On Huppert's rho-sigma conjecture

ON HUPPERT'S RHO-SIGMA CONJECTURE

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Abstract. For an irreducible complex character χ of the finite group G, let $\pi(\chi)$ denote the set of prime divisors of the degree $\chi(1)$ of χ. Denote then by $\rho(G)$ the union of all the sets $\pi(\chi)$ and by $\sigma(G)$ the largest value of $|\pi(\chi)|$, as χ runs in $\mathrm{Irr}(G)$. The *ρ-σ conjecture*, formulated by Bertram Huppert in the 80's, predicts that $|\rho(G)| \leq 3\sigma(G)$ always holds, whereas $|\rho(G)| \leq 2\sigma(G)$ holds if G is solvable; moreover, O. Manz and T.R. Wolf proposed a "strengthened" form of the conjecture in the general case, asking whether $|\rho(G)| \leq 2\sigma(G) + 1$ is true for every finite group G. In this paper we study the strengthened ρ-σ conjecture *for the class of finite groups having a trivial Fitting subgroup*: in this context, we prove that the conjecture is true provided $\sigma(G) \leq 5$, but it is false in general if $\sigma(G) \geq 6$. Instead, we establish that $|\rho(G)| \leq 3\sigma(G) - 4$ holds for every finite group with a trivial Fitting subgroup and with $\sigma(G) \geq 6$ (this being the right, best possible bound). Also, we improve the up-to-date best bound for the solvable case, showing that we have $|\rho(G)| \leq 3\sigma(G)$ whenever G belongs to one particular class including all the finite solvable groups, and we improve the up-to-date best bound obtained in [18] for the the general case.

1. Introduction

The set $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ consisting of the degrees of the irreducible complex characters of a finite group G has been an object of considerable interest since the second part of the 20^{th} century, and the study of the arithmetical structure of this set is a particularly intriguing aspect of Character Theory of finite groups (see for instance [17]). A remarkable question in this research area was posed by Bertram Huppert in the 80's: is it true that at least one of the character degrees is divisible by a "large" portion of the entire set of primes that appear as divisors of some character degree? More precisely, denoting by $\pi(n)$ the set of prime divisors of an integer n, and writing for short $\pi(\chi)$ instead of $\pi(\chi(1))$ when $\chi \in \mathrm{Irr}(G)$, one defines

$$\rho(G) = \bigcup_{\chi \in \mathrm{Irr}(G)} \pi(\chi)$$

and

$$\sigma(G) = \max\{|\pi(\chi)| \mid \chi \in \mathrm{Irr}(G)\} ;$$

Huppert's *ρ-σ conjecture* predicts that $|\rho(G)| \leq 3\sigma(G)$ holds for every finite group G, and that $|\rho(G)| \leq 2\sigma(G)$ if G is solvable. It is worth noting that the bounds are in some sense best possible, as they are attained for the groups A_5 and S_4, respectively.

During the last four decades, several contributions have been given toward the proof of this conjecture. For solvable groups, the first results were obtained by I.M. Isaacs in [16], later improved by D. Gluck and O. Manz in [11]; moreover, the conjecture was proved true by Gluck ([10]) for $\sigma(G) \leq 2$, and also in the case that all degrees in $\mathrm{cd}(G)$ are square-free numbers. The best bound known till now was obtained by O. Manz and T.R. Wolf; they proved in [20] that, if G is solvable, then $|\rho(G)| \leq 3\sigma(G) + 2$.

2000 *Mathematics Subject Classification.* 20C15.

The first author was partially supported by a grant from IPM (No. 99200028). The second and the third authors were partially supported by INDAM-GNSAGA.

As for the non-solvable case, the ρ-σ conjecture was proved true for all finite non-abelian simple groups by D.L. Alvis and M. Barry ([1]), whereas the first general result is due to A. Moretó in [22]: a quadratic bound of $|\rho(G)|$ in terms of $\sigma(G)$. The linear bound $|\rho(G)| \leq 7\sigma(G)$ was obtained by C. Casolo and the second author in [4]; this was then improved to $|\rho(G)| \leq 6\sigma(G) + 1$ by Y. Liu and Z. Lu (see [18, Theorem 1]).

One might wonder whether the factor 3 is the right one for non-solvable groups, or one should instead keep the factor 2 and add a suitable constant for getting a tighter bound. In a recent paper ([24]), H. Tong-Viet studies the so-called *strengthened* ρ-σ conjecture proposed by Manz and Wolf in [20], that is $|\rho(G)| \leq 2\sigma(G) + 1$ for every finite group G. The strengthened ρ-σ conjecture is verified in [24] for all finite almost-simple groups and also for the groups G such that $\sigma(G) \leq 2$.

In this paper we start by considering finite groups with trivial Fitting subgroup and, for these groups, we establish the strengthened ρ-σ conjecture whenever $\sigma(G) \leq 5$. But we remark that the strengthened ρ-σ conjecture is false in general: in fact Example 1.1 of [2], that is recalled below as Example 1.1, provides a sequence of finite groups G (with trivial Fitting subgroup) such that the ratio $|\rho(G)|/\sigma(G)$ tends to 3. The first main result of this paper also establishes the right bound in this setting for the case $\sigma(G) \geq 6$.

Theorem A. Let G be a finite group with trivial Fitting subgroup. Then the following conclusions hold.

(a) If $\sigma(G) \leq 5$, then $|\rho(G)| \leq 2\sigma(G) + 1$.
(b) If $\sigma(G) \geq 6$, then $|\rho(G)| \leq 3\sigma(G) - 4$.

The groups in Example 1.1 show that the bounds in the above theorem are sharp. We note that Theorem A is an improvement of Corollary 3.1 in [18], where the authors show that $|\rho(G)| \leq 3\sigma(G)$ holds for every finite group G having a trivial Fitting subgroup.

On the other hand, the problem of determining whether the restriction about the Fitting subgroup is really needed in Theorem A is left for future investigation (we are not aware of any counterexample): the above statement without that restriction may be thought as a new form of Huppert's ρ-σ conjecture.

We also note that Theorem A has a clear (formal) analogy with Theorem A of [2], where the role of $\sigma(G)$ is played by another invariant of the group, denoted by $\omega(G)$. This is the largest size of a complete subgraph of the character degree graph $\Delta(G)$ of G (recall that $\Delta(G)$ is the simple undirected graph whose vertex set is $\rho(G)$, two vertices p and q being adjacent if and only if there exists a degree in $\mathrm{cd}(G)$ that is divisible by pq). Obviously we have $\sigma(G) \leq \omega(G)$ for every finite group G, and therefore Theorem A immediately implies Theorem A of [2] for groups with trivial Fitting subgroup. At any rate, the tools used when dealing with $\sigma(G)$ are rather different, and the analysis more complicated than that concerning $\omega(G)$.

In the second part of the paper we focus on solvable groups and, using a variation of the arguments in [20], we obtain the following improvement of Manz and Wolf's theorem. (We were informed that, after this paper was submitted, another proof of Theorem B appeared in [19]; we are grateful to the referee for pointing this out.)

Theorem B. Let G be a finite group. If G is solvable, then $|\rho(G)| \leq 3\sigma(G)$.

Actually we can extend the bound given in Theorem B to a wider class of groups. We recall that the generalized Fitting subgroup $\mathbf{F}^*(G)$ of the finite group G is the central product of the Fitting subgroup $\mathbf{F}(G)$ and the *layer* subgroup $\mathbf{E}(G)$, which is the group generated by all the *components* of G (see Section 3). It is well known that $\mathbf{C}_G(\mathbf{F}^*(G)) \leq \mathbf{F}^*(G)$. Hence, $\mathbf{C}_G(\mathbf{F}^*(G)) = \mathbf{Z}(\mathbf{F}(G))$ and, when

$\mathbf{F}(G) = 1$, $\mathbf{C}_G(\mathbf{F}^*(G)) = \mathbf{C}_G(\mathbf{E}(G)) = 1$. We consider, as a generalization of this setting, the case when the centralizer $\mathbf{C}_G(\mathbf{E}(G))$ of the layer subgroup of G is *solvable*. The corresponding family of groups hence contains both the family of the groups with trivial Fitting subgroup and the family of the solvable groups.

Theorem C. Let G be a finite group such that $\mathbf{C}_G(\mathbf{E}(G))$ is solvable. Then

$$|\rho(G)| \leq 3\sigma(G) .$$

Finally, as a consequence of Theorem A and Theorem B, we also get the following improvement of Theorem 1 in [18]. Corollary D is then the best general result concerning Huppert's ρ-σ-conjecture up to date.

Corollary D. Let G be a finite group. Then $|\rho(G)| \leq 5\sigma(G) + 1$ if $\sigma(G) < 6$, and $|\rho(G)| \leq 6\sigma(G) - 4$ otherwise.

For the convenience of the reader, and also because the point of view of the present context is not the same as that of [2], we close this introductory section by recalling Example 1.1 of [2] and some related comments.

Example 1.1. Let $\Pi = \{p_1^{f_1}, ..., p_n^{f_n}\}$ be a set of prime powers where every prime p_i is larger than 5. Assume that, for every $i \in \{1, ..., n\}$, we have $|\pi(p_i^{f_i} - 1) \setminus \{2, 3\}| = |\pi(p_i^{f_i} + 1) \setminus \{2, 3\}| = 1$, and assume further that, for distinct r and s in $\{1, ..., n\}$, the intersection of the sets $\{p_r\} \cup \pi(p_r^{2f_r} - 1)$ and $\{p_s\} \cup \pi(p_s^{2f_s} - 1)$ is $\{2, 3\}$. Now, setting $G_\Pi = \mathrm{PSL}_2(p_1^{f_1}) \times \cdots \times \mathrm{PSL}_2(p_n^{f_n})$ (note that $\mathbf{F}(G_\Pi) = 1$) and taking into account that, for $p > 5$ and $f \geq 1$, we have

$$\mathrm{cd}(\mathrm{PSL}_2(p^f)) = \{1, p^f - 1, p^f, p^f + 1, \frac{1}{2}(p^f + \epsilon)\} \text{ where } \epsilon = (-1)^{\frac{p^f - 1}{2}}$$

(see for instance [25, Theorem 3.2]), it is easy to see that $|\rho(G_\Pi)| = 3n + 2$ and $\sigma(G_\Pi) = n + 2$, thus $|\rho(G_\Pi)| = 3\sigma(G_\Pi) - 4$. As a consequence, if $n \geq 4$, the strengthened ρ-σ conjecture as formulated by Manz and Wolf does not hold for the group G_Π.

Note that $\Pi = \{29, 67, 157, 227\}$ is a set of four prime powers (in fact, of four primes) satisfying the above conditions. This provides a counterexample to the strengthened ρ-σ conjecture in which the size of $\rho(G_\Pi)$ is 14, whereas $\sigma(G_\Pi)$ is 6.

Let Π_n be one particular set of given size n as in Example 1.1. Assuming that such a set exists for arbitrarily large $n \in \mathbb{N}$, we see that the ratio $|\rho(G_{\Pi_n})|/\sigma(G_{\Pi_n})$ converges to 3 as n tends to infinity (actually, machine computation with prime numbers up to 10^6 enables us to construct a group G_Π as in Example 1.1 for which $|\rho(G_\Pi)|/\sigma(G_\Pi) > 2.999$); this leads us to conjecture that, for every positive real number ϵ, there exists a group G (with trivial Fitting subgroup and) with $|\rho(G)|/\sigma(G) > 3 - \epsilon$. In fact, in [9] the authors estimate the asymptotic density of one particular set of primes, whose infinitude yields the existence of a set Π_n as above for every $n \in \mathbb{N}$. This set is actually infinite, if a generalized form of the Hardy-Littlewood conjecture is assumed ([9, Theorem 2.3]).

Every group considered throughout the following discussion is tacitly assumed to be a finite group.

