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Restricted Participation on Financial Markets: A General Equilibrium Approach Using Variational Inequality Methods

Maria Bernadette Donato[1] · **Monica Milasi**[1] · **Antonio Villanacci**[2]

Abstract
We deal with the analysis of a general equilibrium model with restricted participation in financial markets and with numeraire assets. We consider an exchange economy and assume that there are two periods of time and S possible states of nature in the second period. Markets may in principle be complete, but each household has her own specific restricted way to access to it. In particular, we assume that households are allowed to choose portfolios in a closed and convex set containing zero. Our main goal in this work is to provide a proof of existence of equilibria under relatively general assumptions, by assuming that the households may have non-complete or non-transitive preferences, and by using a variational inequality approach. More precisely, we introduce a sequence of generalized quasi-variational inequalities and we show that an associated sequence of solutions converges to an equilibrium.

Keywords General equilibrium · Incomplete financial markets · Restricted participation · Numeraire assets · Variational and quasi-variational inequalities

✉ Monica Milasi
mmilasi@unime.it

Maria Bernadette Donato
mbdonato@unime.it

Antonio Villanacci
antonio.villanacci@unifi.it

1 University of Messina, Messina, Italy

2 University of Firenze, Firenze, Italy

1 Introduction

The object of this paper is the analysis of the existence problem in a general equilibrium economic model with restricted participation on financial markets. The desired goal is accomplished by using a variational inequality approach.

The model describes an exchange economy where time and uncertainty are explicitly taken into account. Agents live for two periods and in the second period a finite number of uncertain scenarios, or states of the world, can occur. We assume that they can have non-complete or non-transitive preferences. It is assumed that agents cannot directly exchange goods in different states, but they can use financial assets to indirectly accomplish that task. A financial asset is a binding contract which in exchange of a payment in the first period gives the right to receive back some resources in each state the second period. It is well known that if the number of available, sufficiently diversified assets is large enough, then an equilibrium exists and it coincides with the standard equilibrium in an exchange economy - see Arrow (1953). If that is not the case, it can been shown that equilibria still exist - see Cass (2006), Werner (1985) and Geneakoplos and Polemarchakis see (1986) - but with dramatically different properties - see Villanacci and others (Villanacci et al. 2002) for a formalization and a proof of the above sentence. Observe that in the above framework, agents or households are assumed to be free to buy any vector of assets, so-called portfolios, in the whole Euclidean space of dimension equal to the number of available assets.

The above model of so-called incomplete financial markets has been criticized under the simple observation that recent years have witnessed the very fast growth of the number of qualitatively different available financial assets. It seems hard to believe that the assumption about the availability of too few assets does hold true in current financially very sophisticated economies.

On the other hand, "while there might be some disagreement over whether, in a modern developed economy, financial markets are actually incomplete, there can hardly be any disagreement over whether at least some economic agents are variously constrained in transacting on those financial markets."[1] It is then important to generalize the incomplete market model adding the restriction that each agent or household can choose her portfolio holdings in a personalized subset of the appropriate Euclidean space of dimension equal to the number of available assets. This subset represents the household specific *portfolio set*, i.e., the set of possible credit transactions available to the corresponding household. The above model of "*restricted participation* constitutes a bona fide generalization of the model with incomplete markets, but the latter potentially embodies far more interesting institutional features (and not just the flavor of restricted participation) since it permits, for instance, modeling short sales bounds or market margin requirements."[2]

[1] See Cass (1992), p. 274.
[2] See Cass (1992), p. 275.

Several contribution on the topic are available in the literature. The first contribution was provided in the research group leaded by David Cass at the University of Pennsylvania and it is the paper by Siconolfi (1988), where existence of equilibria is shown in the case of nominal assets.[3] More recent contributions by Aouani and Cornet (2009, 2011) deal with numeraire assets[4] and more general assumption on the financial side of the economy.

Several other contributions present either extension of the framework of the model or deeper analysis of properties of equilibria. In Gori et al. (2014) and Seghir and Torres-Martinez (2011), authors prove existence in the case of financial constraints depending on some endogenous variables and in Balasko et al. (1990), Gori et al. (2013), and Hoelle et al. (2016), authors prove regularity of equilibria in different specifications of the restricted participation model.

