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Remarks on a class of generalized Liénard planar systems

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"To our friend Pierpaolo Omari on the occasion of his 65th birthday"

Abstract. *We continue the recent investigation [40] about the qualitative properties of the solutions for a class of generalized Liénard systems of the form $\dot{x} = y - F(x, y)$, $\dot{y} = -g(x)$. We present some results on the existence/non-existence of limit cycles depending on different growth assumptions of $F(\cdot, y)$. The case of asymmetric conditions at infinity for $g(x)$ and $F(x, \cdot)$ is also examined. In the second part of the article we consider also a bifurcation result for small limit cycles as well as we discuss the complex dynamics associated to a periodically perturbed reversible system.*

Keywords: Generalized Liénard equations, limit cycles, qualitative theory of planar dynamical systems, averaging and bifurcation.
MS Classification 2020: 34C25, 34C07, 34C23, 34C29, 34C60.

1. Introduction and "state of art"

In the present work we continue the investigation initiated in the recent article [40], dealing with a new class of planar dynamical systems of the form

$$(S) \qquad \begin{cases} \dot{x} = y - F(x, y) \\ \dot{y} = -g(x). \end{cases}$$

For this reason, we believe that it may be appropriate to recall some results recently appeared in [40], because, as far as we know, this is the first case in which this general class of systems was investigated.

For convenience, throughout the paper, we suppose that $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions, in order to guarantee the uniqueness of the solutions for the associated initial value problems. As it is well known, for some specific forms of g and F weaker regularity conditions may be assumed (see, for instance, [1, 32]). We also assume

$$(g0) \qquad g(0) = 0, \ g(x)x > 0 \ \text{ for } \ x \neq 0$$

and

$$(F0) \qquad F_0(y) := y - F(0,y), \text{ vanishes only at } y = 0,$$

therefore the origin is the only singular point of (S) and, moreover, the trajectories move upward in the half-plane with $x < 0$ and downward in the half-plane with $x > 0$. When $F \equiv 0$, we have the planar system associated to the Duffing equation, namely

$$(D) \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -g(x), \end{cases}$$

with is a conservative system (cf. [17, 22]) with first integral (energy) given by

$$H(x,y) = \frac{1}{2}y^2 + G(x), \qquad \text{for } G(x) = \int_0^x g(u)\,du.$$

Under assumption $(g0)$, the origin is a center for (D) and it is a global center if and only if $G(x) \to +\infty$ as $|x| \to +\infty$. Using the same energy function for system (S), we have

$$\dot{H}(x,y) = -F(x,y)g(x).$$

Therefore, the sign of $F(x,y)$ determines the direction of the trajectories with respect to the level lines of H. In the sequel, we will also consider a variant of $(g0)$, namely

$$(g1) \qquad g(0) = 0,\ g(x)x > 0 \ \text{ for } \ x \neq 0,\ \ G(x) \to +\infty \ \text{ for } \ x \to \pm\infty.$$

On the other hand, if $F(x,y) = \mathtt{F}(x)$, then system (S) reduces to

$$\begin{cases} \dot{x} = y - \mathtt{F}(x) \\ \dot{y} = -g(x), \end{cases}$$

which is the usual generalization of the classical planar system

$$(L) \qquad \begin{cases} \dot{x} = y - \lambda\mathtt{F}(x) \\ \dot{y} = -x, \end{cases}$$

introduced by A. Liénard in [23] in his pioneering work about the "oscillations entretenues" motivated by the study of the Van der Pol equation, where

$$\mathtt{F}(x) = \frac{x^3}{6} - x$$

(see also [14, 15, 26]). From this point of view, system (S) can be viewed as a further generalization of the Liénard one (L) and, with this perspective, in the recent article [40] we have initiated the investigation of (S), focusing our attention to the special case in which F splits as

$$F(x,y) = \lambda B(y)A(x), \quad \text{for} \quad \lambda > 0,$$

where $B(y) > 0$ for $y \neq 0$ and $A(x)$ satisfies the standard assumptions on $\mathtt{F}(x)$ in the classical case. In the particular, we have shown in [40] that the choice of a significant case study given by

$$B(y) = |y|^p, \quad A(x) = x^3 - x,$$

already exhibits the mean features of this problem that, as far as we know, have not been investigated before.

In such a framework the following results are proved in [40] for system

$$(H_{p,\lambda}) \qquad \begin{cases} \dot{x} = y - \lambda |y|^p (x^2 - 1)x \\ \dot{y} = -x, \end{cases}$$

with $\lambda > 0$ and $p > 0$.

Theorem 1.1. (The "sublinear" case [40, Theorem 3]) *For every $\lambda > 0$ and $0 < p < 1$, system $(H_{p,\lambda})$ has exactly one limit cycle.*

Notice that for $0 < p < 1$ the uniqueness of the solutions for the initial value problems is still guaranteed, even if the term $B(y) = |y|^p$ is not locally Lipschitz at $y = 0$.

On the other hand, for $p = 1$, we obtain:

Theorem 1.2. ([40, Theorem 1]) *The system $(H_{1,\lambda})$ has a unique limit cycle for $0 < \lambda < \lambda^* = 3\sqrt{3}/2$, while, for $\lambda > \lambda^*$ there are no limit cycles.*

The study of the case when $p > 1$ is more delicate due to different structure of the isoclines. In fact, we have $\dot{x} = 0$ in $(H_{p,\lambda})$ if $y = 0$ or $y = \left(1/(\lambda(x^3 - x))\right)^{1/(p-1)}$ $(y > 0)$, as well as $y = -\left(1/(\lambda(x - x^3))\right)^{1/(p-1)}$ $(y < 0)$. The regions

$$\mathcal{V}_1 := \left\{(x,y) : -1 < x < 0,\, y > 0,\, |y|^{p-1} > \tfrac{1}{\lambda(x^3 - x)}\right\},$$
$$\mathcal{V}_2 := \left\{(x,y) : 0 < x < 1,\, y < 0,\, |y|^{p-1} < -\tfrac{1}{\lambda(x^3 - x)}\right\},$$

which are symmetric with respect to the origin, play a crucial role in the dynamics, because they are are positively invariant and therefore all the trajectories entering such regions become unbounded in the y-component. We also define the negatively invariant regions

$$\mathcal{W}_1 := \left\{(x,y) : x > 1,\, y > 0,\, |y|^{p-1} > \tfrac{1}{\lambda(x^3 - x)}\right\},$$
$$\mathcal{W}_2 := \left\{(x,y) : x < -1,\, y < 0,\, |y|^{p-1} < -\tfrac{1}{\lambda(x^3 - x)}\right\}$$

(see Figure 1), where unboundedness in backward time occurs, as well as possible blow-up in finite negative time. We notice, however, that all the solutions are globally defined in forward time.