2. Preliminaries

If a group G acts on a set Ω, and $\Delta \subseteq \Omega$, we denote by $G_\Delta = \{g \in G \mid \Delta g = \Delta\}$ the stabilizer of Δ in G. Also, if n, m are non-negative integers, we denote by $\pi_{\geq m}(n)$ the set of all prime divisors of n which are greater than or equal to m, whereas $\mathbb{P}_m$ will denote the set of all primes in $\mathbb{N}$ that are smaller than or equal to m. We write $\pi(G)$ and $\pi_{\geq m}(G)$ for $\pi(|G|)$ and $\pi_{\geq m}(|G|)$, respectively. Similarly, if H

is a subgroup of the group G, we use the notation $\pi(G:H)$ and $\pi_{\geq m}(G:H)$. Finally, for $\chi \in \mathrm{Irr}(G)$, as already mentioned we will write $\pi(\chi)$ in place of $\pi(\chi(1))$.

For a group G, we define $\mathrm{m}(G)$ to be the largest integer $m \geq 5$ such that G has a section isomorphic to the alternating group A_m (that is, such that there are subgroups $K \trianglelefteq H \leq G$ with $H/K \cong A_m$), and we set $\mathrm{m}(G) = 0$ if there is no such section in G; note that $\mathbb{P}_{\mathrm{m}(G)}$ is contained in $\pi(G)$, since it is in fact contained in $\pi(A_m)$. We prove next a straightforward property of $\mathrm{m}(G)$.

Lemma 2.1. *Let G be a group, and N a normal subgroup of G. Then, setting $m = \mathrm{m}(G)$, $m_1 = \mathrm{m}(N)$ and $m_2 = \mathrm{m}(G/N)$, we have $m = \max\{m_1, m_2\}$.*

Proof. If $m = 0$, then clearly m_1 and m_2 are both 0 as well, and it is also clear that $m \geq \max\{m_1, m_2\}$ holds for positive values of m as well. Assume now $m \geq 5$, consider a section H/K of G such that $H/K \cong A_m$, and set $\overline{G} = G/N$. Then the section $\overline{H}/\overline{K}$ of $\overline{G}$ is isomorphic either to A_m or to the trivial group, and we have $m = m_2$ in the former case, whereas $m = m_1$ in the latter case. ■

The following proposition concerning permutation groups turns out to be very useful in our arguments.

Proposition 2.2. *Let G be a permutation group on the finite set Ω and $m = \mathrm{m}(G)$. Then there exist $\Gamma,\ \Delta \subseteq \Omega$ such that $\Gamma \cap \Delta = \emptyset$ and*

(a) $\pi_{\geq m}(G) = \pi_{\geq m}(G : G_\Gamma \cap G_\Delta)$,
(b) $|\mathbb{P}_m| \leq 2|\pi(G : G_\Gamma \cap G_\Delta) \cap \mathbb{P}_m|$.

Proof. Observe that, under the additional assumption that the action of G on Ω is transitive, our statement is precisely Proposition 1 of [4].

So, let $\Omega_1, \ldots, \Omega_t$ be the orbits of the action of G on Ω and, for $i \in \{1, \ldots t\}$, denote by K_i the kernel of the action of G on Ω_i; also, set $m_i = \mathrm{m}(G/K_i)$. An application of [4, Proposition 1] to the (transitive) action of G/K_i on the set Ω_i yields that there exist disjoint subsets $\Gamma_i,\ \Delta_i$ of Ω_i such that

(i) $\pi_{\geq m_i}(G/K_i) = \pi_{\geq m_i}(G : G_{\Gamma_i} \cap G_{\Delta_i})$,
(ii) $|\mathbb{P}_{m_i}| \leq 2|\pi(G : G_{\Gamma_i} \cap G_{\Delta_i}) \cap \mathbb{P}_{m_i}|$,

for $i \in \{1, \ldots, t\}$.

Next, define

$$\Gamma = \bigcup_{i=1}^{t} \Gamma_i, \quad \text{and} \quad \Delta = \bigcup_{i=1}^{t} \Delta_i,$$

so that Γ and Δ are clearly disjoint subsets of Ω; it is also immediate to see that $G_\Gamma = \bigcap_{i=1}^{t} G_{\Gamma_i}$ and $G_\Delta = \bigcap_{i=1}^{t} G_{\Delta_i}$. Now, taking into account that $\bigcap_{i=1}^{t} K_i = 1$, we get $m = \max\{m_1, \ldots, m_t\}$ and

$$\pi_{\geq m}(G) \subseteq \bigcup_{i=1}^{t} \pi_{\geq m}(G/K_i) = \bigcup_{i=1}^{t} \pi_{\geq m}(G : G_{\Gamma_i} \cap G_{\Delta_i}) \subseteq \pi_{\geq m}(G : \bigcap_{i=1}^{t} (G_{\Gamma_i} \cap G_{\Delta_i})) =$$

$$= \pi_{\geq m}(G : (\bigcap_{i=1}^{t} G_{\Gamma_i}) \cap (\bigcap_{i=1}^{t} G_{\Delta_i})) = \pi_{\geq m}(G : G_\Gamma \cap G_\Delta),$$

and (a) follows. As for (b), observe that (we may assume $m \neq 0$ and) there exists $i \in \{1, \ldots, m\}$ such that G/K_i has a section isomorphic to A_m; clearly m_i is then equal to m, and our claim easily follows by (ii) observing that $|G : G_{\Gamma_i} \cap G_{\Delta_i}|$ is a divisor of $|G : G_\Gamma \cap G_\Delta|$. ■

Note that, in the previous proposition, we can choose the two subsets Γ and Δ of Ω to be both non-empty unless G is the trivial group. In fact, assume that Δ is empty; then, G being non-trivial, Γ is neither empty nor the whole Ω, and we can replace Δ with $\Omega \setminus \Gamma$.

Next, we sketch the proof of a result concerning almost-simple groups that can be essentially deduced from the proofs of Lemma 2.2 and Theorem 2.3 in [24] (we refer the reader to that paper for the full details).

Denote by $\mathcal{L}(p)$ the class of simple groups of Lie type in characteristic p; also, if q is a prime power, denote by $\ell_n(q)$ a primitive prime divisor of $q^n - 1$ (that is a prime divisor of $q^n - 1$, which does not divide $q^k - 1$ for $1 \leq k < n$), if it exists. By Zsigmondy's theorem ([21, Theorem 6.2]), a primitive prime divisor of $q^n - 1$ always exists unless $n = 2$, or $q = 2$ and $n = 6$.

Proposition 2.3. *Let G be an almost-simple group with socle S. Then there exist two irreducible characters $\chi_1, \chi_2 \in \mathrm{Irr}(S)$ such that the following conclusions hold.*

(a) *If S is either a sporadic simple group, $S \ncong J_1$, or $S \cong A_m$ for $m > 5$, or S is the Tits group, then $\pi(S) = \pi(G) = \pi(\chi_1) \cup \pi(\chi_2)$; if $S \cong J_1$, then $\pi(G) - (\pi(\chi_1) \cup \pi(\chi_2)) = \{11\}$.*

(b) *If $S \in \mathcal{L}(p)$, then $\pi(S) - (\pi(\chi_1) \cup \pi(\chi_2)) \subseteq \{p\}$. Moreover, $\pi(G) - \pi(S)$ is contained in both $\pi(|G : I_G(\chi_1)|)$ and $\pi(|G : I_G(\chi_2)|)$.*

Proof. If S is either a sporadic simple group, or an alternating group, or the Tits group, then we have $\pi(S) = \pi(G)$ and claim (a) follows from [1, Theorem B(i), Theorem C] and by [6] (actually, by [3] there exists $\chi \in \mathrm{Irr}(A_m)$ such that $\pi(A_m) = \pi(\chi)$ for $m \geq 15$). We remark that for A_5 claim (a) does not hold, but this group can be treated as a simple group of Lie type (both in characteristic 2 and 5) and, as such, it satisfies claim (b).

Let now $S \in \mathcal{L}(p)$ (excluding the Tits group). The case $S \cong \mathrm{PSL}_2(2^f)$ (for $f \geq 2$) is treated as Case 1 in [24, proof of Theorem 2.3], so we will henceforth assume that S is not of this kind.

Next, assume that at least one among the primitive prime divisors ℓ_1, ℓ_2 indicated in Table 1 (which summarizes [24, Table 1] and [23, Lemma 2.3]) does not exist; then, it can be seen that S is isomorphic to a group of the kind $\mathrm{PSL}_2(q)$, $\mathrm{PSL}_3(q)$, $\mathrm{PSU}_3(q)$ or $\mathrm{PSp}_4(q)$ for some specific values of $q = p^f$, or S belongs to a finite set of groups (see [24, List $\mathcal{C}$, page 3]). Also, $\pi(G)$ turns out to coincide with $\pi(S)$ in this situation. The four infinite families above are discussed in [24, Cases 1–3, proof of Lemma 2.2], whereas the remaining finite set of groups can be treated via [6].

As regards the cases that are left (for which both ℓ_1 and ℓ_2 exist) we introduce the following setup. Let $\mathscr{G}$ be a simply connected simple algebraic group defined over a field of order q in characteristic p, and let F be a suitable Frobenius map such that $S \cong \mathscr{G}^F/\mathbf{Z}(\mathscr{G}^F)$. Suppose the pair $(\mathscr{G}^*, F^*)$ is dual to $(\mathscr{G}, F)$. Setting $L = \mathscr{G}^F$ and $L^* = (\mathscr{G}^*)^{F^*}$, the irreducible characters of L are partitioned into rational series $\mathscr{E}(L, (s))$ which are indexed by (L^*)-conjugacy classes (s) of semisimple elements $s \in L^*$, by Lusztig theory. Furthermore, if $\gcd(|s|, |\mathbf{Z}(L)|) = 1$, then every $\chi \in \mathscr{E}(L, (s))$ is trivial at $\mathbf{Z}(L)$, thus $\chi \in \mathrm{Irr}(S)$. Observe that $\chi(1)$ is divisible by $|L^* : \mathbf{C}_{L^*}(s)|_{p'}$. Now, let $s_i \in L^*$ be a semisimple element of order ℓ_i for $i \in \{1, 2\}$: then $\mathbf{C}_{L^*}(s_i) = T_i$ (see Table 1) is a maximal torus of L^*. Note that we have $\gcd(\ell_i, |\mathbf{Z}(L)|) = 1$ and also $\pi(\gcd(|T_1|, |T_2|)) \subseteq \pi(|L^* : T_1|_{p'}) \cup \pi(|L^* : T_2|_{p'})$. Let $\chi_i \in \mathscr{E}(L, (s_i))$ for $i \in \{1, 2\}$, so that $\chi_i \in \mathrm{Irr}(S)$ and $\chi_i(1)$ is divisible by $|L^* : T_i|_{p'}$ for $i \in \{1, 2\}$. Then $\pi(S) = \pi(\chi_1) \cup \pi(\chi_2) \cup \{p\}$. If $\pi(G) = \pi(S)$, then we get the desired conclusion. For the case when $\pi = \pi(G) - \pi(S)$ is nonempty, we refer to the discussion in [24, Subcase 3b, proof of Theorem 2.3]). ■

As the last preliminary result we recall the statement of Theorem A in [24], that proves the strengthened ρ-σ conjecture for almost-simple groups.

