G \leq_{\mathbb{P}} K/K_G$ and then $HK_G \leq_{\mathbb{P}} K$ (see Lemma 1.3(3)). Moreover, $H \leq_{\mathbb{P}} HK_G$, by Lemma 1.3(2). It follows that H is a $\mathbb{P}$-subgroup of K.

Step III. The group G is simple and non abelian.
Let N be a minimal normal subgroup of G. By Step II one has that N is not contained in K. In particular, $KN = G$, since K is a maximal subgroup of G. We now have that HN/N is a $\mathbb{P}$-subgroup of G/N (by Lemma 1.3(3)). Hence, there exists a $\mathbb{P}$-chain $HN = H_0 \leq H_1 \leq \ldots \leq H_t = G$. Consider $i \geq 0$ minimal such that $H_{i+1} = G$, that is $H_i < KN = G$. Then $G = KH_i$, because $N \leq H_i$. One

has that $p^a = [G : K] = [KH_i : K] = [H_i : H_i \cap K]$, which implies $H_i \cap K \leq_{\mathbb{P}} H_i$. Clearly, by Lemma 1.3(2), we have $H \leq_{\mathbb{P}} HN \leq_{\mathbb{P}} H_i$ and $H \leq H_i \cap K$. Also, $|H_i| < |G|$ from the choice of i. Hence, by the inductive hypothesis, we infer that H is a $\mathbb{P}$-subgroup of $H_i \cap K$. Finally, note that H_i and K are $\mathbb{P}$-subgroups of G such that $H_iK = G$. We may apply Lemma 1.4(1) to get that $H_i \cap K \leq_{\mathbb{P}} G$. Therefore, $H \leq_{\mathbb{P}} K$. Thus G is a simple group and, clearly, a non abelian one.

Step IV. Conclusion.

Of course, we shall assume that $G \in \mathcal{G}$ otherwise there is nothing to prove. From what we have shown in Step I, K is a maximal subgroup of G having prime power index and the automorphism group of G is transitive on these maximal $\mathbb{P}$-subgroups (by Theorem 1.5). Since H a $\mathbb{P}$-subgroup of one of these, then Lemma 2.11 implies that H is a $\mathbb{P}$-subgroup of K as well. □

CHAPTER 3

Groups in which all soluble subgroups are $\mathbb{P}$-subnormal

In Proposition 2.8 of Chapter 2 we have characterized finite groups all of whose subgroups are $\mathbb{P}$-subnormal. In this chapter we consider the following classes of groups:

$$\mathcal{C} = \{\text{finite groups all of whose Sylow subgroups are } \mathbb{P}\text{-subnormal}\}$$

$$\mathcal{R} = \{\text{finite groups all of whose soluble subgroups are } \mathbb{P}\text{-subnormal}\}.$$

For any given subgroups V, U of G such that $V \trianglelefteq U$, we define by

$$Aut_G(U/V) := N_G(U/V)/C_G(U/V),$$

the group of automorphisms of G induced on the factor U/V, where we set

$$N_G(U/V) = N_G(U) \cap N_G(V)$$

and

$$C_G(U/V) = \left\{g \in N_G(U/V) \,\middle|\, g^{-1}(Vu)\,g = Vu \text{ , for every } u \in U\right\}$$

(sometimes $Aut_G(U/V)$ is called the *automizer* of U/V in G).
We also set $\mathcal{F}$ to be the class of groups in which all non abelian composition factors U/V satisfy the condition $Aut_G(U/V) \simeq U/V \simeq L_2(7)$.
The main result of this chapter is the following.

THEOREM 3.1. *The three classes $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{F}$ coincide.*

This is achieved in steps. In the first section of this chapter we introduce some closure properties for the three classes under consideration. In the subsequent section, we have collected three auxiliary results. More specifically, by applying our classification Theorem 1.7, we first establish a technical lemma for non abelian simple groups (Lemma 3.9) and then we state a sufficient condition for wreath products of groups to belong to $\mathcal{R}$ (Lemma 3.11). Finally, the Lemma 3.14 is a consequence of the classification of primitive groups given by the O'Nan-Scott Theorem 3.13 (we refer the reader to Theorem 1.1.52 in [**4**]). The reader will find the proof of Theorem 3.1 in the last section of the chapter.

3.1. The classes $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{F}_0$

To describe our result, we introduce some standard notation which will be frequently used along all this chapter. Let $S_1, \ldots, S_n$ be groups and let $G = S_1 \times \ldots \times S_n$ be their direct product. The projection map of G onto the i-th component S_i

$$\pi_i : G \to S_i \quad \text{defined by} \quad (s_1, \ldots, s_n) \mapsto s_i$$

is a homomorphism of groups. If $H \leq G$ and $J \subseteq \{1, \ldots, n\}$ let $H^{\Pi_J} = \underset{j \in J}{\times} H^{\pi_j}$ be the subgroup of G generated by the projections of H on the factors $\{S_j\}_{j \in J}$ (for $J = I$ the notation we use is H^{Π}).

REMARK 3.2. Suppose that $G = S_1 \times \ldots \times S_n$ is a direct product of $n \geq 2$ isomorphic copies of a simple group S. We remind the reader that a subgroup H of G is called *subdirect* in G if $H^{\pi_i} = S_i$ for every $i = 1, \ldots, n$. If H is a subdirect subgroup of G then there exists a partition σ of $\{1, \ldots, n\}$ such that $H = \underset{J \in \sigma}{\times} H^{\Pi_J}$, where each $H^{\Pi_J} \simeq S$. In particular, H is isomorphic to a direct product of $|\sigma|$ copies of S (see Lemma (1.4) in [**2**] or Proposition 1.1.39 in [**4**]).

We begin noting that the closure properties for the class $\mathcal{C}$ substantially depend on basics notions of Sylow theory.

LEMMA 3.3. *Let G be a group and let N be a normal subgroup of G. If $G \in \mathcal{C}$ then N, $G/N \in \mathcal{C}$.*

PROOF. Assume that $G \in \mathcal{C}$. Let $N \trianglelefteq G$ and let p be a prime divisor of $|G|$. If P is any Sylow p-subgroup of G then $P \cap N$ and PN/N are Sylow p-subgroups of N and G/N, respectively. Now, the conclusion follows easily by using parts (1) and (2) of Lemma 1.3. □

REMARK 3.4. Let G be a finite group and let p be a prime number dividing the order of G. It is well known that, if P is a given Sylow p-subgroup of G, then there exist elements $g_1 = 1, g_2, \ldots, g_m$ of G such that $\{P, P^{g_2}, \ldots, P^{g_m}\}$ is the set of all Sylow p-subgroups P of G.
Now if $\alpha : P = H_0 \leq \ldots \leq H_n = G$ is a $\mathbb{P}$-subnormal chain from P to G then, the automorphism of conjugation induced by an element $g \in G$, is an index preserving map between any two consecutive terms H_k and H_{k+1} of the chain α; moreover, if $H_k \trianglelefteq H_{k+1}$ then the homomorphic image H_k^g is normal in $H_{k+1}^{g_i}$, whenever $k = 0, \ldots, n-1$. Whence, if we define

$$\alpha^{g_i} : P^{g_i} = H_0^{g_i} \leq \ldots \leq H_n^{g_i} = G, \qquad i = 1, \ldots, m$$

then each α^{g_i} is a $\mathbb{P}$-subnormal chain from P^{g_i} to G.

In a similar way as in Lemma 3.3, we can use basics facts about $\mathbb{P}$-subnormality in order to deduce some closure properties for the class $\mathcal{R}$.

LEMMA 3.5. *Let G be a group.*

(1) *Let N be a soluble normal subgroup of G. Then $G \in \mathcal{R}$ if and only if $G/N \in \mathcal{R}$.*
(2) *Let $G = N \times M$. If $M, N \in \mathcal{R}$ then $G \in \mathcal{R}$.*

PROOF. Let G be a group.

(1) For any given subgroup H of G then H is soluble if and only if HN/N is soluble. Apply now parts (3) and (4) of Lemma 1.3 to conclude as desired.

(2) Assume that G is the direct product of two groups N and M, where $M, N \in \mathcal{R}$. Denote by π_N, π_M the projections maps on the direct factors N and M, respectively.
For any soluble subgroup H of G the homomorphic image $H^{\Pi_N} \times H^{\Pi_M}$ is a soluble subgroup of G. In particular, H is $\mathbb{P}$-subnormal in $H^{\Pi} = H^{\Pi_N} \times H^{\Pi_M}$. Observe

that if $H^{\Pi_N} = N_0 \leq N_1 \leq \ldots \leq N_n = N$ and $H^{\Pi_M} = M_0 \leq M_1 \leq \ldots \leq M_m = M$ are $\mathbb{P}$-subnormal chains in N and in M respectively, then

$$\begin{aligned} H^{\Pi} &= H^{\Pi_N} \times H^{\Pi_M} = \\ &= N_0 \times H^{\Pi_M} \leq N_1 \times H^{\Pi_M} \leq \ldots \leq N_n \times H^{\Pi_M} = \\ &= N \times H^{\Pi_M} = N \times M_0 \leq N \times M_1 \leq \ldots \leq N \times M_m = N \times M = G \end{aligned}$$

is a $\mathbb{P}$-subnormal chain from H^{Π} to G. This proves the result. □

We now need to introduce some additional notation. We reserve the symbol $\mathcal{F}_0$ to denote the class of groups all of whose non abelian composition factors are isomorphic to $L_2(7)$. Of course, $\mathcal{F}_0$ properly contains the class $\mathcal{F}$ and it is immediate that $\mathcal{F}_0$ is closed under homomorphic images. Our interest in considering the class $\mathcal{F}_0$ lies in the fact that $\mathcal{F}_0$ turns out to be closed under subgroups, a property which depends only on the subgroup structure of $L_2(7)$.

Lemma 3.6. *The class $\mathcal{F}_0$ is closed under subgroups.*

Proof. Let G be a group in the class $\mathcal{F}_0$ and let H be a subgroup of G. We proceed by induction on the index $[G : H]$ in order to show that $H \in \mathcal{F}_0$.

When $[G : H] = 1$ the result is true by the assumption. Thus assume that H is a proper subgroup of G. If $H < K < G$, then $[G : K] < [G : H]$ and we conclude $K \in \mathcal{F}_0$ by the inductive hypothesis. Since $[K : H] < [G : H]$, the induction yields $H \in \mathcal{F}_0$, as required. Hence, we have to prove the result for H maximal in G. If $H_G = \bigcap_{g \in G} H^g \neq 1$ then $|G/H_G| < |G|$ and we conclude that $H/H_G \in \mathcal{F}_0$ again by induction. Now $H_G \in \mathcal{F}_0$, whence $H \in \mathcal{F}_0$. As a consequence, we may assume $H_G = 1$.

According to R. Baer's classification of primitive groups (see, for instance, Theorem 1.1.7 in [**4**]), there exists a minimal normal subgroup $M \trianglelefteq G$ such that $G = HM$. We have that $H/(H \cap M) \simeq G/M \in \mathcal{F}_0$. Whence, we need to show that $H \cap M \in \mathcal{F}_0$.