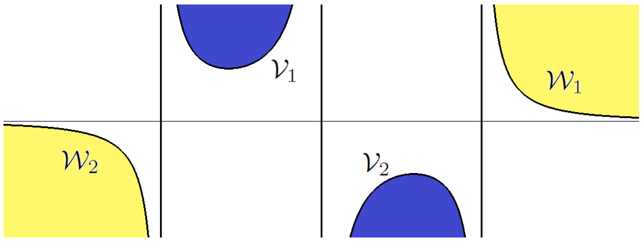


Figure 1: The infinite-isocline and the zero-isocline for system $(H_{2,\lambda})$ and the corresponding regions $\mathcal{V}_i$ and $\mathcal{W}_i$ for $i = 1, 2$. The graph is produced using [42], for a suitable $\lambda > 0$. Different values of λ gives the same qualitative structure; however, increasing the value of λ moves the regions closer to the x-axis and therefore when λ is larger than a critical value, all the nontrivial solutions definitively enter $\mathcal{V}_1 \cup \mathcal{V}_2$ and become unbounded in the future (see [40]).

In this situation, the following result holds.

Theorem 1.3. (The "superlinear" case [40, Theorem 2]) *There is $\lambda_2^* > 0$ such that for every $\lambda \in]0, \lambda_2^*[$ system $(H_{2,\lambda})$ has at least a limit cycle, while for $\lambda > \lambda_2^*$ all the nontrivial trajectories are ultimately unbounded.*

The critical constant in Theorem 1.3 is numerically estimated as

$$1.474 < \lambda_2^* < 1.475.$$

The same result holds for the systems $(H_{p,\lambda})$, for all $p > 1$, providing the existence of a corresponding critical constant λ_p^* (see [40, Remark 3]). We observe also that (as proved in [40]) Theorem 1.1 and Theorem 1.3, even if proved for the special case of systems $(H_{p,\lambda})$, hold for more general systems of the form

$$(S1_\lambda) \qquad \begin{cases} \dot{x} = y - \lambda B(y)A(x) \\ \dot{y} = -g(x), \end{cases}$$

with $B(y)$ positive for $y \neq 0$ and having sublinear/superlinear growth at infinity and $A(x)$ a typical cubic-like function as in the classical Liénard system.

The peculiar features exhibited by systems $(H_{p,\lambda})$, especially in the superlinear case, have raised the attention of other researchers in the field. In [13, Theorem 3.7] Gasull and Giacomini, using the Bendixson-Dulac approach, estimate that $(H_{p,\lambda})$, has no limit cycles for

$$\lambda \geq \frac{3}{\sqrt{2}} \left(\frac{3}{p}\right)^{p/2}, \quad (\text{with } p \in \mathbb{N},\ p \geq 2).$$

As mentioned at the beginning, in the present article, we continue the analysis of system (S) from different points of views. More in details, in Section 2 we present some preliminary results, together with necessary conditions and sufficient ones for the intersection with the ∞-isocline. Such conditions will be used in Section 3 to prove our main result, namely the existence of limit cycles, for a sub-class of (S) not considered before and including some asymmetric conditions. Section 4 is devoted to an averaging/bifurcation approach which complements the results in [40]. Finally, in Section 5 we apply a recent result in [30] to prove the presence of large subharmonic solutions and chaotic-like dynamics for periodic perturbations of system (S) when the associated autonomous system has a mirror symmetry with respect to the y-axis.

2. Preliminary results

We start with some basic facts for the general equation

$$(S) \qquad \begin{cases} \dot{x} = y - F(x,y) \\ \dot{y} = -g(x), \end{cases}$$

assuming the usual regularity conditions for the uniqueness of the solutions of the initial value problems and with $g : \mathbb{R} \to \mathbb{R}$ satisfying $(g0)$. We split this section in some parts, depending on the different class of conditions which will be assumed on $F(x, y)$.

2.1. The case when F_y has a constant sign

Our first result provides a simple criterion of nonexistence of limit cycles when F_Y has a constant sign.

PROPOSITION 2.1. *Assume $(g0)$, $F(0,0) = 0$ and let F be a continuously differentiable function with $F_y(x,y) > 1$, for all $x, y \in \mathbb{R}$. Then system (S) has no limit cycles.*

The proof is straightforward. In fact, observing that the function

$$R(x,y) := y - F(x,y)$$

has $R_y < 0$, elementary phase-plane analysis shows that the trajectories starting on the y-axis with $y > 0$ remain in the second quadrant and, symmetrically, starting with $y < 0$ remain in the fourth quadrant. This, together with $(g0)$ prevents the existence of closed orbits.

We observe that, under the additional assumption that g is strictly monotone increasing, this result is well known in a more general setting from the theory of Hirsch about monotone systems. Indeed, in this case, system (S) turns out to be *competitive* and [19, Theorem 2.3] applies.

On the other hand, in virtue of Dini's theorem, the following proposition holds.

PROPOSITION 2.2. *Assume* $(g0)$, $F(0,0)=0$ *and let* F *be a continuously differentiable function with* $F_y(x,y)<1$, *for all* $x,y\in\mathbb{R}$. *Then the infinite-isocline is the graph of a function* $y=\phi(x)$ *and the trajectories are clockwise.*

The elementary proof is omitted.

The fact that the infinite isocline is a graph allows to better study the problem of intersection with the orbits. This could be performed along the lines of [18, 36, 37]. Another way to attack this problem is given in the next sections concerning the oscillatoriness of all the solutions, because, clearly any oscillatory solution must intersects the isoclines. Moreover, from now on, we assume that

$$F_0(y)y>0, \quad \forall\, y\neq 0,$$

so that the trajectories of system (S) are clockwise. Notice that this assumption was automatically satisfied in [40] due to the special form of $F(x,y)$ considered in the above quoted article.

2.2. Oscillatory solutions

At first we observe that, from

$$\dot{H}(x,y)=-F(x,y)g(x),$$

we immediately see that, if

$$F(x,y)x<0 \ \text{ for } \ xy\neq 0 \text{ in a neighborhood of the origin,} \tag{1}$$

then the origin is a source. This assumption will be crucial in the following. In fact, the following result holds.

THEOREM 2.3. *Under assumptions* $(g1)$, $(F0)$ *and* (1), *if* $|F(x,y)|$ *is bounded, then all solutions of system* (S) *are oscillatory.*

Proof. We observe that there exists a constant $K>0$ such that $|F(x,y)|<K$. Now the proof is divided onto two steps.
If $x\geq 0$, consider the nested ovals, defined by the level lines of

$$\tilde{H}(x,y):=\frac{1}{2}(y+K)^2+G(x), \tag{2}$$

introduced by Ponzo and Wax in [31] and appeared also in [36] and [6]. It is easy to check that the level curves of $\tilde{H}$ are those of the Duffing system shifted

by $-K$ and $\dot{\tilde{H}} = -g(x)[F(x,y)+K] < 0$ for $x > 0$. Observe that, by $(g1)$, the origin of the Duffing system (D) is a global center. Hence the level lines of $\tilde{H}$ are ellipsoid-like curves centered at $(0,-K)$ and filling the plane.
Now, consider a point $P = (0, y_P)$ with $y_P > 0$ and a point $Q = (0, y_Q)$ with $y_Q < 0$ and such that P and Q are on the same level line of $\tilde{H}$. Being $\dot{\tilde{H}} < 0$ for $x > 0$ and using the fact that the origin is a source, it is easy to check that the positive semi-trajectory $\gamma^+(P)$ is guided by the level curve and intersects the y-axis at a point $(0, y_1)$ with $y_Q < y_1 < 0$. In the same way, for $x \le 0$, using the function

