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# Creep, recovery and vibration of an incompressible viscoelastic material of the rate type: Simple tension case

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We consider three-dimensional nonlinear viscoelastic models that account for both stress relaxation and creep/recovery phenomena. These models are based on different frame indifferent time derivatives: the Oldroyd (or upper-convected) derivative, the Jaumann (or co-rotational) derivative and the Cotter–Rivlin (or lower-convected) derivative. Under a simple tension creep process, these constitutive equations predict the same stress relaxation but lead to different situations. The models based on the Oldroyd and the lower-convected derivative require restrictions on the values of the material parameters as well as on the traction/compression stress. The model based on the Jaumann derivatives does not require any restriction. All the constitutive models examined are used to study the finite amplitude, horizontal oscillatory motion of a mass attached to a rate-type viscoelastic string. In this way we generalize the classical results by Beatty and Zhou (1991).

## 1. Introduction

Polymer-based products are commonplace in automotive, aerospace, electronic and biomedical industries. Moreover, nearly all soft tissues are polymeric materials (see, e.g., [1,2]). For these reasons the chemical and structural properties of these materials are studied in details with modern methods and ideas but their mechanical properties are mainly considered in the framework of classical linear elasticity [3]. Nowadays, technical, industrial and scientific applications of polymeric materials are highly sophisticated and thus the development of mechanical and mathematical models accounting for fundamental time-dependent phenomena (such as creep, recovery and stress relaxation) is necessary (see, e.g., [4] for stress relaxation in a nonlinear setting).

In a nonlinear framework, creep and recovery phenomena may be described by minimal nonlinear versions of the usual Kelvin–Voigt model based on the differential theory of viscoelasticity [5]. These models have been proposed by several authors as (see, e.g., [6–9]). However, the possibility to model stress relaxation needs a more complex approach based on rate or integral models [5]. Nonlinear models of viscoelasticity of rate type have been scarcely investigated. For example, Zhou [10] proposed a rate model to study what occurs in simple shear and Filograna et al. [11] proposed a similar model to study the mechanical response of asphalt with respect to some basic deformations.

In a recent article Cormack and Hamilton [12] considered stress relaxation via a one-dimensional empirical model in nonlinear acoustics. This perspective has been re-derived from a general three-dimensional model by Saccomandi and Vianello [13]. In rate models of viscoelasticity a central role is played by objective time derivatives. It is well known that in Continuum Mechanics the constitutive equations are required to be frame indifferent [5]. In the rate theory of viscoelasticity, it is usual to introduce a differential (in time) constitutive equation for the stress tensor and, clearly, the time derivative involved has to be frame indifferent. This opens the problem of the choice of the objective time derivatives to select. Saccomandi and Vianello [13] considered three different frame indifferent time derivatives: the Oldroyd (or upper-convected) derivative, the Jaumann (or co-rotational) derivative and the Cotter–Rivlin (or lower-convected) derivative. They studied the propagation of shear waves and observed that different objective derivatives lead to completely different stress fields. Nevertheless, in the asymptotic limit of small but finite amplitude shear waves, any objective derivative gives the same approximated stress field and the same set of equations governing the wave motions.

This is an important result but the fact that different time derivatives give different descriptions of the same phenomena is puzzling and needs a deeper investigation. Indeed, these indifferent time derivatives are of interest not only in viscoelasticity, but also in non-Newtonian fluid mechanics [14], hypoelasticity and elastoplasticity (see [15–18]).

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Moreover, some thermodynamic issues have been recently pointed out by Morro and Giorgi [19] who studied the thermodynamic consistency of a specific rate type model (the one that in the sequel will be referred to as the "third" model). By following similar arguments as in Morro and Giorgi [19], the thermodynamic consistency of all the models we shall consider can be analyzed. However, this analysis goes beyond the scopes of the manuscript as we are here interested in isothermal processes.

The objective of this study is twofold. By limiting our analysis to simple extension experiments, we investigate the role of three frame indifferent time derivatives (Oldroyd, Jaumann and Cotter–Rivlin) in modeling these deformations. Viscoelasticity is an appropriate framework for such a comparison because the dynamical properties in a quasi-static process are summarized by a single parameter: the *retardation time* (eventually different in creep and relaxation) [20]. By means of the retardation time it is possible to measure the speed of the stress relaxation and of the creep or recovery phenomena which are the standard tests to investigate this kind of material behavior. The second target is to investigate the adequacy of three constitutive models for incompressible, isotropic viscoelastic materials of the rate type (already considered in the literature by Saccomandi and Vianello [13], Zhou [10], Nasseri et al. [21], Phan-Thien et al. [22]) in describing simple extensions. These constitutive equations generalize the standard model in linear viscoelasticity.

Using a notation that allows to handle at same time all the three types of derivatives, we investigate the differences of the use of these objective differential operators and their main features.

We point out that for polymeric melts (i.e. viscoelastic fluids) extensional rheometry has been deeply studied [23], but for solids the situation is quite different. Apart from the classical papers by Gent [24] and Gottenberg et al. [25] (see also the book by Findley et al. [26]), the literature has not advanced significantly. In fact, more recent papers have focused mainly on the linear theory or empirical approaches [27, 28].

The outline of the paper is as follows. In Section 2 we set down the basic equations. In Section 3 we consider stress relaxation and in Section 4 we consider the phenomena of uniaxial strain into detail. In this framework, it is possible to better understand the differences among the three objective time derivatives. In Section 5 we apply the models to the longitudinal oscillations of a free mass and then conclude with some remarks.

## 2. Preliminaries

Here we recall some standard notations used in Continuum Mechanics (see, e.g., [29–31]). Let

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t) \tag{1}$$

be the deformation of a three-dimensional body in the Euclidean space, where $\boldsymbol{x}$ and $\boldsymbol{X}$ are the respective position of a generic particle of the body at time $t$ and at a given initial time $t_o$. Then

$$\mathbb{F} := \frac{\partial \chi(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}, \quad \mathbb{B} := \mathbb{F}\mathbb{F}^{\mathrm{T}}, \quad \mathbb{L} := \dot{\mathbb{F}}\mathbb{F}^{-1}, \tag{2}$$

define, respectively, the deformation gradient, the Cauchy–Green deformation tensor, and the spatial velocity gradient tensor. The superimposed dot identifies the usual material time derivative. The symmetric tensor

$$\mathbb{D} := \frac{1}{2}\left(\mathbb{L} + \mathbb{L}^{\mathrm{T}}\right), \tag{3}$$

is the stretching tensor. In particular, we consider the incompressibility constraint

$$\det \mathbb{F} = 1, \quad \text{or} \quad \mathrm{tr}\mathbb{D} = 0, \tag{4}$$

which entails the introduction of an unknown spherical tensor $-p\mathbb{I}$, with $p$ referred to as pressure [32]. Next, we define the frame time derivative for a given second-order tensor $\mathbb{S}$

$$\mathscr{D}_\gamma(\mathbb{S}) = \dot{\mathbb{S}} + \frac{\gamma - 1}{2}\left(\mathbb{S}\mathbb{L}^{\mathrm{T}} + \mathbb{L}\mathbb{S}\right) + \frac{\gamma + 1}{2}\left(\mathbb{S}\mathbb{L} + \mathbb{L}^{\mathrm{T}}\mathbb{S}\right), \tag{5}$$

where $\gamma$ can be $-1$, 0, or $+1$, identifies the Oldroyd (also referred to as upper-convected or Truesdell rate since $\mathrm{tr}\mathbb{D} = 0$, [5]), the Jaumann (or co-rotational), and the lower derivative (or Cotter–Rivlin rate [33]), respectively.