If M is abelian then $H \cap M \trianglelefteq M$. Since $H \cap M \trianglelefteq H$, it follows that $H \cap M \trianglelefteq HM = G$ and then $H \cap M = 1$, because of the minimality of M. Thus, assume that M is non abelian. Let $M = S_1 \times \ldots \times S_n$ be a direct product of n isomorphic copies of a non abelian simple group $S \simeq L_2(7)$. If $\pi_i : H \cap M \to S_i$ is the projection map on the i-th factor S_i, set $R_i = (H \cap M)^{\pi_i}$ for $i = 1, \ldots, n$. There are now two cases.
Suppose the group $H \cap M$ is not subdirect in M. Since G acts transitively by conjugation on $\{S_1, \ldots, S_n\}$ and $G = HM$ then all the projections $R_i = (H \cap M)^{\pi_i}$ are conjugated by elements of H. In particular, we get that $H \cap M \leq R_1 \times \ldots \times R_n$ is a soluble group (see Remark 1.6). If, otherwise, the group $H \cap M$ is subdirect in M, it follows that $H \cap M$ is isomorphic to a direct product of copies of S (see also Remark 3.2). In both cases, we conclude that $H \cap M \in \mathcal{F}_0$. □

We conclude this section with some observations on the class $\mathcal{F}$.

Remark 3.7. Let N be a normal subgroup of G and let $V \triangleleft U \leq G$ such that $U/V \simeq L_2(7)$ is a composition factor of G. Two possibilities arise for the section U/V: either $UN = VN$ and then $(U \cap N)/(V \cap N) \simeq U/V$, or $UN > VN$ and then $UN/VN \simeq U/V$. In the first case, it is not difficult to check that

$Aut_G((U \cap N)/(V \cap N)) \leq Aut_G(U/V)$. When $UN < VN$ then UN/VN is a composition factor of G/N and, now we have the reverse inclusion $Aut_G(U/V) \lesssim Aut_{G/N}(UN/VN)$.

According to our definition, if we consider any group G in the class $\mathcal{F}$, then $Aut_G(U/V) \simeq L_2(7)$ for every non abelian composition factor U/V of G. We have to be careful, since the condition $Aut_G(U/V) \simeq L_2(7)$ for the composition factor U/V may depend upon the the particular composition series chosen. Consider the following example.

EXAMPLE 3.8. Let $A = PGL_2(7)$ be the automorphism group of $L_2(7)$. Then $A = X : \langle a \rangle$, where $X = L_2(7)$ and a is the inverse-transpose automorphism of $L_2(7)$ of order 2. Let N be an elementary abelian group of order 4 with basis $\{u, v\}$ on which A acts with the following rules: $w^x = w$ for every $w \in N$ and $x \in X$, $u^a = u$ and $v^a = uv$. Now, take the semidirect product $G = NA$. Note that G is not in $\mathcal{F}$, since G is an homomorphic image of $PGL_2(7)$ which is not in $\mathcal{F}$. Consider

$$1 < \langle u \rangle < \langle u \rangle X < NX < G.$$

This is a composition series of G (which is actually a chief series) whose unique non abelian composition factor $\langle u \rangle X / \langle u \rangle$ satisfies $Aut_G(\langle u \rangle X / \langle u \rangle) = NA/N \simeq PGL_2(7)$. However, this is not true if we choose a different composition series

$$1 < \langle v \rangle < \langle v \rangle X < NX < G.$$

Indeed, in this case one has $\langle v \rangle X / \langle v \rangle \simeq L_2(7)$ but $a \notin N_G(\langle v \rangle X) \cap N_G(\langle v \rangle)$, so that now $Aut_G(\langle v \rangle X / \langle v \rangle) = NX/N \simeq L_2(7)$.

3.2. Auxiliary results

The set of $\mathbb{P}$-subnormal subgroups in a finite simple group has been completely determined by the main result of Chapter 1 (Theorem 1.7 in Section 1.2), which, in turn, depends on the classification of the finite simple groups. As we have noted in the first chapter (Remark 1.6), all proper subgroups of $L_2(7)$ are soluble and each of them is $\mathbb{P}$-subnormal in S. Because of this exceptional behavior we have that $L_2(7) \in \mathcal{C}$ and, among all non abelian simple groups, it is in fact the only one which lies in this class.

LEMMA 3.9. *Let $S \neq L_2(7)$ be a finite non abelian simple group and let H be a soluble $\mathbb{P}$-subnormal subgroup of S. Then there exists a prime divisor p of $|S|$ such that $p \nmid |H|$.*

PROOF. Let S be a non abelian simple group not isomorphic to $L_2(7)$ and let H be a soluble $\mathbb{P}$-subnormal subgroup of S. By Theorem 1.7(2), we may assume that $S \in \mathcal{G}$.

If S is non classical, that is one of the groups listed in Lemma 1.14, then it is straightforward to check that $H = 1$ is the only soluble $\mathbb{P}$-subnormal subgroup of S.

Assume S classical and $S \in \mathcal{G}_0$. By Lemma 2.2, there exists a prime number $p \in \pi(S)$ and a maximal $\mathbb{P}$-subgroup $P < S$ such that $H \leq_{\mathbb{P}} P$ and $[S : P] = p^a$ for some $a \geq 1$. We distinguish two cases. If $S \neq U_4(2)$ then P is a p'-Hall subgroup of S, by Theorem 1.5. Thus, $p \nmid |H|$ and we are done. If $S = U_4(2)$ then 5 does not divide the order of H, by Theorem 1.7(4). Whence, $5 \in \pi(S) \setminus \pi(H)$ and we are done.

Consider now the case of $S = L_n(q) \in \mathcal{G} \setminus \mathcal{G}_0$, where $q = r^k$ for some $r \in \mathbb{P}$, $n \geq 3$ and $(n, q) \notin \{(3,2), (3,3), (3,8)\}$. We recall that

$$|S| = \frac{q^{n(n-1)/2}}{gcd(n, q-1)} \cdot \prod_{k=2}^{n} \left(q^k - 1\right)$$

(see, for instance, Table 2.1.C in [**19**]). We may use the classification in Theorem 1.7(3f) to get that $\pi(H) \subseteq \{r\} \cup \pi(c)$, where we have set $c = (q-1)/gcd(n, q-1)$. By using the Zsygmondy's Theorem ([**25**]), there exists a prime number t such that t is a divisor of $q^3 - 1$ but t does not divide $\left(q^2 - 1\right) = (q+1)(q-1)$. As a consequence, $t \in \pi(S) \setminus \pi(H)$. □

We are interested in the following implication of the previous result.

Corollary 3.10. *The class $\mathcal{C}$ is contained in $\mathcal{F}_0$.*

Proof. Consider $G \in \mathcal{C}$ and argue by induction on $|G|$ in order to show that $G \in \mathcal{F}_0$.

Suppose that G is non simple and let M be a non trivial proper normal subgroup of G. Combining together Corollary 3.3 and the inductive hypothesis, we conclude that $M \in \mathcal{F}_0$ and $G/M \in \mathcal{F}_0$. Then, $G \in \mathcal{F}_0$ as desired.

Now let G be a simple group. Clearly, we may assume that G is non abelian and that G is not isomorphic to $L_2(7)$. By applying Lemma 3.9, one infers the existence of an element $g \in G$ of prime order p such that $\langle g \rangle$ is not a $\mathbb{P}$-subnormal subgroup of G. Thus, since Sylow p-subgroups of G are all conjugated in G, we deduce that each of them is not $\mathbb{P}$-subnormal in G. □

In order to show that $\mathcal{R}$ and $\mathcal{F}$ define the same class, we now need to investigate the behavior of $\mathbb{P}$-subnormal chains with regard to the extension problem. For instance, the group $PGL_2(7)$ is a split extension of $L_2(7) \in \mathcal{R}$ (by Remark 1.6) by a cyclic group of order 2 but $PGL_2(7) \notin \mathcal{R}$, since its Sylow 2-subgroups are maximal subgroups and, of course, not of prime power index.
The key step for the proof of Theorem 3.1 is a result for wreath products.

Lemma 3.11. *Let H be an element of $\mathcal{R}$ and let $S = L_2(7)$. If ρ is a permutation representation for H, then the associated wreath product $G = S \wr_\rho H$ belongs to $\mathcal{R}$.*

Proof. Write $G = B \rtimes H$, where the base group $B = S_1 \times \ldots \times S_n$ is a direct product of isomorphic copies of S. Given $\rho : H \to Sym(n)$ with $h \mapsto h^\rho$ a group homomorphism from H into the symmetric group on $I = \{1, \ldots, n\}$, we set $h^\rho : i \mapsto i^h$ for $h \in H$ and $i = 1, \ldots, n$. Also, we make use of the notations we have already introduced at the beginning of Section 3.1: for every $i \in \{1, \ldots, n\}$ each projection map $\pi_i : B \to S_i$ is a homomorphism such that $\left(x^b\right)^{\pi_i} = \left(x^{\pi_i}\right)^b$ and $\pi_{i^h} = h^{-1}\pi_i h$, whenever $h \in H$ and $x, b \in B$.

Let X be a soluble subgroup of G. If X is contained in B, we deduce the $\mathbb{P}$-subnormality of X in B by Lemma 3.5(2). Therefore, as $B \trianglelefteq G$, $X \trianglelefteq\trianglelefteq_{\mathbb{P}} G$. Also, it is clear that X is $\mathbb{P}$-subnormal in any soluble subgroup containing it. As a consequence, there is no loss of generality in assuming that X is a maximal soluble subgroup such that $B < BX$.

Since $X/(X\cap B)\simeq XB/B\leq G/B\simeq H\in\mathcal{R}$, by Lemma 1.3(3) we get that XB is $\mathbb{P}$-subnormal in G. Now,

$$XB=XB\cap HB=(XB\cap H)B=B\rtimes R\simeq S\wr_{\rho_R}R,$$

where $R=XB\cap H$ and $\rho_R:R\to Sym(n)$ denotes the restriction of the representation ρ to R. When $XB<G$ the inductive argument on the group order assures that X is $\mathbb{P}$-subnormal in XB. We deduce that $X\trianglelefteq\trianglelefteq_{\mathbb{P}}G$.