Our model continues the line of research by Aouani and Cornet (2009), Aouani and Cornet (2011), and Siconolfi (1988). By using a variational inequality approach, we show the existence of equilibria in a model with numeraire assets and restricted participation. More precisely, we provide a different and, we believe, easier existence proof under assumptions at the same level of economic generality as in Aouani and Cornet (2009). It is well known that a variational inequality problem provides a general formulation that encompasses many mathematical problems, including, among others, nonlinear equations, optimization, complementarity and fixed point problems. The theory of variational inequalities was introduced in the seventies by Stampacchia (1964), as an innovative and effective method to solve equilibrium problems arising in mathematical physics. Nowadays, the variational inequality, with all its generalizations and extensions, has developed as a powerful tool for the analysis of several classes of equilibrium problems arising in different branches of applied sciences. For the state of the art about this topic, see Allevi et al. (2019), Aussel (2014), Aussel and Dutta (2008), Barbagallo et al. (2014), Berglund and Kwon (2014), Daniele et al. (2014), Donato et al. (2016), Donato et al. (2018a), Donato et al. (2018b), Friesz et al. (2001), Hamdouch et al. (2016), Jofrè et al. (2007), Milasi (2013), and Scrimali (2014) and references therein.

The paper is organized as follows. In Section 2, we describe the set-up of the model and present and discuss our assumptions. To better appreciate the contribution of our paper, we compare our assumptions on the financial side of economy with those used in Aouani and Cornet (2009), Aouani and Cornet (2011), and Siconolfi (1988). In Section 3, we prove some preliminary results on the main ingredients of the model: the set of no-arbitrage asset prices, the portfolio sets and the budget constraint set-valued functions. In Section 4, we introduce a sequence of variational inequalities and we show that the associated sequence of solutions converges to an equilibrium.

[3] Assets mainly differ in terms of the nature of their returns: if they are measured in terms of units of account, say euros or dollars, then assets are called nominal assets.

[4] Returns of numeraire assets are measured in terms of a given, so-called numeraire, good.

The Appendix lists some well known results on variational inequalities needed in our framework.

2 Set-Up of the Model

We consider a model of restricted participation with numeraire assets. We assume that there are 2 periods of time, say today and tomorrow: the state of the world today is known to individuals, and it is called state 0; in the following period, S states of the world, with $S > 1$, are possible. We label each state of the world, or spot, by s, where $s = 0$ corresponds to the first period, and we set $\mathcal{S}^0 := \{0\} \cup \mathcal{S}$ and $\mathcal{S} := \{1, ..., S\}$. In this framework, a commodity may be defined in terms not only of its physical or chemical characteristics, but also in terms of the period or the state of nature in which it is available. Spot commodity markets open in the first and second period, and there are C commodities or goods, with $C > 1$, in each spot, labelled by $c \in \mathcal{C} := \{1, ..., C\}$ and the total number of commodities available in the economy is $G := (S + 1)C$. There are H households, with $H > 1$, labelled by $h \in \mathcal{H} := \{1, ..., H\}$, and A assets, with $A \geq 1$, labelled by $a \in \mathcal{A} := \{1, ..., A\}$. An asset is an $S + 1$ dimensional vector whose first component is the price of the asset, and the other S components are the yields of that asset in each state, i.e., the amount of units of a given good, called the numeraire good, that a household has the right to receive in each state if she purchased one unit of that asset in state 0. We choose good C as the numeraire.

The time structure of the model is the following one. In the first period, commodities and assets are exchanged and first period consumption takes place. Then uncertainty is resolved, households fulfill their financial commitments and, finally, exchange and consume second-period commodities. Following the standard notation, we have that x_h^{sc} is the consumption of commodity c in state s by household h, e_h^{sc} is the endowment of commodity c in state s owned by household h and p^{sc} denotes the price of commodity c in state s and, moreover we set