We remark that the $\gamma$ is not a constitutive parameter since it can take only three values ($-1$, 0, or $+1$) and so it selects the specific objective time derivative.

We consider three constitutive laws for the Cauchy stress $\mathbb{T}$ of an incompressible viscoelastic material of the rate type. For a better understanding of differences among these three models, it is useful to refer to the deviatoric (or shear) part of the Cauchy stress $\widehat{\mathbb{T}} = \mathbb{T} + p\mathbb{I}$

- **First model** [10]

$$\frac{1}{\phi}\mathscr{D}_\gamma(\mathbb{T}) + \mathbb{T} = \underbrace{\mathbb{T} - \widehat{\mathbb{T}}}_{-p\mathbb{I}} + \beta_1\mathbb{B} + \beta_{-1}\mathbb{B}^{-1} + 2\eta\mathbb{D}. \tag{6}$$

- **Second model** (derived by the general constitutive laws presented by Saccomandi and Vianello [13])

$$\frac{1}{\phi}\mathscr{D}_\gamma\left(\widehat{\mathbb{T}}\right) + \widehat{\mathbb{T}} = \beta_1\mathbb{B} + \beta_{-1}\mathbb{B}^{-1} + 2\eta\mathbb{D}. \tag{7}$$

- **Third model** [21,22]

$$\begin{cases} \widehat{\mathbb{T}} = \mathbb{T}^V + \beta_1\mathbb{B} + \beta_{-1}\mathbb{B}^{-1}, \\ \dfrac{1}{\phi}\mathscr{D}_\gamma\left(\mathbb{T}^V\right) + \mathbb{T}^V = 2\eta\mathbb{D}. \end{cases} \tag{8}$$

It is evident from these definitions that the difference among them consists, essentially, in which part of the Cauchy stress the operator $\mathscr{D}_\gamma$ is applied to.

In the above models $\beta_1$, $\beta_{-1}$, are, in general, material functions while $\eta$ and $\phi$ are constant parameters representing the material viscosity and a characteristic time, respectively. In particular, we take $\phi$ and $\eta$ non-negative as prescribed by simple thermodynamic issues.

To simplify the algebra $\beta_1$ and $\beta_{-1}$, are set to be constant parameters. However this hypothesis it is not strictly necessary and many of our results may be extended to the general case [10]. In particular, we take

$$\beta_1 = \frac{G}{1+\alpha}, \qquad \beta_{-1} = -\frac{\alpha G}{1+\alpha}, \tag{9}$$

with $G = \left(\beta_1 - \beta_{-1}\right) > 0$, material elastic modulus and $\alpha \in [0, 1]$. In the elastic case, when $\alpha = 0$, we recover the neo-Hookean case, while for $\alpha \neq 0$, we have the incompressible Mooney–Rivlin model.

We point out that in the linear limit the three models collapse to the same model, i.e. the three dimensional version of standard linear solid also known as the Zener model. The one dimensional empirical version of such model is given, in the Maxwell representation, by

$$\sigma + \frac{\nu}{E_2}\dot{\sigma} = E_1\varepsilon + \nu\frac{E_1 + E_2}{E_2}\dot{\varepsilon}, \tag{10}$$

where $\sigma$ is the one dimensional applied stress and $\varepsilon$ is the one dimensional strain. In the Maxwell representation we are indeed considering a spring and spring plus a dashpot in parallel. The parameters $E_1$ and $E_2$ are the Young's modulus of the two springs and $\nu$ is the viscosity of the dashpot.

## 3. Stress relaxation

Stress relaxation phenomena have been observed in all viscoelastic materials and consist in a decay process of the stress $\mathbb{T}$ under constant deformation (and, consequently, constant pressure[1]), starting from an initial stress $\mathbb{T}^{in}$. We now show that, for the three models, these processes are characterized by the same universal equation, regardless of the material response functions. If we look at the equilibrium stress $\mathbb{T}_e$ corresponding to a given deformation $\mathbb{F}_e$, with $\det \mathbb{F}_e = 1$, i.e.

$$\mathbb{T}_e = -p_e\mathbb{I} + \beta_1\mathbb{B}_e + \beta_{-1}\mathbb{B}_e^{-1}, \tag{11}$$

in all cases we get $\mathbb{T} = \mathbb{T}_e$, meaning that the equilibrium stress corresponding to a given deformation is completely determined by the constitutive equation for incompressible hyper-elastic materials.

Concerning the **first model**, i.e. (6), we have

$$\begin{cases} \dot{\mathbb{T}} = -\phi\ \left[(\mathbb{T}+p\mathbb{I}) - \left(\beta_1\mathbb{B}_e + \beta_{-1}\mathbb{B}_e^{-1}\right)\right] = -\phi\ \left[(\mathbb{T}+p\mathbb{I}) - \left(\mathbb{T}_e + p_e\mathbb{I}\right)\right], \\ \mathbb{T}|_{t=0} = \mathbb{T}^{in}\ , \end{cases} \tag{12}$$

yielding, since $p = p_e$, to such a Cauchy problem

$$\begin{cases} \dot{T}_{ij} = -\phi\left(T_{ij} - T_{e,ij}\right), \qquad i,j = 1,2,3 \\ T_{ij}\Big|_{t=0} = T_{ij}^{in}\ , \end{cases} \tag{13}$$

i.e. problem (26) in Zhou [10]. The constant $\phi^{-1}$ is often referred to as *stress relaxation time* or simply *relaxation time*. However, in the visco-elasticity literature, the term retardation time, usually denoted as $t_r$, is often used as a measure of this property, so that the identification $t_r = \phi^{-1}$ is frequently encountered.

Focusing on the **second model**, i.e. (7), and considering, as before, $p = p_e$, we obtain again the Cauchy problem (13).

Concerning the **third model**, i.e. (8), we have $\mathbb{T} - \mathbb{T}_e = \mathbb{T}^V$, since $p = p_e$, and thus $\dot{\mathbb{T}} = -\phi\ \left[\mathbb{T} - \mathbb{T}_e\right]$, i.e. Cauchy problem (13).

We can therefore conclude that the three models are indistinguishable by performing only stress relaxation experiments. In order to highlight possible differences between the dynamics predicted by the three constitutive models, it is necessary to consider a time dependent deformation. In particular in the next section a uniaxial creep process is investigated.

## 4. Uniaxial strain and creep process

We represent the body deformation (1) as

$$x = \lambda X, \quad y = \frac{1}{\sqrt{\lambda}}Y, \quad z = \frac{1}{\sqrt{\lambda}}Z, \tag{14}$$

where constraint (4) is taken into account, and $\lambda(t)$ is the stretch (dimensionless), always positive. Next, we consider a creep process, i.e. the evolution of the deformation under the action of a constant stress until reaching an asymptotic equilibrium state. In particular, the initial state is the undeformed one and the applied stress takes the form

$$\mathbb{T} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

Before proceeding, a brief remark is in order. While deformation (14) and stress (15) are in one-to-one correspondence in case of elastic materials [34], it has not been shown (at least as in our knowledge) that a simple tensile load produces a simple extension when the material is modeled as an incompressible viscoelastic continuum of rate type. Therefore the results that we shall illustrate in the sequel are not definitive, but should be taken as indicative since, in principle, the axial stress (15) could produce deformations different from the simple extension.

