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ON THE REGULARITY OF A GRAPH RELATED TO CONJUGACY CLASSES OF GROUPS

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Dedicated to Professor Antonio Machì on the occasion of his 70th birthday

ABSTRACT. Given a finite group G, denote by $\Gamma(G)$ the simple undirected graph whose vertices are the (distinct) non-central conjugacy class sizes of G, and two vertices of $\Gamma(G)$ are adjacent if and only if they are not coprime numbers. In this note we prove that $\Gamma(G)$ is a 2-regular graph if and only if it is a complete graph with three vertices, and $\Gamma(G)$ is a 3-regular graph if and only if it is a complete graph with four vertices.
Keywords: finite groups, conjugacy class sizes.

1. Introduction

Let G be a finite group, and let $\mathrm{cs}(G)$ denote the set whose elements are the sizes of the conjugacy classes of G. The literature is rich in results that show the mutual influence between the group structure of G and the arithmetical structure of $\mathrm{cs}(G)$, and several variations on this subject have been considered. A recent survey paper by A.R. Camina and R.D. Camina ([2]) is a very useful reference about this research field.

In order to have a better understanding of the set $\mathrm{cs}(G)$ (as well as of any finite set of positive integers), it is useful to introduce some particular graphs. The so-called *common divisor graph* $\Gamma(G)$ on $\mathrm{cs}(G)$ is defined as the simple undirected graph whose vertex set is $\mathrm{cs}(G) \setminus \{1\}$, and two non-central class sizes are adjacent (i.e. joined by an edge) if and only if they are not coprime numbers. Recalling that, given a positive integer k, a graph is called k-regular if every vertex is adjacent to exactly k vertices, we can state the main result of this note.

Theorem. *Let G be a finite group. The graph $\Gamma(G)$ is 3-regular if and only if it is a complete graph with four vertices.*

The following question may be a natural one: is it true that, given a positive integer k, the graph $\Gamma(G)$ is k-regular if and only if it is a complete graph with $k+1$ vertices?

This certainly holds for $k = 1$ (taking into account Theorem 2.2(a)), and by Theorem 3.3 also for $k = 2$. The main theorem in this paper provides an affirmative answer for $k = 3$, and the answer is also affirmative for any positive integer k if G is a finite nonabelian simple group (in fact, as shown in [6], $\Gamma(G)$ is always a complete graph in this case). But, at the time of this writing, the above problem in its full generality is open.

To close with, every group considered in the following discussion is assumed to be a finite group.

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2. Preliminaries

In this section we recall some definitions and known results that will be relevant for our discussion, and we introduce some notation. Also, we prove a number of preliminary lemmas.

Let $\mathcal{G}$ be a graph. Given two vertices x and y of $\mathcal{G}$, a *path of length* n joining x and y is a set of vertices $\{v_0 = x, v_1, ..., v_n = y\}$ such that v_i and v_{i+1} are adjacent for every $i \in \{0, ..., n-1\}$. The *distance* $\mathrm{d}(x, y)$ is the minimum length of a path joining x and y (whereas $\mathrm{d}(x, y)$ is not defined if there does not exist any such path), and the *diameter* $\mathrm{diam}(\mathcal{G})$ of $\mathcal{G}$ is the maximum distance between two vertices of $\mathcal{G}$ lying in the same connected component.

As already mentioned, for a positive integer k, the graph $\mathcal{G}$ is called k*-regular* if every vertex of $\mathcal{G}$ is adjacent to exactly k vertices.

Let W be a subset of the vertex set of $\mathcal{G}$. The graph whose vertex set is W and such that two vertices are adjacent if and only if they are adjacent in $\mathcal{G}$ is called the *subgraph of* $\mathcal{G}$ *induced by* W.

We shall denote by C_n a connected 2-regular graph with n vertices (that is, a cycle with n vertices), and by K_n the complete graph with n vertices.

Finally, an *independent set* in $\mathcal{G}$ is a nonempty set of pairwise non-adjacent vertices.

We state next an obvious lemma.