$$
\begin{aligned}
&x_h^s := (x_h^{sc})_{c\in\mathcal{C}} \in \mathbb{R}^C, \quad x_h := (x_h^s)_{s\in\mathcal{S}^0} \in \mathbb{R}^G, \qquad x := (x_h)_{h\in\mathcal{H}} \in \mathbb{R}^{GH},\\
&e_h^s := (e_h^{sc})_{c\in\mathcal{C}} \in \mathbb{R}_+^C, \quad e_h := (e_h^s)_{s\in\mathcal{S}^0} \in \mathbb{R}_+^G, \qquad e := (e_h)_{h\in\mathcal{H}} \in \mathbb{R}_+^{GH},\\
&p^s := (p^{sc})_{c\in\mathcal{C}} \in \mathbb{R}_+^C, \quad \mathbf{p^1} := (p^s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{SC}, \quad p := (p^0, \mathbf{p^1}) = (p^s)_{s\in\mathcal{S}^0} \in \mathbb{R}_+^G.
\end{aligned}
$$

and $D := (p^{sj})_{s,j\in\mathcal{S}}$ is a $S \times S$ matrix such that $p^{sj} = 0$ for all $s \neq j$ and $p^{ss} = p^{sC}$ for $s \in \mathcal{S}$. Each individual h is characterized by a binary relation $\succeq_h$ on $\mathbb{R}^G$ which describes the taste of household: $x \succeq_h y$ denotes that *the consumption y is at least as desired by the consumer as x*; the strict inequality $x \succ_h y$ means that *x is strictly preferred to y*, i.e., $x \succeq_h y$ *but not* $y \succeq_h x$. Finally, $x \sim_h y$ means that *x is indifferent to y*, that is, $x \succeq_h y$ and $y \succeq_h x$. Given the preference relation $\succeq_h$, we define the strictly preference set-valued function P_h as follows $P_h : \mathbb{R}^G \rightrightarrows \mathbb{R}^G$,

$$
P_h(x_h) = \left\{ z \in \mathbb{R}^G : z \succ_h x \right\}.
$$

We assume that assets pay in each state in units of good C, the so-called numeraire good. q^a represents the price of asset a and b_h^a represents the demand of asset a by household h; we set

$$q := (q^a)_{a\in\mathcal{A}} \in \mathbb{R}^A, \qquad b_h := \left(b_h^a\right)_{a\in\mathcal{A}}, \qquad b := (b_h)_{h\in\mathcal{H}} .$$

The $S \times A$ matrix $Y := (y^{sa})_{s\in\mathcal{S},a\in\mathcal{A}}$ is called yield matrix and for any $a \in \mathcal{A}$ and $s \in \mathcal{S}$, y^{sa} is the yield of asset a in state s, i.e., the number of units of good C delivered by asset a in state s.

Households are allowed to choose portfolios in a personalized subset B_h of $\mathbb{R}^A$; B_h is the financial constrained set of household h; define $B = \prod_{h\in\mathcal{H}} B_h$.

An economy in a financial economy model with numeraire assets and restricted participation is an element $\Sigma := (e, P, Y, B) \in \mathbb{R}^{GH}_{++} \times \mathcal{P} \times \mathcal{M}_{S,A} \times \mathcal{B}$, where $\mathcal{M}_{S,A}$ is the set of $S \times A$ dimensional matrices, $\mathcal{P}$ is the set of the set-valued functions $P = (P_h)_{h\in\mathcal{H}}$ and $\mathcal{B}$ is the set of all lists of financial constrained sets of households. The aim of each household is to have an optimal consumption under the constraints that in period 0 expenditure for goods and assets is smaller than the value of wealth in that period and, similarly, in each state in the future, expenditure for consumption is smaller than wealth increased by the value of the assets yields. For any $h \in \mathcal{H}$, we define the budget set of h at prices $(q, p^0, p^{\mathbf{1}})$ as follows:

$$\begin{aligned}\Gamma_h(q, p^0, p^{\mathbf{1}}) : = \{&(x_h, b_h) \in \mathbb{R}^G_+ \times B_h : \langle p^0, x_h^0 - e_h^0\rangle_C + \langle q, b_h\rangle_A \leq 0,\\ &\times\langle p^s, x_h^s - e_h^s\rangle_C - p^{sC}\langle y^s, b_h\rangle_A \leq 0 \quad \forall s \in \mathcal{S}\}.\end{aligned}$$

The formal definition of equilibrium is presented below.

