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1 Introduction

Throughout this note G will be a finite group and $\mathrm{cd}(G)$ will be the set of degrees of irreducible characters of G. A theorem of Thompson says that if every degree in $\mathrm{cd}(G) - \{1\}$ is divisible by some prime p, then G has a normal p-complement (see Corollary 12.2 of [5] or Theorem 23.3 of [4]). In [2], Berkovich showed that more can be said in this situation. In particular, he proved that if p divides every degree in $\mathrm{cd}(G) - \{1\}$, then G is solvable. In this paper, we will generalize Berkovich's result.

We define $\Gamma(G)$ to be the graph whose vertex set is $\mathrm{cd}(G) - \{1\}$. There is an edge between a and b if $(a, b) > 1$. This is the common-divisor character degree graph. If p divides every degree in $\mathrm{cd}(G) - \{1\}$, then $\Gamma(G)$ is a complete graph. We will show that Berkovich's conclusion can be obtained by assuming only that $\Gamma(G)$ is a complete graph.

Main Theorem. *If $\Gamma(G)$ is a complete graph, then G is a solvable group.*

This result illustrates one difference between the character degree sets of solvable and nonsolvable groups since there do exist solvable groups G with $\Gamma(G)$ a complete graph. First, if G is a

p-group, then $\Gamma(G)$ is a complete graph. Let p, q, and r be distinct primes. In Theorem B of [12], Turull constructs for every p-group A of order p^l, a finite $\{q, r\}$-group H and an action of A on H so that $\mathbf{C}_H(A) = 1$ and the Fitting height of H is l. Taking G to be the resulting semi-direct product, it follows from Glauberman correspondence, Theorem 13.1 of [5] or Theorem 18.15 of [4], that the principal character of H is the only G-invariant character in $\mathrm{Irr}(H)$, and it follows that p divides every degree in $\mathrm{cd}(G)$. Finally, the third author along with Moretó and Wolf has constructed in [7] for every pair of odd primes (p, q) where p is congruent to 1 modulo 3 and q divides $p+1$ a solvable group G with $\mathrm{cd}(G) = \{1, 3q, p^2q, 3p^3\}$. Taking direct products of G with itself we can get groups with many character degrees that have $\Gamma(G)$ a complete graph, but no prime divides every degree.

To understand the situation when $\Gamma(G)$ is a complete graph and G is a solvable group there seem to be two cases that need to be studied. The first is the case where some prime p divides every degree in $\mathrm{cd}(G) - \{1\}$, and the second is the case where no such prime p exists. We have seen that if G is a group from the first case, then there is no bound on either the derived length or the Fitting height of G. On the other hand, Berkovich does obtain some additional information regarding the structure of G in [2]. In the case where there is no prime p dividing all the degrees in $\mathrm{cd}(G) - \{1\}$, we have little information since we have few examples.

When looking at the literature, there have been few results about the graph $\Gamma(G)$, and the only result we know of regarding $\Gamma(G)$ when G is nonsolvable is due to McVey. In [9], he proves that if G is a nonsolvable group, and $\Gamma(G)$ is connected, then $\Gamma(G)$ has diameter at most 3. Notice that our result says in this situation that $\Gamma(G)$ has diameter at least 2. Examples show that both diameters occur among nonsolvable groups.

In the literature, usually another graph $\Delta(G)$ has been attached to $\mathrm{cd}(G)$. This graph takes the primes dividing degrees in $\mathrm{cd}(G)$ to be its vertices, and there is an edge between p and q if pq divides some degree $a \in \mathrm{cd}(G)$. It is not surprising that these two graphs are closely related. It is not difficult to show that one is connected if and only if the other is, and in this case that their diameters differ by at most 1 (see [8]). Recently, it has been shown that $\Delta(G)$ is a complete graph for most simple groups (see [13], [14], and [15]), which perhaps makes it surprising that $\Gamma(G)$ is never a complete graph for a nonsolvable group G.

Using tensor-induction the theorem is reduced to the case where G is almost simple. If G is a group of Lie-type, the Steinberg character has prime power degree. With the Steinberg character in hand, Thompson's result implies the theorem. The alternating and sporadic cases are handled by explicitly listing two characters of relatively prime degree.