$$\hat{H}(x,y) := \frac{1}{2}(y-K)^2 + G(x), \tag{3}$$

where the level curves of H are those of the Duffing system shifted by $+K$, and considering two points $R = (0, y_R)$ with $y_R < 0$ and $S = (0, y_S)$ with $y_S > 0$ which belongs to the same level line of $\hat{H}$, we get that $\gamma^+(R)$ intersects the y-axis at a point $(0, y_2)$ with $0 < y_2 < y_S$. In fact, now we have

$$\dot{\hat{H}} = g(x)[K - F(x,y)] < 0 \text{ for } \ x < 0. \tag{4}$$

Therefore, all trajectories are oscillatory.
As a side remark we observe that the existence/nonexistence of limit cycles is not relevant for the proof. On the other hand, the assumption (1) cannot be avoided, otherwise we cannot exclude the presence of non-oscillatory solutions tending to/escaping from the origin, as for instance, in the case in which the origin is a stable node or a homoclinic point. □

2.3. Necessary and sufficient conditions for the intersection with the ∞-isocline

In order to attack the problem of the existence of a limit cycle, a critical step is the intersection with the ∞-isocline, namely to prove that the trajectories intersect the set

$$y = F(x,y).$$

In this light, we prove some necessary conditions for the intersection, as well as some sufficient ones. We consider in detail the case in which $x > 0$. The symmetric case $x < 0$ can be treated in the same way with slight modifications. We show that a modification of the argument introduced in [36] works also in this situation.

THEOREM 2.4. *Assume here exists a continuous function* $\phi_1 : [0, +\infty) \to \mathbb{R}$ *such that*

$$F(x,y) \le \phi_1(x), \quad \forall\, x \ge 0 \textit{ and } \forall\, y.$$

Then, a necessary condition for the intersection with the infinite isocline is that

$$\limsup_{x\to+\infty} (G(x) + \phi_1(x)) = +\infty. \tag{5}$$

Notice that in (5) the "lim sup" concerns only the function ϕ_1 as $G(+\infty)$ always exists because G is monotone increasing.

Proof. Assume, by contradiction, that $G(x)+\phi_1(x)$ is upper bounded on $x \geq 0$. Therefore, there exists two positive constants, L, K such that

$$G(x) < L, \quad \phi_1(x) < K, \quad \forall\, x \geq 0$$

and, therefore $F(x, y) < K$.

Arguing as in [36], consider again the function $\hat{H}$ defined in (3). Now we have $\dot{\hat{H}} = g(x)[K - F(x, y)] > 0$ for $x > 0$. Compare with (4) and observe that now $g(x) > 0$. Therefore, the trajectory starting from any point $(0, y_0)$ with $y_0 > K + \sqrt{2L}$ is bounded away from the line $\hat{H}(x, y) = L$, which, being above the line $y = K$, separates the trajectory from the ∞-isocline. □

In the same way, the following result holds.

Theorem 2.5. *Assume there exists a continuous function $\phi_2 : (-\infty, 0] \to \mathbb{R}$ such that*

$$F(x, y) \geq \phi_2(x), \quad \forall\, x \leq 0 \text{ and } \forall\, y.$$

Then, a necessary condition for the intersection with the infinite isocline is that

$$\limsup_{x\to-\infty} (G(x) - \phi_2(x)) = +\infty. \tag{6}$$

Now we treat the sufficient conditions and we have the following theorem for the case $x > 0$.

Theorem 2.6. *Assume*

i) $G(x) \to +\infty$ *as* $x \to +\infty$ *and let* $F(x, y)$ *be bounded below on* $x \geq 0$,

or, alternatively,

ii) $G(x) < L$ *for all* $x > 0$ *and there exists* $\phi : [0, +\infty) \to \mathbb{R}$ *continuous and such that* $F(x, y) \geq \phi(x), \forall\, x \geq 0, \forall\, y$ *with* $\limsup_{x\to+\infty} \phi(x) = +\infty$ *and* ϕ *bounded below.*

Then any trajectory starting from a point $(0, y_0)$ *with* $y_0 > 0$ *intersects the* ∞*-isocline.*

Proof. The first case was already treated in Theorem 2.3 of Section 2.2. We just observe that the proof of Theorem 2.3 clearly works the same if we assume $F(x,y)$ bounded below for $x \geq 0$ and bounded above for $x \leq 0$.

For the second case, we need to develop a more detailed analysis of the vector field of system (S). Arguing as in Section 2.2 we introduce the function $\tilde{H}$ defined in (2), where K is a lower bound for ϕ. Note that $\dot{\tilde{H}}(x,y) < 0$, because $g(x) > 0$. Therefore, trajectories of system (S) enters the curve $\tilde{H}(x,y) = L$ (see [36] and also [6]). Figure 2 shows the situation.

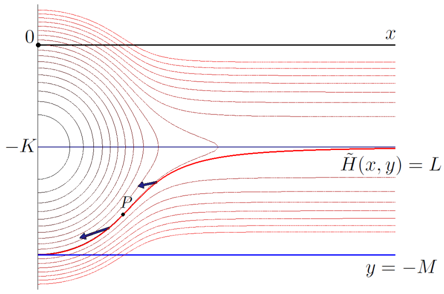


Figure 2: The figure represents the level lines of $\tilde{H}$ with $K = 1$ and $g(x) = \frac{x}{1+x^8}$, with $L = \frac{\pi\sqrt{2}}{8}$. The arrows of the vector field indicate that a trajectory of system (S) departing from a point P on the level line of $\tilde{H}$ passing through $(0, -M)$ (with $M = K + \sqrt{2L}$), enters the region $\tilde{H} < L$ and is bounded away from the x-axis in the backward time. The graphs are produced using [42].

Consider a point $(0, y_0)$ with $y_0 > 0$ and therefore above the ∞-isocline. The slope of the trajectory, namely $y' = \frac{-g(x)}{y - F(x,y)}$ is negative; hence, if the trajectory does not intersects the ∞-isocline, then it is the graph of a decreasing function, bounded from below by the line $y = -M := -K - \sqrt{2L}$ and hence it must have a horizontal asymptote. This is not possible because if we compare with the solution of the auxiliary system

$$\dot{x} = y - \phi(x), \quad \dot{y} = -g(x),$$

we have that, according to [36], if $\limsup_{x \to +\infty} \phi(x) = +\infty$, then the trajectories of such system starting from $(0, y)$ with $y > 0$ intersects the corresponding ∞-isocline, namely $y = \phi(x)$ and thus the possibility of a horizontal asymptote is prevented. Being $F(x,y) \geq \phi(x)$, a comparison of the respective slopes shows that the trajectories of system (S) are guided by the ones of the auxiliary system and this concludes the proof. □

In a similar manner, we have also the following result.