The **first model**, i.e. constitutive Eq. (6), gives rise to this system

$$\begin{cases} T_{11} = -p + \beta_1\lambda^2 + \beta_{-1}\lambda^{-2} - \dfrac{1}{\phi}\left(\dot{T}_{11} + 2\gamma\dfrac{\dot\lambda}{\lambda}T_{11}\right) + 2\eta\dfrac{\dot\lambda}{\lambda}, \\ T_{22} = -p + \beta_1\lambda^{-1} + \beta_{-1}\lambda - \dfrac{1}{\phi}\left(\dot{T}_{22} - \gamma\dfrac{\dot\lambda}{\lambda}T_{22}\right) - \eta\dfrac{\dot\lambda}{\lambda}, \end{cases} \tag{16}$$

with $T_{33}$ fulfilling identical to the one governing $T_{22}$.

Considering the **second model**, we have $T_{ii} = -p + \widehat{T}_{ii}$ $(i = 1,2,3)$, where $\widehat{T}_{ii}$ obeys Eq. (7), i.e.

$$\begin{cases} \widehat{T}_{11} = \beta_1\lambda^2 + \beta_{-1}\lambda^{-2} - \dfrac{1}{\phi}\left(\dot{\widehat{T}}_{11} + 2\gamma\dfrac{\dot\lambda}{\lambda}\widehat{T}_{11}\right) + 2\eta\dfrac{\dot\lambda}{\lambda}, \\ \widehat{T}_{22} = \beta_1\lambda^{-1} + \beta_{-1}\lambda - \dfrac{1}{\phi}\left(\dot{\widehat{T}}_{22} - \gamma\dfrac{\dot\lambda}{\lambda}\widehat{T}_{22}\right) - \eta\dfrac{\dot\lambda}{\lambda}. \end{cases} \tag{17}$$

The equation for $\widehat{T}_{33}$ is identical to the one for $\widehat{T}_{22}$.

Concerning the **third model**, i.e. constitutive Eq. (8), we obtain

$$\begin{cases} T_{11} = -p + \beta_1\lambda^2 + \beta_{-1}\lambda^{-2} + T_{11}^V, \\ T_{22} = -p + \beta_1\lambda^{-1} + \beta_{-1}\lambda + T_{22}^V, \\ T_{11}^V = -\dfrac{1}{\phi}\left(\dot{T}_{11}^V + 2\gamma\dfrac{\dot\lambda}{\lambda}T_{11}^V\right) + 2\eta\dfrac{\dot\lambda}{\lambda}, \\ T_{22}^V = -\dfrac{1}{\phi}\left(\dot{T}_{22}^V - \gamma\dfrac{\dot\lambda}{\lambda}T_{22}^V\right) - \eta\dfrac{\dot\lambda}{\lambda}, \end{cases} \tag{18}$$

where the equations for $T_{33}$ and $T_{33}^V$ are formally identical to the ones for $T_{22}$ and $T_{22}^V$.

Because of (11), the stress (15) gives rise, for all models, to the equilibrium stretch $\lambda_e$

$$\sigma = \left(\beta_1 - \frac{\beta_{-1}}{\lambda_e}\right)\left(\lambda_e^2 - \frac{1}{\lambda_e}\right) \underset{(9)}{=} G\frac{\lambda_e + \alpha}{1+\alpha}\left(\lambda_e - \frac{1}{\lambda_e^2}\right), \tag{19}$$

and to the equilibrium pressure[2]

$$p_e = \beta_1\lambda_e^{-1} + \beta_{-1}\lambda_e \underset{(9)}{=} G\frac{1 - \alpha\lambda_e^2}{\lambda_e(1+\alpha)}. \tag{20}$$

In particular we get, $\lambda_e > 1$ (extension) when $\sigma > 0$, $\lambda_e = 1$ if $\sigma = 0$ and $\lambda_e \in (0,1)$ (compression) if $\sigma < 0$.

### 4.1. First model

The first model, but only when $\gamma = 1$, has been already investigated in Zhou [10]. Here we also consider the cases $\gamma = 0, -1$. Focusing on (16) where we set $T_{11} = \sigma$ and $T_{22} = T_{33} = 0$, we obtain, after eliminating $p$ and recalling (19), the following Cauchy problem for $\lambda(t)$

$$\begin{cases} \left(3\eta - 2\sigma\dfrac{\gamma}{\phi}\right)\dfrac{\dot\lambda}{\lambda} = \dfrac{G}{(1+\alpha)}\left[\left(\lambda_e + \alpha\right)\left(\lambda_e - \dfrac{1}{\lambda_e^2}\right) - (\lambda + \alpha)\left(\lambda - \dfrac{1}{\lambda^2}\right)\right], \\ \lambda(0) = 1\ . \end{cases} \tag{21}$$

As in Zhou [10] we introduce the *retardation time*

$$t_r = \frac{(1+\alpha)}{G}\left(3\eta - 2\sigma\frac{\gamma}{\phi}\right), \tag{22}$$

and we immediately realize that $\gamma$ plays a fundamental role in the sign of such a quantity. In other words, if $\gamma = -1$, $\operatorname{sign} t_r$ is positive provided

[1] The stress due to the incompressibility constraint does not vary as the material undergoes static deformation.

[2] We remark that $\lambda_e = 1$ entails $\sigma = 0$, and $p_e = G\frac{1-\alpha}{1+\alpha}$.

$\sigma > 0$. On the contrary, for $\gamma = +1$, $t_r$ may become negative for $\sigma > \sigma^*$ where $\sigma^* = \frac{3\eta\phi}{2}$, is a critical value of applied tension. The situation is reversed if we take $\sigma < 0$. In this case $t_r$ could change sign when $\gamma = -1$. When $\gamma = 0$ the retardation time $t_r$ is independent of $\sigma$.

The requirement $t_r > 0$ may be considered in the framework of the one-dimensional empirical Zener model (10). This is not a rigorous argument but it may give a sort of rule of thumb to understand this inequality. If we use the empirical model it is possible to make the following correspondence among the various parameter of (10) and the **first model** (in case of $\alpha = 0$, i.e. neo-Hookean model, and $\gamma = 1$)

$$\eta = \frac{\nu(E_1+E_2)}{E_2}, \qquad \phi = \frac{E_2}{\nu}, \qquad E_1 = 3G. \tag{23}$$

Therefore requiring $t_r > 0$ entails

$$E_1 + E_2 > \frac{2}{3}\sigma,$$

but (19) and (23) relate $\sigma$ to $G$ and $\lambda_e$ and $G$ to $E_1$, so that

$$\frac{1}{2}\left(1+9\frac{E_2}{E_1}\right) > \lambda_e^2 - \frac{1}{\lambda_e}.$$

This gives a quite strict bound on $\lambda_e$ that shows that the model maybe used for finite but moderate deformations. This is not truly a problem because it is well known that both the neo-Hookean and the Mooney–Rivlin material model are valid only for moderate deformations also in the purely elastic case. This bound is relaxed for $\alpha \neq 0$, and significantly relaxed if we use strain hardening elastic models [35].