2 Results

Before we actually prove the Main Theorem, we gather some facts. Our proof will be based on the classification of finite simple groups. The main point of the classification is that if S is a nonabelian simple group, then S is either a simple group of Lie type, an alternating group, one of 26 sporadic simple groups, or the Tits group. We gather information on each of these types of simple groups.

We begin with the simple groups of Lie type. For these groups, we look at a particular character called the Steinberg character. The Steinberg character and its properties can be found in many places, we will use Chapter 6 of [3] for our reference. For us, the important property of the Steinberg character is that its degree is a prime power, which is the following result found as Theorem 6.4.7 of [3].

Theorem 1. *Let G be a nonabelian simple group of Lie type with defining characteristic p. If χ is the Steinberg character for G, then $\chi(1) = |G|_p$.*

We also need the following result regarding the Steinberg character which was proved in [10] and [11].

Theorem 2 (Schmid)**.** *Let N be a normal subgroup of a group G, and suppose that N is isomorphic to a finite simple group of Lie type. If θ is the Steinberg character for N, then θ extends to G.*

We next consider the alternating groups. We make the observation that $(n-1)(n-2)/2 = (n^2-3n+2)/2 = (n^2-3n)/2 + 2/2 = n(n-3)/2 + 1$, and so, the numbers $n(n-3)/2$ and $(n-1)(n-2)/2$ are relatively prime.

Theorem 3. *If $n \geq 6$, then $\mathrm{Irr}(\mathrm{Alt}(n))$ contains characters of degree $n(n-3)/2$ and $(n-1)(n-2)/2$ that extend to $\mathrm{Sym}(n)$.*

Proof. Taking into account that $\mathrm{Alt}(n)$ is 4-transitive for n greater than 5, we can define two irreducible characters σ and τ for $\mathrm{Alt}(n)$, of degree $n(n-3)/2$ and $(n-1)(n-2)/2$ respectively, as in Theorem 11.9 of [4]. It is clear by the definition that both these characters allow an extension to $\mathrm{Sym}(n)$. □

For the sporadic groups, the following fact can be found in the atlas, [1]. Often the Tits group, which we denote by ${}^2F_2(4)'$, is lumped in with the groups of Lie type, and the arguments we use for the groups of Lie type could be adapted for the Tits group. Since the Tits group is in the atlas, it is easier to include it with the sporadic groups. Table 1 will list the atlas characters that we need.

Theorem 4. *Let S be a sporadic simple group or the Tits group, and let A be the automorphism group of S. Then there exist nonlinear characters $\chi_m, \chi_n \in \mathrm{Irr}(S)$ so that $(\chi_m(1), \chi_n(1)) = 1$ and both χ_m and χ_n extend to A.*

For the last of the preliminary facts, we will need tensor induction of characters. There are several sources for tensor induction. We will be using [6]. Let H be a subgroup of a group G, and let T be a right transversal for H in G. If $g \in G$ and $t \in T$, then define $t \cdot g$ to be the element of T that lies in Htg. This defines an action of G on T. Take T_0 to be a set of orbit representatives for the action of $\langle g \rangle$ on T via $\cdot$, and let $s(t)$ denote the size of the $\langle g \rangle$-orbit containing t. If θ is a character of H, we define $\theta^{\otimes G}$ as a function of G by $\theta^{\otimes G}(g) = \prod_{t \in T_0} \theta(tg^{s(t)}t^{-1})$. It is proved in [6] that $\theta^{\otimes G}$ is a character. In addition, if N is a normal subgroup of G so that $N \subseteq H$, then it is proved in Lemma 4.1 of [6] for $n \in N$ that $\theta^{\otimes G}(n) = \prod_{t \in T} \theta(tnt^{-1})$. We note that tensor induction is also described in Theorem 25.3 of [4].

Using tensor induction, we prove the following lemma. This lemma should be compared with Lemma 25.5 of [4].