Definition 2.1 The vector $(\widetilde{x}, \widetilde{b}, \widetilde{q}, \widetilde{p}) \in \mathbb{R}^{GH} \times \mathbb{R}^{AH} \times \mathbb{R}^A \times \mathbb{R}^G_+$ is an **equilibrium** vector for the economy Σ if

1. for any $h \in \mathcal{H}$,

$$(\widetilde{x}_h, \widetilde{b}_h) \in \Gamma_h(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^1)$$

$$(P_h(\widetilde{x}_h) \times B_h) \cap \Gamma_h(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^1) = \emptyset; \tag{1}$$

2. for any $s \in \mathcal{S}^0$ and $c \in \mathcal{C}$,

$$\sum_{h\in\mathcal{H}} \widetilde{x}_h^{sc} \leq \sum_{h\in\mathcal{H}} e_h^{sc} \quad \text{if } \widetilde{p}^{sc} = 0,$$

$$\sum_{h\in\mathcal{H}} \widetilde{x}_h^{sc} = \sum_{h\in\mathcal{H}} e_h^{sc} \quad \text{if } \widetilde{p}^{sc} > 0;$$

3. for any $a \in \mathcal{A}$,

$$\sum_{h\in\mathcal{H}} \widetilde{b}_h^a = 0.$$

Household h choice variables are her consumption vector $x_h \in \mathbb{R}^G$ and her constrained portfolio $b_h \in B_h$. We then say that a consumption, portfolio holding, commodity and asset price vector is an equilibrium vector for the economy Σ if at those prices, $(\widetilde{x}_h, \widetilde{b}_h)$ is optimal in household h's budget set and market clears, i.e., commodities demand is smaller than or equal to commodities supply and assets demand is equal to zero.

The description of the set of no free lunch good prices and no arbitrage assets prices is a convenient preliminary step in the process of proving existence of equilibrium prices: prices outside that set cannot be equilibrium prices. In the case of unrestricted financial participation and numeraire asset, the set of no-arbitrage asset prices[5] for household h (see e.g. Carosi et al. (2009)) is given by[6]

$$Q^u(D,Y) := \left\{ q \in \mathbb{R}^A : \text{ there is no } b_h \in \mathbb{R}^A \text{ such that } \begin{bmatrix} -q \\ DY \end{bmatrix} b_h > 0 \right\}$$
$$= \left\{ q \in \mathbb{R}^A : \ \forall b_h \in \mathbb{R}^A \text{ such that } DYb_h > 0 \text{ we have } \langle q, b_h \rangle_A > 0 \right\}.$$

By using, a form of the Alternative Lemma (for details, see Lemma 14, page 297, in Villanacci et al. (2002)), one has that

$$Q^u(D,Y) = \left\{ q \in \mathbb{R}^A : \ \exists \nu \in \mathbb{R}^S_{++} \text{ such that } q = \nu DY \right\}.$$

Observe that if $\left(p^{sC}\right)_{s\in\mathcal{S}} >> 0$, then

$$Q^u\,(D,Y) = Q^u\,(Y) := \left\{ q \in \mathbb{R}^A : \exists \nu \in \mathbb{R}^S_{++} \text{ such that } q = \nu Y \right\}.$$

In the case of restricted participation, it may be that there is $b_h^* \in \mathbb{R}^A$ such that $\begin{bmatrix} -q \\ DY \end{bmatrix} b_h^* > 0$, but if B_h is bounded in the direction of b_h^*, then household h is not allowed to demand an unbounded amount of that portfolio. Therefore, in the case of presence of financial restriction, for given $\left(p^{sC}\right)_{s\in\mathcal{S}} \in \mathbb{R}^S_+$, $Y \in \mathcal{M}_{S,A}$ and $B \in \mathcal{B}$, we define the set[7] of no-arbitrage asset prices for household h as

$$Q_h(D,Y,B_h) := \left\{ q \in \mathbb{R}^A : \text{ there is no } b_h \in \text{rec } B_h \text{ such that } \begin{bmatrix} -q \\ DY \end{bmatrix} b_h > 0 \right\}$$