Theorem 2.7. *Assume*

i) $G(x) \to +\infty$ *as* $x \to -\infty$ *and let* $F(x,y)$ *be bounded above on* $x \le 0$,

or, alternatively,

ii) $G(x) < L$ *for all* $x < 0$ *and there exists* $\phi : (-\infty, 0] \to \mathbb{R}$ *continuous and such that* $F(x,y) \le \phi(x), \forall\, x \le 0, \forall\, y$ *with* $\liminf_{x\to-\infty} \phi(x) = -\infty$ *and* ϕ *bounded above.*

Then any trajectory starting from a point $(0, y_0)$ *with* $y_0 < 0$ *intersects the* ∞*-isocline.*

3. The non-symmetric case. Existence of limit cycles

The results proved in the previous sections allow us to obtain theorems on the existence of limit cycles for system (S). In this light we also consider, like in [40], the special case in which F splits as $F(x,y) = B(y)A(x)$, namely

$$(S1) \qquad \begin{cases} \dot{x} = y - B(y)A(x) \\ \dot{y} = -g(x), \end{cases}$$

with g satisfying $(g0)$ and $A(0) = 0$. An example in this context will be presented in the sequel. On the other hand, differently than in [40], where the case $G(x) \to +\infty$ for $x \to \pm\infty$ and $F(x,y)$ of positive definite sign outside a vertical strip was analyzed, here we investigate a situation in which a certain degree of asymmetry is allowed in the functions G and F.

With this respect, we preliminarily observe that if $G(x) \to +\infty$ as $x \to \pm\infty$, then, as proved in the preceding section, a condition of the form

$$F(x,y) > -K \;\text{ for }\; x > 0 \;\text{ and }\; F(x,y) < K \;\text{ for }\; x < 0,$$

is sufficient to have the large solutions to wind/unwind around the origin. This however is not enough to guarantee the existence of limit cycles, but only to have the oscillatoriness of the solutions. On the other hand, if we do not assume the condition of divergence of $G(x)$ from both sides, the situation is completely different. For sake of simplicity, we consider the case of $G(x)$ such

$$G(-\infty) = +\infty, \quad G(+\infty) = L, \;\text{ for some } L > 0.$$

The opposite case can be considered in the same manner. As usual, in order to get the existence of a periodic orbit, via the Poincaré-Bendixson theorem, we need to introduce a winding trajectory. The idea is the following: we use Theorem 2.6 in the case *ii)* to produce a trajectory which does not intersect the positive x-axis back in time. For $x < 0$ we apply Theorem 2.7 in the case i) (see also Theorem 2.3) in order to prove that such a trajectory intersects the y-axis at some point $y_0 > 0$. Finally, for $x > 0$, we apply again Theorem 2.6 in the case ii). Accordingly, the following conditions are now assumed.

THEOREM 3.1. *Assume* $(g0)$, $F_0(y)y > 0$ *for* $y \neq 0$ *and*

$$F(x, y)x < 0 \;\; \textit{for } xy \neq 0 \;\; \textit{in a neighborhood of the origin.}$$

Suppose that

$$G(-\infty) = +\infty, \quad G(+\infty) = L, \;\; \textit{for some } L > 0$$

and there is $\phi : [0, +\infty) \to \mathbb{R}$ *continuous, with* $F(x, y) \geq \phi(x), \forall\, x \geq 0, \forall\, y$ *such that* $\limsup_{x\to+\infty} \phi(x) = +\infty$ *and* ϕ *bounded below. Finally, assume that* $F(x, y)$ *is bounded above on* $x \leq 0$. *Then system* (S) *has at least a stable limit cycle.*

Proof. With reference to Figure 2, consider a point P on the level curve $\tilde{H} = L$ (with $x > 0$). As proved above, the negative semi-trajectory passing through P does not intersect the x-axis, while the the positive semi-trajectory intersects the negative y-axis at a point $(0, -y_1)$ above the point $(0, -M)$ with $M := K + \sqrt{2L}$. In view of Theorem 2.7, case $-i$), such positive trajectory intersects the negative x-axis and then the positive y-axis. At this point, using Theorem 2.6, case $-ii$), the positive trajectory intersects the x-axis and again the negative y-axis at a point $(0, -y_2)$ with $0 < y_2 < y_1$. Hence we have proved the existence of a winding trajectory. As observed above, the assumption that $F(x, y)x < 0$, for $xy \neq 0$ in a neighborhood of the origin, implies that the origin is a source. As usual, the Poincaré-Bendixson theorem ensures the existence of at least a stable limit cycle. □

REMARK 3.2. Conversely, if we assume

$$G(+\infty) = +\infty, \quad G(-\infty) = L, \;\; \text{for some } L > 0,$$

we can have, in the same way, a dual result, using Theorem 2.6 and Theorem 2.7 in the reverse order.

In the same light we can treat the case in which $G(x)$ is bounded and $F(x, y)$ is bounded below on $x > 0$ and bounded above on $x < 0$. The proofs require only obvious modifications and are omitted.

The applicability of the above theorem can be verified by several examples. As a first case, let us consider system $(S1)$ with

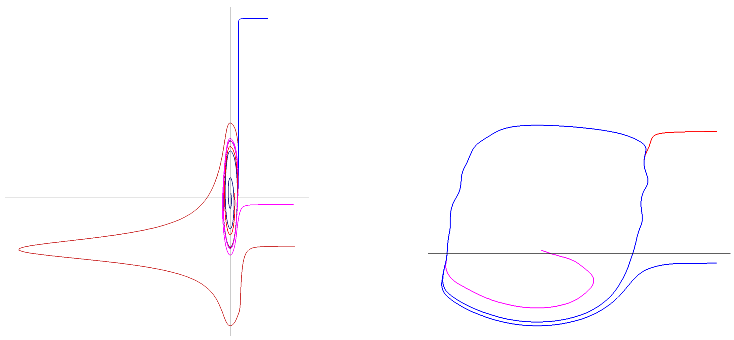
$$B(y) = \frac{(y + \frac{2}{\sqrt{3}})(y^2 + 1)}{y^2 + 3}, \quad A(x) = x(x^2 - 1), \quad g(x) = \min\left\{x, \frac{1}{x^2}\right\}. \tag{7}$$

Observe that $G(-\infty) = +\infty$ and $G(+\infty) = \frac{3}{2}$. Notice that $F(x, y)$ is not always positive outside the strip $[-1, 1]$. Therefore, the results in [40] cannot be applied. On the other hand, we are in the environment of Theorem 3.1 and therefore the existence of a limit cycle is granted. Figure 3 of Pinocchio [1](left panel) illustrates the situation. Moreover, as appears in the figure, the existence of a separatrix, asymptotic to the line $y = -\frac{2}{\sqrt{3}}$, can be easily proved.

As a second example, we consider the case in which $F(x, y)$ is not split as $B(y)A(x)$, namely,

$$F(x, y) = \frac{(y + \frac{2}{\sqrt{3}})(y - \cos(10x)(\cos(10y) + 2.1)x(x - 1)}{y^2 + 1.1 + \sin(x^3)} \tag{8}$$

and $g(x)$ as in (7). Also in this case, Theorem 3.1 applies. Figure 3 (right panel) illustrates the situation.