Clearly, the identification (23) may be discussed and the only rigorous observation is that when $\phi \to \infty$ we recover the retardation time of a viscoelastic solid of differential type for any value of $\gamma$. This means that when the stress relaxation time, i.e. $\phi^{-1}$, is very small, larger values of applied tension are admissible and therefore the critical tension $\sigma^*$ may approach the failure load. Moreover, when $\sigma > 0$ (i.e. in case of extension) we have that:

- if $\gamma = 1$, the creeping speed is smaller than in the pure differential viscoelastic model;
- if $\gamma = 0$, the creeping speed is exactly the same of the differential model;
- if $\gamma = -1$, the creeping speed is larger.

Therefore we have pointed out a striking difference about the role of the different objectives derivatives. It is therefore that the selection of the derivative is a constitutive issue and not just a mathematical issue. However, we remark that if $\gamma \neq 0$, restrictions must be imposed on the values of the material parameters $\eta$, $\phi$, $G$, $\alpha$ as well as on the applied stress $\sigma$ so that the Cauchy problem (21) does not give rise to meaningless solutions. Moreover, we remark that, if $\gamma = 0$, the stress relaxation time $\phi^{-1}$, does not appear in the (22), i.e. the definition of the retardation time. This is formally correct since, in creep processes, the stress remains unchanged over time.

Let us end this section pointing out that taking $\gamma = 0$, and scaling the time with $t_r$ which now is given by $3(1+\alpha)\eta/G$, problem (21) rewrites as

$$\begin{cases} \dfrac{\dot\lambda}{\lambda} = \lambda_e\left(\lambda_e - \dfrac{1}{\lambda_e^2}\right) - \lambda\left(\lambda - \dfrac{1}{\lambda^2}\right), \\ \lambda(0) = 1. \end{cases} \tag{24}$$

Problem (24) can be solved analytically and an implicit formula for $\lambda$ can be obtained. However, this formula is very complicated and for this reason we prefer to solve (24) numerically. In Fig. 1 we show solutions to (24) for several choices of the equilibrium stretch larger as well as smaller than one.

### 4.2. Second model

Recalling that $T_{11} = \sigma$ and $T_{22} = T_{33} = 0$, we have

$$\widehat{T}_{11} = \sigma + p, \qquad \widehat{T}_{22} = \widehat{T}_{33} = p,$$

which, once replaced in (17), gives raise, with the help of (9) and (19), to the following system

$$\begin{cases} \left[3\eta - 2\dfrac{\gamma}{\phi}\left(\sigma + \dfrac{p}{2}\right)\right]\dfrac{\dot\lambda}{\lambda} = \dfrac{G}{(1+\alpha)}\left[(\lambda_e+\alpha)\left(\lambda_e - \dfrac{1}{\lambda_e^2}\right) - (\lambda+\alpha)\left(\lambda - \dfrac{1}{\lambda^2}\right)\right], \\ \dot p + \left(\phi - \gamma\dfrac{\dot\lambda}{\lambda}\right)p = \dfrac{\phi G}{1+\alpha}\left(\dfrac{1}{\lambda} - \alpha\lambda\right) - \eta\phi\dfrac{\dot\lambda}{\lambda}, \end{cases} \tag{25}$$

whose initial conditions are $\lambda(0) = 1$ and $p(0)$ given by (20) with $\lambda_e = 1$, i.e. $p(0) = G\frac{1-\alpha}{1+\alpha}$. Again, restrictions must be imposed on $\phi$, $G$, $\alpha$ and on the stress $\sigma$ in order to get meaningful solutions if $\gamma = \pm 1$. Indeed, a sufficiently large tensile stress (i.e. $\sigma > 0$) can give rise to a negative $\dot\lambda$ if $\gamma = 1$, while a sufficiently large compression stress (i.e. $\sigma < 0$) can give rise to a positive $\dot\lambda$ if $\gamma = -1$. In Fig. 2 we report $\lambda$ solution to (25) for $\lambda_e$ in a physically meaningful range (for example, 10% of the initial value). The time has been normalized with $3\eta/G$, $\alpha = 0$, $\eta\phi/G = 1$ and, according to (19), $\sigma/G = \lambda_e^2 - 1/\lambda_e$.

Next, as remarked in Section 4.1, we have that the creeping speed depends on $\sigma$ (as well as on $p$) when $\gamma = \pm 1$. In other words, the creeping speed in compression can be different from the one in extension. No difference occurs when $\gamma = 0$. Moreover, if $\gamma = 0$, no restrictions on $\sigma$ are needed and in problem (25) two different time scales are evident: $\phi^{-1}$ for the stress (or better for the pressure) and the usual $t_r = 3(1+\alpha)\eta/G$ for the stretch. Moreover, taking $\gamma = 0$, the stretch evolution equation is decoupled from the pressure and we recover problem (24). This is very interesting since the evolution of $p$ (which, in principle, is defined up to a constant) should have no effect on $\lambda$, which is the actually measurable quantity.

### 4.3. Third model

Setting $T_{11} = \sigma$ and $T_{22} = T_{33} = 0$ in system (18) and introducing

$$A(\lambda) = \beta_1\lambda^2 + \beta_{-1}\lambda^{-2} = \frac{G}{1+\alpha}\left(\lambda^2 - \alpha\lambda^{-2}\right),$$

$$[0.04in] B(\lambda) = \beta_1\lambda^{-1} + \beta_{-1}\lambda = \frac{G}{1+\alpha}\left(\lambda^{-1} - \alpha\lambda\right),$$

we have

$$\begin{cases} \sigma + p - A(\lambda) = T^V_{11}, \\ p - B(\lambda) = T^V_{22}, \\ T^V_{11} = -\dfrac{1}{\phi}\left(\dot T^V_{11} + 2\gamma\dfrac{\dot\lambda}{\lambda}T^V_{11}\right) + 2\eta\dfrac{\dot\lambda}{\lambda}, \\ T^V_{22} = -\dfrac{1}{\phi}\left(\dot T^V_{22} - \gamma\dfrac{\dot\lambda}{\lambda}T^V_{22}\right) - \eta\dfrac{\dot\lambda}{\lambda}, \end{cases}$$

which gives rise to the following differential equation for the unknown function $\lambda(t)$

$$\frac{\dot\lambda}{\lambda}\left[3\eta - \frac{\gamma}{\phi}(2\sigma + 3p - 2A - B) - \frac{\lambda}{\phi}\left(B' - A'\right)\right]$$
$$= \underbrace{\frac{G}{(1+\alpha)}\left[(\lambda_e+\alpha)\left(\lambda_e - \frac{1}{\lambda_e^2}\right) - (\lambda+\alpha)\left(\lambda - \frac{1}{\lambda^2}\right)\right]}_{\sigma-(A-B)},$$

where (19) has been exploited and where $'$ denotes $d/d\lambda$. Also in this case the physically meaningless evolutions of $\lambda(t)$ may occur when $\gamma = \pm 1$, if restrictions on $\phi$, $G$, $\alpha$ and $\sigma$ are not imposed. In this case, taking as time scale $t_r = 3(1+\alpha)\eta/G$, we have,

$$\begin{aligned} &\frac{\dot\lambda}{\lambda}\left[1 + \frac{\phi^{-1}}{t_r}\left(\alpha\lambda + 2\lambda^2 + \frac{1}{\lambda} + \frac{2\alpha}{\lambda^2}\right)\right] \\ &= (\lambda_e+\alpha)\left(\lambda_e - \frac{1}{\lambda_e^2}\right) - (\lambda+\alpha)\left(\lambda - \frac{1}{\lambda^2}\right), \end{aligned} \tag{26}$$

where now $\phi^{-1}$, i.e. the stress relaxation time introduced in Section 3, enters the dynamics of $\lambda$. However, if $\phi^{-1} \ll t_r$, the **third model**