Lemma 2.1. *Let $\mathcal{G}$ be a graph and n an integer greater than 5. If $\mathcal{G}$ contains an induced subgraph isomorphic to C_n, then it contains an independent set of three vertices.*

It is well known that, given a set of positive integers X, we can construct the *common divisor graph* $\Gamma(X)$, and the *prime graph* $\Delta(X)$. The former is the graph whose vertex set is $X \setminus \{1\}$, and two vertices are adjacent if and only if they are not coprime numbers, whereas the latter is the graph that has

$$\rho(X) := \{p \text{ prime } : \exists x \in X \text{ with } p \mid x\}$$

as vertex set, and there is an edge between $p, q \in \rho(X)$ if and only if pq is a divisor of x for some $x \in X$.

Let G be a group, and set $\mathrm{cs}(G) = \{|C| : C \text{ is a conjugacy class of } G\}$. In what follows, we shall write $\Gamma(G)$ and $\Delta(G)$ for $\Gamma(\mathrm{cs}(G))$ and $\Delta(\mathrm{cs}(G))$ respectively.

The next theorem collects some known results that we shall use in our discussion.

Theorem 2.2. *Let G be a group. Then the following conclusions hold.*

(a) *The graph $\Gamma(G)$ has at most two connected components and, if it is disconnected, both connected components are isolated vertices* ([1, 7]).
(b) *The diameter of $\Gamma(G)$ is at most 3* ([4]).
(c) *The size of an independent set of $\Delta(G)$ is at most 2* ([5]).

Also the next few lemmas will play a role in our proof of the main result.

Lemma 2.3. *Let X be a set of positive integers, and n an integer greater than 3. If $\Gamma(X)$ is 3-regular and it contains an induced subgraph isomorphic to C_n, then $\Delta(X)$ contains an induced subgraph isomorphic to C_n as well.*

Proof. Let $v_0, v_1, ..., v_{n-1}$ be the vertices of an induced subgraph of $\Gamma(X)$ isomorphic to C_n (in what follows, the indices are meant modulo n). We know that there exist $p_0, p_1, ..., p_{n-1} \in \rho(X)$ such that $p_i \mid v_i$ and $p_i \mid v_{i+1}$. Observe that, for $i \neq j$, we have $p_i \neq p_j$: otherwise, either v_i is adjacent to v_{j+1} or v_{i+1} is adjacent to v_j, and this is a contradiction, because the v_i are vertices of C_n and $n \geq 4$.

For the same reason, we have $p_i \nmid v_j$ for every $j \notin \{i, i+1\}$, otherwise v_j is adjacent both to v_i and to v_{i+1}.

We claim that the p_i are vertices of an induced subgraph of $\Delta(X)$ isomorphic to C_n. In fact, each p_i is adjacent to p_{i-1} and to p_{i+1}. Moreover, if there exists $j \notin \{i-1, i, i+1\}$ and $v \in X$ such that p_i and p_j divide v, it follows from the above paragraph that $v \notin \{v_0, ..., v_{n-1}\}$. But now v is adjacent to v_i, v_{i+1}, v_j and v_{j+1} (which are distinct vertices), against the assumption of 3-regularity for $\Gamma(X)$. □

Lemma 2.4. *Let X be a set of positive integers. Assume that $\Gamma(X)$ is 3-regular of diameter at most 3, and that $\Gamma(X)$ has no induced subgraph isomorphic to K_4. If $\Gamma(X)$ has an induced subgraph with four vertices a, b, c, d such that a, b, c are pairwise adjacent, and d is adjacent to a and to b, then $\Gamma(X)$ contains an induced subgraph isomorphic to C_n with $n \geq 6$.*

Proof. Let $e \notin \{a, b\}$ be a vertex adjacent to c (note that e can not be d). If e is adjacent to d, then take $x \notin \{c, d\}$ adjacent to e and $y \neq e$ adjacent to x: we get a contradiction, because $\mathrm{d}(a, y) = 4$ (see Figure 1).

Figure/Figure1.pdf

FIGURE 1

We deduce that e is not adjacent to d, therefore there exists $f \notin \{a, b, e\}$ adjacent to d. We claim that e and f are not adjacent. Assuming, for a proof by contradiction, that this is not the case, we distinguish two situations.

(a) *There exists a vertex g adjacent to both e and f.* Take $x \notin \{e, f\}$ adjacent to g: we get $\mathrm{d}(a, x) = 4$, a contradiction (see Figure 2).

Figure/Figure2.pdf

FIGURE 2

(b) *There does not exist any vertex adjacent to both e and f.* Let $g \notin \{c, f\}$ and $h \notin \{d, e\}$ be two distinct vertices adjacent to e and f respectively, and take $x \notin \{e, h\}$ adjacent to g. We get a contradiction in this case as well, because $\mathrm{d}(a, x) = 4$ (see Figure 3).