Theorem 2.4. *Let G be an almost-simple group. Then $|\pi(G)| \leq 2\sigma(G)$ unless $G \cong \mathrm{PSL}_2(2^f)$ with $f \geq 2$ and $|\pi(2^f - 1)| = |\pi(2^f + 1)|$. For the exceptions, we have $|\pi(G)| \leq 2\sigma(G) + 1$.*

Table 1. Two tori for groups of Lie type in characteristic p

$G = G(q)$, $q = p^f$	$\|T_1\|$	$\|T_2\|$	ℓ_1	ℓ_2
$A_n(q)$	$(q^{n+1}-1)/(q-1)$	q^n-1	$\ell_{n+1}(q)$	$\ell_n(q)$
${}^2A_n(q), n \equiv 0(4)$	$(q^{n+1}+1)/(q+1)$	q^n-1	$\ell_{2n+2}(q)$	$\ell_n(q)$
${}^2A_n(q), n \equiv 1(4)$	$(q^{n+1}-1)/(q+1)$	q^n+1	$\ell_{(n+1)/2}(q)$	$\ell_{2n}(q)$
${}^2A_n(q), n \equiv 2(4)$	$(q^{n+1}+1)/(q+1)$	q^n-1	$\ell_{2n+2}(q)$	$\ell_{n/2}(q)$
${}^2A_n(q), n \equiv 3(4)$	$(q^{n+1}-1)/(q+1)$	q^n+1	$\ell_{n+1}(q)$	$\ell_{2n}(q)$
$B_n(q), C_n(q), n \geq 3$ *odd*	q^n+1	q^n-1	$\ell_{2n}(q)$	$\ell_n(q)$
$B_n(q), C_n(q), n \geq 2$ *even*	q^n+1	$(q^{n-1}+1)(q+1)$	$\ell_{2n}(q)$	$\ell_{2n-2}(q)$
$D_n(q), n \geq 5$ *odd*	$(q^{n-1}+1)(q+1)$	q^n-1	$\ell_{2n-2}(q)$	$\ell_n(q)$
$D_n(q), n \geq 4$ *even*	$(q^{n-1}+1)(q+1)$	$(q^{n-1}-1)(q-1)$	$\ell_{2n-2}(q)$	$\ell_{n-1}(q)$
${}^2D_n(q)$	q^n+1	$(q^{n-1}+1)(q-1)$	$\ell_{2n}(q)$	$\ell_{2n-2}(q)$
${}^2B_2(q)$	$q \pm \sqrt{2q}+1$	$q-1$	$\ell_4(q)$	$\ell_1(q)$
${}^2G_2(q)$	$q \pm \sqrt{3q}+1$	$q-1$	$\ell_6(q)$	$\ell_1(q)$
${}^2F_4(q)$	$q^2+q+1 \pm (\sqrt{3q}+\sqrt{2q})$	q^2-q+1	$\ell_{12}(q)$	$\ell_6(q)$
${}^3D_4(q)$	q^4-q^2+1	$(q^3+1)(q+1)$	$\ell_{12}(q)$	$\ell_6(q)$
$G_2(q)$	$\Phi_6(q)$	$\Phi_3(q)$	$\ell_6(q)$	$\ell_3(q)$
$F_4(q)$	$\Phi_{12}(q)$	$\Phi_8(q)$	$\ell_{12}(q)$	$\ell_8(q)$
$E_8(q)$	$\Phi_{30}(q)$	$\Phi_{24}(q)$	$\ell_{30}(q)$	$\ell_{24}(q)$
$E_6(q)$	$\Phi_9(q)$	$\Phi_{12}(q)\Phi_3(q)$	$\ell_9(q)$	$\ell_{12}(q)$
${}^2E_6(q)$	$\Phi_{18}(q)$	$\Phi_{12}(q)\Phi_6(q)$	$\ell_{18}(q)$	$\ell_{12}(q)$
$E_7(q)$	$\Phi_{18}(q)\Phi_2(q)$	$\Phi_{14}(q)\Phi_2(q)$	$\ell_{18}(q)$	$\ell_{14}(q)$

3. The ρ-σ conjecture for groups with trivial Fitting subgroup

In this section we present a proof of Theorem A, that was stated in the Introduction. The result is obtained by combining Theorem 3.3 and Theorem 3.4, which provide the bounds in the case $\sigma(G) \leq 5$ and $\sigma(G) \geq 6$ respectively.

Recall that a *component* of a group G is a non-trivial subnormal subgroup of G which is *quasi-simple* (i.e., a perfect group whose factor group over its centre is simple). Denoting by $\mathbf{E}(G)$ the subgroup generated by all the components of G, the *generalized Fitting subgroup* $\mathbf{F}^*(G)$ of G is then defined as the product $\mathbf{F}(G)\mathbf{E}(G)$. In the proof of Theorem 3.1 it will be useful to take into account that, if $\mathbf{F}(G)$ is trivial, then $\mathbf{F}^*(G)$ is the product of all the minimal normal subgroups of G (see for instance [12, 6.5.5(b)]), and also that distinct components of G centralize each other ([12, 6.5.3]). The following result should be compared with [4, Proposition 4] and [18, Proposition 3.2].

Theorem 3.1. *Let G be a group with $\mathbf{F}(G) = 1$, and assume that G does not have any simple characteristic subgroup. Then there exist $\psi_1, \psi_2 \in \mathrm{Irr}(\mathbf{F}^*(G))$ (not necessarily distinct) such that, setting $m = \mathrm{m}(G/\mathbf{F}^*(G))$, the following conclusions hold.*

(a) $\pi_{\geq m}(G) \subseteq \bigcup_{i=1}^{2} \left(\pi(\psi_i) \cup \pi(G : I_G(\psi_i))\right)$;
(b) $|\mathbb{P}_m| \leq 2|\pi(G : I_G(\psi_i)) \cap \mathbb{P}_m|$ *for* $i = 1, 2$.

Proof. We argue by induction on $|G|$. Let M be a minimal characteristic subgroup of G, and define $N = \mathbf{C}_G(M)$ (note that N is a characteristic subgroup of G as well). Then $\mathbf{F}(G/N) = 1$ and $\mathbf{F}^*(G/N) = MN/N$, so the factor group of G/N over its generalized Fitting subgroup is isomorphic to G/MN; also, it is easy to see that G/N does not contain any simple characteristic subgroup. Assume first that $N \neq 1$, and set $m_1 = \mathrm{m}(G/MN)$. By induction, there exist $\overline{\lambda}_1, \overline{\lambda}_2 \in \mathrm{Irr}(MN/N)$ such that

$$\pi_{\geq m_1}(G/N) \subseteq \bigcup_{i=1}^{2} \left(\pi(\overline{\lambda_i}) \cup \pi(G/N : I_{G/N}(\overline{\lambda_i}))\right),$$

and also, $|\mathbb{P}_{m_1}| \leq 2|\pi(G/N : I_{G/N}(\overline{\lambda_i})) \cap \mathbb{P}_{m_1}|$ for $i \in \{1, 2\}$. Regarding $\overline{\lambda_i}$ as a character of MN by inflation, let $\lambda_i \in \mathrm{Irr}(M)$ be such that $\lambda_i \times 1_N = \overline{\lambda_i}$. Then clearly $I_G(\lambda_i)/N = I_{G/N}(\overline{\lambda_i})$.

Observe now that $\mathbf{F}(N) = 1$. Moreover, taking into account the paragraph preceding this theorem, we have $\mathbf{F}^*(G) = M \times \mathbf{F}^*(N)$. Note also that, obviously, N does not contain any simple characteristic subgroup. Let $m_2 = \mathrm{m}(N/\mathbf{F}^*(N))$. By induction, there exist μ_1 and $\mu_2 \in \mathrm{Irr}(\mathbf{F}^*(N))$, such that

$$\pi_{\geq m_2}(N) \subseteq \bigcup_{i=1}^{2} \left(\pi(\mu_i) \cup \pi(N : I_N(\mu_i))\right)$$

and $|\mathbb{P}_{m_2}| \leq 2|\pi(N : I_N(\mu_i)) \cap \mathbb{P}_{m_2}|$, for $i \in \{1, 2\}$.

Take now $\psi_1 = \lambda_1 \times \mu_1$ and $\psi_2 = \lambda_2 \times \mu_2$ in $\mathrm{Irr}(\mathbf{F}^*(G))$. Since $G/\mathbf{F}^*(G)$ is an extension of $MN/\mathbf{F}^*(G) \cong N/\mathbf{F}^*(N)$, by G/MN, we have $m = \max\{m_1, m_2\}$ by Lemma 2.1. As $I_G(\psi_i) = I_G(\lambda_i) \cap I_G(\mu_i)$, we conclude that $\pi(G : I_G(\lambda_i)) \cup \pi(N : I_N(\mu_i)) \subseteq \pi(G : I_G(\psi_i))$, and so (a) holds. Moreover, since $\mathbb{P}_m$ coincides either with $\mathbb{P}_{m_1}$ or with $\mathbb{P}_{m_2}$, claim (b) follows as well.

Hence we may assume $N = 1$, so $M = \mathbf{F}^*(G)$ is the socle of G. Now, M is the direct product of a set Ω of subgroups that are all isomorphic to a suitable non-abelian simple group S, and that are permuted by conjugation by G. Denoting by K the kernel of this action, we remark that the permutation group G/K on Ω is non-trivial because M is not a simple group. Note also that K is a characteristic subgroup of G, and that $\mathrm{Aut}(G)$ transitively permutes the set $\{\mathbf{C}_K(S) \mid S \in \Omega\}$; as a consequence, the factor groups $K/\mathbf{C}_K(S)$ are pairwise isomorphic when S ranges over Ω.

Noting that K/M is solvable, since it is a subgroup of a direct product of copies of $\mathrm{Out}(S)$, we see that $\mathrm{m}(G/K) = \mathrm{m}(G/M) = m$. By Proposition 2.2, there exist two disjoint subsets $\Gamma, \Delta \subseteq \Omega$ such that

$$\pi_{\geq m}(G/K) = \pi_{\geq m}(G : G_\Gamma \cap G_\Delta)$$

and

$$|\mathbb{P}_m| \leq 2|\pi(G : G_\Gamma \cap G_\Delta) \cap \mathbb{P}_m|;$$

as observed in the paragraph following Proposition 2.2, we can assume that both Γ and Δ are non-empty.

Now, take $S \in \Omega$ and define $C = \mathbf{C}_K(S)$. An application of Proposition 2.3 to the almost-simple group K/C (of socle $SC/C \simeq S$) yields the existence of χ_1, χ_2, ξ in $\mathrm{Irr}(S)$ such that $\pi(\chi_1) \cup \pi(\chi_2)$ contains all the primes in $\pi(S)$, except possibly a single prime p that is recovered as an element of $\pi(\xi)$, and both $|K : I_K(\chi_1)|$, $|K : I_K(\chi_2)|$ contain all the primes in $\pi(K/C) - \pi(S)$. Writing $\Omega = \{S_1, \ldots S_k\}$, we choose $\chi_1^{(i)}, \chi_2^{(i)}, \xi^{(i)} \in \mathrm{Irr}(S_i)$ in this way; since the groups $K/\mathbf{C}_K(S_i)$ are pairwise isomorphic, we can assume $\chi_1^{(i)}(1) = \chi_1^{(j)}(1)$ and $\chi_2^{(i)}(1) = \chi_2^{(j)}(1)$ for all $i, j \in \{1, \ldots, k\}$.

Next, we define $\psi_1 \in \mathrm{Irr}(M)$ as follows. For S_i belonging to Γ we take the character $\chi_1^{(i)}$, for S_j in Δ we take $\chi_2^{(j)}$, and for S_t in $\Omega - (\Gamma \cup \Delta)$ we take the trivial character; then, we set ψ_1 to be the direct product of all these characters. Similarly we define $\psi_2 \in \mathrm{Irr}(M)$, just replacing $\chi_2^{(j)}$ with $\xi^{(j)}$ for all $S_j \in \Delta$, so we clearly have $\pi(M) \subseteq \pi(\psi_1) \cup \pi(\psi_2)$. Moreover, as $I_K(\psi_1)$ and $I_K(\psi_2)$ both lie in $\bigcap_{i=1}^k I_K(\chi_1^{(i)})$, we see that $\pi(G : I_G(\psi_1))$ and $\pi(G : I_G(\psi_2))$ (which clearly contain $\pi(K : I_K(\psi_1))$ and $\pi(K : I_K(\psi_2))$, respectively) both contain $\bigcup_{i=1}^k \pi(K/\mathbf{C}_K(S_i)) - \pi(S)$. In view of the fact that $\bigcap_{i=1}^k \mathbf{C}_K(S_i)$ is trivial, so that K embeds in the direct product of the groups $K/\mathbf{C}_K(S_i)$, we conclude that $\bigcup_{i=1}^2 \pi(\psi_i) \cup \pi(G : I_G(\psi_i))$ contains $(\pi(K) - \pi(M)) \cup \pi(M) = \pi(K)$.