(a) Phase-portrait of the system $(S1)$ with $B(y)$, $A(x)$ and $g(x)$ as in (7) exhibiting an unexpected "Pinocchio" shape.

(b) Phase-portrait of the system (S) with $F(x, y)$ as in (8) and $g(x)$ as in (7).

Figure 3: The figures summarize the two examples of application of Theorem 3.1, considered above. The graphs are produced using [42].

4. The superlinear case: an averaging-bifurcation approach

In this section, we show the effectiveness of a classical bifurcation technique, dating back to Poincaré (according to Lefschetz [22, pp. 314–320]) and Liénard

[1]Pinocchio, is a fictional character and the protagonist of the children's novel "The Adventures of Pinocchio" by the Italian writer Carlo Collodi (pen name of Carlo Lorenzini) from Florence. The novel has been translated in almost all languages and the title of the figure will be evident to the reader.

to prove the existence of limit cycles bifurcating form a periodic orbit of a center when a parameter multiplying the nonlinear terms is small (see also the introduction and the references in [7] for more information and historical remarks on the averaging-bifurcation method in the qualitative analysis of planar systems). This approach was successfully applied by Duff and Levinson [12] (see also [34]) to produce multiple limit cycles bifurcating from circular orbits of the harmonic oscillator and represents a very useful technique to construct specific examples of multiplicity results for Liénard or Rayleigh equations. We refer also to [4, Remark 2.4] for another application of this method. As pointed out by the Referee, this approach has been used in very many papers, even in the more general case $\dot{x} = y - \lambda P(x, y)$, $\dot{y} = -x + \lambda Q(x, y)$, with $|\lambda|$ a small parameter and P and Q arbitrary polynomials. Actually, in [7] the method is generalized to the small perturbations of a planar Hamiltonian system like $\dot{x} = -yG(x, y) + \varepsilon A(x, y)$, $\dot{y} = xG(x, y) + \varepsilon B(x, y)$. Hence, from this point of view, our application to systems of the form $(H_{p,\lambda})$ could be merely considered as an exercise. On the other hand, we hope that presenting few examples in this direction can be of some interest, also as a comparison to the global approach considered in [40]. We give now some details for the reader's convenience.

As a first step we introduce some polar coordinates. Consider a general planar system of the form

$$(S\mathcal{F}_\varepsilon) \qquad \dot{x} = y - \varepsilon\mathcal{F}(x, y), \quad \dot{y} = -x.$$

where we suppose that $\mathcal{F}(x, y)$ is a locally Lipschitz continuous function with $\mathcal{F}(0, 0) = 0$. System $(S\mathcal{F}_\varepsilon)$ can be seen as a perturbation of the linear equation

$$(Lin) \qquad \dot{x} = y, \quad \dot{y} = -x.$$

which represents a global center with all the orbits being concentric circumferences and having 2π as fundamental period. In this context, it is natural to express the solutions of (Lin) and $(S\mathcal{F}_\varepsilon)$ in polar coordinates. For convenience, due to the fact that the trajectories rotates clockwise around the origin, we propose a modified (but equivalent) polar coordinates system with the angles counted positively oriented in the clockwise sense starting from the positive y-axis. In this manner, we have that $x = \rho \sin\theta$ and $y = \rho\cos\theta$ and a general nontrivial solution $(x(t), y(t))$ of a planar system $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$ (having the origin as equilibrium point) will satisfy the equations

$$\begin{cases} \dot{\rho} = \frac{xX+yY}{\sqrt{x^2+y^2}} = X(\rho\sin\theta, \rho\cos\theta)\sin\theta + Y(\rho\sin\theta, \rho\cos\theta)\cos\theta \\ \dot{\theta} = \frac{yX-xY}{x^2+y^2} = \rho^{-1}\big(X(\rho\sin\theta, \rho\cos\theta)\cos\theta - Y(\rho\sin\theta, \rho\cos\theta)\sin\theta\big). \end{cases}$$

This system, in case of $(S\mathcal{F}_\varepsilon)$, reduces to

$$\begin{cases} \dot{\rho} = -\varepsilon\mathcal{F}(\rho\sin\theta, \rho\cos\theta)\sin\theta \\ \dot{\theta} = 1 - \varepsilon\rho^{-1}\mathcal{F}(\rho\sin\theta, \rho\cos\theta)\cos\theta. \end{cases}$$

Now, let $B(0,R)$ be an open disc of center the origin and radius $R>0$ (with R sufficiently large) in the phase-plane containing the origin. For $\varepsilon>0$ small enough, say $0<\varepsilon<\varepsilon_0$, we have that for all initial points in $B(0,R)\setminus\{0\}$ the solutions of $(S\mathcal{F}_\varepsilon)$ are defined in $[0,4\pi]$ with values in the larger disc $B(0,2R)$ and satisfy $\dot\theta>0$ for all $t\in[0,4\pi]$. Hence (for ε sufficiently small), we can express the solutions in the (θ,ρ)-plane, as solutions of the equation

$$\frac{d\rho}{d\theta}=-\frac{-\varepsilon\mathcal{F}(\rho\sin\theta,\rho\cos\theta)\sin\theta}{1-\varepsilon\rho^{-1}\mathcal{F}(\rho\sin\theta,\rho\cos\theta)\cos\theta}\,.$$

The general method as described in [22, 34] usually considers nonlinear terms which are polynomials [34] or analytic functions [22], in order to represents the solutions as series with respect the parameter ε. In our case, we just need to introduce the first order terms of the expansion, thus, assuming a sufficiently smooth function $\mathcal{F}(x,y)$ (e.g., of class C^1) appears to be sufficient for our purposes (in any case, in our examples of application the nonlinearities will be of polynomial type).

Taking the first order expansion of $\rho(\theta)$ as a function of $\varepsilon>0$ (for $\varepsilon\to0^+$) (cf. [22] for the details) we obtain that $\rho(2\pi)-\rho(0)=-\varepsilon\Psi(\rho)+o(\varepsilon,\rho,\theta)$, where

$$(P)\qquad\qquad \Psi(r):=\int_0^{2\pi}\mathcal{F}(r\sin\theta,r\cos\theta)\sin\theta\,d\theta$$

and with $o(\varepsilon,\rho,\theta)/\varepsilon\to0^+$ as $\varepsilon\to0^+$, uniformly with respect (ρ,θ) in a compact set.

Then, according to [22, Theorem 5.5], we can state the following result [2].