$$= \{ q \in \mathbb{R}^A : \ \forall b_h \in \text{rec } B_h \text{such that} DYb_h > 0 \text{ we have } \langle q, b_h \rangle_A > 0 \},$$

and the set of no-arbitrage asset prices as

$$Q(D,Y,B) := \bigcap_{h\in\mathcal{H}} Q_h(D,Y,B_h)$$

$$= \{ q \in \mathbb{R}^A : \ \forall b \in \cup_{h\in\mathcal{H}} \text{rec } B_h \text{such that } DYb > 0 \text{ we have } \langle q, b \rangle_A > 0 \}.$$

From an economic viewpoint, prices in Q_h are such that if there exists a portfolio b_h which gives a positive return in some state and non-negative return in each state tomorrow, i.e., such that $DYb_h > 0$, and which can bought in an unbounded amount

[5] In the symbol Q^u, the superscript u stays for "unrestricted".

[6] For vectors $y, z \in \mathbb{R}^n$, $y \geq z$ means that for $i = 1, ..., n$, $y_i \geq z_i$; $y >> z$ means that for $i = 1, ..., n$, $y_i > z_i$ and $y > z$ means that $y \geq z$ but $y \neq z$.

[7] rec B_h is the recession cone of B_h; see the Appendix for definition and simple facts.

by household h, i.e., $b_h \in \text{rec } B_h$, then that portfolio must cost a positive amount today, i.e., $\langle q, b_h \rangle_A > 0$. Moreover, define

$$Q_h(Y, B_h) := \left\{ q \in \mathbb{R}^A : \text{ there is no } b_h \in \text{rec } B_h \text{ such that } \begin{bmatrix} -q \\ Y \end{bmatrix} b_h > 0 \right\},$$

$$Q(Y, B) := \bigcap_{h \in \mathcal{H}} Q_h(Y, B_h).$$

Remark 2.1 For any $Y \in \mathcal{M}_{S,A}$, $B \in \mathcal{B}$, and $\left(p^{sC}\right)_{s \in \mathcal{S}} \in \mathbb{R}^S_{++}$, one has that $Q_h(Y, B_h) = Q_h(D, Y, B_h)$. Indeed, that result follows immediately from the fact that if $\left(p^{sC}\right)_{s \in \mathcal{S}} \in \mathbb{R}^S_{++}$, then

$$\begin{bmatrix} -q \\ Y \end{bmatrix} b > 0 \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} -q \\ Y \end{bmatrix} b = \begin{bmatrix} -q \\ DY \end{bmatrix} b > 0.$$

From now on, we make the following Assumptions.

Assumption 1 *For any $h \in \mathcal{H}$, $e_h >> 0$.*

Assumption 2 *For any $h \in \mathcal{H}$, the preference set-valued function P_h is*

2.1 *lower semicontinuous, with open and convex valued;*
2.2 *strictly increasing in the numeraire good sC, for every $s \in \mathcal{S}$, i.e.,*

$$\forall \widehat{x}_h, \widehat{\widehat{x}}_h \in \mathbb{R}^G_+ : \widehat{x}_h \geq \widehat{\widehat{x}}_h \ \ with \ \ \widehat{x}^{sC}_h > \widehat{\widehat{x}}^{sC}_h \quad \Rightarrow \quad x_h \in P_h(\widehat{\widehat{x}}_h);$$

2.3 *locally nonsatiated in state 0, i.e.,*

$$\forall x_h = (x^0_h, x^1_h, \ldots, x^S_h) \in \mathbb{R}^G_+ \ \ and \ \ \forall \varepsilon > 0, \ \ \exists \widehat{x}_h = (\widehat{x}^0_h, x^1_h, \ldots, x^S_h) \in \mathbb{R}^G_+$$
$$such \ that \ \ \|\widehat{x}^0_h - x^0_h\| < \varepsilon \ and \ \ \widehat{x}_h \in P_h(x_h).$$