Moreover, *assuming that the degrees of* χ_1,χ_2 *are distinct* (and swapping the role of χ_1, χ_2 if $\chi_1(1) = \xi(1)$), it is easy to see that both $I_G(\psi_1)$ and $I_G(\psi_2)$ lie in $G_\Gamma \cap G_\Delta$, therefore $\pi(G : I_G(\psi_1))$ and $\pi(G : I_G(\psi_2))$ contain $\pi_{\geq m}(G/K)$. Thus claim (a) is proved, and also claim (b) follows immediately.

If $\chi_1(1) = \chi_2(1) \neq \xi(1)$, then one character (namely ψ_2) is enough to obtain the desired conclusions. If, finally, we have $\chi_1(1) = \chi_2(1) = \xi(1)$, then the argument needs a small adjustment: namey, in the definition of ψ_1, for the groups S_j belonging to Δ we choose any non-linear irreducible character whose degree is different from $\chi_1(1)$ in place of $\chi_2^{(j)}$, and ψ_1 so modified is enough to obtain the desired conclusions. ∎

Corollary 3.2. *If G is a group such that $\mathbf{F}(G) = 1$ and G does not have any simple characteristic subgroup, then $|\pi(G)| \leq 2\sigma(G)$.*

Proof. Let ψ_1, ψ_2 be the characters in $\mathrm{Irr}(\mathbf{F}^*(G))$ provided by Theorem 3.1 and, for $i \in \{1, 2\}$, take $\chi_i \in \mathrm{Irr}(G|\psi_i)$. By Clifford Theory we have $\pi(\psi_i) \cup \pi(G : I_G(\psi_i)) \subseteq \pi(\chi_i)$, whence, for some $i \in \{1, 2\}$, we get $2|\pi(\chi_i)| \geq |\pi(G)|$ by Theorem 3.1. ∎

We remark that the argument of Theorem 3.1 can be easily adapted to prove the following statement. *Let G be a group with $\mathbf{F}(G) = 1$, and assume that G does not have any simple characteristic subgroup. Assume further that $|G|$ is not divisible by 3. Then there exist $\psi \in \mathrm{Irr}(\mathbf{F}^*(G))$ such that $\pi(G) = \pi(\psi) \cup \pi(G : I_G(\psi))$.*

To this end, one should take into account that the assumption of $|G|$ not being a multiple of 3 yields $\mathrm{m}(G) = 0$; moreover, the same assumption yields that every simple subnormal subgroup S of G is a Suzuki group, and so it is possible to find χ_1, χ_2 in $\mathrm{Irr}(S)$ such that $\pi(\chi_1) \cup \pi(\chi_2) = \pi(S)$ (see [14, Page 182]).

As a consequence it is immediate to see that, under the same hypotheses, we have $|\pi(G)| = \sigma(G)$. We omit the full proofs of these facts because they amount to a straightforward modification of what already seen in Theorem 3.1 and Corollary 3.2, and also because these results are not essential for the rest of this paper.

We are now in a position to prove the two theorems that together constitute Theorem A.

Theorem 3.3. *Let G be a group with trivial Fitting subgroup. If $\sigma(G) \leq 5$, then we have $|\pi(G)| \leq 2\sigma(G) + 1$.*

Proof. Let G be a counterexample of minimal order to the statement. We start by observing that, by Corollary 3.2, the set of simple normal subgroups of G is non-empty. So, let us consider any simple normal subgroup S of G, and set $C = \mathbf{C}_G(S)$; note that C cannot be trivial, as otherwise G would be an almost-simple group and, in view of Theorem 2.4, it would not be a counterexample. Moreover, C clearly satisfies our assumptions and, being strictly smaller than G, it also satisfies the conclusion $|\pi(C)| \leq 2\sigma(C) + 1$.

Let us fix the following convention. If the simple group S is in $\mathcal{L}(p)$, then we call p the *associated prime* of S (in the cases when S has multiple characteristic, we choose p to be the smallest among them) and, if $S \cong J_1$, then we set 11 to be the associated prime of S; for any other isomorphism type of S, we say that S has no associated prime. Now we apply Proposition 2.3 to the almost-simple group G/C, whose socle is isomorphic to S: there exist χ_1 and χ_2 in $\mathrm{Irr}(S)$ such that $\pi(S) = \pi(\chi_1) \cup \pi(\chi_2)$ unless S has an associated prime p, in which case we have $\pi(S) - (\pi(\chi_1) \cup \pi(\chi_2)) \subseteq \{p\}$. Moreover, defining $\pi_0 = \pi(G/C) - \pi(S)$, we have $\pi_0 \subseteq \pi(|G : I_G(\chi_i)|)$ for $i \in \{1, 2\}$. If we set $\pi_1 = \pi(\chi_1) \cup \pi_0$,

$\pi_2 = \pi(\chi_2) \cup \pi_0$, and π_3 to be either $\{p\}$ or $\emptyset$ according on whether S has an associated prime p or not, we can write $\pi(G) = \pi(C) \cup \bigcup_{i=1}^{3}(\pi_i - \pi(C))$, and therefore

$$|\pi(G)| \leq |\pi(C)| + \sum_{i=1}^{3} |\pi_i - \pi(C)|. \tag{1}$$

(Note that $|\pi_3 - \pi(C)| \leq 1$.) If S has an associated prime p, then we take $\chi_3 \in \mathrm{Irr}(S)$ whose degree is divisible by p; otherwise we just take $\chi_3 = 1_S$. Let $\delta \in \mathrm{Irr}(C)$ such that $|\pi(\delta)| = \sigma(C)$, for $i \in \{1, 2, 3\}$ we can consider irreducible characters of G lying above $\delta \times \chi_i$, and we deduce that

$$\sigma(G) \geq \sigma(C) + \max_{i \in \{1,2,3\}} \{|\pi_i - \pi(C)|\}. \tag{2}$$

As the next step, we will prove a number of claims.

(i) *For $i \neq j \in \{1, 2\}$, $\pi(\chi_i) - \pi(C)$ is not contained in $\pi_j \cup \pi_3$, and $\pi_3 - \pi(C)$ is not contained in $\pi_1 \cup \pi_2$; in particular, $\pi(\chi_1) - \pi(C)$, $\pi(\chi_2) - \pi(C)$, $\pi_3 - \pi(C)$ are non-empty, and $\pi(S)$ contains at least two primes (one in $\pi(\chi_1) - \pi(\chi_2)$ and the other in $\pi(\chi_2) - \pi(\chi_1)$) that are not in $\pi(C) \cup \pi_0 \cup \pi_3$.*

For a proof by contradiction, assume that $\pi(\chi_1) - \pi(C)$ is contained in $\pi_2 \cup \pi_3$; then, taking into account that π_0 lies in π_2, we get $\pi(G) = \pi(C) \cup \bigcup_{i=2}^{3}(\pi_i - \pi(C))$, and Inequality (1) gives

$$|\pi(G)| \leq |\pi(C)| + 2 \max_{i \in \{2,3\}} \{|\pi_i - \pi(C)|\}.$$

Recalling that $|\pi(C)| \leq 2\sigma(C) + 1$, we obtain

$$|\pi(G)| \leq 2(\sigma(C) + \max_{i \in \{2,3\}} \{|\pi_i - \pi(C)|\}) + 1$$

and therefore, in view of Inequality (2), we get the contradiction $|\pi(G)| \leq 2\sigma(G) + 1$. The same argument of course works for the other inclusions as well, and the remaining parts of the claim can be easily deduced.

The fact that $\pi_3 - \pi(C)$ is non-empty yields, on one hand, that $\sigma(G)$ is strictly larger than $\sigma(C)$ (taking into account Inequality (2)). On the other hand, S must have an associated prime which does not lie in $\pi(C)$:

(ii) *S is either of Lie type in odd characteristic, or $S \cong J_1$. In particular, $|S|$ is divisible by 3.*

Here, the only thing to observe is that S cannot lie in $\mathcal{L}(2)$ because C has even order, as $C \neq 1$ and $\mathbf{F}(C) = 1$.

Since two (distinct) simple normal subgroups of G centralize each other, we also deduce the following:

(iii) *The associated primes of the simple normal subgroups of G are pairwise distinct.*

Some other remarks concerning the centralizer C of a simple normal subgroup S of G:

(iv) *$|\pi(C)| = 2\sigma(C) + 1$. In addition, C has a simple characteristic subgroup and $|C|$ is divisible by 3; furthermore, C is not an almost-simple group.*

In fact, we know that $|\pi(C)| \leq 2\sigma(C)+1$, so, if the claim is false, then we have $|\pi(C)| \leq 2\sigma(C)$. Now, using again inequalities (1) and (2) as in **(i)** and recalling that $|\pi_3 - \pi(C)| = 1$, we get $|\pi(G)| \leq 2\sigma(G)+1$, which is not the case. Given that, C must have a simple characteristic subgroup by Corollary 3.2 and, as this subgroup is a simple normal subgroup of G, by **(ii)** its order is divisible by 3. Finally, if C is an almost-simple group, its socle (which is a simple normal subgroup of G) in this situation must be then isomorphic to a projective special linear group in characteristic 2 by Theorem 2.4, and this is against claim **(ii)**.

Since we have seen that 3 divides $|C|$, claim **(i)** (namely, $\pi_3 - \pi(C) \neq \emptyset$) yields also:

(v) $S \notin \mathcal{L}(3)$.

Next,

(vi) *Both* $\sigma(S)$ *and* $\sigma(C)$ *lie in* $\{3, 4\}$.

To the end of showing $\sigma(S) \geq 3$, by Lemma 2.4 in [24] we only have to rule out the case $S \cong \mathrm{PSL}_2(q)$ (where q is a p-power with $p > 3$) and $|\pi(q \pm 1)| \leq 2$; but in this situation the degree of (say) χ_1, as defined in the second paragraph of this proof, is divisible only by 2 and 3, so $\pi(\chi_1) - \pi(C)$ is empty, against **(i)**. As for $\sigma(C) \leq 4$, this follows at once by the fact that, as observed, we have $\sigma(C) < \sigma(G)$; the remaining conclusions also follow easily, taking into account that (by **(iv)**) C contains a simple normal subgroup of G.

Consider now the set $\mathcal{K} = \{S_1, S_2, \ldots, S_k\}$ of all the simple normal subgroups of G. As our last preliminary claim, we show the following:

(vii) $G = (S_1 \times S_2 \times \cdots \times S_k) \times U$, *where* $|\pi(U)| \leq 2\sigma(U)$.

Let $S \in \mathcal{K}$. Recalling that (by **(vi)**) we have $\sigma(C) \in \{3, 4\}$, assume first $\sigma(C) = 3$ (which implies $|\pi(C)| = 2 \cdot 3 + 1 = 7$) and $\sigma(G) = 5$: then Inequality (2) yields $|\pi_i - \pi(C)| \leq 2$ for $i \in \{1, 2, 3\}$. Now, $\pi_1 - \pi(C)$ cannot contain any element u of π_0, as otherwise u would lie in $\pi_2 - \pi(C)$ as well, and the set $X = (\pi_1 - \pi(C)) \cup (\pi_2 - \pi(C))$ would contain at most three elements. As a consequence of the fact that $\pi(G)$ is covered by $\pi(C) \cup X \cup (\pi_3 - \pi(C))$ we would then obtain $|\pi(G)| \leq 7 + 3 + 1 = 11 = 2\sigma(G) + 1$, against our assumption. We conclude that $\pi(G) = \pi(SC)$ in this case.

But the same holds also whenever $\sigma(G) - \sigma(C) = 1$ (which covers all the remaining possibilities): in fact, in the latter situation, $\pi_i - \pi(C)$ is a singleton and (by **(i)**) it must be contained in $\pi(\chi_i)$ for $i \in \{1, 2\}$.