Consequently, we are reduced to the case $G=XB$. The set I admits a partition Δ such that $B=\underset{\delta\in\Delta}{\times}S_\delta$, where each $S_\delta:=\underset{i\in\delta}{\times}S_i$ is a minimal normal subgroup of G. Since $N_G(S_i)=B(N_G(S_i)\cap X)=BN_X(S_i)$, for any fixed $\delta\in\Delta$ and $i\in\delta$ we may write $S_\delta=\underset{t\in T_\delta}{\times}S_i^t$, for some right transversal T_δ of $N_X(S_i)$ in X. By decomposing each $x=bh\in X$ with $b\in B$ and $h\in H$, one has

$$\begin{aligned}(B\cap X)^{\pi_i x}&=(B\cap X)^{\pi_i bh}=(B\cap X)^{b\pi_i h}\\&=(B\cap X)^{bh\pi_{ih}}=(B\cap X)^{x\pi_{ih}}\\&=(B\cap X)^{\pi_{ih}}.\end{aligned}$$

The last equality holds as $B\cap X\trianglelefteq X$. As a result, $(B\cap X)^{\Pi}X=X(B\cap X)^{\Pi}$. Since each map π_i is a homomorphism, $(B\cap X)^{\Pi}X$ is a soluble subgroup of G. From the maximal choice of X we are forced to conclude that $(B\cap X)^{\Pi}X=X$, which gives

$$B\cap X=\underset{\delta\in\Delta}{\times}\ \underset{t\in T_\delta}{\times}(S_i\cap X)^t=\underset{\delta\in\Delta}{\times}(B\cap X)^{\Pi_\delta}<B. \tag{11}$$

Choose a subset $\delta\in\Delta$ and an index $i\in\delta$. Consider $S_i\cap X\leq V_i<S_i$ and note that V_i is soluble (by Remark 1.6 as each $S_i\simeq L_2(7)$). Now, one has $N_G(S_i)=BC_H(S_i)=BC_G(S_i)$ thus we may write

$$(B\cap X)C_X(S_i)=BC_X(S_i)\cap X=B(C_G(S_i)\cap X)\cap X=BC_G(S_i)\cap X=N_X(S_i).$$

By (11), we get

$$N_X(S_i)=(B\cap X)C_X(S_i)=(S_i\cap X)C_X(S_i),$$

since $[S_i\cap X,S_j\cap X]\leq[S_i,S_j]=1$ for $i=1,\dots,n$ and $i\neq j$. It follows that $V_\delta=\underset{t\in T_\delta}{\times}V_i^t$ is normalized by $N_X(S_i)$. Now $X=N_X(S_i)T_\delta$, so that $V_\delta X=XV_\delta$ is a subgroup of G and it is soluble. The maximality of X implies $X=V_\delta X$ and then $V_i=S_i\cap X$, proving that each $S_i\cap X$ is a maximal subgroup of S_i. In particular, there exists a partition $\Delta=\Delta_1\cup\Delta_2$ such that $[S_i:S_i\cap X]=8$ whenever $i\in\delta$ and $\delta\in\Delta_1$, while $[S_i:S_i\cap X]=7$ whenever $i\in\delta$ and $\delta\in\Delta_2$.
Therefore,

$$\begin{aligned}W_0=B\cap X\quad&<W_1=\left(\underset{\delta\in\Delta_1}{\times}(B\cap X)^{\Pi_\delta}\right)\times\left(\underset{\delta\in\Delta_2}{\times}S_\delta\right)\\&<\left(\underset{\delta\in\Delta_1}{\times}S_\delta\right)\times\left(\underset{\delta\in\Delta_2}{\times}S_\delta\right)=B\end{aligned}$$

is a $\mathbb{P}$-subnormal chain in B whose terms are normalized by X. By lifting them to the chain

$$X = XW_0 < XW_1 < XB = G$$

we obtain that X is $\mathbb{P}$-subnormal in G and the proof is complete. □

REMARK 3.12. The conclusion in Lemma 3.11 shows that the class $\mathcal{R}$ is not closed under subgroups. For instance, when the homomorphism ρ is the regular representation for the cyclic group $R = C_2$, the standard wreath product $G = L_2(7) \wr_\rho C_2$ belongs to $\mathcal{R}$. However, if we write $G = B \rtimes R$ then we may consider $D < B$ such that $D \simeq L_2(7)$ is a diagonal subgroup of B and D is normalized by R. Thus $Y = DR \simeq PGL_2(7) \notin \mathcal{R}$.

Let G be any finite group. If M is a non abelian minimal normal subgroup of G then M is a direct product of, say n, conjugates S_i of a non abelian simple group S. A result of F. Gross and L. G. Kovács ([**13**] or Theorem 1.1.35 in [**4**]) shows that the structure of G is completely determined by knowledge of the groups G/M and of $N_G(S_1)/(S_2 \times \ldots \times S_n) \lesssim Aut(S)$. In particular, it will be convenient for our purposes to write the well-known O'Nan-Scott Theorem, which classifies all primitive groups with a unique non abelian minimal normal subgroup (we refer the reader to Theorem 1.1.52 in [**4**]).

THEOREM 3.13 (O'Nan-Scott Theorem). *Let G be a primitive group with a unique non abelian minimal normal subgroup $M = S_1 \times \ldots \times S_n$ which is the direct product of $n \geq 1$ isomorphic copies of a simple group $S \simeq S_i$, and let H be a core-free maximal subgroup of G. Also, set $N = C_G(S_1)$ and $C = C_G(S_1)$. Then one of the following hold.*

(1) *G is an almost simple group.*
(2) *(G, H) is equivalent to a primitive pair with simple diagonal action; in this case, $H \cap M$ is a full diagonal subgroup of M.*
(3) *(G, H) is equivalent to a primitive pair with product action such that $H \cap M = D_1 \times \ldots \times D_l$ is a direct product of $l > 1$ subgroups such that, for each $j = 1, \ldots, l$, the subgroup D_j is a full diagonal subgroup of a direct product $\times_{i \in I_j} S_i$, and $\{I_1, \ldots, I_l\}$ is a minimal non trivial G-invariant partition of $I = \{1, \ldots, n\}$ in blocks for the action of U on I.*
(4) *(G, H) is equivalent to a primitive pair with product action such that the projection $R_1 = (M \cap H)^{\pi_1}$ is a non trivial proper subgroup of S_1; in this case, set $V = H \cap N$, then $R_1 = VC \cap S_1$ and VC/C is a maximal subgroup of N/C.*
(5) *(G, H) is equivalent to a primitive pair with twisted wreath product action; in this case $H \cap M = 1$.*

We also introduce some additional notation. For any group G let

$$\mathscr{D}(G) = \{H < \cdot\, G \mid 2 \notin \pi([G:H]) \text{ and } |\pi([G:H])| \geq 2\},$$

where the symbol $H < \cdot\, G$ means that H is a maximal subgroup of G.

LEMMA 3.14. *Assume that the group G has a unique minimal normal subgroup $M = S_1 \times \ldots \times S_n$, where $S_i \simeq L_2(7)$ for $i = 1, \ldots, n$. Let H be a maximal subgroup of G such that $G = HM$ (the existence of such a subgroup is guaranteed by the Frattini's argument).*

If $H \in \mathscr{D}(G)$ *then* $N_G(S_1)/C_G(S_1) \simeq PGL_2(7)$.
Conversely, if $N_G(S_1)/C_G(S_1) \simeq PGL_2(7)$ *and the index* $[G : H]$ *is an odd number, then* $H \in \mathscr{D}(G)$.

Proof. Set $N = N_G(S_1)$ and $C = C_G(S_1)$, one has that N/C is almost simple with $L_2(7) \leq N/C \leq PGL_2(7) \simeq L_2(7) : 2$. Therefore, either $N/C \simeq L_2(7)$ or $N/C \simeq PGL_2(7)$. Also, we set $V = N_G(S_1) \cap H = N \cap H$ and $R = \times_{i=1,\ldots,n} (M \cap H)^{\pi_i}$.

Assume $H \in \mathscr{D}(G)$. Suppose that $R = M$. By Remark 3.2, one can find a proper subset $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, n\}$ such that $H \cap M \simeq S_{i_1} \times \ldots \times S_{i_m}$. Whence $[M : H \cap M] = |S|^{n-m} \geq |S|$ is an even number. On the other hand, note that $[M : H \cap M] = [HM : H] = [G : H]$, a contradiction. Therefore, there must be $i \in \{1, \ldots, n\}$ such that $(M \cap H)^{\pi_i}$ is a proper subgroup of S_i.
Since G acts transitively by conjugation on $\{S_1, \ldots, S_n\}$ and $G = HM$, then all the projections $(M \cap H)^{\pi_i}$ are conjugated by elements of H. It follows that $(M \cap H)^{\pi_1} \simeq \ldots \simeq (M \cap H)^{\pi_n}$, and then $[M : R] = [S_1 : (M \cap H)^{\pi_1}]^n$ is a divisor of $[G : H]$, since $M \cap H \leq R$. We are in the situation of item (4) of Theorem 3.13, therefore VC/C is maximal in N/C, supplementing $S_1C/C = M/C$ and $S_1 \cap VC = (M \cap H)^{\pi_1}$. Then

$$[N : VC] = [S_1VC : VC] = [S_1 : S_1 \cap VC] = [S_1 : (M \cap H)^{\pi_1}], \tag{12}$$

which is an odd number. We deduce that $VC/C \in \mathscr{D}(N/C)$. This implies that N/C is not isomorphic to $L_2(7)$, as every maximal subgroup of $L_2(7)$ has index a power of a prime number. Thus $N/C \simeq PGL_2(7)$.

Conversely, assume that $N/C \simeq PGL_2(7)$ and let H be a maximal subgroup of G with $HM = G$ and $[G : H]$ odd. To prove the result we only need to show that $|\pi([G : H])| = 2$. By contradiction, assume that $[G : H] = t^b$, for some prime number $t \neq 2$ and $b \geq 1$. Arguing in the same way as in the previous case, if $H \cap M$ were subdirect in M then $H \cap M$ would be isomorphic to a direct product of m isomorphic copies of the simple group S, where $1 \leq m < n$ (see Remark 3.2). As a consequence, $[M : H \cap M] = |S|^{n-m} > 1$ is a divisor of $[M : H \cap M] = [G : H] = t^b$, which is impossible. Thus, $M \cap H$ is not subdirect in M and we may assume that $(M \cap H)^{\pi_1} < S_1$. In particular, we deduce that $[S_1 : S_1 \cap H]$ is a power of t. Another application of the O'Nan-Scott Theorem (Theorem 3.13(4)) yields that VC/C is a maximal subgroup of N/C, supplementing $S_1C/C = M/C$ and $S_1 \cap VC = (M \cap H)^{\pi_1}$. As before, condition (12) holds and VC/C has index t^c in N/C for some $c \geq 1$. However, one checks that the group $N/C \simeq PGL_2(7)$ does not admit any maximal subgroup whose index is the power of a prime $t \neq 2$. Whence, $|\pi([G : H])| = 2$ and $H \in \mathscr{D}(G)$.

$\square$

3.3. Proof of Theorem 3.1

Part of the main characterization is contained in the following consequence of Lemma 3.11.

Lemma 3.15. *If* $G \in \mathcal{F}$ *then* $G \in \mathcal{R}$.

Proof. Let $G \in \mathcal{F} \setminus \mathcal{R}$ be a minimal counterexample. Then there exists a soluble subgroup R of G such that R is not $\mathbb{P}$-subnormal in G.

Clearly, G can not be simple, otherwise $G = L_2(7)$ but then $G \in \mathcal{R}$ and this is not the case. Let M be a minimal normal subgroup of G. Of course, we have that $G/M \in \mathcal{F}$. For, if $(U/M)/(V/M)$ is any non abelian composition factor of G/M then $U/V \simeq (U/M)/(V/M)$ is a composition factor of G such that

$$Aut_{G/M}\left((U/M)/(V/M)\right) \simeq Aut_G(U/V).$$

Hence, $G/M \in \mathcal{R}$ by the minimal choice of G. Set $L = RM$, then $L/M \simeq R/RM$ is $\mathbb{P}$-subnormal in G/M, which implies $L \trianglelefteq\trianglelefteq_{\mathbb{P}} G$ (by Lemma 1.3(3)).

Assume that $L < G$. Let U/V be a non abelian composition factor of L. Since L/M is soluble, then it must be $(U \cap M)/(V \cap M) \simeq U/V$ and, furthermore, $Aut_L((U \cap M)/(V \cap M)) = Aut_G(U/V) \simeq L_2(7)$, by the assumptions on G (see Remark 3.7). This proves that $L \in \mathcal{F}$. By the inductive hypothesis we deduce that $L \in \mathcal{R}$. Whence, R is $\mathbb{P}$-subnormal in L and then R is $\mathbb{P}$-subnormal in G, which is false.

Then $L = G$. We proceed by steps to reach a contradiction.