Assumption 3 *For every $s \in \mathcal{S}^0$, $c \in \mathcal{C}$, there exists $\varepsilon > 0$ and $h' \in \mathcal{H}$ such that for every $x'_h \in \mathbb{R}^G_+$, $P_{h'|\mathcal{B}(x_{h'}, \varepsilon)}$[8] is strictly increasing in sc, i.e.,*

$$\forall \widehat{x}_{h'}, \widehat{\widehat{x}}_{h'} \in \mathcal{B}(x_{h'}, \varepsilon) : \widehat{x}_{h'} \geq \widehat{\widehat{x}}_{h'} \ \ with \ \ \widehat{x}^{sc}_{h'} > \widehat{\widehat{x}}^{sc}_{h'} \quad \Rightarrow \quad \widehat{x}_{h'} \in P_{h'}(\widehat{\widehat{x}}_{h'}).$$

Assumption 4 *For any $h \in \mathcal{H}$,*

4.1 *B_h is a convex and closed subset of $\mathbb{R}^A$ and $0_A \in B_h$;*
4.2 *$KerY \cap rec \ B_h = \{0_A\}$;*
4.3 *for any $p \in \mathbb{R}^G_+$ such that $p^0 = 0_C$ and for any $q \in \mathrm{Cl}(Q_h(D, Y, B)) \setminus \{0_A\}$ there exists $b_h \in B_h$ such that $\langle -q, b_h \rangle_A > 0$.*

Assumption 1 is a survival assumption on the commodity side of the economy: it helps insuring households are able to buy, and consume, some good in each state of the world.

[8] $\mathcal{B}(x_{h'}, \varepsilon)$ is the ball centered at $x_{h'}$ and radius ε

Assumption 2.1 is relatively general and standard in the general equilibrium literature.[9]

Assumption 2.2 is based on the fact that, by construction of the model, households agreed upon choosing the numeraire good as the unit of measure of asset yields and therefore "they strongly like that good".

Assumption 2.3 simply says that households care about consumption in period zero.

Assumption 3 stresses the fact that each good is appreciated at least by one household.

Assumption 4.1 is quite general and implies that households are allowed to stay out of the financial market.

Assumption 4.2 is crucial in several steps in the proofs below and it is implied by any of the following conditions: B_h is bounded (which implies that rec $B_h = \{0_A\}$); there are no redundant assets, i.e., rank $Y = A$ (which implies that Ker$Y = \{0_A\}$).

Assumption 4.3 is a survival assumption on the financial side of the economy: it insures that even if the available endowment at time zero has no value, then there exists an admissible portfolio which generates positive wealth in state zero itself.

Several reasonable conditions are indeed sufficient for Assumption 4.3 (see Proposition 2, page 776, in Aouani and Cornet (2009)). For example, it is enough that for any households there is a lower bond on some asset demand, or the origin of $\mathbb{R}^A$ is an interior point of the portfolio set.

We can observe that our assumptions are at least as general as Siconolfi's ones (see Siconolfi (1988)). More precisely, Assumptions 4.1 and 4.3 are identical to those assumed by the most relevant contributions which are related to our work, i.e. Aouani and Cornet (2009) and Aouani and Cornet (2011). Assumption 4.2 is not logically comparable with the corresponding one in Aouani and Cornet (2009) and it is indeed less general than that one in Aouani and Cornet (2011); on the other, all the economically interesting conditions proposed in Aouani and Cornet (2011) which imply their assumption do imply ours as well. Moreover, the proof of existence in Aouani and Cornet (2011) is a further elaboration on the already not trivial proof in Aouani and Cornet (2009).

With respect to the consumption side of the model, we point out that we analyze the case of preferences represented by a continuous, quasiconcave utility function satisfying some monotonicity assumptions. This approach is line with other contributions in the literature and it is more general than Siconolfi's approach (see Siconolfi (1988), page 276). Our assumptions on preferences are definitely less general than those analyzed in Aouani and Cornet (2009) and Aouani and Cornet (2011).

3 Preliminary properties

Proposition 3.1 1. $\emptyset \neq Q^u(D, Y) \subseteq Q(D, Y, B) \subseteq \mathrm{Cl}(Q(D, Y, B))$;
2. *for any* $\gamma \in \mathbb{R}_{++}$, *one has* $\gamma\, Q(D, Y, B) \subseteq Q(D, Y, B)$;

[9] This general way to describe the tastes of the households encompasses the case where the household h has a preference relation $\succeq_h$ which is a complete preorder.