PROPOSITION 4.1. *Let $r_0>0$ be a simple zero of Ψ such that $\Psi'(r_0)>0$. Then there exists $\varepsilon^*>0$ such that for each $0<\varepsilon<\varepsilon^*$ there is an orbitally asymptotically stable limit cycle Γ^ε in a neighborhood of the circumference $C(r_0):=\{(x,y):x^2+y^2=r_0\}$. Moreover, Γ^ε tends to $C(r_0)$ as $\varepsilon\to0^+$.*

Proof. We give only a sketch of the proof, leaving the details to [22, 34]. Assuming $\Psi(r_0)=0$ and $-\Psi'(r_0)<0$, there exists an open neighborhood U of r_0 such that for $\rho\in U$ (and $\varepsilon>0$ sufficiently small), we have $\rho(2\pi)-\rho(0)>0$ when $\rho<r_0$ and $\rho(2\pi)-\rho(0)<0$ when $\rho>r_0$. This proves that in a narrow annular neighborhood of $C(r_0)$ of the form

$$V(r_0,\delta_\varepsilon):=\{(x,y)\in\mathbb{R}^2: r_0-\delta_\varepsilon<(x^2+y^2)^{1/2}<r_0+\delta_\varepsilon\}$$

there is a stable limit cycle of system $(S\mathcal{F}_\varepsilon)$. Moreover, by construction, $\delta_\varepsilon\to0^+$ as $\varepsilon\to0^+$ from which we find that Γ^ε tends to $C(r_0)$. □

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[2]Our presentation is not exactly the same (verbatim) as in [22, Theorem 5.5], but it is substantially equivalent. Moreover, the sign of our function Ψ is the opposite of the equivalent function considered in [22].

To show the consistency of our result with the classical ones, consider for a moment the Van der Pol equation $\ddot{x}+\mu(x^2-1)\dot{x}+x=0$ in the Liénard plane $\dot{x}=y-\mu\big(\frac{x^3}{3}-x\big)$, $\dot{y}=-x$. In this case, we have $\varepsilon=\mu>0$ and $\mathcal{F}(x,y)=\frac{x^3}{3}-x$. An elementary computation of the integral in (P) yields to $\Psi(r)=\frac{r^3\pi}{4}-\pi r$, and so there is a unique positive zero with positive derivative at $r_0=2$. In this manner, we re-obtain the classical result in [22, p. 320].

We consider now an application of Proposition 4.1 to the superlinear case. In fact, it seems that the superlinear case is the most interesting one to analyze from the point of view of the bifurcation with respect to small parameters, because we have already proved that limit cycles do exist only for small values of the parameter λ. As a first case, we reconsider the system $(H_{2,\lambda})$ for which we already proved the existence/nonexistence result in Theorem 1.3. An elementary computation of the integral in (P) yields to $\Psi(r)=r^5\frac{\pi}{8}-r^3\frac{\pi}{4}$. This function has a unique positive zero with positive derivative at $r_0=\sqrt{2}$. Thus Proposition 4.1 guarantees that when $\lambda>0$ is small there is a (unique) limit cycle approximating the circumference of center the origin and radius $\sqrt{2}$.

To show the effectiveness of our approach to other superlinear cases ($p>1$), we consider another system related to $(H_{4,\lambda})$, namely

$$(H'_{4,\lambda}) \qquad \dot{x}=y-\lambda y^4(x^4-1)x, \quad \dot{y}=-x,$$

which fits in the frame of equation $(S\mathcal{F}_\varepsilon)$ with $\varepsilon=\lambda$ and $\mathcal{F}(x,y)=y^4(x^4-1)x$. An elementary computation of the integral in (P) yields to $\Psi(r)=r^9\frac{3\pi}{128}-r^5\frac{\pi}{8}$. This function has a unique positive zero with positive derivative at $r_0=(16/3)^{1/4}\approx 1.519671371$. Figure 4 illustrates this situation. The numerical simulation is performed for $\lambda=0.3$.

Clearly, the same approach can be extended to system $(S1_\lambda)$ with $g(x)=x$. In particular, as suggested by the Referee, it might be interesting to apply Proposition 4.1 to the case when $B(y)=|y|^p$ and $A(x)$ is an arbitrary polynomial, namely to the system

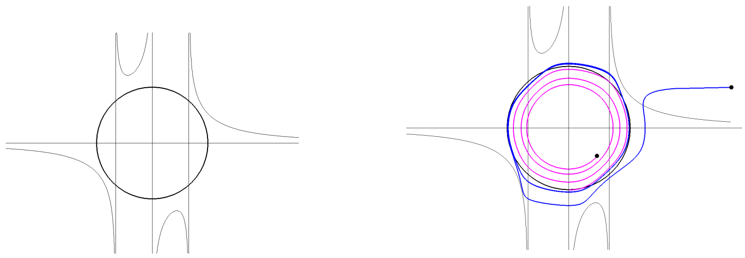
$$\dot{x}=y-\lambda|y|^pA(x), \quad \dot{y}=-x,$$

with $A(x)$ a polynomial of degree $n\geq 2$ such that $A(0)=0$, that we represent as $A(x)=x\tilde{A}(x)$, with $\tilde{A}(x)=a_0+a_1x+\dots a_{n-1}x^{n-1}$. In this case, the function Ψ in (P) takes the form of

$$\Psi(r)=r^{p+1}\int_0^{2\pi}(\sin^2\theta)|\cos\theta|^p\tilde{A}(r\sin\theta)\,d\theta=r^{p+1}\mathscr{P}(r),$$

for $\mathscr{P}(r):=\sum_{\substack{k=0,\dots,n-1\\ k\text{ even}}}a_kc_kr^k$, where $c_k:=\int_0^{2\pi}|\sin\theta|^{2+k}|\cos\theta|^p\,d\theta$ (k even). Now, if we assume $a_0<0$ and $a_{k*}>0$ with k^* the larger even integer in

$\tilde{A}(x)$ such that $a_k \neq 0$, we know that there is a first positive zero for Ψ where $\Psi'(r_0) \geq 0$ and Ψ changes sign from negative to positive values. Checking whether $\Psi'(r_0) > 0$, in order to enter in the setting of Proposition 4.1, may be achieved by a more precise analysis on the coefficients of $\mathscr{P}(r)$.



(a) Case of system $(H'_{4,\lambda})$ for $\lambda = 0.3$, putting in evidence the circumference $C(r_0)$ for $r_0 = (16/3)^{1/4}$ and the part of the infinite-isocline in the range $[-4,4]\times[-3,3]$. Note that all the regions $\mathcal{W}_i$ and $\mathcal{V}_i$ for $i = 1,2$ are visible.

(b) The phase-portrait of system $(H'_{4,\lambda})$ for $\lambda = 0.3$, showing the fact that the solutions tend to a limit cycle close to the circumference $C(r_0)$ for $r_0 = (16/3)^{1/4}$. The circumference partially overlaps with the limit cycle. We have indicated with a dot the initial point of the orbits.

Figure 4: The figures summarize the essential information for system $(H'_{4,\lambda})$ with $\lambda > 0$ and small, according to Proposition 4.1. The simulation also suggest the fact that the critical value λ^* such that there are limit cycles for $0 < \lambda < \lambda^*$, must be larger but near to $\lambda = 0.3$, since our external trajectory moves very close to the escaping region $\mathcal{V}_2$. The graphs are produced using [42].

Remark 4.2. We observe that the technique exposed in this Section can also be applied to a more general system of the form $\dot{x} = y - \varepsilon\mathcal{G}(x,y)$, $\dot{y} = -g(x)$ with g and $\mathcal{G}$ sufficiently regular functions and $g : \mathbb{R} \to \mathbb{R}$ satisfying $(g1)$. In this case, the change of variables $u = z(x) := \sqrt{2G(x)}\,\mathrm{sign}(x)$, introduced in [9], leads to the equivalent system $\dot{u} = y - \varepsilon\mathcal{G}(z^{-1}(u), y)$, $\dot{y} = -u$ which is of the form of $(S\mathcal{F}_\varepsilon)$.