Step I. M is the unique minimal normal subgroup of G and M is non abelian.
Let N be another minimal normal subgroup of G. Note that N can not be abelian, by Lemma 3.5(1). Since $G/M = L/M \simeq R/(R \cap M)$ is soluble, then NM/M must be trivial, contradiction. It follows that M is the unique minimal normal subgroup of G.

Step II. $\mathscr{D}(G)$ is empty.
If not, take $H \in \mathscr{D}(G)$. If $M \leq H$ then H/M is a maximal subgroup of G/M. This is clearly impossible, since the quotient group $G/M = L/M$ is soluble and then all its maximal subgroups have prime power index (see Theorem 5.3 Chapter IV in [**29**]). Therefore, $M \nleq H$ and then $HM = G$, by the maximality of H. Also, M is the unique minimal normal subgroup of G, by Step I. As a consequence of Lemma 3.14 we get $G \notin \mathcal{F}$, contrary to our assumption.

Step III. Final contradiction.
Let H be a maximal subgroup of G such that $HM = G$. Then $H \notin \mathscr{D}(G)$ by Step II. Furthermore, $M = S_1 \times \ldots \times S_n$ where $S_i \simeq L_2(7)$ for $i \in \{1, \ldots, n\}$ and Lemma 3.14 holds (use Step I): in particular, the condition $N_G(S_1)/C_G(S_1) \simeq L_2(7)$ assures that G/M acts on M by permuting the S_i's. Since G/M is soluble, the result in Lemma 3.11 yields $G \in \mathcal{R}$. □

PROPOSITION 3.16. *Let $G \in \mathcal{F}_0$. If every Sylow 2-subgroup of G is $\mathbb{P}$-subnormal in G, then $G \in \mathcal{F}$.*

PROOF. Let G be a counterexample of smallest order. The group G is not simple, otherwise $G = L_2(7) \in \mathcal{F}$. By the assumptions, there exists a composition factor U/V of G such that $U/V \simeq L_2(7)$ and $Aut_G(U/V) \simeq PGL_2(7)$.

Let M be a minimal normal subgroup of G. If M were abelian then we immediately get a contradiction by Lemma 3.5(1). Whence, $M = S_1 \times \ldots \times S_n$, where $S_i = L_2(7)$ for $i \in \{1, \ldots, n\}$ and $n \geq 1$. Now note that $G/M \in \mathcal{F}$, by the minimality of G (Lemma 1.3(3)). Therefore, $UM = VM$ and then $(U \cap M)/(V \cap M) \simeq U/V$ is a composition factor of G contained in M (see Remark 3.7). In particular, up to conjugation, we get that

$$Aut_G((U \cap M)/(V \cap M)) \simeq Aut_G(S_1/1) = N_G(S_1)/C_G(S_1) \simeq PGL_2(7).$$

Suppose that T is another minimal normal subgroup of G and $T \neq M$. Arguing in the same way as before, we obtain that G/T is in $\mathcal{F}$. However, $M \simeq$

$M/(M\cap T)\simeq MT/T\leq G/T$ which implies $Aut_{G/T}((U\cap M)T/(V\cap M)T)\simeq Aut_G((U\cap M)/(V\cap M))\simeq L_2(7)$, contrary our assumptions.

Hence, we are reduced to the case in which M is the unique minimal normal subgroup of G and $N_G(S_1)/C_G(S_1)\simeq PGL_2(7)$. Thus, $M=F^*(G)$ and then $C_G(M)\leq M$, by the theorem of Bender-Fitting (see Theorem 9.8 in [**16**]). In particular $C_G(M)=1$, since M is non abelian. Set $N=N_G(S_1)$ and $C=C_G(S_1)$. Since $[PGL_2(7):L_2(7)]=2$, one has $N=D_0S_1/C$, where D_0 is a Sylow 2-subgroup in $N_G(S_1)$. Write $D_0=D\cap N_G(S_1)$, for a suitable Sylow 2-subgroup D of G. Let H be the last proper term of a maximal $\mathbb{P}$-subnormal chain from D to G. Assume that H is normal in G. Then, $H\in\mathcal{F}_0$ and every Sylow 2-subgroup of G is contained in H. Further, since $D\trianglelefteq\trianglelefteq_{\mathbb{P}} H$ every Sylow 2-subgroup of H is $\mathbb{P}$-subnormal in H, by Lemma 1.3(1). We conclude that $H\in\mathcal{F}$ by the minimality of G. Of course, since G/H has odd order then G/H has no composition factors isomorphic to $L_2(7)$. Hence, $G\in\mathcal{F}$ a contradiction.
Therefore, H is not a normal subgroup of G, and thus H is maximal in G with index $[G:H]=q^a$, for some odd prime q and $a\geq 1$. Hence, we have that $H\leq HM\leq G$ and $HM\neq G$, otherwise Lemma 3.14 implies $H\in\mathscr{D}(G)$, which is impossible. Thus $M\leq H$. Since $C_G(M)=1$, it follows that every minimal normal subgroups of H is contained in M.

Also, we have $H\in\mathcal{F}_0$ by Lemma 3.6. In particular, by using the same notations as in Remark 3.4, if α is any $\mathbb{P}$-subnormal chain from D to H and $\left\{D,D^{h_2},\dots,D^{h_k}\,\middle|\,h_i\in H\right\}$ is the set of all Sylow 2-subgroups of H, then α^{h_i} are maximal $\mathbb{P}$-subnormal chains from D^{h_i} to H, for all $i=1,\dots,k$. We conclude that $H\in\mathcal{F}$ by minimality of G. This is clearly impossible, since it implies $D_0=D\cap N_G(S_1)\leq H\cap N_G(S_1)=N_H(S_1)$, which means $D_0S_1=N_H(S_1)$, and hence $N_H(S_1)/C_H(S_1)\simeq PGL_2(7)$, contrary to the fact that $H\in\mathcal{F}$. $\square$

We can finally prove Theorem 3.1.

Proof of Theorem 3.1. It is trivial that $\mathcal{R}\subseteq\mathcal{C}$. By Lemma 3.15, we get $\mathcal{F}\subseteq\mathcal{R}$. Finally, by Corollary 3.10 and Proposition 3.16 we establish the inclusion $\mathcal{C}\subseteq\mathcal{F}$. This completes the proof. $\square$

CHAPTER 4

Some invariants associated to $\mathbb{P}$-subnormal subgroups

In the first part of this chapter we show how to associate to every $\mathbb{P}$-subnormal subgroup H of G a set of prime numbers, denoted by $\mathbb{P}_G(H)$, which is independent from the choice of $\mathbb{P}$-subnormal chain from H to G (see Theorem 4.1 and Definition 4.2). The primes in $\mathbb{P}_G(H)$ are divisors of the index $[G : H]$. In Proposition 4.4, we use this invariant to generalize a well-known result of H. Wielandt (for subnormal subgroups, see Theorem 4.1.2 in [**21**]), in the context of $\mathbb{P}$-subnormality; namely we prove a sufficient condition for the intersection and the join of two $\mathbb{P}$-subnormal subgroups to be again $\mathbb{P}$-subnormal subgroups.

Then in the second part we adapt some results of R. W. Carter and A. Mann in the context of soluble groups to our situation and we prove that for a particular class of groups (called $\mathcal{G}$-free groups) we can associate to every $\mathbb{P}$-subnormal subgroup H an invariant $z(H)$ related to normal links of particular maximal $\mathbb{P}$-subnormal chains (Theorem 4.10 and Corollary 4.11).

4.1. The set $\mathbb{P}_G(H)$

As defined in Chapter 1, for a $\mathbb{P}$-subnormal subgroup H of G and a $\mathbb{P}$-subnormal chain $\alpha:\ H = H_0 < \ldots < H_m = G$, the set $\mathbb{P}_G(H,\alpha)$ is defined by

$$\mathbb{P}_G(H,\alpha) = \{p \in \mathbb{P} \mid \exists\, i \in \{0,\ldots,m-1\} \text{ such that } [H_{i+1} : H_i] = p^a\}.$$

We remind the reader that, as a consequence of the results given in Lemma 2.4 and Theorem 2.5 of Chapter 2, we have proved that if S is any finite simple group then the set $\mathbb{P}_S(1,\alpha)$ is independent on the choice of the maximal chain $\alpha \in \mathscr{M}(1,S)$. In particular, the values for $\mathbb{P}_S(1,\alpha)$ have been summarized in Table 1 at the end of Section 2.1. In fact, this result can be generalized to any $\mathbb{P}$-subnormal subgroup of an arbitrary group G, as the following theorem shows.

THEOREM 4.1. *Let G be a group and H, K be two $\mathbb{P}$-subnormal subgroups of G such that $H \leq K$. Assume that α and β are maximal $\mathbb{P}$-subnormal chains respectively, from H and K to G. Then $\mathbb{P}_G(K,\beta) \subseteq \mathbb{P}_G(H,\alpha)$.*
In particular, when $H = K$ then $\mathbb{P}_G(H,\alpha) = \mathbb{P}_G(H,\beta)$.

PROOF. We proceed by induction on the order of G. Let N be a maximal normal subgroup of G.
By Lemma 1.3(1), one has that $H \cap N$ and $K \cap N$ are $\mathbb{P}$-subnormal subgroups of N. Similarly, HN/N and KN/N are $\mathbb{P}$-subnormal subgroups of G/N, by using Lemma 1.3(2). The relation (6) of Lemma 1.24 implies that

$$\mathbb{P}_G(H,\alpha) = \mathbb{P}_N(H \cap N,\alpha_N) \cup \mathbb{P}_{G/N}\left(HN/N,\alpha^{G/N}\right) \tag{13}$$

and

$$\mathbb{P}_G(K,\beta) = \mathbb{P}_N(K\cap N,\beta_N) \cup \mathbb{P}_{G/N}\left(KN/N,\beta^{G/N}\right). \tag{14}$$

In particular, by part (1) of the same Lemma 1.24 we get $\mathbb{P}_N(H\cap N,\alpha_N) = \mathbb{P}_N(H\cap N,\widetilde{\alpha})$ and $\mathbb{P}_N(K\cap N,\beta_N) = \mathbb{P}_N(K\cap N,\widetilde{\beta})$, where $\widetilde{\alpha}$ and $\widetilde{\beta}$ are maximal $\mathbb{P}$-subnormal refinements respectively of α_N and β_N.
If $N\neq 1$ the inductive hypothesis assures that $\mathbb{P}_N(K\cap N,\widetilde{\beta}) \subseteq \mathbb{P}_N(H\cap N,\widetilde{\alpha})$ and $\mathbb{P}_{G/N}(KN/N,\beta^{G/N}) \subseteq \mathbb{P}_{G/N}(HN/N,\alpha^{G/N})$. By (13) and (14), it follows that $\mathbb{P}_G(K,\beta) \subseteq \mathbb{P}_G(H,\alpha)$, which is the result. Thus $N=1$ and G is a non abelian simple group (when G has prime order the result is trivial).

Of course, if $G\notin\mathcal{G}$ then trivially we have $\mathbb{P}_G(H,\alpha)=\mathbb{P}_G(K,\beta)=\emptyset$. Whence, assume that $G\in\mathcal{G}$.

If either $G=A_n$ with $n\geq 7$ or $G\in\{M_{11},M_{23}\}$, then a direct inspection to Lemma 1.14 shows that the result is true.

If $G\in\mathcal{G}_0$, then H and K are $\mathbb{P}$-subgroups of G, by Lemma 2.2. Hence, $\mathbb{P}_G(K,\beta)=\pi([G:K])$ and $\mathbb{P}_G(H,\alpha)=\pi([G:H])$ (see Remark 2.3). As a consequence, the condition $H\leq K$ implies $\mathbb{P}_G(K,\beta)\subseteq\mathbb{P}_G(H,\alpha)$, as required.