Now, if SC is strictly contained in G, our minimality assumption leads again to the contradiction $|\pi(G)| = |\pi(SC)| \leq 2\sigma(SC) + 1 \leq 2\sigma(G) + 1$. Thus we have $G = SC$ and, since this holds for all the elements in $\mathcal{K}$, we easily deduce that $G = (S_1 \times S_2 \times \cdots \times S_k) \times U$ where $U = \bigcap_{i=1}^{k} \mathbf{C}_G(S_i)$. Finally, observe that either U is trivial, or it is a group with trivial Fitting subgroup and no simple characteristic subgroups. Therefore, by Corollary 3.2, the desired conclusion follows.

We observed in **(iv)** that, for $S \in \mathcal{K}$, there is an element of $\mathcal{K}$ lying in $\mathbf{C}_G(S)$, thus $|\mathcal{K}| \neq 1$. We work next to exclude also $|\mathcal{K}| > 1$, thus obtaining a contradiction.

Note that, if S_i is in $\mathcal{K}$, then the two characters $\chi_1^{(i)}$, $\chi_2^{(i)} \in \mathrm{Irr}(S_i)$ given by Proposition 2.3 "cover" together all the prime divisors of $|S_i|$ except possibly the associated prime p_i of S_i, as explained in the second paragraph of this proof (recall that, by **(ii)** and **(v)**, we have $p_i \notin \{2, 3\}$); in particular, taking into account that $\pi(\chi_1^{(i)}) - \pi(\mathbf{C}_G(S_i))$ and $\pi(\chi_2^{(i)}) - \pi(\mathbf{C}_G(S_i))$ are non-empty by **(i)**, we see that one among $\chi_1^{(i)}$, $\chi_2^{(i)}$ has a degree divisible by $2r_i$, and another (possibly the same) has a degree divisible by $3s_i$, where r_i and s_i are (possibly equal) primes not dividing $|\mathbf{C}_G(S_i)|$. Note that hence, clearly, $\{r_i, s_i\} \cap \{r_j, s_j\} = \emptyset$ for $i \neq j$. Given that, assume $|\mathcal{K}| \geq 4$ and take four distinct elements S_i, $i \in \{1, \ldots, 4\}$, in $\mathcal{K}$: considering irreducible characters ξ_i of S_i such that $\{2, r_1\} \subseteq \pi(\xi_1)$, $\{3, s_2\} \subseteq \pi(\xi_2)$, $p_3 \in \pi(\xi_3)$ and $p_4 \in \pi(\xi_4)$, we see that $|\pi(\xi_1 \times \cdots \times \xi_4)| \geq 6$, against the assumption $\sigma(G) \leq 5$. Our conclusion so far is that $|\mathcal{K}| \leq 3$.

Assume then $\mathcal{K} = \{S_1, S_2, S_3\}$, and observe that in this case we have $\sigma(S_i) = 3$ for every $i \in \{1, 2, 3\}$: in fact, if (say) S_1 has an irreducible character ξ_1 with $|\pi(\xi_1)| \geq 4$ then, considering $\xi_i \in \mathrm{Irr}(S_i)$ with $p_i \in \pi(\xi_i)$ for $i \in \{2, 3\}$, by **(iii)** we get $|\pi(\xi_1 \times \xi_2 \times \xi_3)| \geq 6$, again a contradiction. Furthermore, assume that an element S of $\mathcal{K}$ is not a 2-dimensional projective special linear group; then, we claim that there exists an irreducible character ξ of S whose degree is divisible by p and by a prime $t \notin \{2, 3\}$. Once this will be established, setting $S_3 = S$, we can choose $\xi_3 = \xi \in \mathrm{Irr}(S_3)$, and $\xi_1 \in \mathrm{Irr}(S_1)$, $\xi_2 \in \mathrm{Irr}(S_2)$ as in the paragraph above, getting the contradiction $|\pi(\xi_1 \times \xi_2 \times \xi_3)| \geq 6$. So, we work to prove this claim.

According to Theorem 6.3 of [17] and recalling that, by **(i)**, $\pi(S)$ contains at least two primes r, s other than 2, 3 and p, the only cases that must be treated are when either $S \cong \mathrm{PSL}_3(p^f)$ or $S \cong \mathrm{PSU}_3(p^f)$ for some positive integer f. In the former case, looking at Theorem 3.2 of [25], our claim is true unless (setting $q = p^f$) we have $\pi((q+1)(q^2+q+1)) \subseteq \{2,3\}$; this forces the odd number q^2+q+1 to be a power of 3, and the primes r, s to lie in $\pi(q-1)$; but S has an irreducible character ψ of degree $(q-1)(q^2+q+1)$, which implies $\pi(\psi) \supseteq \{2,3,r,s\}$ and contradicts $\sigma(S) = 3$. The case $S \cong \mathrm{PSU}_3(p^f)$ can be treated similarly, referring to [25, Theorem 3.4]. This establishes the claim in the above paragraph.

In order to rule out the case $\mathcal{K} = \{S_1, S_2, S_3\}$, it remains then to treat the situation when S_i is a 2-dimensional projective special linear group (in characteristic larger than 3) for every $i \in \{1,2,3\}$. The set of character degrees of $\mathrm{PSL}_2(q)$, where $q > 5$ is a prime power (note that $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$ does not show up here by **(ii)**), is well known (see Example 1.1). In our situation, taking into account **(i)** and the fact that $\sigma(S_i) = 3$ for all $i \in \{1,2,3\}$, each S_i has two irreducible characters $\chi_1^{(i)}$, $\chi_2^{(i)}$ such that $\pi(\chi_1^{(i)}) \supseteq \{2,3,r_i\}$ and $\pi(\chi_2^{(i)}) \supseteq \{2,s_i\} \cup T_i$, where r_i, s_i are primes not lying in $|\mathbf{C}_G(S_i)|$ and T_i is either empty or a singleton $\{t_i\}$. But every T_i must be in fact empty: if (say) $t_1 \in T_1$, then $\chi_2^{(1)} \times \chi_1^{(2)} \in \mathrm{Irr}(S_1 \times S_2)$ has a degree divisible by $2, 3, s_1, t_1, r_2$, and choosing an irreducible character of S_3 whose degree is divisible by p_3 we get the contradiction $\sigma(S_1 \times S_2 \times S_3) \geq 6$.

Recall that $G = (S_1 \times S_2 \times S_3) \times U$, where $U = \bigcap_{i=1}^{3} \mathbf{C}_G(S_i)$; but, as $\sigma(S_1 \times S_2 \times S_3) = 5 = \sigma(G)$, clearly the set $\pi(U) - \pi(S_1 \times S_2 \times S_3)$ must be empty, and again by minimality we conclude that $G = S_1 \times S_2 \times S_3$. At this stage, we see that $|\pi(G)|$ is 11, so it equals $2\sigma(G)+1$, and this is the contradiction which excludes $|\mathcal{K}| = 3$.

Finally, assume $\mathcal{K} = \{S_1, S_2\}$ and write $G = S_1 \times S_2 \times U$. Denoting by $\chi_1^{(1)}, \chi_2^{(1)}$ the characters of S_1 given by Proposition 2.3 (as in the second paragraph of this proof) and, similarly, by $\chi_1^{(2)}, \chi_2^{(2)}$ those of S_2, we consider the irreducible characters $\chi_1 = \chi_1^{(1)} \times \chi_1^{(2)}$ and $\chi_2 = \chi_2^{(1)} \times \chi_2^{(2)}$ of $S_1 \times S_2$. Note that, up to swapping $\chi_1^{(2)}$ and $\chi_2^{(2)}$, we may assume $\{2,3\} \subseteq \pi(\chi_1)$. In this setting, we have

$$\pi(G) = \pi(U) \cup (\pi(\chi_1) - \pi(U)) \cup (\pi(\chi_2) - \pi(U)) \cup \{p_1, p_2\}, \tag{3}$$

and $\sigma(G) \geq \sigma(U) + |\pi(\chi_i) - \pi(U)|$ for both $i = 1$ and $i = 2$.

Let us consider first the case $U \neq 1$. Since $\mathbf{F}(U) = 1$, any minimal normal subgroup of U is a direct product of isomorphic non-abelian simple groups, and the number of factors in this product has to be larger than 1; in fact, U being a direct factor of G, a normal subgroup of U is also normal in G, thus a simple normal subgroup of U would be an element of $\mathcal{K}$, clearly a contradiction. As a consequence, we have $\sigma(U) \geq 2$; on the other hand $\sigma(U) \leq 3$, as otherwise, taking $\delta \in \mathrm{Irr}(U)$ such that $\pi(\delta) = \sigma(U)$ and $\psi \in \mathrm{Irr}(S_1 \times S_2)$ with $\pi(\psi) \supseteq \{p_1, p_2\}$, we would get $|\pi(\psi \times \delta)| \geq 6$.

If $\sigma(U) = 2$, by (**vii**) we have $|\pi(U)| \leq 4$, and therefore Equation (3) yields that both $|\pi(\chi_1) - \pi(U)|$ and $|\pi(\chi_2) - \pi(U)|$ must be 3 (otherwise we would get the contradiction $|\pi(G)| < 12$). In particular there exist primes r, s, t, all lying outside $\pi(U)$, such that $\pi(\chi_1) \supseteq \{2,3,r,s,t\}$. Now, as usual, we can produce an irreducible character of G whose degree is divisible by six primes taking the product of χ_1 and $\xi \in \mathrm{Irr}(U)$ with $\xi(1)$ divisible by an odd prime different from 3.

Assume finally $\sigma(U) = 3$, so $|\pi(U)| \leq 6$. By Equation (3), we see that $|\pi(U)|$ must be 6 and $|\pi(\chi_i) - \pi(U)|$ is 2 for $i \in \{1,2\}$. Now, there exist primes r, s, both outside $\pi(U)$, such that $\pi(\chi_1) \supseteq \{2,3,r,s\}$; moreover, by the main result of [23], we can find an irreducible character of U whose degree is divisible by two primes in $\pi(U) - \{2,3\}$ (because this set contains four elements). Taking the product of this character with χ_1 we reach again the contradiction $\sigma(G) \geq 6$.

To rule out the case $|\mathcal{K}| = 2$ and finish the proof, it remains to consider the case $U = 1$, so, $G = S_1 \times S_2$. In this situation, $\mathbf{C}_G(S_1) = S_2$ is (almost-)simple, against what observed in **(iv)**; thus also in this last case we reached a contradiction, and our minimal counterexample G does not exist. ∎

Theorem 3.4. *Let G be a group with trivial Fitting subgroup. If $\sigma(G) > 5$, then we have $|\pi(G)| \leq 3\sigma(G) - 4$.*

Proof. As in the proof of Theorem 3.3, we assume the existence of a counterexample to the statement, and we take G of minimal order among these counterexamples. By Corollary 3.2 and Theorem 2.4 (taking also into account that $2\sigma(G)+1$ is smaller than $3\sigma(G)-4$ whenever $\sigma(G) > 5$), the set of simple normal subgroups of G is non-empty and G is not an almost-simple group. So, let us consider a simple normal subgroup S of G, and set $C = \mathbf{C}_G(S)$ (note that $C \neq 1$, because G is not almost-simple). We see that C has a trivial Fitting subgroup and clearly it is a proper subgroup of G.

Recall the second paragraph in the proof of Theorem 3.3: an application of Proposition 2.3 to the almost-simple group G/C yields that there exist χ_1 and χ_2 in $\mathrm{Irr}(S)$ such that $\pi(S) = \pi(\chi_1) \cup \pi(\chi_2)$ unless S has an associated prime p, in which case we have $\pi(S) - (\pi(\chi_1) \cup \pi(\chi_2)) \subseteq \{p\}$. Defining $\pi_0 = \pi(G/C) - \pi(S)$, $\pi_1 = \pi(\chi_1) \cup \pi_0$, $\pi_2 = \pi(\chi_2) \cup \pi_0$, and $\pi_3 = \{p\}$ or $\pi_3 = \emptyset$ according to whether S has an associated prime or not, we have $\pi(G) = \pi(C) \cup \bigcup_{i=1}^{3}(\pi_i - \pi(C))$, and therefore

$$|\pi(G)| \leq |\pi(C)| + \sum_{i=1}^{3} |\pi_i - \pi(C)| \leq |\pi(C)| + 3 \max_{i \in \{1,2,3\}} \{|\pi_i - \pi(C)|\}.$$

(Note that $|\pi_3 - \pi(C)| \leq 1$.) Moreover, as seen in the previous theorem,

$$\sigma(G) \geq \sigma(C) + m,$$

where $m = \max_{i \in \{1,2,3\}} \{|\pi_i - \pi(C)|\}$.