3. *$Q(D, Y, B) \cup \{0_A\}$ is a cone;*
4. *$Q(Y, B)$ is open and $Q(D, Y, B)$ is convex;*
5. $\mathrm{Cl}\,(Q(D, Y, B) \cup \{0_A\}) = \mathrm{Cl}\,(Q(D, Y, B))$*;*
6. $\mathrm{Cl}\,(Q(D, Y, B))$ *is a convex and closed cone;*
7. *for any $\alpha \in \mathbb{R} \setminus \{0\}$,*

$$\alpha Q\,(D, Y, B) = Q\left(D, \alpha\, Y, \frac{B}{\alpha}\right);$$

8. *for any $\alpha \in \mathbb{R} \setminus \{0\}$,*

$$\alpha \mathrm{Cl}\,(Q\,(D, Y, B)) = \mathrm{Cl}\left(Q\left(D, \alpha\, Y, \frac{B}{\alpha}\right)\right).$$

Proof 1.-2. As observed in Section 2, from a form of the Alternative Lemma,

$$Q^u(D, Y) = \left\{q \in \mathbb{R}^A : \ \forall b_h \in \mathbb{R}^A \text{ s.t. } DYb_h > 0 \text{ we have } \langle q, b_h\rangle_A > 0\right\}.$$

Then, the desired conclusions follow from the fact that rec $B_h \subseteq \mathbb{R}^A$ and from the Definition of $Q(D, Y, B)$.

3. Take $q \in Q(D, Y, B) \cup \{0_A\}$; we want to show that for any $\lambda \in \mathbb{R}_+$, we have $\lambda q \in Q(D, Y, B) \cup \{0_A\}$. If $q = 0_A$ or $\lambda = 0$, we are done. If $q \in Q(D, Y, B) \setminus \{0_A\}$ and $\lambda > 0$, from item above it follows that $\lambda q \in Q(D, Y, B)$.
4. Convexity follows easily from the definition of $Q\,(D, Y, B)$. About the openness of $Q\,(Y, B)$ observe what follows. Suppose otherwise; then there exists $\overline{q} \in Q(Y, B)$ such that for any $n \in \mathbb{N}$, there exists $q^n \notin Q(Y, B)$ and such that $\|\overline{q} - q^n\| < \frac{1}{n}$ and therefore $\lim_{n\to+\infty} q^n = \overline{q}$. Since for any $n \in \mathbb{N}$, $q^n \notin Q(Y, B)$, then, there exists $b_h^n \in \mathrm{rec} B_h$ such that $Yb_h^n > 0$ and $\langle q^n, b_h^n\rangle_A \leq 0$. Then, for any $n \in \mathbb{N}$, $b_h^n \neq 0_A$ and, without loss of generality, $\lim_{n\to+\infty} \frac{b_h^n}{\|b_h^n\|} = \widetilde{b}_h^0 \neq 0_A$. Then, taking also into account that, from Proposition 6.1 in Appendix, rec B_h is a cone, we have $\frac{b_h^n}{\|b_h^n\|} \in \mathrm{rec}\, B_h$, $Y\frac{b_h^n}{\|b_h^n\|} > 0$ and $\langle q^n, \frac{b_h^n}{\|b_h^n\|}\rangle_A \leq 0$. Taking limits for $n \to +\infty$, we get $\widetilde{b}_h^0 \in \mathrm{rec} B_h$, $Y\widetilde{b}_h^0 \geq 0$ and $\langle \overline{q}, \widetilde{b}_h^0\rangle_A \leq 0$. We now claim that $Y\widetilde{b}_h^0 > 0$; indeed if we suppose otherwise, we have $\widetilde{b}_h^0 \in \mathrm{Ker} Y$ and then, since $\widetilde{b}_h^0 \neq 0_A$, Assumption 4.2 is violated. Then, we showed that there exists $\widetilde{b}_h^0 \in \mathrm{rec} B_h \subseteq B_h$ such that $Y\widetilde{b}_h^0 > 0$ and $\langle \overline{q}, \widetilde{b}_h^0\rangle_A \leq 0$, contradicting the assumption that $\overline{q} \in Q(Y, B)$.
5. Clearly, $\mathrm{Cl}\,(Q(D, Y, B)) \subseteq \mathrm{Cl}\,(Q(D, Y, B) \cup \{0_A\})$. We prove that