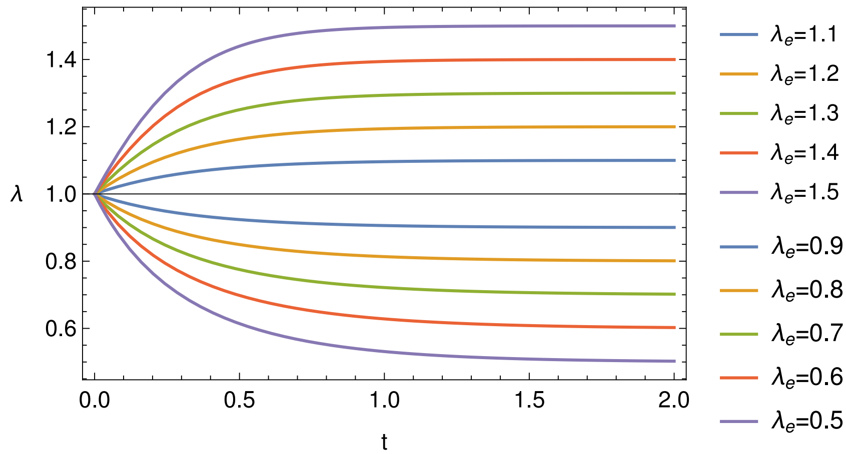


**Fig. 1.** Solutions to (24) for various $\lambda_e$.

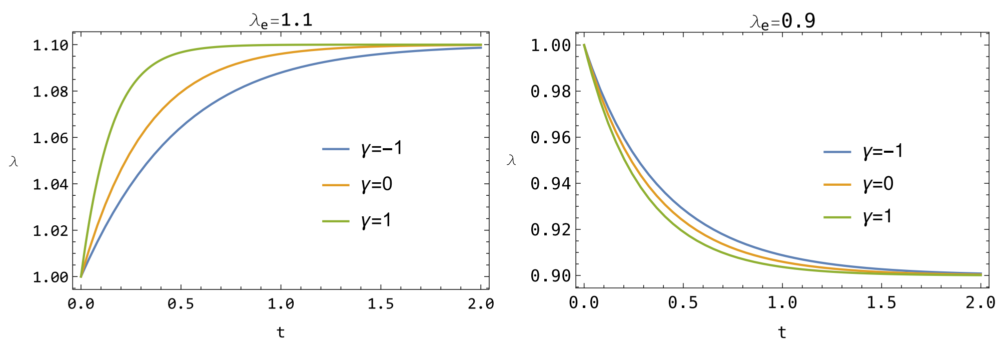


**Fig. 2.** $\lambda(t)$ solution to (25) for $\alpha = 0$, $\sigma/G = \lambda_e^2 - 1/\lambda_e$ and $\eta\phi/G = 1$. The left panel shows the extension case with $\lambda_e = 1.1$. The right panel shows the compression case with $\lambda_e = 0.9$. Time is normalized with $3\eta/G$.

converges to the **first model** and to the **second model**, provided $\gamma = 0$. Moreover, in this case the creeping speed is unaffected by the sign of $\sigma$ (see Fig. 3).

## 5. Free oscillations of a mass

We consider the mechanical system schematically depicted in Fig. 4, i.e. a body of mass $m$ constrained to move on the horizontal $x$ axis under the action of a string, whose material obeys the constitutive equations given by the **first**, the **second** and the **third model**, respectively. We denote with $\ell(t)$ the string length and with $\ell(0)$ its non-stretched length. We assume that: the guide is frictionless, the loads acting on the lateral string sides vanish, the string deforms according to (14) and its mass is definitely smaller than $m$ (thus we can safely neglect the string inertia).

The problem of the finite amplitude, horizontal oscillatory motion of a mass attached to a neo-Hookean rubber spring and supported by a fixed, ideally smooth and rigid horizontal surface has been solved exactly by Beatty [36]. The undamped, large amplitude, vibrations of a body supported symmetrically by isotropic elastic shear mounts has been studied by Beatty [37]. The damped case has been analyzed by Beatty and Zhou [38] and Zhou [10].

Concerning the motion equation of the body, we have

$$m\ell(0)\ddot{\lambda} = -T_{11}A, \tag{27}$$

where $A$ is the area of the contact surface between the body and the string, which we consider fixed. This causes a deviation from the deformation (14) near the string end in contact with the body. However, we neglect this edge effect.

We analyze the problem whose initial datum is the equilibrium stretch $\lambda(0) = \lambda_o > 1$ corresponding to: (i) the stretch caused the load $\sigma_o > 0$, i.e. recalling (19)

$$\frac{\sigma_o}{G} = \frac{\lambda_o + \alpha}{1+\alpha}\left(\lambda_o - \frac{1}{\lambda_o^2}\right), \tag{28}$$

(ii) $\dot{\lambda}(0) = 0$, (iii) $\ddot{\lambda}(0) = -\sigma_o A/m\ell(0)$ . In particular, the equilibrium state corresponds to $\sigma_0 = 0$ since it entails $\lambda_o = 1$ and vice-versa.

The purpose of this section is to highlight the difference in the dynamics predicted by the three models. In particular, we shall see that the **first** and the **second model** give rise to the same motion equation for the body of mass $m$ (provided $\gamma = 0$). Furthermore, when we model the string with Eq. (8), i.e. with the **third model**, we obtain a motion equation different from the previous ones.

### 5.1. First and second model

Focusing at first on the **first model**, i.e. on system (16), we follow the same procedure illustrated in Section 4 and take $\gamma = 0$. After eliminating $p$, we obtain

$$\frac{1}{\phi}\dot{T}_{11} + T_{11} = G\frac{\lambda+\alpha}{1+\alpha}\left(\lambda - \frac{1}{\lambda^2}\right) + 3\eta\frac{\dot{\lambda}}{\lambda}. \tag{29}$$

Setting then

$$\tilde{T}_{11} = \frac{T_{11}}{G}, \qquad \tilde{t} = \frac{t}{t_c}, \quad \text{with} \quad t_c = \sqrt{\frac{m\ell(0)}{GA}}, \tag{30}$$

and

$$a = \frac{3\eta\phi}{G}, \qquad b = \phi t_c, \tag{31}$$

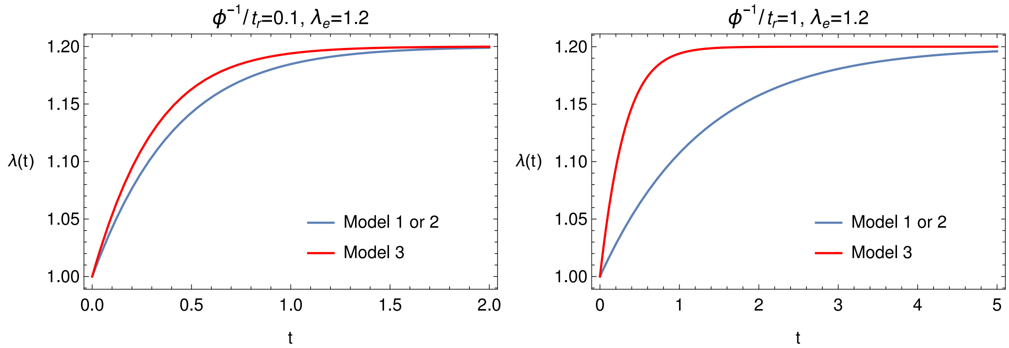


**Fig. 3.** Solution to (26) for $\alpha = \gamma = 0$ compared with that to (24) for $\lambda_e = 1.2$, and two different values of the ratio $\phi^{-1}/t_r$.