Now, if $\sigma(C) > 5$, then C satisfies our assumptions and, by minimality, we get $|\pi(C)| \leq 3\sigma(C) - 4$. Thus $|\pi(G)| \leq (3\sigma(C) - 4) + 3m = 3(\sigma(C) + m) - 4 \leq 3\sigma(G) - 4$, and G is not a counterexample.

Therefore we have $\sigma(C) \leq 5$. In this case, by Theorem 3.3, we have $|\pi(C)| \leq 2\sigma(C) + 1$, hence $|\pi(G)| \leq (2\sigma(C) + 1) + 2m + 1 \leq 2(\sigma(C) + m) + 2$. If $\sigma(C) + m \leq 5$, then we also get $\sigma(C) + m \leq \sigma(G) - 1$, whence $|\pi(G)| \leq 2\sigma(G)$, which is impossible. On the other hand, $\sigma(C) + m \geq 6$ implies $2(\sigma(C) + m) + 2 \leq 3(\sigma(C) + m) - 4 \leq 3\sigma(G) - 4$, the final contradiction that completes the proof. ∎

4. The solvable case: a proof of Theorem B

We begin now our discussion concerning the ρ-σ conjecture for solvable groups. After two general preliminary statements, we will prove a proposition on solvable permutation groups that will be a key step.

Lemma 4.1 ([4, Lemma 3])**.** *Let $K \leq L \leq H$ be groups with $K \trianglelefteq H$. Then, for every $S \leq H$, we have*

$$\pi(H : S) - \pi(H/K) = \pi(L : L \cap S) - \pi(H/K) .$$

Lemma 4.2. *Let K be a group such that $K/\mathbf{F}(K)$ is nilpotent. Then there exists $\theta \in \mathrm{Irr}(K)$ such that $\pi(K/\mathbf{F}(K))) \subseteq \pi(\theta)$.*

Proof. This follows from [21, Proposition 17.3]. ∎

Proposition 4.3. *Let H be a solvable permutation group on a finite non-empty set Ω. Then there exists a non-empty subset $\Delta \subseteq \Omega$ such that*

(a) $\pi(H) \subseteq \pi(H : H_\Delta) \cup \{p\}$*, for a suitable* $p \in \{2, 3\}$*;*

(b) $\pi(H) = \pi(H : H_\Delta)$ *if* $|H|$ *is odd.*

Proof. Part (b) is part (b) of [21, Corollary 5.7].

In order to prove part (a), assume first that the action of H on Ω is primitive. In this case we prove that there are two non-empty subsets $\Delta_1, \Delta_2 \subseteq \Omega$ *of distinct sizes* such that both set-stabilizers H_{Δ_i} verify claim (a) in the statement, for the same prime p.

Clearly, if there exists a subset Δ of Ω such that $H_\Delta = 1$ and $|\Delta| \neq |\Omega|/2$, then we set $\Delta_1 = \Delta$ and $\Delta_2 = \Omega - \Delta$. If no such Δ exists (in the terminology of [21], this amounts to saying that there are no *strongly regular* orbits in the action of H on the power set of Ω), then by [21, Theorem 5.6] we are in one of the following cases.

(a): $|\Omega| = 2$: then we take any subset of size 1 as Δ_1 and $\Delta_2 = \Omega$.
(b): $|\Omega| = 3$, $H \cong S_3$: take $|\Delta_1| = 1$, $|\Delta_2| = 2$.
(c): $|\Omega| = 4$, $H \cong A_4$ or S_4; take $|\Delta_1| = 1$, $|\Delta_2| = 3$.
(d): $|\Omega| = 5$: H is a Frobenius group of order 10 or 20; take $|\Delta_1| = 3$, $|\Delta_2| = 4$.
(e): $|\Omega| = 7$: H is a Frobenius group of order 42; take $|\Delta_1| = 2$, $|\Delta_2| = 5$.
(f): $|\Omega| = 8$: $H \cong A\Gamma(2^3)$; take $|\Delta_1| = 2$, $|\Delta_2| = 6$.
(g): $|\Omega| = 9$ and H is a subgroup of the semidirect product of $\mathrm{GL}_2(3)$ with its natural module V; here we take $|\Delta_1| = 1$, $|\Delta_2| = 8$ (note that H is a $\{2,3\}$-group and that V is a regular normal subgroup).

Next, we assume that H is transitive but imprimitive on Ω. Let $\Omega = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_m$ be an imprimitive decomposition of Ω, with minimal $|\Sigma_i| \neq 1$. Let $\Sigma = \Sigma_1$ and write $\Sigma_i = \Sigma^{x_i}$ for suitable elements $x_1 = 1, x_2, \ldots, x_m \in H$. Let $L = H_\Sigma$ be the stabilizer of Σ in H and let $K = \bigcap_{i=1}^m L^{x_i}$ be the kernel of the action of H on the set $\overline{\Omega} = \{\Sigma_1, \ldots, \Sigma_m\}$. So $\overline{H} = H/K$ is a solvable (transitive) permutation group on $\overline{\Omega}$, and by [7, Corollary 3] (or by Proposition 2.2) there exist two disjoint subsets Γ_1, Γ_2 of $\overline{\Omega}$ such that $\overline{H}_{\Gamma_1} \cap \overline{H}_{\Gamma_2}$ contains no Sylow subgroup of $\overline{H}$. Let C be the kernel of the action of L on Σ; so L/C is a *primitive* permutation group on Σ (by the minimality of $|\Sigma|$) and there exist two non-empty subsets $\Delta_1, \Delta_2 \subseteq \Sigma$ such that $|\Delta_1| \neq |\Delta_2|$ and $\pi(L : L_{\Delta_i}) \supseteq \pi(L/C) - \{p\}$, $i = 1, 2$, with $p \in \{2, 3\}$. We can also assume that $\Sigma_1 \in \Gamma_1$.

Define now

$$\Delta = \bigcup_{\Sigma^{x_i} \in \Gamma_1} \Delta_1^{x_i} \cup \bigcup_{\Sigma^{x_i} \in \Gamma_2} \Delta_2^{x_i} .$$

Since $|\Delta_1| \neq |\Delta_2|$, it follows that $H_\Delta K/K = \overline{H_\Delta} \leq \overline{H}_{\Gamma_1} \cap \overline{H}_{\Gamma_2}$. Hence $\pi(H/K) \subseteq \pi(H : H_\Delta)$. Moreover, by Lemma 4.1, $\pi(H : H_\Delta) - \pi(H/K) = \pi(L : L_\Delta) - \pi(H/K)$. As L_Δ is a subgroup of L_{Δ_1} (since $\Sigma_1 \in \Gamma_1$) and $\pi(K) \subseteq \pi(L/C)$ (because K is isomorphic to a subgroup of a direct product of groups isomorphic to L/C), we deduce that $\pi(H) = \pi(K) \cup \pi(H/K) \subseteq \pi(H : H_\Delta) \cup \{p\}$.

Finally, we assume that H is not transitive on Ω; let $\Omega_1, \Omega_2, \ldots, \Omega_k$, $k \geq 2$, be the orbits of H on Ω and let K_i be the kernel of the action of H on Ω_i, for $i = 1, 2, \ldots, k$. By what we have proved till now, for each i there exists a non-empty $\Delta_i \subseteq \Omega_i$ and a prime $p_i \in \{2, 3\}$ such that $\pi(H/K_i) \subseteq \pi(H/K_i : (H/K_i)_{\Delta_i}) \cup \{p_i\} = \pi(H : H_{\Delta_i}) \cup \{p_i\}$. By part (b), if $2 \notin \pi(H/K_i)$, then we can choose $\Delta_i \subseteq \Omega_i$ such that $\pi(H/K_i) = \pi(H : H_{\Delta_i})$.

As $\bigcap_i K_i = 1$, we have $\pi(H) = \bigcup_i \pi(H/K_i)$. Hence, setting $\Delta = \bigcup_i \Delta_i$, we conclude that $\pi(H) \subseteq \pi(H : H_\Delta) \cup \{p\}$, where $p = 2$ if $p_i = 2$ for each i and $p = 3$ if $p_i = 3$ for some i. ∎

Next, we introduce some terminology. If H is a group and V is an irreducible H-module, then it is possible to find a pair (L, W), where L is a subgroup of H and W is a primitive L-module, such that we have $V = W^H$ (here $L = \mathbf{N}_H(W)$); if V is primitive, then $L = H$. We say that a faithful irreducible

H-module V is of *type 1* if there exists a pair (L, W) of this kind, such that the normal core L_H of L in H is metabelian.

In the situation described in the previous paragraph, setting $R = L/\mathbf{C}_L(W)$, we have $\mathbf{F}(R) = EZ$ where $Z = \mathbf{Z}(\mathbf{F}(R))$ is cyclic, $E \trianglelefteq R$ and E is a direct product of extraspecial groups. Then one sets $e_W = \sqrt{|E/\mathbf{Z}(E)|}$ ([26, Definition 2.1]). We remark that $e_W = 1$ if and only if $\mathbf{F}(R)$ is cyclic and R acts as a group of semilinear maps on W. Hence, if $e_W = 1$, then L_H is metabelian (because it embeds in the direct product of groups isomorphic to the metabelian group $L/\mathbf{C}_L(W)$) and V is of type 1.

Proposition 4.4. *Let G be a solvable group whose Frattini subgroup $\mathbf{\Phi}(G)$ is trivial. Then there exist $\chi, \psi \in \mathrm{Irr}(G)$ and a prime $p \in \{2, 3\}$ such that $\rho(G) \subseteq \pi(\chi) \cup \pi(\psi) \cup \{p\}$. Furthermore, if $|G/\mathbf{F}(G)|$ is odd, then there exist $\chi, \psi \in \mathrm{Irr}(G)$ such that $\rho(G) = \pi(\chi) \cup \pi(\psi)$.*

Proof. As $\mathbf{\Phi}(G)$ is trivial, $F = \mathbf{F}(G)$ is a direct product of minimal normal subgroups of G and it has a complement H in G (see [13, III.4.4 and III.4.5]). Write $F = M_1 \times \cdots \times M_n$, where the M_i are minimal normal subgroups of G, and let V_i be the dual group $\widehat{M_i}$ for $i \in \{1, \ldots, n\}$. So V_i is an irreducible H-module, and $V = \widehat{F} = V_1 \times \cdots \times V_n$ is a completely reducible and faithful H-module. We will work by induction on $|G|$.

We start by assuming that $V = V_1$ is an irreducible H-module (of course faithful), and we write $V = W^H$, where W is a primitive L-module for $L = \mathbf{N}_H(W)$. Let $K = L_H$ be the normal core of L in H; then $\overline{H} = H/K$ is a transitive permutation group on the set $\Omega = \{W^{x_1}, \ldots, W^{x_m}\}$, where $\{x_1 = 1, \ldots, x_m\}$ is a right transversal of L in H. By Proposition 4.3, there exists a non-empty subset Δ of Ω and a prime $p \in \{2, 3\}$ such that $\pi(\overline{H} : \overline{H}_\Delta) \supseteq \pi(\overline{H}) - \{p\}$. Moreover, we can assume that either $\pi(\overline{H} : \overline{H}_\Delta) = \pi(\overline{H})$ or $2 \in \pi(\overline{H})$, and there is no loss of generality in assuming $W = W_1 \in \Delta$.