Lemma 5. *Let N be a minimal normal subgroup of G so that $N = S_1 \times \cdots \times S_t$, where $S_i \cong S$, a nonabelian simple group. Let A be the automorphism group of S. If $\sigma \in \mathrm{Irr}(S)$ extends to A, then $\sigma \times \cdots \times \sigma \in \mathrm{Irr}(N)$ extends to G.*

Proof. Let $H = \mathbf{N}_G(S_1)$ and $C = \mathbf{C}_G(S_1)$. We know that G acts transitively on $\{S_1, \ldots, S_t\}$, so $|G : H| = t$. Take T to be a right transversal for H in G, and label the elements in $T = \{x_1, \ldots, x_t\}$

so that $S_1^{x_i} = S_i$. Let $\sigma_i \in \mathrm{Irr}(N)$ be the character whose ith component is σ, and the other components are 1_S. Observe that $\sigma_1^{x_i} = \sigma_i$. Also, $\prod_{i=1}^t \sigma_i = \sigma \times \cdots \times \sigma \in \mathrm{Irr}(N)$.

We know that H/C is isomorphic to a subgroup of the automorphism group of S_1, and so, H/C is isomorphic to a subgroup of A. Also, we know that $S_1 \cap C = \mathbf{Z}(S_1) = 1$, so $S_1C = S_1 \times C \subseteq H$. Observe that $N \cap C = S_2 \times \cdots \times S_t$, so σ_1 extends to $\sigma \times 1_C \in \mathrm{Irr}(S_1C)$. Now, we know that $\sigma \times 1_C$ viewed as a character of $S_1C/C \cong S_1$ extends to the automorphism group of S_1, so $\sigma \times 1_C$ extends to $\theta \in \mathrm{Irr}(H/C)$. We let $\chi = \theta^{\otimes G}$.

We will show that χ is an extension of $\sigma \times \cdots \times \sigma$. Given an element $n \in N$, we have

$$\chi(n) = \theta^{\otimes G}(n) = \prod_{i=1}^t \theta(x_i n x_i^{-1}) = \prod_{i=1}^t \sigma_1(x_i n x_i^{-1}) = \prod_{i=1}^t \sigma_1^{x_i}(n) = \prod_{i=1}^t \sigma_i(n) = (\sigma \times \cdots \times \sigma)(n).$$

It follows that $\chi_N = \sigma \times \cdots \times \sigma$ as desired. □

We now are ready to prove the main theorem.

Proof of Main Theorem. We will prove the contrapositive. In other words, we assume that G is not solvable, and we show that $\Gamma(G)$ is not a complete graph. Since G is not solvable, we can find normal subgroups M and N in G so that N/M is a nonabelian chief factor for G. Given elements $a, b \in \mathrm{cd}(G/M)$, it is easy to see that a and b are adjacent in $\Gamma(G/M)$ if and only if they are adjacent in $\Gamma(G)$. Thus, it suffices to show that $\Gamma(G/M)$ is not a complete graph, and so, we may assume that $M = 1$.

Now, N is a nonabelian minimal normal subgroup of G, so $N = S_1 \times \cdots \times S_t$ where $S_i \cong S$ for some nonabelian simple group S. First, suppose that S is simple group of Lie type with characteristic p for some prime p. Take σ to be the Steinberg character of S so $\sigma(1)$ is a power of p. In light of Theorem 2, we can use Lemma 5 to see that $(\sigma(1))^t$ lies in $\mathrm{cd}(G)$. Since G does not have a normal p-complement, we can use Thompson's theorem (Corollary 12.2 of [5]) to see that $\mathrm{cd}(G) - \{1\}$ must have a degree a that is not divisible by p. Hence, $(\sigma(1))^t$ and a are relatively prime, so $\Gamma(G)$ is not a complete graph.

Suppose that S is an alternating group on n items with $n \geq 7$ or S is a sporadic simple group or the Tits group. (Note that $\mathrm{Alt}(5) \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$ and $\mathrm{Alt}(6) \cong \mathrm{PSL}_2(9)$, so these two cases have already been handled.) For $n \geq 7$, we know that the automorphism group of $\mathrm{Alt}(n)$ is $\mathrm{Sym}(n)$. Let A be the automorphism group of S. We use Theorem 3 or Theorem 4 to find nonlinear characters $\sigma, \tau \in \mathrm{Irr}(S)$ so that $(\sigma(1), \tau(1)) = 1$ and σ and τ both extend to A. Thus, we may use Lemma 5 to see that $(\sigma(1))^t$ and $(\tau(1))^t$ both lie in $\mathrm{cd}(G)$. Obviously, these degrees are relatively prime, so $\Gamma(G)$ is not a complete graph. □