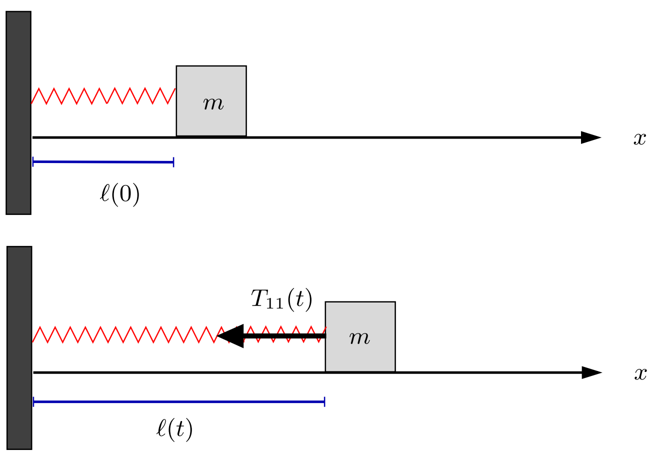


**Fig. 4.** Horizontal uniaxial oscillations of a rigid block under the action of the stress $T_{11}(t)$ due to a deformed ideal viscoelastic string.

we obtain such a system (where we omit "$\sim$" to keep the notation as light as possible)

$$\begin{cases} \ddot{\lambda} = -T_{11}, \\ \dot{T}_{11} + b\,T_{11} = b\,\dfrac{\lambda+\alpha}{1+\alpha}\left(\lambda - \dfrac{1}{\lambda^2}\right) + a\,\dfrac{\dot{\lambda}}{\lambda}, \\ \lambda(0) = \lambda_o, \quad \dot{\lambda}(0) = 0, \quad T_{11}(0) \underset{(28)}{=} \dfrac{\lambda_o+\alpha}{1+\alpha}\left(\lambda_o - \dfrac{1}{\lambda_o^2}\right), \end{cases} \tag{32}$$

that we rewrite as

$$\begin{cases} \dddot{\lambda} + b\,\ddot{\lambda} + a\,\dfrac{\dot{\lambda}}{\lambda} + b\,\dfrac{\lambda+\alpha}{1+\alpha}\left(\lambda - \dfrac{1}{\lambda^2}\right) = 0, \\ \lambda(0) = \lambda_o, \quad \dot{\lambda}(0) = 0, \quad \ddot{\lambda}(0) = -\dfrac{\lambda_o+\alpha}{1+\alpha}\left(\lambda_o - \dfrac{1}{\lambda_o^2}\right). \end{cases} \tag{33}$$

In particular, we remark that, once prescribed $\lambda_o$, i.e. the load $\sigma_o$, system (33) is characterized only by three dimensionless parameters, namely $a$, $b$, and $\alpha$.

Focusing now on the **second model**, i.e. Eq. (7), and recalling that $T_{22} = T_{33} = 0$, we have $\widehat{T}_{22} = \widehat{T}_{33} = p$, while $\widehat{T}_{11} = T_{11} + p$. Using (17) with $\gamma = 0$, we get such a system

$$\begin{cases} T_{11} + p = \beta_1\lambda^2 + \beta_{-1}\lambda^{-2} - \dfrac{1}{\phi}\left(\dot{T}_{11} + \dot{p}\right) + 2\eta\dfrac{\dot{\lambda}}{\lambda}, \\ p = \beta_1\lambda^{-1} + \beta_{-1}\lambda - \dfrac{1}{\phi}\dot{p} - \eta\dfrac{\dot{\lambda}}{\lambda}. \end{cases}$$

Combining the two equations we again obtain (29). Consequently the **second model** gives rise, after performing the non-dimensionalization (30), to system (32) or (33), provided $\gamma = 0$. Hence, the **first** and the **second model** identify if and only if $\gamma = 0$ (Jaumann derivative).

Let us now analyze the equilibrium state $\lambda = 1$. We remark that $\lambda = 1$ is reached by the solutions of the first as well as the second model only if $a$ is not too small. This is made evident by a simple perturbation analysis. Indeed, Eq. $(33)_1$, has $\lambda = 1$ as the sole equilibrium solution for any choice of $a$, $b$, and $\alpha$. System (33) is equivalent to $\dot{\boldsymbol{u}} = \boldsymbol{F}(\boldsymbol{u})$ with $\boldsymbol{u}^T = (u_1, u_2, u_3) = (\lambda, \dot{\lambda}, \ddot{\lambda})$ and

$$\boldsymbol{F}^T = \left(u_2,\ u_3,\ -bu_3 - a\frac{u_2}{u_1} - b\frac{u_1+\alpha}{1+\alpha}\left(u_1 - \frac{1}{u_1^2}\right)\right).$$

Thus, by linearizing around the steady state $(1, 0, 0)$, we get

$$\dot{\boldsymbol{u}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3b & -a & -b \end{pmatrix}\boldsymbol{u}.$$

The eigenvalues are given by

$$\xi^3 + b\,\xi^2 + a\,\xi + 3\,b = 0. \tag{34}$$

Then we can apply the Routh–Hurwitz criterion [39] which guarantees that all roots of (34) have strictly negative real part if and only if $a > 3$. For completeness, we show in Fig. 5 the real part of the three eigenvalues as functions of $a$ for given $b$. Of course, changing $\alpha$ has no effect on this conclusion.

### 5.2. Third model

Using (18) with $\gamma = 0$, and setting

$$T^V = T^V_{11} - T^V_{22}, \qquad T^E(\lambda) = G\frac{\lambda+\alpha}{1+\alpha}\left(\lambda - \frac{1}{\lambda^2}\right),$$

get such a system

$$\begin{cases} \ddot{\lambda} = -\dfrac{A}{m\ell(0)}T_{11}, \\ T_{11} = T^E(\lambda) + T^V, \\ \dot{T}^V = -\phi T^V + 3\eta\phi\dfrac{\dot{\lambda}}{\lambda}, \end{cases}$$

which reduces to this third order equation

$$\dddot{\lambda} + \phi\ddot{\lambda} + \frac{A}{m\ell(0)}\left(3\eta\phi + \frac{dT^E}{d\lambda}\lambda\right)\frac{\dot{\lambda}}{\lambda} + \frac{A\phi}{m\ell(0)}T^E(\lambda) = 0. \tag{35}$$

Recalling (30), (31) and scaling, as usual, $T^E$ and $T^V$ with $G$, the dimensionless version of (35) is

$$\dddot{\lambda} + b\ddot{\lambda} + \left(a + \frac{dT^E}{d\lambda}\lambda\right)\frac{\dot{\lambda}}{\lambda} + bT^E(\lambda) = 0. \tag{36}$$

and the corresponding Cauchy problem to solve is

$$\begin{cases} \dddot{\lambda} + b\ddot{\lambda} + \left(a + \dfrac{dT^E}{d\lambda}\lambda\right)\dfrac{\dot{\lambda}}{\lambda} + bT^E(\lambda) = 0, \\ \lambda(0) = \lambda_o, \quad \dot{\lambda}(0) = 0, \quad \ddot{\lambda}(0) = -\dfrac{\lambda_o+\alpha}{1+\alpha}\left(\lambda_o - \dfrac{1}{\lambda_o^2}\right). \end{cases} \tag{37}$$