We are left with the cases of $G=L_n(q)$ that are described in Lemma 1.13, where q is the power of a prime number r and $n\geq 2$. If P and Q denote the last proper terms of the chains α and β respectively, consider $\overline{\alpha}$ and $\overline{\beta}$ the $\mathbb{P}$-subnormal chains obtained from α and β by removing the last term G. Since $[G:P]=[G:Q]$ is the power of the same prime number p (see Theorem 1.5(5)), then $\mathbb{P}_G(H,\alpha)=\{p\}\cup\mathbb{P}_P(H,\overline{\alpha})$ and $\mathbb{P}_G(H,\beta)=\{p\}\cup\mathbb{P}_Q\left(K,\overline{\beta}\right)$. Set $U=O_r(P)$ and $V=O_r(Q)$ the unipotent radical subgroups of P and Q, the relation (8) in Lemma 1.24 yields

$$\begin{aligned}
&\mathbb{P}_G(H,\alpha)=\{p\}\cup\mathbb{P}_U(H\cap U,\overline{\alpha}_U)\cup\mathbb{P}_{P/U}\left(HU/U,\overline{\alpha}^{P/U}\right)\\
&\mathbb{P}_G(K,\beta)=\{p\}\cup\mathbb{P}_V\left(K\cap V,\overline{\beta}_V\right)\cup\mathbb{P}_{Q/V}\left(KV/V,\overline{\beta}^{Q/V}\right),
\end{aligned}\tag{15}$$

where $\mathbb{P}_U(H\cap U,\overline{\alpha}_U)\cup\mathbb{P}_V\left(K\cap V,\overline{\beta}_V\right)\subseteq\{r\}$, because both U and V are r-groups. Clearly, $\mathbb{P}_U(H\cap U,\overline{\alpha}_U)=\emptyset$ if and only if $U\leq H$. Note that the condition $U\leq H$ yields $U\cap V\neq 1$. If not, $VU=V\rtimes U$ is an r-subgroup of Q and there exists $1\neq x\in V$ such that $[U,x]=1$. Since $C_G(U)=U$ (see Lemma 1.8(3)), it follows that $x\in C_G(U)\cap V=U\cap V=1$, a contradiction. Thus $U\cap V\neq 1$. Since $U\cap V\leq K\cap V$ the result in Lemma 1.13(2) implies that $Q'\leq K\leq Q$. Hence, $V\leq K$, which is $\mathbb{P}_V\left(K\cap V,\overline{\beta}_V\right)=\emptyset$. We deduce that if $r\in\mathbb{P}_Q\left(K,\overline{\beta}\right)$ then $r\in\mathbb{P}_P(H,\overline{\alpha})$. Consequently, by (15), to complete the proof it remains to show the inclusion $\mathbb{P}_{Q/V}\left(KV/V,\overline{\beta}^{Q/V}\right)\subseteq\mathbb{P}_{P/U}\left(HU/U,\overline{\alpha}^{P/U}\right)$.
Now, by Lemma 1.13, one of these situations may arise: either $KQ\leq S(Q)$ or $Q'\leq KV$.

(1) $KV\leq S(Q)$.
Then $HU\leq S(P)$, because $H\leq K$ is soluble. Also, we recall that both $S(P)/U$ and $S(Q)/V$ are r'-groups (see also Lemma 1.8(2)). Since $H/(H\cap U)\simeq HU/U\leq S(P)/U$, we deduce that $U\cap H=O_r(H)$. Similarly, we have that $K\cap V=O_r(K)$ and $K/K\cap V$ is an r'-group. Note that $H\cap V\leq H\cap U$, because $H\cap V$ is a normal r-subgroup of H. By the fact that

$$(H\cap U)/(H\cap V)\simeq(K\cap V)(H\cap U)/(K\cap V)\leq K/(K\cap V)\simeq KV/V,$$

we deduce that $H \cap U = H \cap V$ and then $HU/U \simeq HV/V$. As a consequence,

$$\begin{aligned}\mathbb{P}_{Q/V}\left(KV/V, \overline{\beta}^{Q/V}\right) &= \pi\left([Q : Q'S(Q)]\right) \cup \pi\left([S(Q) : KV]\right) \\ &\subseteq \pi\left([Q : Q'S(Q)]\right) \cup \pi\left([S(Q) : HV]\right) \\ &= \pi\left([P : P'S(P)]\right) \cup \pi\left([S(P) : HU]\right) \\ &= \mathbb{P}_{P/U}\left(HU/U, \overline{\alpha}^{P/U}\right),\end{aligned}$$

where the first and last equalities follow by Lemma 2.4.

(2) $Q' \leq KV$.

We distinguish two subcases: either $HU \leq S(P)$ or $P' \leq HU$.

If $HU \leq S(P)$ then, again by Lemma 2.4, it follows

$$\pi\left([P : P']\right) \subseteq \pi\left([P : P'S(P)]\right) \cup \pi\left([S(P) : HU]\right) = \mathbb{P}_{P/U}\left(HU/U, \overline{\alpha}^{P/U}\right),$$

and further,

$$\mathbb{P}_{Q/V}\left(KV/V, \overline{\beta}^{Q/V}\right) = \pi\left([Q : KV]\right) \subseteq \pi\left([Q : Q']\right),$$

because $\overline{\beta}^{Q/V}$ is a $\mathbb{P}$-chain. Therefore,

$$\mathbb{P}_{Q/V}\left(KV/V, \overline{\beta}^{Q/V}\right) \subseteq \mathbb{P}_{P/U}\left(HU/U, \overline{\alpha}^{P/U}\right)$$

and we are done.

If $P' \leq HU$ we conclude that both H and K are non soluble groups, by Lemma 1.13. Whence, the result in Lemma 2.4 implies that α and β are $\mathbb{P}$-chains, that is, $\mathbb{P}_G(H, \alpha) = \pi([G : H])$ and $\mathbb{P}_G(K, \beta) = \pi([G : K])$. Since we are assuming $H \leq K$, it follows that $\pi([G : K]) \subseteq \pi([G : H])$, and therefore $\mathbb{P}_G(K, \beta) \subseteq \mathbb{P}_G(H, \alpha)$, as wanted.

When $H = K$, by interchanging the roles of α and β, we get that $\mathbb{P}_G(H, \alpha) = \mathbb{P}_G(H, \beta)$. The proof is complete. □

As a consequence of the previous result, the following definition makes sense.

Definition 4.2. For every $\mathbb{P}$-subnormal subgroup H of G set

$$\mathbb{P}_G(H) := \mathbb{P}_G(H, \alpha),$$

where α is a maximal $\mathbb{P}$-subnormal chain from H to G.

Note that $\mathbb{P}_G(H) \subseteq \pi([G : H])$ and by the previous theorem that $\mathbb{P}_G(K) \subseteq \mathbb{P}_G(H)$ for $H, K \trianglelefteq\trianglelefteq_{\mathbb{P}} G$ and $H \leq K$. Moreover, if $H \trianglelefteq\trianglelefteq_{\mathbb{P}} G$ with $\mathbb{P}_G(H) = \emptyset$ then clearly H is subnormal in G.

4.2. A criterion of permutability for ℙ-subnormal subgroups

A well known result of H. Wielandt (Theorem 4.1.2 in **[21]**) is the following.

Theorem (H. Wielandt). *If H and K are subnormal subgroups of a group G with $\pi([H : H']) \cap \pi([K : K']) = \emptyset$ then H and K permute.*

As a consequence of our previous results, we shall establish a similar criterion of permutability for $\mathbb{P}$-subnormal subgroups, which is the content of Proposition 4.4. For its proof we need the following lemma.

LEMMA 4.3. *Let $G = HN$, where $N \trianglelefteq G$ is a direct product of n isomorphic copies of a non abelian simple group S. Assume that $S \notin \mathcal{G}$ and that H is $\mathbb{P}$-subnormal in G. Then H is subnormal in G.*

PROOF. Write $N = S_1 \times \ldots \times S_n$, where $S_i \simeq S$ for each $i = 1, \ldots, n$ and $n \geq 1$. By induction on $|G|$, we prove that H is a subnormal subgroup of G. Consider $\alpha : H = H_0 \leq H_1 \leq \ldots \leq H_m = G$ a $\mathbb{P}$-subnormal chain from H to G with $m \geq 1$ and let $W = H_{m-1} < G$.

Assume that $W \ntrianglelefteq G$. Then there exists a prime number p such that $[G : W] = p^a$ and $a \geq 1$. It follows that $[N : W \cap N] = [WN : W] = [G : W] = p^a$.
For every $i = 1, \ldots, n$ set $R_i = (W \cap N)^{\pi_i}$, where each $\pi_i : W \cap N \to S_i$ denotes the projection map on the i-th factor S_i. Let $L = R_1 \times \ldots \times R_n$. Since $W \cap N \leq L$ then $[N : L]$ is a divisor of $[N : W \cap N]$. In particular, there exists $i \in \{1, \ldots, n\}$ such that $R_i < S_i$ and $[S_i : R_i]$ is a positive power of p. This violates the hypothesis $S \notin \mathcal{G}$, so that this situation never happens.

Therefore, W is a normal subgroup of G. It follows that $W \cap N \lhd N$ and then $W \cap N \simeq S^k$, for some $1 \leq k < n$. Now $(W \cap N) H = W \cap HN = W$ and clearly H is $\mathbb{P}$-subnormal in W. By the inductive hypothesis we conclude that H is subnormal in W. Since $W \lhd G$ then H is subnormal in G. $\square$

PROPOSITION 4.4. *Let G be a group and let H and K be two $\mathbb{P}$-subnormal subgroups of G. If $\mathbb{P}_G(H) \cap \mathbb{P}_G(K) = \emptyset$ then H and K permute and HK is $\mathbb{P}$-subnormal in G.*

PROOF. In a minimal counterexample G, let H and K be subgroups of G as in the hypotheses and such that $[G : H][G : K]$ is minimal with respect to $\langle H, K\rangle$ not being $\mathbb{P}$-subnormal in G.

First we prove that G can not be a simple group.
Assume the contrary, and of course that G is a non abelian simple group not in $\mathcal{G}$. If G is not isomorphic to $L_2(7)$ then there exists a unique prime number t such that both H and K lie (as $\mathbb{P}$-subnormal subgroups) in maximal subgroups of G of index a power of t. Then $t \in \mathbb{P}_G(H) \cap \mathbb{P}_G(K)$, which is a contradiction. When $G \simeq L_2(7)$ then every subgroup of G is $\mathbb{P}$-subnormal in G (see Remark 1.6), and this a contradiction again. Hence, G is non simple.