Table 1: Degrees of Sporadic Groups and the Tits Group

Group	Chars.	Degrees
M_{11}	χ_2	$10 = 2 \cdot 5$
	χ_5	11
M_{12}	χ_7	$54 = 2 \cdot 3^3$
	χ_8	$55 = 5 \cdot 11$
J_1	χ_4	$76 = 2^2 \cdot 19$
	χ_6	$77 = 7 \cdot 11$
M_{22}	χ_2	$21 = 3 \cdot 7$
	χ_5	$55 = 5 \cdot 11$
J_2	χ_6	$36 = 2^2 \cdot 3^2$
	χ_{13}	$175 = 5^2 \cdot 7$
M_{23}	χ_2	$22 = 2 \cdot 11$
	χ_3	$45 = 3^2 \cdot 5$
${}^2F_4(2)'$	χ_4	$27 = 3^3$
	χ_8	$325 = 5^2 \cdot 13$
HS	χ_2	$22 = 2 \cdot 11$
	χ_7	$175 = 5^2 \cdot 7$
J_3	χ_6	$324 = 2^3 \cdot 3^4$
	χ_{13}	$1615 = 5 \cdot 17 \cdot 19$
M_{24}	χ_2	23
	χ_3	$45 = 3^2 \cdot 5$
M^cL	χ_2	$22 = 2 \cdot 11$
	χ_{14}	$5103 = 3^6 \cdot 7$
He	χ_9	$1275 = 3 \cdot 5^2 \cdot 17$
	χ_{15}	$6272 = 2^7 \cdot 7^2$
Ru	χ_5	$783 = 3^3 \cdot 29$
	χ_{20}	$45500 = 2^2 \cdot 5^3 \cdot 7 \cdot 13$
Suz	χ_2	$143 = 11 \cdot 13$
	χ_{43}	$248832 = 2^{10} \cdot 3^5$
$O'N$	χ_2	$10944 = 2^6 \cdot 3^2 \cdot 19$
	χ_{19}	$116963 = 7^3 \cdot 11 \cdot 31$
Co_3	χ_2	23
	χ_5	$275 = 5^2 \cdot 11$
Co_2	χ_2	23
	χ_4	$275 = 5^2 \cdot 11$
Fi_{22}	χ_{56}	$1441792 = 2^{17} \cdot 11$
	χ_{57}	$1791153 = 3^9 \cdot 7 \cdot 13$
HN	χ_{10}	$16929 = 3^4 \cdot 11 \cdot 19$
	χ_{45}	$3200000 = 2^{10} \cdot 5^5$
Ly	χ_7	$120064 = 2^8 \cdot 7 \cdot 67$
	χ_{50}	$53765625 = 3 \cdot 5^6 \cdot 31 \cdot 37$
Th	χ_2	$248 = 2^3 \cdot 31$
	χ_7	$30875 = 5^3 \cdot 13 \cdot 19$
Fi_{23}	χ_4	$5083 = 13 \cdot 17 \cdot 23$
	χ_{94}	$504627200 = 2^{18} \cdot 5^2 \cdot 7 \cdot 11$
Co_1	χ_3	$299 = 13 \cdot 23$
	χ_{17}	$673750 = 2 \cdot 5^4 \cdot 7^2 \cdot 11$
J_4	χ_2	$1333 = 31 \cdot 43$
	χ_{11}	$1776888 = 2^3 \cdot 3^2 \cdot 23 \cdot 29 \cdot 37$
Fi_{24}'	χ_2	$8671 = 23 \cdot 29 \cdot 13$
	χ_6	$1603525 = 5^2 \cdot 7^3 \cdot 11 \cdot 17$
B	χ_2	$4371 = 3 \cdot 31 \cdot 47$
	χ_{119}	$2642676197359616 = 2^{39} \cdot 11 \cdot 19 \cdot 23$
M	χ_2	$196883 = 59 \cdot 71 \cdot 47$
	χ_{16}	$8980616927734375 = 5^9 \cdot 7^6 \cdot 11^2 \cdot 17 \cdot 19$

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