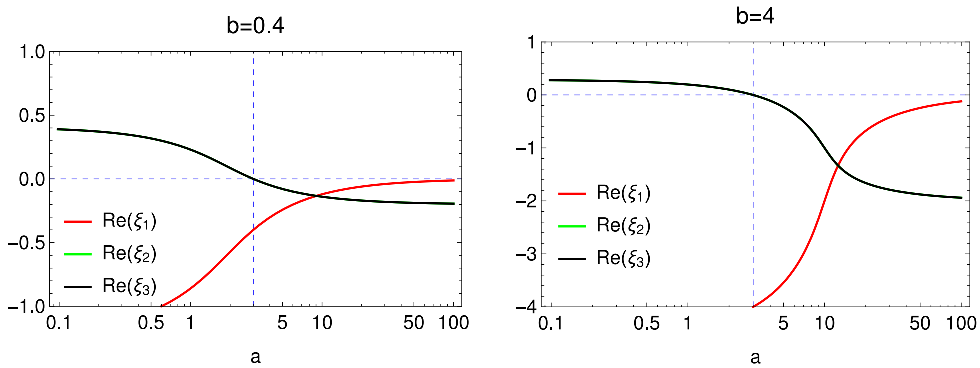


**Fig. 5.** Real part of solutions $\xi$ to (34) for $b = 0.4$ (left) and $b = 4$ (right). The critical value $a = 3$ is emphasized in both plots. The curve $\mathrm{Re}\left(\xi_1\right)$ and $\mathrm{Re}\left(\xi_2\right)$ coincide.

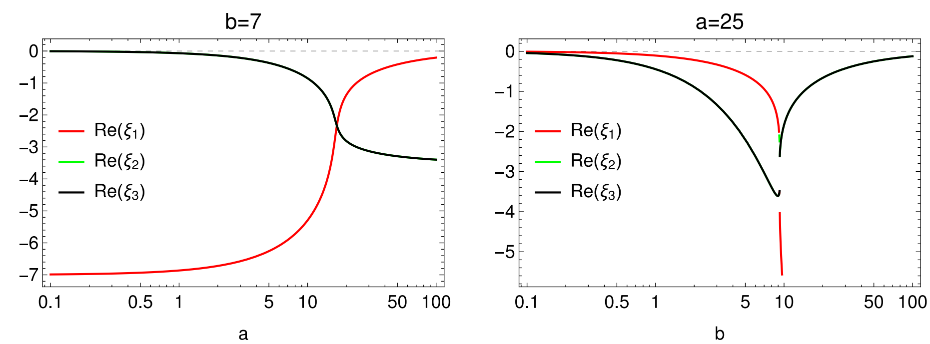


**Fig. 6.** Real part of solutions to (38) for $b = 7$ (left) and $a = 25$ (right). The curve $\mathrm{Re}\left(\xi_1\right)$ and $\mathrm{Re}\left(\xi_2\right)$ practically coincide.

We proceed as before and analyze the equilibrium state $\lambda = 1$. In this case we have

$$\boldsymbol{F}^T = \left(u_2,\ u_3,\ -bu_3 - u_2\left(\frac{dT^E}{du_1} + \frac{a}{u_1}\right) - bT^E\right),$$

and by linearizing around the steady state $(1, 0, 0)$, we obtain

$$\dot{\boldsymbol{u}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3b & -a-3 & -b \end{pmatrix} \boldsymbol{u}.$$

The eigenvalues are the solutions to

$$\xi^3 + b\xi^2 + \xi(3 + a) + 3b = 0. \tag{38}$$

Once more, we apply the Routh–Hurwitz criterion which guarantees that all roots of (34) have always negative real part. Fig. 6 shows the behavior of the real part of the eigenvalues by keeping, alternatively, one parameter fixed and varying the other one.

Concerning the stability of the configuration $\lambda = 1$ (i.e. the configuration in which the string is not deformed) we have a difference between the three constitutive models. The **third model** predicts that $\lambda = 1$ is asymptotically stable for each value of parameters $a$ and $b$. In addition, if we model the string with the **first** or the **second model**, we obtain that $\lambda = 1$ is asymptotically stable when $a > 3$.

In the next section we analyze the differences between the three constitutive models when we consider the dynamics, that is, the oscillations of the mass $m$.

### *5.3. Oscillations*

We now compare the solutions to (27) predicted by the three models for the same value of the material parameters. As shown by Fig. 7, $\lambda(t)$ displays only "little differences" between the third model and the second (or first) model. Furthermore, we note that the response of the three models to the variation of the parameter $a$ is similar: by increasing $a$, all models show an increase of the dumping effect. In general, what emerges from the simulations is that the three models give rise to similar dynamics. However, the evolution of $\lambda(t)$, while remaining similar when skipping from one constitutive model to another, can change significantly if the parameters and $\lambda_o$ are varied. This fact is highlighted by Figs. 8–10 where the string is modeled with the **third model** (we point out, however, that the first and second model give similar results) and the initial condition $\lambda_o$ is changed, considering both $\lambda_o > 1$ and $\lambda_o < 1$.

Fig. 8 shows $\lambda(t)$ for $a = 2$ by changing $b$. Figs. 9 and 10 show a comparison between the solutions corresponding to two different initial conditions $\lambda_o$: the blue line corresponds to $\lambda_o$ above the equilibrium while the red line to $\lambda_o$ below the equilibrium. The observed general feature is that both solutions are almost symmetric during the whole evolution, although by decreasing $a$ and $b$ there is a slight change in phase.

## 6. Concluding remarks

It is well known that there are many objective "time-like" tensor derivatives which are acceptable for formulating rates of change in constitutive equations. An important result of our study is that the choice of the objective derivative is a true constitutive matter. The retardation time in creep experiments depends strongly on such a choice. Therefore only an experimental test can elucidate the kind of derivative needed.

Moreover, we have confirmed the result by Prager [40] who emphasized that the Jaumann derivative is somehow the best choice from a mathematical perspective as it exhibits all the formal algebraic properties of the ordinary time derivative. Here we have pointed out that the models based on the Jaumann derivative are the simplest to handle from a mathematical point of view. When we consider a creep process and the deformation obeys (14), the models based on the Oldroyd or the lower-convected derivative, have be handled with care to avoid nonphysical situations (see Table 1).