Let N be a minimal normal subgroup of G. Then $HN/N \trianglelefteq\trianglelefteq_{\mathbb{P}} G/N$ and $KN/N \trianglelefteq\trianglelefteq_{\mathbb{P}} G/N$, by Lemma 1.3(2). If $N \leq H$, or if $N \leq K$, then we obtain

$$\langle HN/N, KN/N\rangle = HK/N \trianglelefteq\trianglelefteq_{\mathbb{P}} G/N,$$

by the minimal choice of G. Lemma 1.3(4) implies that HK is $\mathbb{P}$-subnormal in G. Thus we assume that $H < HN$ and $K < KN$. Since $[G : HN][G : KN] < [G : H][G : K]$, the subgroups HN and KN permute and HNK is $\mathbb{P}$-subnormal in G. In particular, $\mathbb{P}_G(HNK) \subseteq \mathbb{P}_G(H) \cap \mathbb{P}_G(K) = \emptyset$ (by Theorem 4.1). This means that HNK is subnormal in G. By Lemma 1.3(1), we deduce that H and K are $\mathbb{P}$-subnormal in HNK. If HNK is a proper subgroup of G then a contradiction follows immediately. Whence

$$HNK = G. \tag{16}$$

Now if N is an elementary abelian t-subgroup, for some prime t, then t divides both $[HN : H] > 1$ and $[KN : K] > 1$. By Theorem 4.1 and Definition 4.2, it follows that $t \in \mathbb{P}_{HN}(H) \cap \mathbb{P}_{KN}(K) \subseteq \mathbb{P}_G(H) \cap \mathbb{P}_G(K) = \emptyset$, which is impossible.

Hence N is a direct product of isomorphic copies of a non abelian simple group S. Write $N = S_1 \times \ldots \times S_n$, where $S_k \simeq S$ for every $k = 1, \ldots, n$. Let respectively $\alpha : \ H = H_0 < \ldots < H_u = G$ and $\beta : \ K = K_0 < \ldots < K_v = G$ be two maximal $\mathbb{P}$-subnormal chains from H to G and from K to G. Consider $i \in \{1, \ldots, u\}$ and $j \in \{1, \ldots, v\}$ minimal such that $N \leq H_{i+1} \cap K_{j+1}$. From the choice of i, one has $H_i < H_i N \leq H_{i+1}$. If $H_i \lhd H_{i+1}$ then $H_i N = H_{i+1}$ (because $H_i N / H_i \lhd H_{i+1}/H_i$ and H_{i+1}/H_i is simple, by the maximality of α). Otherwise H_i is a maximal subgroup of H_{i+1} and then again $H_i N = H_{i+1}$. Similarly, one has $K_j N = K_{j+1}$.

We claim that H_i and K_j are both maximal subgroups respectively of H_{i+1} and K_{j+1} with prime power indices. Otherwise, $H_i \lhd H_{i+1}$ and $K_j \lhd K_{j+1}$ where H_{i+1}/H_i and K_{j+1}/K_j are two non abelian simple groups not in $\mathcal{G}$. Since both H_i and N are normal in H_{i+1} then $H_i \cap N \trianglelefteq H_{i+1}$. It follows that $H_i \cap N = S_1 \times \ldots \times S_{n-1} \simeq S^{n-1}$ and then $H_{i+1}/H_i = H_i N / H_i \simeq N / H_i \cap N \simeq S \notin \mathcal{G}$. Since H is $\mathbb{P}$-subnormal in HN, we may apply Lemma 4.3 to deduce that H is subnormal in HN. Similarly, K is subnormal in KN. It follows that the subgroup $L = (H \cap N)(K \cap N) \trianglelefteq\trianglelefteq N$ is normalized by both H and K. On the other hand, $N \leq N_G(L)$, because N is a direct product of simple components and $L \trianglelefteq\trianglelefteq N$ (see, for instance, 6.5.2 in [**20**]). Consequently $N_G(L) = G$, by (16). Being N a minimal normal subgroup then either $L = 1$ or $L = N$. If we suppose $(H \cap N)(K \cap N) = N$, we get in contradiction with (16). We are forced to assume $(H \cap N)(K \cap N) = 1$. Since $H \trianglelefteq\trianglelefteq HN$ and $K \trianglelefteq\trianglelefteq KN$, we conclude that H and K centralize each component S_k for every $k = 1, \ldots, n$ (use the result 6.5.2 in [**20**] again). This means that $H, K \leq C_G(N)$. Let $V \coloneqq C_G(N)$. Since $N \lhd G$, also $V \trianglelefteq G$, whence H and K are $\mathbb{P}$-subnormal in V by Lemma 1.3(1). Since N is non abelian, one has $V < G$. By the minimality of G, we have that $HK = KH$ and HK is $\mathbb{P}$-subnormal in V, hence in G, a contradiction.

Therefore H_i is a maximal subgroup of H_{i+1} such that $p^a = [H_{i+1} : H_i] = [N : N \cap H_i]$, for some prime p. Hence, there exists $k \in \{1, \ldots, n\}$ such that $(N \cap H_i)^{\pi_k}$ is a subgroup of index a power of p in S_k. Note that this implies $p \in \mathbb{P}_{H_{i+1}}(H) \subseteq \mathbb{P}_G(H)$. Similarly, one can find a prime number q such that $[K_{j+1} : K_j] = [N : N \cap K_j] = q^b$, for some $b \geq 1$. With the same argument as before, there exists $l \in \{1, \ldots, n\}$ for which $(N \cap K_j)^{\pi_l}$ is a subgroup of index a power of q in S_l. In particular, $q \in \mathbb{P}_{K_{j+1}}(K) \subseteq \mathbb{P}_G(K)$. If $p = q$ then $p \in \mathbb{P}_G(H) \cap \mathbb{P}_G(K) \neq \emptyset$, but this is impossible. Thus, $p \neq q$ and this implies that $S \simeq L_2(7)$ (by Remark 1.6). This yields that $\alpha_N = \{H_k \cap N\}_{k=0,\ldots,u}$ is a $\mathbb{P}$-chain from $H \cap N$ to N, as a consequence of Theorem 2.5. By the equality (8) in Lemma 1.24, we infer that

$$\pi([N : H \cap N]) = \mathbb{P}_N(H \cap N, \alpha_N) \subseteq \mathbb{P}_G(H, \alpha) = \mathbb{P}_G(H).$$

Similarly, we get that $\beta_N = \{K_l \cap N\}_{l=0,\ldots,v}$ is a $\mathbb{P}$-chain from $K \cap N$ to N such that

$$\pi([N : K \cap N]) = \mathbb{P}_N(K \cap N, \beta_N) \subseteq \mathbb{P}_G(K, \beta) = \mathbb{P}_G(K).$$

Since by hypothesis

$$\pi([N : H \cap N]) \cap \pi([N : K \cap N]) \subseteq \mathbb{P}_G(H) \cap \mathbb{P}_G(K) = \emptyset,$$

it follows that $(H \cap N)(K \cap N) = N$. In particular, $G = HK$ a contradiction which completes the proof. $\square$

As an immediate application of Lemma 1.4(1) we have the following.

Corollary 4.5. *Let G be a group and let H, K be $\mathbb{P}$-subnormal subgroups of G. If $\pi(G : H) \cap \pi(G : K) = \emptyset$, then $H \cap K$ is $\mathbb{P}$-subnormal in G.*

Proof. We make induction on $|G|$. Apply Proposition 4.4 to deduce that $\langle H, K\rangle = HK$ is $\mathbb{P}$-subnormal in G. In particular, HK is subnormal in G, because $\mathbb{P}_G(HK) \subseteq \mathbb{P}_G(H) \cap \mathbb{P}_G(K) = \emptyset$ (by Theorem 4.1). By Lemma 1.3(1) it follows that H and K are $\mathbb{P}$-subnormal subgroups of HK, where of course $\mathbb{P}_{HK}(H) \cap \mathbb{P}_{HK}(K) \subseteq \mathbb{P}_G(H) \cap \mathbb{P}_G(K) = \emptyset$. If $HK < G$, the inductive hypothesis on HK yields the thesis. Otherwise $HK = G$, and we may apply Lemma 1.4(1) to get the result. □

4.3. Counting the normal links in $\mathbb{P}$-subnormal chains

We remind the reader that for any given $\mathbb{P}$-subnormal chain $H = H_0 < \ldots < H_m = G$ a pair of consecutive terms (H_i, H_{i+1}) such that $H_i \trianglelefteq H_{i+1}$ is called a normal link of the chain (see Definition 1.19).

As already pointed out, for soluble groups every chain of subgroups is a $\mathbb{P}$-subnormal chain. The behavior of normal links in maximal chains of soluble groups has been object of investigation by R. W. Carter in some of his works ([**7**]), which are now of fundamental importance in the theory of these groups. By definition, if H is a subnormal subgroup of G then there exists a chain from H to G all of whose links are normal. On the other hand, if H is a Carter subgroup (which means a nilpotent self-normalizing subgroup) then it can be shown that any maximal chain from H to G has no normal links.

We briefly describe an idea of R. W. Carter in §2 of [**8**] that associates to every subgroup H of a soluble group G a set of invariants, which depend on the abstract structure of H and on the way in which H is embedded in G. These invariants have a clear interpretation in terms of reducibility of Hall systems (see [**6**] and [**15**]) and they are obtained by evaluating the product of the indices of all the normal links in special kinds of unrefinable chains of subgroups from H to G. In order to clarify the structure of such chains, we remind the reader some standard facts about operator groups, which will be needed also in the sequel (see Chapter II, §3 in [**28**]).

If G is any (not necessarily soluble) group and H is a subgroup of G, it is possible to regard G as a group with operator domain the conjugations induced by elements of H. An *H-subgroup* of G is a subgroup of G which is normalized by H, and the series

$$1 = A_0 < A_1 < \ldots < A_n = G \tag{17}$$

is an H-composition series of G if each A_i is a maximal normal H-subgroup of A_{i+1}. The factor groups A_{i+1}/A_i are called *H-composition factors* of G. An H-composition factor has no proper characteristic subgroups. The set of H-composition factors is divided into two types. We say that A_{i+1}/A_i of G is a *central* H-composition factor if all the operators act trivially on A_{i+1}/A_i, that is $[A_{i+1}, H] \leq A_i$. Otherwise, the factor is said to be *eccentric*. It is worth remarking that the number of central H-composition factors does not depend on the choice of the H-composition series, by the Theorem of Jordan-Hölder (Corollary 1 in Chapter II, §3 in [**28**]).

For any H-composition series (17) of a soluble group G and every $H \leq G$, Carter considers the chain

$$\gamma: \; H = HA_0 \leq HA_1 \leq \ldots \leq HA_n = G,$$

and notices that it is always unrefinable, that is, it defines a sequence of H-subgroups each maximal in the next. This is easy to see, since H either covers or avoids each H-composition factor of G. We remind that H *covers* the factor A_{i+1}/A_i if $(A_{i+1} \cap H)\, A_i = A_{i+1}$ and that H *avoids* the factor A_{i+1}/A_i if $(A_{i+1} \cap H)\, A_i = A_i$. This cover-avoidance property of H gives another possible way of dividing the H-composition factors into two classes. Specifically, there are four different types of H-composition factors: the central factors covered by H, the eccentric factors covered by H, the central factors avoided by H and the eccentric factors avoided by H. Carter has shown that the product of the orders of all normal links in a maximal chain such as γ does not depend on the choice of the H-composition series (17). As a consequence, to every subgroup H of a soluble group G, there is associated the value $z\,(H)$ of such a product. It is possible to give interpretations to the two invariants $z\,(H)$ and $[G:H]\,/z\,(H)$. It turns out that the integer $[G:H]\,/z\,(H)$ coincides with the total number of Hall systems of G divided by the number of the ones reducible into H. While, if for any maximal chain α from H to G we denote by $z\,(\alpha)$ the product of the indices of all normal links of α, then $z\,(H) = max\,\{z\,(\alpha) \mid \alpha\}$. Subsequently, A. Mann proved a sharpered characterization of $z\,(H)$, by showing that it is a multiple of $z\,(\alpha)$, whenever α is an unrefinable chain of subgroups from H to G (Theorem 1 in [**24**]). In particular, the chains γ are, among all unrefinable chains from H to G, those in which the number of normal links is maximal.