Figure/Figure3.pdf

FIGURE 3

Therefore, we proved that e and f are not adjacent. Now, there exists a path from e to f whose length is at most 3. Let $e, v_1, \dots, v_h, f$ be a path of minimum length (note that $h \in \{1, 2\}$). The vertices $e, v_1, \dots, v_h, f, d, b, c$ induce a subgraph of $\Gamma(X)$ isomorphic to C_n with $n \geq 6$, as desired. □

Lemma 2.5. *Let X be a set of positive integers, and $n \in \{4, 5\}$. If $\Gamma(X)$ is 3-regular, it has an induced subgraph isomorphic to C_n, and it has an edge with no vertex in common with this subgraph, then $\Delta(X)$ has an independent set of three vertices.*

Proof. Let $v_0, v_1, ..., v_{n-1}$ be the vertices of the induced subgraph of $\Gamma(X)$ isomorphic to C_n (again, the indices are meant modulo n). As observed in the proof of Lemma 2.3, there exist $p_0, p_1, ..., p_{n-1} \in \rho(X)$ such that $p_i \mid v_i$ and $p_i \mid v_{i+1}$, and these primes are vertices of $\Delta(X)$ inducing a subgraph isomorphic to C_n. Also, let y and z be the two adjacent vertices of $\Gamma(X)$ (through a prime q, say) such that $\{y, z\} \cap \{v_0, ..., v_{n-1}\} = \emptyset$.

We claim that, for every $i \in \{0, ..., n-1\}$ and for every $v \in X \setminus \{y, z\}$, we have $qp_i \nmid v$. In fact, if q divides v_i, then v_i is adjacent to v_{i-1}, v_{i+1}, y and z, contradicting the assumption of 3-regularity. Moreover, if $v \in X \setminus \{v_0, v_1, ..., v_{n-1}, y, z\}$ is such that qp_i divides v, then v is adjacent to v_i, v_{i+1}, y and z, again a contradiction.

Now, we know that p_0 and p_2 are not adjacent in $\Delta(X)$. If q is not adjacent to any of them, then we are done. So, assume that there exists $v \in X$ such that $qp_0 \mid v$. By the above paragraph we get $v \in \{y, z\}$, and we can assume $v = y$. Observe that now y is adjacent to v_0, v_1 and z. In this situation, we claim that $\{p_1, p_{n-1}, q\}$ is an independent set of $\Delta(X)$. Certainly p_1 and p_{n-1} are not adjacent. Also, neither p_1 nor p_{n-1} can divide y, otherwise y would be adjacent to v_2 or to v_{n-1} respectively. Similarly, neither p_1 nor p_{n-1} can divide z, otherwise either v_1 would be adjacent to v_0, v_2, y and z, or, respectively, v_0 would be adjacent to v_1, v_{n-1}, y and z. But, by the above paragraph, every element of $X \setminus \{y, z\}$ is not divisible by qp_i, for $i \in \{1, n-1\}$, whence $\{p_1, p_{n-1}, q\}$ is an independent set of $\Delta(X)$, as claimed.

An entirely similar argument shows that, if there exists $v \in X$ such that $qp_2 \mid v$, then $\{p_1, p_3, q\}$ is an independent set of $\Delta(X)$. The proof is now complete. □

3. COMPLETENESS OF 2-REGULAR GRAPHS

In [3], A.R. Camina and R.D. Camina proved the following theorem.

Theorem 3.1 ([3]). *Let G be a group such that, given any three distinct conjugacy class sizes of G, two of them are coprime. Then $\Gamma(G)$ has at most three vertices.*

Using this result, we shall show that $\Gamma(G)$ is a 2-regular graph if and only if it has three vertices and it is complete (i.e., it is a triangle). First, we prove the following proposition.