Now, take any set $\{\mu_1, \ldots, \mu_m\}$ where μ_i lies in W^{x_i} for $i \in \{1, \ldots, m\}$ and μ_i is non-trivial if and only if $W^{x_i} \in \Delta$. Define then $\lambda = \mu_1 \times \cdots \times \mu_m \in V$. Taking into account the imprimitivity decomposition of V and the choice of the μ_i, setting $I = \mathbf{C}_H(\lambda)$ we see that $\overline{I} = IK/K$ is contained in the stabilizer $\overline{H}_\Delta$, whence $\pi(H : I) \supseteq \pi(H/K) - \{p\}$.

Consider first the case when V is an H-module of type 1, so (for a suitable choice of the pair (L, W)) K is metabelian, and let $\lambda \in V$ be as above. Since λ is not the trivial character of F (because $\Delta \neq \emptyset$), we claim that $\pi(H : I) \supseteq \pi(\mathbf{F}(K))$ as well. In fact, if $q \in \pi(\mathbf{F}(K))$ and Q is a Sylow q-subgroup of $\mathbf{F}(K)$, then $1 \neq Q \trianglelefteq H$ and hence (by Clifford Theorem) V_Q has no trivial irreducible constituent; in particular, $\langle \lambda \rangle$ can not be a trivial Q-module, hence Q does not lie in I. Thus, for any $\chi \in \mathrm{Irr}(G|\lambda)$, we deduce that $\pi(\chi) \supseteq (\pi(H/K) \cup \pi(\mathbf{F}(K))) - \{p\}$. Moreover, by Lemma 4.2 there exists $\theta \in \mathrm{Irr}(K)$ such that $\pi(K/\mathbf{F}(K))) \subseteq \pi(\theta)$. So, taking a character $\psi \in \mathrm{Irr}(H|\theta)$ and viewing ψ as an irreducible character of G by inflation, we get $\rho(G) = \pi(H) \subseteq \pi(\chi) \cup \pi(\psi) \cup \{p\}$, where $p \in \{2, 3\}$ and $\rho(G) = \pi(H) = \pi(\chi) \cup \pi(\psi)$ if $|H|$ is odd.

Assume next that V is irreducible and not of type 1 as an H-module, thus $e = e_W \geq 2$. Here we aim to prove a stronger property: there exists $\lambda \in V$ such that, for all $\chi \in \mathrm{Irr}(G|\lambda)$, we have $\rho(G) = \pi(\chi) \cup \{p\}$ where $p \in \{2, 3\}$.

By Theorem 3.1 of [26] and Theorem 3.1 of [27], either $R = L/\mathbf{C}_L(W)$ has a regular orbit on W or e belongs to the set $\{1, 2, 3, 4, 8, 9, 16\}$. In the latter case, we see that the center $\mathbf{Z}(E)$ of E is a non-trivial q-group where $q \in \{2, 3\}$, and $\mathbf{Z}(E)$ acts fixed-point freely on W. Moreover, by Theorem 11.3 of [21] there exists a (non-trivial) $\mu \in W$ such that $\pi(R : \mathbf{C}_R(\mu)) \supseteq \pi(R) - \{2, 3\}$. As $\mathbf{Z}(E)$ does not fix μ, we conclude that $\pi(R : \mathbf{C}_R(\mu)) \supseteq \pi(R) - \{p_0\}$ for a suitable $p_0 \in \{2, 3\}$. In fact we remark that, if $\pi(R : \mathbf{C}_R(\mu))$ does not contain $\pi(R) - \{2\}$, then 3 does not divide $|\mathbf{Z}(E)|$ and so $\mathbf{F}(R)$ is a central product of an extraspecial 2-group and a cyclic group. Finally, if $2 \notin \pi(R)$ then by [8, Theorem 2.2] we can assume that $\pi(R:\mathbf{C}_R(\mu))=\pi(R)$. Note that the last equality clearly holds also if R does have a regular orbit on W, provided μ is chosen in such an orbit.

Define $\lambda=\nu_1\times\cdots\times\nu_m\in V$, where $\nu_i=\mu^{x_i}$ if $W^{x_i}\in\Delta$ and ν_i is trivial otherwise. Set also $I=\mathbf{C}_H(\lambda)$, as above. Then we already know that $\pi(H:I)\supseteq\pi(H/K)-\{p\}$. On the other hand, since $L\cap I=\mathbf{C}_L(\lambda)\leq\mathbf{C}_L(\mu)$, and since we have $\pi(R)\supseteq\pi(K)$ because K embeds in the direct product of groups isomorphic to R, Lemma 4.1 yields

$$\pi(H:I)-\pi(H/K)=\pi(L:L\cap I)-\pi(H/K)\supseteq\pi(R)-(\pi(H/K)\cup\{p_0\})\supseteq\pi(K)-(\pi(H/K)\cup\{p_0\})\,.$$

Let now χ be in $\mathrm{Irr}(G|\lambda)$, and observe first that if $p_0=p$, then we clearly have $\rho(G)\subseteq\pi(\chi)\cup\{p_0\}$. Assume then $p_0=2$ and $p=3$. If Δ can be chosen so that $\pi(H:I)\supseteq\pi(H/K)$, then we get $\rho(G)\subseteq\pi(\chi)\cup\{2\}$; on the other hand, if such a Δ does not exist, then 2 lies in $\pi(H/K)$ and we have $\rho(G)\subseteq\pi(\chi)\cup\{3\}$. The only case that still needs to be treated is when 3 divides $|K|$ but not $|R:\mathbf{C}_R(\mu)|$ (so, $p_0=3$ and $p=2$). In this setting, $\pi(R:\mathbf{C}_R(\mu))$ does not contain $\pi(R)-\{2\}$, and therefore (as observed in the paragraph above) the 2-complement of $\mathbf{F}(R)$ is cyclic; we claim that such a situation forces 2 to lie in $\pi(K)$. In fact, assuming the contrary and setting $C=\mathbf{C}_L(W)$, we see that $\mathbf{F}(KC/C)$ is contained in the 2-complement of $\mathbf{F}(R)$ and it is therefore cyclic. It follows that the factor group of KC/C over its Fitting subgroup (which embeds in the automorphism group of a cyclic group of odd order) is cyclic as well, so that KC/C is metabelian. But this contradicts the fact that V is not of type 1, because K now embeds in the direct product of groups isomorphic to KC/C and it is then metabelian as well. This contradiction yields $2\in\pi(H:I)$, hence we have $\rho(G)=\pi(\chi)\cup\{3\}$ for any $\chi\in\mathrm{Irr}(G|\lambda)$.

Finally, if $|H|$ is odd, then $\pi(\overline{H}:\overline{H}_\Delta)=\pi(\overline{H})$ and $\pi(R:\mathbf{C}_R(\mu))=\pi(R)$, so we get $\rho(G)=\pi(\chi)$ for any $\chi\in\mathrm{Irr}(G|\lambda)$.

It remains to consider the case when V is not irreducible, so, we assume $n\geq 2$. Suppose first that one of the modules V_i, say V_1, is not of type 1 as an $H/\mathbf{C}_H(V_1)$-module. Let L be a complement of V_1 in G and $C=\mathbf{C}_L(V_1)\trianglelefteq G$. Write $U=\mathbf{F}(C)=\mathbf{F}(G)\cap C$. Applying to the L/C-module V_1 (not of type 1) the discussion carried out above in the irreducible case, we can find $\lambda_0\in V_1$ such that, for all $\theta\in\mathrm{Irr}(G|\lambda_0\times 1_C)$, we have $\pi(L/C)\subseteq\pi(\theta)\cup\{p_0\}$ for a suitable $p_0\in\{2,3\}$. Furthermore, if $|L/C|$ is odd, then $\pi(L/C)=\pi(\theta)$.

Since $\mathbf{\Phi}(C)=1$, by inductive hypothesis there exist $\alpha,\beta\in\mathrm{Irr}(C)$ such that $\pi(C/U)=\rho(C)\subseteq\pi(\alpha)\cup\pi(\beta)\cup\{p_1\}$, for a suitable $p_1\in\{2,3\}$; and $\pi(C/U)=\pi(\alpha)\cup\pi(\beta)$ if $|C/U|$ is odd. Let now $\chi\in\mathrm{Irr}(G|\lambda_0\times\alpha)$ and $\psi\in\mathrm{Irr}(G|\lambda_0\times\beta)$. Observe that $\rho(G)=\pi(L/C)\cup\pi(C/U)$, that if $\pi(L/C)-\pi(\theta)=\{3\}$ then $2\in\pi(L/C)$ and if $\pi(C/U)-(\pi(\alpha)\cup\pi(\beta))=\{3\}$ then $2\in\pi(C/U)$. Hence, setting $p=\max\{p_0,p_1\}$, we have $\rho(G)\subseteq\pi(\chi)\cup\pi(\psi)\cup\{p\}$ with $p\in\{2,3\}$. Moreover, if $|G/\mathbf{F}(G)|=|L/U|$ is odd, then $\rho(G)\subseteq\pi(\chi)\cup\pi(\psi)$.

Therefore, we can assume that each of the V_i is an irreducible $H/\mathbf{C}_H(V_i)$-module of type 1. Write $V_i=(W_i)^H$, where W_i is a primitive L_i-module for a subgroup L_i of H and, setting $K_i=(L_i)_H$, the factor group $K_i/\mathbf{C}_H(V_i)$ is metabelian. Let $K=\bigcap_i K_i$. Then $\overline{H}=H/K$ is a permutation group on the set $\Omega=\bigcup_i\Omega_i$, where Ω_i is the set consisting of the conjugates of the module W_i by the action of H (so the sets Ω_i are the orbits of $\overline{H}$ on Ω). By Proposition 4.3, we can choose a (non-empty) subset $\Delta\subseteq\Omega$, such that $\pi(\overline{H})\subseteq\pi(\overline{H}:\overline{H}_\Delta)\cup\{p\}$, where $p\in\{2,3\}$, and $\pi(\overline{H})\subseteq\pi(\overline{H}:\overline{H}_\Delta)$, if $|\overline{H}|$ is odd. We can also clearly assume that Δ a has non-empty intersection with every orbit Ω_i.

For every $W\in\Omega$, we now choose a $\mu_W\in W$ such that $\mu_W\neq 1_W$ if and only if $W\in\Delta$. We define $\lambda=\prod_{W\in\Omega}\mu_W\in V$ and $I=\mathbf{C}_H(\lambda)$. Arguing as in the irreducible case of type 1, we conclude that, for every $\chi\in\mathrm{Irr}(G|\lambda)$, $\pi(\chi)\supseteq(\pi(H/K)\cup\pi(\mathbf{F}(K)))-\{p\}$, and $\pi(\chi)\supseteq(\pi(H/K)\cup\pi(\mathbf{F}(K)))$ if $|H/K|$ is odd. Since K is metabelian (as every group $K_i/\mathbf{C}_H(V_i)$ is metabelian and $\bigcap_i\mathbf{C}_H(V_i)=1$), by Lemma 4.2

there exists $\theta \in \mathrm{Irr}(K)$ such that $\pi(K/\mathbf{F}(K))) \subseteq \pi(\theta)$. So, taking a character $\psi \in \mathrm{Irr}(H|\theta)$ and viewing ψ as an irreducible character of G by inflation, we get that $\rho(G) = \pi(H) \subseteq \pi(\chi) \cup \pi(\psi) \cup \{p\}$, where $p \in \{2,3\}$, and that $\rho(G) = \pi(\chi) \cup \pi(\psi)$ if $|H|$ is odd. ∎

We are ready to prove a result that will imply Theorem B.

Theorem 4.5. *If G is a solvable group, then there exist $\beta_1, \beta_2, \beta_3 \in \mathrm{Irr}(G)$ such that*

$$|\rho(G)| \leq |\pi(\beta_1)| + |\pi(\beta_2)| + |\pi(\beta_3)|$$

and

$$\rho(G) = \pi(\beta_1) \cup \pi(\beta_2) \cup \pi(\beta_3) \cup \{p\}$$

for a suitable $p \in \{2,3\}$.