We now implement these ideas of Carter and Mann in our context of $\mathbb{P}$-subnormality, for every finite abstract group. Extending the terminology of Carter, we give the following definition.

Definition 4.6. Let G be a group. Assume that H is a $\mathbb{P}$-subnormal subgroup of G and let $\alpha: \; H = H_0 \leq \ldots \leq H_m = G$ be a $\mathbb{P}$-subnormal chain.
We define

$$z\,(\alpha) = \prod [H_{i+1} : H_i],$$

where the product is taken over all normal links of α.
We also set

$$z_0\,(\alpha) = \prod [H_{i+1} : H_i],$$

where the product is taken over all abelian normal links of α, that are all those normal links (H_i, H_{i+1}) whose quotient H_{i+1}/H_i is abelian (see Definition 1.19 in Chapter 1).

We now define an equivalence relation on the set $\mathscr{M}\,(H,G)$ of maximal $\mathbb{P}$-subnormal chains from H to G. For every $\alpha, \beta \in \mathscr{M}\,(H,G)$ we set

$$\alpha \asymp \beta \quad \text{iff} \quad z\,(\alpha) = z\,(\beta)\,.$$

The quotient set $\mathscr{M}\,(H,G)\,/\asymp$ is partially ordered with respect to

$$[\alpha] \leq [\beta] \quad \text{iff} \quad z\,(\alpha) \mid z\,(\beta)\,,$$

for $[\alpha]\,, [\beta] \in \mathscr{M}\,(H,G)\,/\asymp$. We point out the following questions:

- Q_1**:** For $\alpha \in \mathscr{M}\,(H,G)$, does $z\,(\alpha)$ admit some "natural" combinatorial interpretation?

Q_2**:** How many maximal elements does the poset $\mathscr{M}(H,G)/\asymp$ contain?

As partially noted above, when G is a soluble group and α is a maximal chain from a subgroup H to G, Carter gives a combinatorial interpretation of $z(\alpha)$ by showing that it coincides with the product of $[G:H]$ by the number of Hall systems which are reducible into the chain α, divided by the total number of Hall systems of G (p. 541 in [**8**]).

We are unable to answer to question Q_1 when G is not soluble.
In the next (and final) section we answer question Q_2 when G lies in a special class of groups.

4.3.1. $\mathcal{G}$-free groups. We extend the aforementioned results of Carter and Mann to the following class of groups.

Definition 4.7. A group having no composition factors belonging to $\mathcal{G}$ is called $\mathcal{G}$-free.

We also introduce the terminology below.

Definition 4.8. Let G be a group, H a subgroup of G and $1 = A_0 < A_1 < \ldots < A_n = G$ an H-composition series of G. The chain

$$H = HA_0 \leq HA_1 \leq \ldots \leq HA_n = G$$

is called a *C-chain* from H to G.

A crucial key is the fact that if G is $\mathcal{G}$-free, the condition $H \trianglelefteq\trianglelefteq_{\mathbb{P}} G$ implies the cover-avoidance property with respect to the H-composition factors, namely the following lemma.

Lemma 4.9 (cover-avoidance property). *Let G be a $\mathcal{G}$-free group and let H be a $\mathbb{P}$-subnormal subgroup of G. Assume that*

$$\sigma:\ 1 = A_0 < A_1 < \ldots < A_n = G$$

is an H-composition series in G. Then H covers or avoids each H-composition factor of σ. In particular, if A_{i+1}/A_i is a non abelian H-composition factor avoided by H then $HA_i \lhd HA_{i+1}$ and the group $HA_{i+1}/HA_i \simeq A_{i+1}/A_i$ is simple (not in $\mathcal{G}$).

Proof. For every $i \in \{0,\ldots,n-1\}$ the H-composition factor A_{i+1}/A_i is a direct product of, say, m isomorphic copies of a simple group S which is not in $\mathcal{G}$, by assumption. We distinguish the two cases: either S is abelian or not.

Suppose S is abelian. Then $A_i(H\cap A_{i+1})$ is an H-subgroup between A_i and A_{i+1} which refines σ. Thus either $A_i(H\cap A_{i+1}) = A_i$ and $H\cap A_i = H\cap A_{i+1}$ or $A_i(H\cap A_{i+1}) = A_{i+1}$ and $HA_i = HA_{i+1}$. This means that H covers or avoids the H-composition factor A_{i+1}/A_i.

Suppose S is non abelian. By Lemma 1.4(2), we have that HA_i/A_i is a $\mathbb{P}$-subnormal subgroup of HA_{i+1}/A_i. Whence, HA_i/A_i is subnormal in HA_{i+1}/A_i, by Lemma 4.3. If H does not cover A_{i+1}/A_i then let M/A_i be a maximal normal subgroup of HA_{i+1}/A_i containing HA_i/A_i. We have that $M\cap A_{i+1}$ is a proper H-invariant subgroup of A_{i+1} containing A_i. Since σ is unrefinable (as an H-series) it follows that $M\cap A_{i+1} = A_i$, and then $M = HA_i$. This argument shows that either $A_iH\cap A_{i+1} = A_i$ or $A_iH\cap A_{i+1} = A_{i+1}$, that is to say that H either covers or avoids A_{i+1}/A_i.

Note in particular that we also proved, when A_{i+1}/A_i is non abelian and $HA_i < HA_{i+1}$, that HA_i is normal in HA_{i+1} and $HA_{i+1}/HA_i \simeq A_{i+1}/A_i$ is simple. $\square$

The proof of the following theorem uses a variation of arguments in §2 of [**8**] and Theorem 1 of [**24**].

THEOREM 4.10. *Let G be a $\mathcal{G}$-free group and let H be a $\mathbb{P}$-subnormal subgroup of G.*

(1) *Every C-chain from H to G is a maximal $\mathbb{P}$-subnormal chain.*
(2) *If γ_1 and γ_2 are two C-chains from H to G, then $z(\gamma_1) = z(\gamma_2)$. Moreover, $z_0(\gamma_1) = z_0(\gamma_2)$ and this coincides with the product of the indices of the central H-composition factors of G that are avoided by H.*
(3) *For every maximal $\mathbb{P}$-subnormal chain α from H to G and every C-chain γ from H to G, the value $z(\alpha)$ divides $z(\gamma)$. Also, $z_0(\alpha)$ divides $z_0(\gamma)$ and*

$$z(\alpha)/z_0(\alpha) = z(\gamma)/z_0(\gamma).$$

PROOF. Let $\gamma : H = HA_0 \leq HA_1 \leq \ldots \leq HA_n = G$ be a C-chain from H to G defined by an H-composition series $\sigma : 1 = A_0 < A_1 < \ldots < A_n = G$.

(1) By induction on $[G : H]$ we show that γ is a maximal $\mathbb{P}$-subnormal chain. If $[G : H] = 1$ there is nothing to prove, thus suppose $H < G$. Consider k minimal such that $A_{k+1} \not\leq H$. By Lemma 1.4(2) we have that $H \trianglelefteq\trianglelefteq_{\mathbb{P}} HA_{k+1} \trianglelefteq\trianglelefteq_{\mathbb{P}} G$. Since $[G : H] > [G : HA_{k+1}]$, the inductive hypothesis yields that

$$HA_{k+1} \leq \ldots \leq HA_n = G \tag{18}$$

is a maximal $\mathbb{P}$-subnormal chain. By Lemma 4.9, we know that H covers or avoids the H-composition factor A_{k+1}/A_k. But $H = HA_k < HA_{k+1}$ so that we have $A_k(H \cap A_{k+1}) = A_k$, which implies $H \cap A_k = H \cap A_{k+1}$. Now two possibilities arise: either A_{k+1}/A_k is abelian or not.

If A_{k+1}/A_k is abelian then H is a maximal subgroup of HA_{k+1}. For, if there were K with $H < K < HA_{k+1}$ then $K \cap A_{k+1}$ would be a proper H-subgroup between A_k and A_{k+1}, which is impossible. Furthermore, we have that $A_k \leq H \cap A_{k+1}$. It follows that $[HA_{k+1} : H] = [A_{k+1} : H \cap A_{k+1}]$ is the power of a prime number, since it divides the order of A_{k+1}/A_k, which is elementary abelian. Thus, H is maximal in HA_{k+1} with prime power index. Since (18) is a maximal $\mathbb{P}$-subnormal chain then γ is a maximal $\mathbb{P}$-subnormal chain.

If A_{k+1}/A_k is non abelian then $HA_k \triangleleft HA_{k+1}$ and $HA_{k+1}/HA_k \simeq A_{k+1}/A_k$ is a simple group not in $\mathcal{G}$ (again by Lemma 4.9). This completes the proof that γ is a maximal $\mathbb{P}$-subnormal chain from H to G.

(2) We prove that the function z is constant on the set of C-chains from H to G.

We first claim that (HA_i, HA_{i+1}) is a proper normal link in γ if and only if A_{i+1}/A_i is an H-composition factor avoided by H, that is either central or non abelian.

Since (HA_i, HA_{i+1}) is a proper normal link in γ, H does not cover A_{i+1}/A_i, whence by Lemma 4.9 we have that $HA_{i+1}/HA_i \simeq A_{i+1}/A_i$. Assume that A_{i+1}/A_i is abelian. Then

$$[A_{i+1}, H] \leq [HA_{i+1}, HA_i] \leq HA_i \cap A_{i+1} = A_i(H \cap A_{i+1}) = A_i,$$

and so A_{i+1}/A_i is central. Conversely, assume that A_{i+1}/A_i is avoided by H and is central or non abelian. Then $HA_i < HA_{i+1}$. Moreover, if A_{i+1}/A_i is central then

$[HA_i, A_{i+1}] = [H, A_{i+1}]\,[A_i, A_{i+1}] \leq A_i$, forcing $HA_i \lhd HA_{i+1}$. When A_{i+1}/A_i is non abelian, Lemma 4.9 gives the same conclusion, and this completes the proof of the claim. Note that A_{i+1}/A_i is avoided by H if and only if $[HA_{i+1} : HA_i] = [A_{i+1} : A_i] > 1$. Therefore, we have that

$$z(\gamma) = f_0(\sigma)\, f_1(\sigma),$$

where $f_0(\sigma) = \prod [A_{i+1} : A_i]$, the product being taken over all the i such that A_{i+1}/A_i are central and avoided by H, and $f_1(\sigma) = \prod [A_{i+1} : A_i]$, where the product now is over all the i such that A_{i+1}/A_i are non abelian and avoided by H. To conclude the proof of this part it is enough to show that the functions f_0 and f_1 are constant on the set of H-composition series of G.