5. Reversible systems and chaotic dynamics for the periodically perturbed equation

In this section we consider a class of systems of the form

$$(S1_\lambda) \qquad \begin{cases} \dot{x} = y - \lambda B(y)A(x) \\ \dot{y} = -g(x), \end{cases}$$

for which the orbits are symmetric with respect to the y-axis. Systems with a symmetry with respect to a line are important in the study of centers and were considered by professor Roberto Conti in his seminal article [10] (see also [8] for previous important work on planar centers). For more recent contributions to this topic, see [33] and the references therein.

Such equations with a line of symmetry are a particular case of the *reversible systems*, also studied by Arnold [2] and Moser [27]. A thorough treatment of the KAM theory for time-dependent reversible systems is contained in the monograph of Sevryuk [35]. An application of these results to the problem of the boundedness of the solutions for a class of Liénard equations can be found in [24]. Consistently with the observation that "there is a very close similarity between the behaviour of solutions of reversible systems and that of Hamiltonian ones" (see [35, p. 3]), we have recently extended in [30] some approaches for the existence of chaotic-like dynamics for periodically perturbed Hamiltonian systems [25], to a class of forced Liénard equations

$$\dot{x} = y - F(x) + E(t), \quad \dot{y} = -g(x),$$

with F even and g odd. See also [16, 28, 29, 38, 39] and the references quoted therein for other results concerning the forced case.

Our aim now is to further extend this type of approach to the perturbed system

$$(S2_\lambda) \qquad \begin{cases} \dot{x} = y - \lambda B(y)A(x) + E(t), \\ \dot{y} = -g(x), \end{cases}$$

where $E : \mathbb{R} \to \mathbb{R}$ is a piecewise continuous and T-periodic forcing term. For other results related to [30], see [20].

As a preliminary analysis, we consider system $(S1_\lambda)$ with $B(y), A(x)$ even and $g(x)$ odd functions. In this situation, it is immediate to check that the autonomous system is invariant with respect to the composition of a planar involution $R : (x, y) \mapsto (-x, y)$ (i.e., the symmetry with respect to the y-axis) and time-reversal. In fact, setting $u(t) := -x(-t)$ and $v(t) := y(-t)$ we obtain the same system as the original one. Notice that the symmetry of the trajectories with respect to the y-axis does not depend on our choice of $B(y)$ as an even function. Any choice of $B(y)$ would be fine, provided that $A(x)$ is even and $g(x)$ is odd. To make a comparison with the results in [40] we will focus pour attention to the superlinear case and will take $B(y) = |y|^p$ with $p > 1$. Moreover, we will concentrate, as in [40] to a "case study", by taking for $A(x)$ a function, which is negative for $0 < x < 1$ and positive for $x > 1$. In this manner, we study the model equation

$$(R_{p,\lambda}) \qquad \begin{cases} \dot{x} = y - \lambda|y|^p(x^2 - |x|) \\ \dot{y} = -x, \end{cases}$$

with $\lambda > 0$ and $p > 1$. Analogously as in [40] we introduce now the regions

$$\mathcal{V}_1 := \left\{(x,y) : -1 < x < 0,\, y < 0,\, |y|^{p-1} < -\tfrac{1}{\lambda(x^2-|x|)}\right\},$$
$$\mathcal{V}_2 := \left\{(x,y) : 0 < x < 1,\, y < 0,\, |y|^{p-1} < -\tfrac{1}{\lambda(x^2-|x|)}\right\},$$

which are symmetric with respect to y-axis and are positively invariant, so that all the trajectories entering such regions become unbounded in the y-component. We also define the negatively invariant regions

$$\mathcal{W}_1 := \left\{(x,y) : x > 1,\, y > 0,\, |y|^{p-1} > \tfrac{1}{\lambda(x^2-|x|)}\right\},$$
$$\mathcal{W}_2 := \left\{(x,y) : x < -1,\;\, y > 0,\, |y|^{p-1} > \tfrac{1}{\lambda(x^2-|x|)}\right\},$$

which are symmetric with respect to the y-axis, where unboundedness in backward time occurs, as well as possible blow-up in finite negative time. However, as in case of system $(H_{p,\lambda})$, all the solutions of $(R_{p,\lambda})$ are globally defined in forward time (see Figure 5).

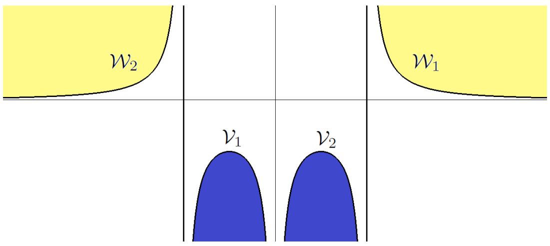
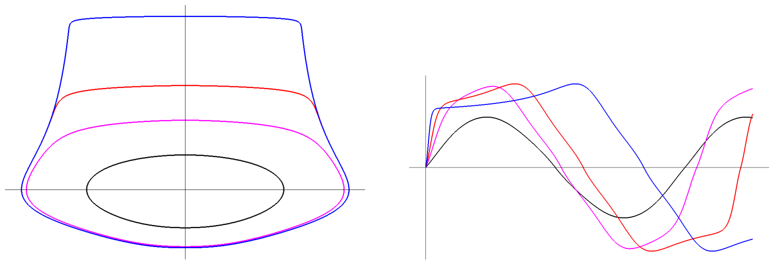


Figure 5: The infinite-isocline and the zero-isocline for system $(R_{2,\lambda})$ and the corresponding regions $\mathcal{V}_i$ and $\mathcal{W}_i$ for $i = 1, 2$. The graph is produced using [42], for a suitable $\lambda > 0$. Different values of λ gives the same qualitative structure; however, increasing the value of λ moves the regions closer to the x-axis and therefore when λ is larger than a critical value, all the nontrivial solutions definitively enter $\mathcal{V}_1 \cup \mathcal{V}_2$ and become unbounded in the future. It may be interesting to compare Figure 1 with the present one. The structure of the regions $\mathcal{V}_i$ and $\mathcal{W}_i$ for systems $(H_{p,\lambda})$ and $(R_{p,\lambda})$ (with $p > 1$), as well as their dynamical properties are the same. The only difference, due to reversibility of system $(R_{p,\lambda})$ is the symmetry of the orbits with respect to the y-axis.