Proposition 3.2. *Let G be a group, and k an integer greater than 1. If $\Gamma(G)$ is k-regular, then $\Gamma(G)$ is connected and it contains a triangle.*

Proof. First of all we observe that $\Gamma(G)$ is connected, otherwise it would consist of two isolated vertices (see Theorem 2.2(a)). Since $\Gamma(G)$ is k-regular with $k \geq 2$, it has at least three vertices. If it has more than three vertices, then Theorem 3.1 yields that $\Gamma(G)$ contains a triangle. If it has exactly three vertices, then it is 2-regular and it is a triangle. □

As a corollary, we obtain:

Theorem 3.3. *Let G be a group. The graph $\Gamma(G)$ is 2-regular if and only if it is a triangle.*

Proof. Clearly, we need only to prove the "only if" direction. By Proposition 3.2, $\Gamma(G)$ is connected and contains a triangle. If $\Gamma(G)$ contains more than three vertices, then a vertex outside the triangle is adjacent to one of the vertices of the triangle, in contradiction to $\Gamma(G)$ being 2-regular. Hence $\Gamma(G)$ is a triangle. □

4. Completeness of 3-regular graphs

This section is devoted to the proof of the main result in this note, which we state again.

Theorem 4.1. *Let G be a group. The graph $\Gamma(G)$ is 3-regular if and only if it is a complete graph with four vertices.*

Proof. Clearly, what has to be proved is the "only if" part of the statement. First of all we observe that $\Gamma(G)$ is connected, otherwise it would consist of two isolated vertices (see Theorem 2.2(a)). Assume that $\Gamma(G)$ is not complete. Thus, since it is 3-regular, it does not contain an induced subgraph isomorphic to K_4. We are in the hypothesis of Proposition 3.2, so we find a triangle in $\Gamma(G)$. Let us call a, b and c the vertices of this triangle. If there exists a vertex d of $\Gamma(G)$ which is adjacent to two vertices among a, b and c, then $\Gamma(G)$ satisfies the hypothesis of Lemma 2.4 (we have $\mathrm{diam}(\Gamma(G)) \leq 3$ in view of Theorem 2.2(b)), and we find in $\Gamma(G)$ an induced subgraph isomorphic to C_n, where $n \geq 6$. We can now use Lemma 2.3 to find an induced subgraph isomorphic to C_n in $\Delta(G)$. Thus $\Delta(G)$ contains an independent set of three vertices (see Lemma 2.1), and this is in contrast with Theorem 2.2(c).

In view of the previous paragraph, we can find three vertices d, e, f of $\Gamma(G)$ which are adjacent to a, b and c respectively (and such that $\{a, b, c, \} \cap \{d, e, f\} = \emptyset$). We claim that d and e are not adjacent. Otherwise $\Gamma(G)$ contains an induced subgraph isomorphic to C_4 which, together with the vertices c and f, verifies the hypothesis of Lemma 2.5; thus we reach a contradiction, because we find in $\Delta(G)$ an independent set of three vertices (see Figure 4).
For the same reason, e is not adjacent to f. As $\Gamma(G)$ is 3-regular, there exist two distinct vertices, g and h, which are different from b and adjacent to e. We claim that d is not adjacent to g. Otherwise a, b, e, g and d are the vertices of an induced subgraph isomorphic to C_5 and, together with c and f, they satisfy the hypothesis of Lemma 2.5 (see Figure 5).

Thus $\Delta(G)$ contains an independent set of three vertices, a contradiction. Similarly, d is not adjacent to h. Since $\mathrm{diam}(\Gamma(G)) \leq 3$, there exists a path joining d to g whose length is at most 3. Let $d, v_1, \ldots, v_h, g$ be such a path. We observe that

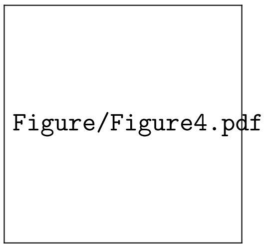

Figure 4

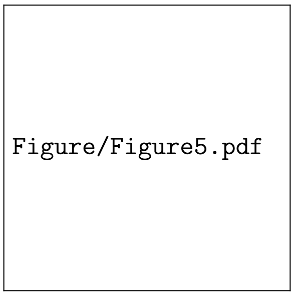

Figure 5

e is not one of the v_i, because otherwise h is one of the v_i too, and the length of the path would be greater than 3 since d and h are not adjacent. It follows that $a, d, v_1, \ldots, v_h, g, e, b$ are the vertices of an induced subgraph isomorphic to C_n in $\Gamma(G)$, where $n \geq 6$. We can use Lemma 2.3 which, together with Lemma 2.1, yields that $\Delta(G)$ contains an independent set of three vertices, the final contradiction. $\square$

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