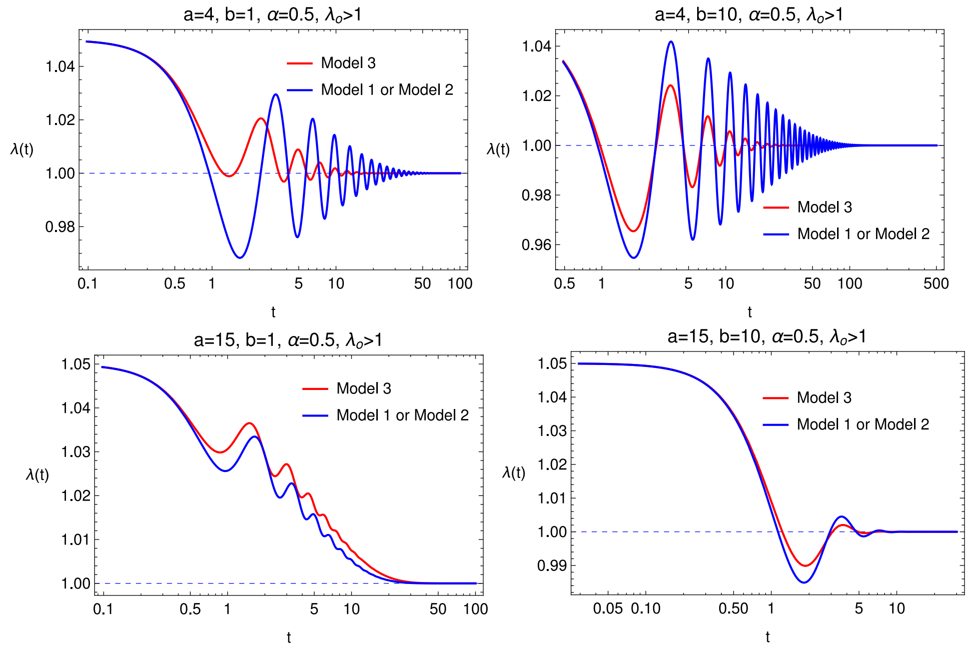


**Fig. 7.** Model comparison for the same material parameters, provided that $a > 3$.

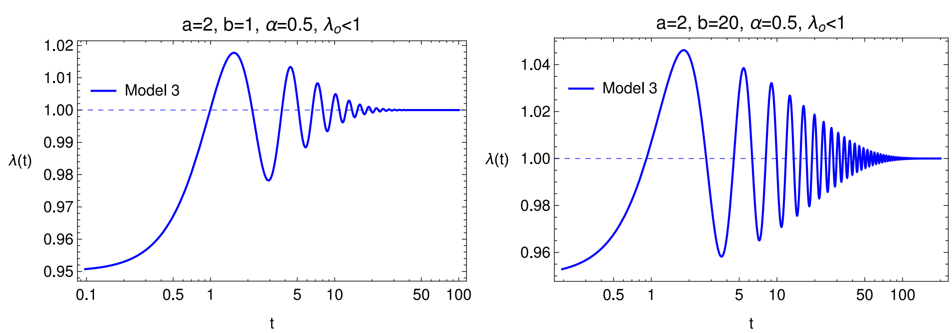


**Fig. 8.** Response of the **third model**, i.e. model (8), when $a = 2$, for $b = 2$ (left) and $b = 20$ (right).

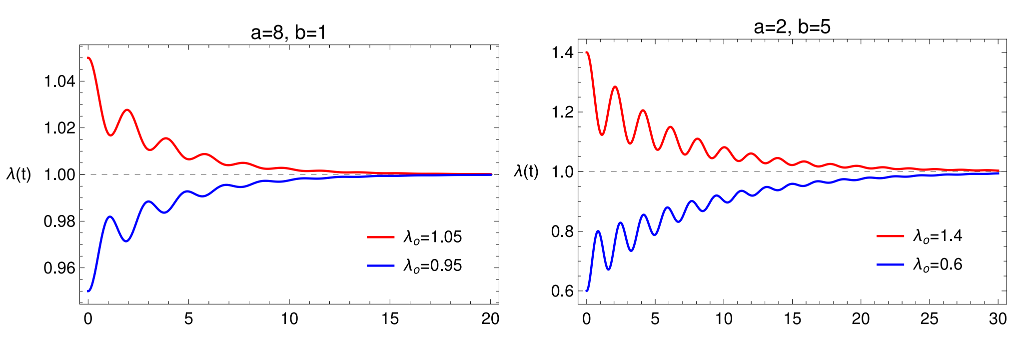


**Fig. 9.** Comparison between solutions to system (37) for $(a, b) = (8, 1)$ (left) and $(a, b) = (2, 5)$ (right), by changing the initial condition $\lambda_0$ close to 1, both larger and smaller than 1.

Another constitutive issue we encounter in the generalization of the standard linear model is posed by the evolution equation for the stress rate. Indeed, in some approaches this evolution equation is written for the total Cauchy stress, in other cases only for the Cauchy extra-stress and in other situations only for the viscous part of the Cauchy stress. For this reason the **first model** is a possible mechanical setting that we can imagine to generalize the classical Zener model.

In considering the finite amplitude free damped vibration of a mass, we have obtained another result. Only for the model considering the evolution equation in the viscous part of the Cauchy stress, i.e. **third model** , the oscillations are asymptotically stable for any value of the constitutive parameters (see Table 2). Therefore this model seems the more natural choice.

We point out that our results have been derived considering for the elastic part of the stress a Mooney–Rivlin constitutive strain-energy density function. This choice has been dictated only by the sake of algebraic simplicity. All our results can be easily generalized to more complex forms of strain–energy density functions.

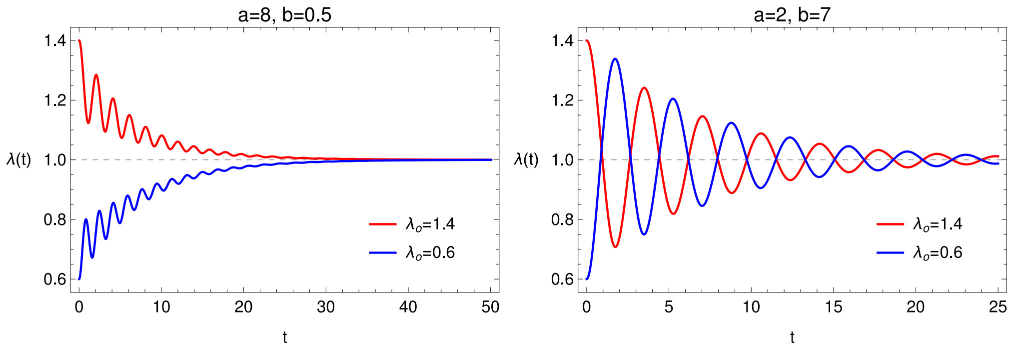


**Fig. 10.** Comparison between solutions to system (37) for $(a, b) = (8, 0.5)$ (left) and $(a, b) = (2, 7)$ (right), by changing the initial condition $\lambda_0$.

**Table 1**
Comparison among the three models and the three material derivatives for the uniaxial creep experiment. Regardless of the selected model, only the Jaumann derivative does not need any restrictions upon material parameters and the stress in order to avoid non-physical solutions.

| Model | Oldroyd | Jaumann | Cotter–Rivlin |
|---|---|---|---|
| First | restrictions needed | no restrictions | restrictions needed |
| Second | restrictions needed | no restrictions | restrictions needed |
| Third | restrictions needed | no restrictions | restrictions needed |

**Table 2**
Comparison among the three models and the three material time derivatives for the oscillation experiment. Regardless of the derivative used, only the third model provides asymptotically stable solutions for any value of the material parameters.

| Model | Oldroyd | Jaumann | Cotter–Rivlin |
|---|---|---|---|
| First | restrictions needed | restrictions needed | restrictions needed |
| Second | restrictions needed | restrictions needed | restrictions needed |
| Third | no restrictions | no restrictions | no restrictions |

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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