We claim that A_{i+1}/A_i is a (central) H-composition factor covered by H if and only if $(A_{i+1} \cap H)/(A_i \cap H)$ is a (central) chief factor of H. Clearly, $H \cap A_i$ and $H \cap A_{i+1}$ are normal in H. Assume that A_{i+1}/A_i is covered by H. By way of contradiction suppose there exists $K \lhd H$ between $H \cap A_i$ and $H \cap A_{i+1}$. Then both subgroups A_i and $H \cap A_{i+1}$ normalize KA_i. Thus, $KA_i \trianglelefteq A_{i+1} = (H \cap A_{i+1})\, A_i$ and therefore KA_i would be an H-subgroup that lies strictly between A_i and A_{i+1}, a contradiction. Therefore, $(A_{i+1} \cap H)/(A_i \cap H)$ is a chief factor of H. Moreover, if A_{i+1}/A_i is central then $[A_{i+1}, H] \leq A_i$ and this implies that $(A_{i+1} \cap H)/(A_i \cap H)$ is central. Conversely, let $(A_{i+1} \cap H)/(A_i \cap H)$ be a chief factor of H. Then $(A_{i+1} \cap H)/(A_i \cap H) > 1$ and so A_{i+1}/A_i is covered by H. Also, when $(A_{i+1} \cap H)/(A_i \cap H)$ is central then $[A_{i+1}, H] = [A_i\,(H \cap A_{i+1}), H] = [A_i, H]\,[H \cap A_{i+1}, H] \leq A_i$ and A_{i+1}/A_i is central. This completes the proof of the claim.

By the Theorem of Jordan-Hölder, there is a one-to-one correspondence between the chief factors of any two chief series of H. Moreover, this correspondence preserves central factors as well as non abelian factors (see 9.13, Chapter A in [**11**]). By this fact and the previous claim we conclude that the functions f_0 and f_1 are constant on the H-composition series of G.

(3) Let $\alpha : \; H = H_0 < H_1 < \ldots < H_m = G$ and $K_i = HA_i$ for $i = 0, \ldots, n$. We show that there exists an injective index-preserving map f from the set of the normal links of α to the set of the normal links of γ. Moreover, we prove that the restriction of f to the set of the non abelian normal links of α is a bijection into the set of non abelian normal links of γ. This will imply that $z(\alpha) | \, z(\gamma)$ and $z(\alpha)/z_0(\alpha) = z(\gamma)/z_0(\gamma)$; as a consequence we obtain also that $z_0(\alpha) | \, z_0(\gamma)$.

We form the Zassenhaus refinements of the chains α and γ, that is, we insert the terms $H_{i,j} = H_i\,(K_j \cap H_{i+1})$ in α and $K_{j,i} = K_j\,(H_i \cap K_{j+1})$ in γ. In general, $H_{i,j}$ and $K_{j,i}$ are subsets of G and not subgroups. However, when $H_i \lhd H_{i+1}$ then $H_{i,j} \leq G$, and similarly when $K_j \lhd K_{j+1}$ then $K_{j,i} \leq G$. Moreover, by order reasons we have

$$|K_{j,i+1}| \,/\, |K_{j,i}| = |H_{i,j+1}| \,/\, |H_{i,j}|\,. \tag{19}$$

Suppose that (H_i, H_{i+1}) is a normal link of α. Then each $H_{i,j}$, for $j = 0, ..., n$, is a subgroup of G and $H_{i,0} = H_i$, $H_{i,n} = H_{i+1}$. Let $j = j(i)$ be the first index such that $H_{i,j+1} \nleq H_i$. We claim that $H_{i,j+1} = H_{i+1}$. Since $H_{i+1} \cap A_{j+1}$ is subnormal in H_{i+1} and $H_i \lhd H_{i+1}$ then

$$H_{i,j+1} = H_i\,(H_{i+1} \cap K_{j+1}) = H_i\,(H_{i+1} \cap HA_{j+1}) = H_i\,(H_{i+1} \cap A_{j+1})$$

is a subnormal subgroup of H_{i+1}, by a well-known result of Wielandt (see, for instance, 6.7.1 in [**20**]). Therefore, $H_{i,j+1} = H_{j+1}$ by the maximality of α.

Now we show that (K_j, K_{j+1}) is a normal link of γ, and this will allow us to define $f((H_i, H_{i+1})) = (K_j, K_{j+1})$. Since $H_i = H_i(H_{i+1} \cap A_j)$ and $H_{i+1} = H_i(H_{i+1} \cap A_{j+1})$ and they are distinct, if we set $A_{j,i} = (H_i \cap A_{j+1}) A_j$ and $A_{j,i+1} = (H_{i+1} \cap A_{j+1}) A_j$, then we have

$$A_{j,i} \neq A_{j,i+1}.$$

We prove that

$$\begin{aligned} A_{j,i} &= A_j \\ A_{j,i+1} &= A_{j+1}. \end{aligned} \tag{20}$$

Note first that both $A_{j,i}$ and $A_{j,i+1}$ are $\mathbb{P}$-subnormal H-subgroups of A_{j+1} that contain A_j. This follows by Lemma 1.4(2), since $H_i \cap A_{j+1}$ and $H_{i+1} \cap A_{j+1}$ are $\mathbb{P}$-subnormal in A_{j+1} (Lemma 1.3(1)) and A_j is normal in A_{j+1}. Now, if A_{j+1}/A_j is abelian then clearly $A_{j,i}$ and $A_{j,i+1}$ are H-invariant subnormal subgroups of G. We reach the same conclusion even if A_{j+1}/A_j is non abelian, since in this case A_{j+1}/A_j is a direct product of isomorphic copies of a simple group not in $\mathcal{G}$ and we can use Lemma 4.3. It follows that $A_{j,i} = A_j$ and $A_{j,i+1} = A_{j+1}$ since the H-series σ is unrefinable. Consequently, we have that

$$\begin{aligned} [H, A_{j+1}] = &\ [H, (H_{i+1} \cap A_{j+1}) A_j] \\ = &\ [H, H_{i+1} \cap A_{j+1}] [H, A_j] \leq [H, H_{i+1}] [H, A_j] \leq H_i A_j, \end{aligned}$$

and thus $[H, A_{j+1}] \leq A_{j+1} \cap H_i A_j = (H_i \cap A_{j+1}) A_j = A_j$. We conclude that

$$[K_j, K_{j+1}] = [HA_j, HA_{j+1}] \leq HA_j = K_j,$$

proving that (K_j, K_{j+1}) is a normal link of γ. Furthermore, by (20) we obtain that

$$K_j = K_{j,i} \text{ and } K_{j+1} = K_{j,i+1}.$$

Equation (19) yields that $[K_{j+1} : K_j] = [H_{i+1} : H_i]$, proving that f is an index-preserving map.

The proof that f is injective is straightforward from the fact that if $t \leq i$ then $K_{j,t} = K_j$ while if $t > i$ then $K_{j,t} = K_{j+1}$.

We now show that the restriction of f to the set of non abelian normal links of α is surjective onto the set of non abelian normal links of γ. Consider (K_j, K_{j+1}) a non abelian normal link of γ. Then each $K_{j,i}$, for $i = 0, ..., m$, is a subgroup of G and $K_{j,0} = K_j$, $K_{j,m} = K_{j+1}$. Let $i = i(j)$ be the first index such that $K_{j,i+1} \not\leq K_j$. We claim that $K_{j,i+1} = K_{j+1}$. As before, note that $K_j = K_{j,i} \neq K_{j,i+1}$ from the identity (19). Now $K_{j,i+1} = K_j(H_{i+1} \cap K_{j+1}) = K_j(H_{i+1} \cap A_{j+1})$ is a $\mathbb{P}$-subnormal subgroup of K_{j+1} containing K_j, by the fact that $H_{i+1} \cap A_{j+1}$ is $\mathbb{P}$-subnormal in K_{j+1} (Lemma 1.3(1) and $A_{j+1} \trianglelefteq\trianglelefteq K_{j+1}$) and Lemma 1.4(2). It follows that $K_{j,i+1} = K_{j+1}$, since γ is a maximal $\mathbb{P}$-subnormal chain, by point (1) of this theorem. Thus, by combining together the relation in (19) and the maximality of α, we are forced to assume that $H_i = H_{i,j}$ and $H_{i+1} = H_{i,j+1}$. This is to say that (H_i, H_{i+1}) is a non abelian normal link of α with $[H_{i+1} : H_i] = [K_{j+1} : K_j]$. This completes the proof of the theorem. $\square$

As an immediate consequence, we can now answer question Q_2 for $\mathcal{G}$-free groups.

Corollary 4.11. *Let G be a $\mathcal{G}$-free group and let $H \trianglelefteq\trianglelefteq_{\mathbb{P}} G$. Then there exists a unique maximal element in $\mathscr{M}(H,G)/\asymp$.*

Moreover, for a $\mathbb{P}$-subnormal subgroup H in a $\mathcal{G}$-free group G, we define the following values:

$$z(H) = z(\gamma) \text{ and } z_0(H) = z_0(\gamma),$$

where γ is any C-chain from H to G.

Note that, even in the soluble case in part (3) of Theorem 4.10, we could not expect that $z(H)$ is equal to $z(\alpha)$, for every $\alpha \in \mathscr{M}(H,G)$.

Example 4.12. Let $G = S_3$. If we consider the trivial subgroup, then clearly $z(1) = 6$ while $\alpha: 1 < \langle(12)\rangle < G$ is a maximal $\mathbb{P}$-subnormal chain from 1 to G and, of course, $z(\alpha) = 2$. Thus $z(\alpha)$ strictly divides $z(1)$.

Note also that if we define $\mathscr{M}^*(H,G)$ to be the subset of all maximal $\mathbb{P}$-subnormal chains from H to G having maximal number of normal links, we can prove the following.

Corollary 4.13. *Let G be a $\mathcal{G}$-free group, $H \trianglelefteq\trianglelefteq_{\mathbb{P}} G$ and $\mu \in \mathscr{M}^*(H,G)$. Then $z(\mu) = z(H)$.*

Proof. Let $\mu \in \mathscr{M}^*(H,G)$. By Theorem 4.10(3) it is enough to show that $z_0(\mu) = z_0(H)$. Let γ be any C-chain, so that $z_0(\gamma) = z_0(H)$ which we assume to be

$$z_0(\gamma) = p_1^{a_1} \dots p_n^{a_n}$$

for p_i distinct prime numbers. Note that since $\gamma \in \mathscr{M}(H,G)$ by Theorem 4.10(1), the sum $\sum_{i=1}^n a_i$ is the number of the normal abelian links of γ. Since $z_0(\mu) \mid z_0(\gamma)$ then we can write $z_0(\mu) = p_1^{b_1} \dots p_n^{b_n}$, where each $b_i \le a_i$ and the sum $\sum_{i=1}^n b_i$ is the number of the normal abelian links of μ. The assumption $\mu \in \mathscr{M}^*(H,G)$ implies that $\sum_{i=1}^n b_i = \sum_{i=1}^n a_i$ and therefore $z_0(\mu) = z_0(\gamma)$, which means $\mu \asymp \gamma$. $\square$

When G is not $\mathcal{G}$-free and H is a proper $\mathbb{P}$-subnormal subgroup of G then, in general a C-chain from H to G is not a $\mathbb{P}$-subnormal chain, as it may be seen by taking G to be any simple group in $\mathcal{G}$ and H a subgroup which is not maximal in G. Moreover, it is not difficult to find examples for which the condition of Theorem 4.10(3) does not hold.

Example 4.14. Let $G = L_2(7)$ and let α be an arbitrary maximal $\mathbb{P}$-subnormal chain from 1 to G. If M is the last proper term of α, then either $M \simeq S_4$ or $M \simeq 7:3$ (by Remark 1.6). Note that, if τ is any composition series of M then $\tau \in \mathscr{M}(1,M)$ and $z(\tau) = |M|$. Since (M,G) is not a normal link of α, it follows that $z(\alpha)$ is a divisor of $z(\tau)$. We conclude that in $\mathscr{M}(1,G)/\asymp$ there exist two maximal elements $[\mu_1]$ and $[\mu_2]$, where $z(\mu_1) = 24$ and $z(\mu_2) = 21$.

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