System $(R_{p,\lambda})$ has a center at the origin. In fact, for $|y|$ small enough, the orbits are close to those of the harmonic oscillator $\dot{x} = y$, $\dot{y} = -x$. Hence, a trajectory departing from a point $P_0 = (0, y_0)$ with $y_0 > 0$ small enough, will traverse the right-half plane and cross the y-axis again at a point $P_1 = (0, y_1)$ with $y_1 < 0$. Hence, by the y-axis mirror symmetry of the orbits, such a trajectory will come bach to P_0 after traversing the left-half plane. On the other hand, a trajectory departing from a point $Q_0 = (0, y_0)$ with $y_0 > 0$ large, will move quickly toward the region $\mathcal{W}_1$ (without entering it) and then will hit the vertical line $x = 1$ at a point $Q_1 = (1, y_1)$ with $y_1 < 0$. From now on, either the orbits enter the region $\mathcal{V}_2$ and become ultimately unbounded, or it will cross again the y-axis at a point $Q_2 = (0, y_2)$, with $y_2 < 0$ (and avoiding the region $\mathcal{V}_2$). In this latter case, by symmetry, we obtain again a periodic orbit, typically with a larger period than the small one, if $\lambda > 0$ is small enough. Notice that, as in [40], a condition of the form $\lambda < \lambda_p^*$ will guarantee that larger orbits of system $(R_{p,\lambda})$ do not enter the region $\mathcal{V}_2$. An illustration of this situation is given in Figure 6.



(a) Structure of the center at the origin for system $(R_{2,\lambda})$. The orbits are consider for an initial point $P_0 = (0, y_0)$, with $y_0 = 1, 2, 3, 5$.

(b) Solutions $(t, x(t))$ for system $(R_{2,\lambda})$, with $x(0) = 0$ and $\dot{x}(0)) = y(0) = y_0$, with $y_0 = 1, 2, 3, 5$. The fact that the period increases from smaller to larger orbits is apparent.

Figure 6: The figure illustrates the dynamics associated to system $(R_{2,\lambda})$ for $\lambda = 3/4$. The graph is produced using [42].

The gap in the periods between the smaller and the larger orbits of the center allows us to enter in a framework considered in [30], for the study of the chaotic behavior of the solutions to the periodically perturbed system $(S2_\lambda)$. Indeed, continuing our analysis to the case study $(R_{p,\lambda})$ as a paradigmatic model of $(S1_\lambda)$, we can produce a *topological horseshoe*, namely a compact invariant set Λ for the Poincaré Φ map associated to system

$$(S_{p,\lambda}) \qquad \begin{cases} \dot{x} = y - \lambda |y|^p (x^2 - |x|) + E(t) \\ \dot{y} = -x, \end{cases}$$

such that Φ on Λ is semi-conjugate to the two-sided Bernoulli shift on $m \geq 2$ symbols and such that to *any* k-periodic sequence of symbols it corresponds the existence of a k-periodic point P for Φ in Λ, so that the solution of $(S_{p,\lambda})$ starting at the point P is a kT-periodic solution of the system.

To enter in the setting described in [30] we need to find two *linked annular regions* $\mathcal{A}_1$ and $\mathcal{A}_2$ for the autonomous systems

$$(S^i_{p,\lambda}) \qquad \begin{cases} \dot{x} = y - \lambda |y|^p (x^2 - |x|) + E_i \\ \dot{y} = -x, \end{cases}$$

with $i = 1, 2$, such that there is a period gap between the inner and the outer closed orbits which are the boundaries of A_i. The geometric configuration is illustrated in Figure 7.

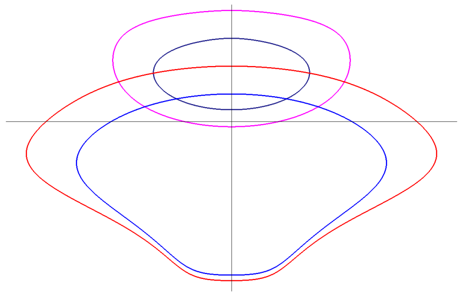
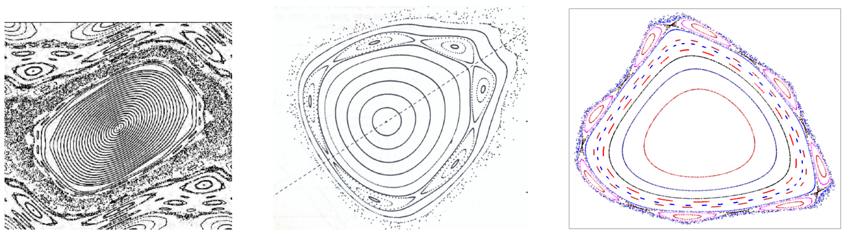


Figure 7: Phase-portraits of the systems $(S^i_{2,\lambda})$ $(i = 1, 2)$ for $\lambda = 3/4$. The two lower closed orbits are obtained for $E_1 = A = 1$ and initial points $(0, 0.5)$ and $(0, 1)$. The two upper closed orbits are obtained for $E_2 = -A = -1$ and initial points $(0, 1.5)$ and $(0, 2)$. The analysis of the period shows that there is still a (small) gap between the periods of the two orbits bounding each of the annular regions $\mathcal{A}_1$ and $\mathcal{A}_2$. The graph is produced using [42]. For the presentation, the aspect-ratio of the figure has been slightly modified (with a compression along the y-axis).

More precisely, if we denote by $\mathcal{A}_i$ the annular region in the plane bounded by two closed orbits of system E_i and we assume that the annuli $\mathcal{A}_1$ and $\mathcal{A}_2$ are *topologically linked* as in Figure 7 (see [30, Definition 3.2] for the technical condition), then a gap for the periods of the bounding orbits allows us to enter in a variant of the theory of the linked twist maps ([11, 41], as well as [25] and there references therein) and have the existence of chaotic-like dynamics (in the sense described above) and infinitely many subharmonics, if we take as $E(t)$ a T-periodic forcing term which is close to a stepwise function of sufficiently large period. For our purposes, we will take

$$E(t) = A \tanh(n \sin(\omega t + \alpha)), \quad A > 0, n \geq 1, \omega > 0, \alpha \in [0, 2\pi[\,, \tag{9}$$

which is a smooth function, close (in the L^1-norm) to a stepwise function of period $T = 2\pi/\omega$, and such that $E(t)$ oscillates between the values $E_1 = A$ and $E_2 = -A$.



(a) Dynamics associated with the classical *standard map* when the parameter ε gets close to 1 (from [5, Fig.1.5, p.14]).

(b) Dynamics associated with an area preserving *quadratic map* considered by Henon (1969) (from [3, Fig.1.39, p.52]).

(c) Poincaré map (for a period of the forcing term $T = 4$) for the system $(S_{2,\lambda})$ for $\lambda = 3/4$ and $E(t)$ as in (9), with $A = 1$, $\alpha = 0$, $n = 6$, $\omega = \pi/2$.. The graph is produced using [42].

Figure 8: The portrait (Figure (c) at the right) after 800 iterations of the Poincaré map, for different initial points, suggests the presence of invariant curves and quasi-periodic solutions, as well as the existence of large subharmonic solutions, *librational curves* (according to [5, Fig. 1.5, pag.14] - see Figure (a) at the left), *island chains* [3, Fig. 1.39, pag. 52] (see Figure (b) at the center) and more complex solutions, perhaps of chaotic-type; a situation analogous to that encountered for planar area-preserving homeomorphisms associated to Hamiltonian systems when the variation of some parameters moves the system outside the integrability case [5, 27]. The graphs are produced using [42].

Acknowledgments

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