Proof. Set $\overline{G} = G/\mathbf{\Phi}(G)$; by Proposition 4.4 there exist (by inflation) $\beta_1, \beta_2 \in \mathrm{Irr}(G)$ and a suitable $p \in \{2,3\}$ such that, setting $\pi_i = \pi(\beta_i)$ for $i = 1,2$, we have $\pi(G/\mathbf{F}(G)) = \rho(\overline{G}) \subseteq \pi_1 \cup \pi_2 \cup \{p\}$. Define also ν as the set of primes in $\pi(G)$ for which G has a normal non-abelian Sylow subgroup, so that $\rho(G) = \nu \cup \rho(\overline{G})$, and let N be a Hall ν-subgroup of G. Now, N is a normal subgroup of G and there exists $\varphi \in \mathrm{Irr}(N)$ such that $\pi(\varphi) = \nu$. Clearly $|\pi_1|, |\pi_2|, |\nu| \leq \sigma(G)$ and

$$\rho(G) = \nu \cup \rho(\overline{G}) \subseteq \nu \cup \pi_1 \cup \pi_2 \cup \{p\}\ .$$

If $|\pi_1 \cap \pi_2| \geq 1$, then $|\pi_1 \cup \pi_2 \cup \{p\}| \leq |\pi_1 \cup \pi_2| + 1 \leq |\pi_1| + |\pi_2|$ and hence

$$|\rho(G)| \leq |\nu| + |\pi_1| + |\pi_2| \leq 3\max\{|\nu|, |\pi_1|, |\pi_2|\} \leq 3\sigma(G)\ .$$

Hence we can assume that π_1 and π_2 are disjoint sets. By a similar argument, we can assume $\nu \cap (\pi_1 \cup \pi_2 \cup \{p\}) = \emptyset$.

Assume now that, for every $\chi \in \mathrm{Irr}(G|\varphi)$ (so, $\nu \subseteq \pi(\chi)$), we have $\pi(\chi) = \nu$. Since $(|N|, |G/N|) = 1$, we have that φ is G-invariant (by Clifford correspondence) and there exists an extension $\theta \in \mathrm{Irr}(G)$ of φ. Thus Gallagher theorem yields that G/N is abelian and result easily follows by Lemma 4.2.

Therefore we can assume that there exists $\beta_3 \in \mathrm{Irr}(G|\varphi)$ such that ν is a proper subset of $\pi(\beta_3)$. Hence, $\pi(\beta_3) \cap (\pi_1 \cup \pi_2 \cup \{p\}) \neq \emptyset$ and then, arguing along the same line as above, we conclude that $|\rho(G)| \leq |\pi(\beta_1)| + |\pi(\beta_2)| + |\pi(\beta_3)|$. ∎

From the first inequality Theorem 4.5 we hence deduce the following corollary, which is Theorem B of the Introduction.

Corollary 4.6. *If G is solvable group, then $|\rho(G)| \leq 3\sigma(G)$.*

5. A generalization of the solvable case

We conclude our discussion by proving Theorem C, which extends the inequality $\rho(G) \leq 3\sigma(G)$ to all groups G such that $\mathbf{C}_G(\mathbf{E}(G))$ is solvable. As mentioned in the Introduction, this class of groups includes both the solvable groups and the groups with trivial Fitting subgroup.

First, we need an application of Theorem 3.1 which takes also into account Proposition 2.3. As Theorem 3.1, also the following result is clearly related to [4, Proposition 4] and [18, Proposition 3.2].

Theorem 5.1. *Let G be a group with $\mathbf{F}(G) = 1$. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Irr}(\mathbf{F}^*(G))$ (not necessarily distinct) such that, setting $m = \mathrm{m}(G/\mathbf{F}^*(G))$, the following conclusions hold.*

(a) $\pi_{\geq m}(G) \subseteq \bigcup_{i=1}^{2} \big(\pi(\alpha_i) \cup \pi(G : I_G(\alpha_i))\big) \cup \pi(\alpha_3)$.
(b) $|\mathbb{P}_m| \leq 2|\pi(G : I_G(\alpha_i)) \cap \mathbb{P}_m|$ *for $i = 1,2$.*

Proof. Let us denote by M the subgroup generated by all the simple characteristic subgroups of G, so that $M = S_1 \times \cdots \times S_t$ where the S_j are non-abelian simple groups, and set $C = \mathbf{C}_G(M)$. Since $\mathbf{F}(C) = 1$ and clearly C does not have any simple characteristic subgroup, we can apply Theorem 3.1 to the group C: defining $m_0 = \mathrm{m}(C/\mathbf{F}^*(C))$, there exist two irreducible characters $\psi_1, \psi_2 \in \mathrm{Irr}(\mathbf{F}^*(C))$ such that

$$\pi_{\geq m_0}(C) \subseteq \bigcup_{i=1}^{2} \pi(\psi_i) \cup \pi(C : I_C(\psi_i)) \quad \text{and} \quad |\mathbb{P}_{m_0}| \leq 2|\pi(C : I_C(\psi_i)) \cap \mathbb{P}_{m_0}|$$

for $i = 1, 2$.

Next we use Proposition 2.3 to get, for all $j \in \{1, \ldots, t\}$, three irreducible characters $\chi_1^{(j)}, \chi_2^{(j)}, \xi^{(j)}$ of S_j such that

$$\pi(S_j) \subseteq \pi(\chi_1^{(j)}) \cup \pi(\chi_2^{(j)}) \cup \pi(\xi^{(j)}) \quad \text{and} \quad \pi(G/\mathbf{C}_G(S_j)) - \pi(S_j) \subseteq \pi(G : I_G(\chi_i^{(j)})),\ i = 1, 2.$$

Consider now the irreducible characters $\chi_i = \chi_i^{(1)} \times \cdots \times \chi_i^{(t)}$ $(i \in \{1, 2\})$ and $\xi = \xi^{(1)} \times \cdots \times \xi^{(t)}$ of M; then, observing that $\mathbf{F}^*(G) = M \times \mathbf{F}^*(C)$, define $\alpha_1 = \chi_1 \times \psi_1$, $\alpha_2 = \chi_2 \times \psi_2$, $\alpha_3 = \xi \times 1_{\mathbf{F}^*(C)}$. It is routine to check that $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Irr}(\mathbf{F}^*(G))$ satisfy (a); moreover, taking into account that either $m = m_0$ or $\mathbb{P}_{m_0}$ is contained in $\pi(G/MC)$, the characters α_1 and α_2 are easily verified to satisfy (b) as well. ■

Finally, we prove Theorem C.

Theorem C. Let G be a group such that $\mathbf{C}_G(\mathbf{E}(G))$ is solvable. Then $|\rho(G)| \leq 3\sigma(G)$.

Proof. Set $E = \mathbf{E}(G)$ and $F = \mathbf{F}(G)$. We know (see [12, 6.5.2]) that every solvable normal subgroup of G centralizes E. Hence, by assumption we have $\mathbf{C}_G(E) = R$, where R is the solvable radical (i.e. the largest solvable normal subgroup) of G. Let $Z = E \cap F = \mathbf{Z}(E)$.

Our first claim is that $\rho(G/Z) = \rho(G)$. So, take $p \in \rho(G)$ and denote by P a Sylow p-subgroup of G. We have $Z \leq \mathbf{Z}(E) \cap E' \leq \mathbf{\Phi}(E)$, so $Z \leq \mathbf{\Phi}(G)$ and hence $F/Z = \mathbf{F}(G/Z)$; it follows that if P is not normal in G, then PZ/Z is not normal in G/Z and hence p lies in $\rho(G/Z)$. On the other hand, if P is normal in G (and non-abelian), then $E \cap P$ is contained in $E \cap F = Z$, whence p does not divide $|E/Z|$. As a consequence, p does not divide the order of the Schur multiplier M of E/Z and, in particular, $p \nmid |Z|$ (in fact, note that E is a central extension of E/Z whose kernel Z lies in E', so Z embeds in M by [15, Corollary 11.20(b)]). We conclude that $PZ/Z \simeq P$ is non-abelian, so $p \in \rho(G/Z)$.

Furthermore, by Lemma 3.2 of [5] we have $E/Z \subseteq \mathbf{E}(G/Z)$ (it can be easily checked that actually equality holds) and $\mathbf{C}_{G/Z}(E/Z) = R/Z$; it follows that $\mathbf{C}_{G/Z}(\mathbf{E}(G/Z)) \subseteq R/Z$ is solvable. Our conclusion after the last two paragraphs is that we can assume $Z = 1$.

Note that now $Z = E \cap R = 1$, so $ER = E \times R$. An application of Theorem 5.1 to the factor group G/R (whose Fitting subgroup is clearly trivial, and whose generalized Fitting subgroup is ER/R) yields the existence of $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Irr}(E)$ such that, setting $\pi_i = \pi(\alpha_i) \cup \pi(G : I_G(\alpha_i))$ for $i \in \{1, 2\}$ and $\pi_3 = \pi(\alpha_3)$, we have

$$\pi_{\geq m}(G/R) \subseteq \pi_1 \cup \pi_2 \cup \pi_3 \quad \text{and} \quad |\mathbb{P}_m| \leq 2|\pi(G : I_G(\alpha_i)) \cap \mathbb{P}_m| \quad \text{for } i = 1, 2$$

where $m = \mathrm{m}(G/ER)$. Observe that, in general, the sets π_1, π_2 and π_3 do not cover together the whole $\pi(G/R)$, but they clearly do so in the case $m = 0$; at any rate, it is not difficult to see that the two displayed formulas above imply $|\pi(G/R)| \leq |\pi_1| + |\pi_2| + |\pi_3|$.

Also, consider the characters $\beta_1, \beta_2, \beta_3 \in \mathrm{Irr}(R)$ provided by an application of Theorem 4.5 to R, such that

$$\rho(R) \subseteq \pi(\beta_1) \cup \pi(\beta_2) \cup \pi(\beta_3) \cup \{p\} \quad \text{and} \quad |\rho(R)| \leq |\pi(\beta_1)| + |\pi(\beta_2)| + |\pi(\beta_3)|.$$

Note that p lies in $\mathbb{P}_m$ if $m \neq 0$, as then $m \geq 5$.

We are now ready to finish our proof. If $m \neq 0$, we take $\alpha = \alpha_i$ such that $|\pi_i| = \max_{j=1,2,3}\{|\pi_j|\}$, and $\beta = \beta_i$ such that $|\pi(\beta) - \pi(G/R)| = \max_{j=1,2,3}\{|\pi(\beta_j) - \pi(G/R)|\}$. If $m = 0$, we take $\beta = \beta_i$ such that $|\pi(\beta)| = \max_{j=1,2,3}\{|\pi(\beta_j)|\}$ and $\alpha = \alpha_i$ such that $|\pi_i - \pi(R)| = \max_{j=1,2,3}\{|\pi_j - \pi(R)|\}$. One easily sees that, for any $\chi \in \mathrm{Irr}(G|\alpha \times \beta)$, in both cases we have $|\rho(G)| \leq 3\pi(\chi) \leq 3\sigma(G)$. ∎

We conclude with a proof of Corollary D, stating it again for convenience.

Corollary D. Let G be a group. Then $|\rho(G)| \leq 5\sigma(G) + 1$ if $\sigma(G) < 6$, and $|\rho(G)| \leq 6\sigma(G) - 4$ otherwise.

Proof. Let R be the solvable radical of G. Observe that we have $\rho(G/R) = \pi(G/R)$, and $\rho(G) = \rho(R) \cup \rho(G/R)$; also, we have $\max\{\sigma(R), \sigma(G/R)\} \leq \sigma(G)$. Since $\mathbf{F}(G/R) = 1$, the result follows at once by Theorem A and Theorem B. ∎

Acknowledgment. The authors wish to express their gratitude to the referee for several valuable comments that improved the exposition of the material in this paper, in particular for pointing out some important references and for suggesting to add Corollary D.

Some part of this research has been carried out during a visit of the first author at the Dipartimento di Matematica e Informatica "U. Dini" (DIMAI) of the University of Firenze. She wishes to thank the DIMAI for the hospitality.

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