$$\mathrm{Cl}\,(Q(D, Y, B) \cup \{0_A\}) \subseteq \mathrm{Cl}\, Q(D, Y, B).$$

Let $q \in \mathrm{Cl}\,(Q(D, Y, B) \cup \{0_A\})$. If $q = 0_A$, we can take an arbitrary $q^* \in Q(D, Y, B)$ and $q_n = \frac{1}{n} q^*$. Since $Q(D, Y, B) \cup \{0_A\}$ is a cone, we have $\{q_n\}_{n\in\mathbb{N}} \subseteq Q(D, Y, B)$ and $\lim_{n\to+\infty} q_n = 0_A = q$; that is $q \in \mathrm{Cl}\, Q(D, Y, B)$. If $q \neq 0_A$. There exists a sequence $\{q_n\}_{n\in\mathbb{N}} \subseteq (Q(D, Y, B) \cup \{0_A\})$ such that

$\lim_{n\to+\infty} q_n = q$. Since $q \neq 0_A$, there exists $\nu \in \mathbb{N}$ such that for any $n > \nu$, $q_n \neq 0_A$. Hence $\{q_n\}_{n\in\mathbb{N}} \subseteq Q(D, Y, B)$, and $q \in \mathrm{Cl}\,(Q(D, Y, B))$.

6. Since $Q(D, Y, B) \cup \{0_A\}$ is a cone, then $\mathrm{Cl}\,(Q(D, Y, B) \cup \{0_A\})$ is a cone. Hence, from $\mathrm{Cl}\,(Q(D, Y, B)) = \mathrm{Cl}\,(Q(D, Y, B) \cup \{0_A\})$ and taking into account that $Q(D, Y, B)$ is convex, $\mathrm{Cl}\,(Q(D, Y, B))$ is a closed and convex cone.
7. Firstly, we prove that $\alpha Q(D, Y, B) \subseteq Q(D, \alpha Y, \frac{B}{\alpha})$. Taken $q \in \alpha Q(D, Y, B)$, then there exists $q' \in Q(D, Y, B)$ such that $q = \alpha q'$, that is $q' = \frac{q}{\alpha}$. From $\frac{q}{\alpha} \in Q(D, Y, B)$, we have that $\forall b \in \cup_{h\in\mathcal{H}} \mathrm{rec}\, B_h$ such that $DYb > 0$ it follows that $\langle \frac{q}{\alpha}, b\rangle_A > 0$. We have to prove that $\forall \widehat{b} \in \cup_{h\in\mathcal{H}} \mathrm{rec}\, \frac{B_h}{\alpha}$ such that $\alpha DY\widehat{b} > 0$, it follows that $\langle q, \widehat{b}\rangle_A > 0$. Taken $\widehat{b} \in \cup_{h\in\mathcal{H}} \mathrm{rec}\, \frac{B_h}{\alpha}$ such that $\alpha DY\widehat{b} > 0$, then there exists $h' \in \mathcal{H}$ such that $\widehat{b} \in \mathrm{rec}\, \frac{B_{h'}}{\alpha} = sgn\left(\frac{1}{\alpha}\right) \mathrm{rec}\, B_{h'}$, where last equality follows from Proposition 6.1.2 in the Appendix. Observe that $\alpha\widehat{b} \in \alpha\, sgn(\frac{1}{\alpha})\mathrm{rec}\, B_{h'} = |\alpha|\mathrm{rec}\, B_{h'} = \mathrm{rec}\, B_{h'}$, then, by assumption, $\langle \frac{q}{\alpha}, \alpha\widehat{b}\rangle_A > 0$. Then $\langle q, \widehat{b}\rangle_A > 0$ as desired. In order to prove that $Q(D, \alpha Y, \frac{B}{\alpha}) \subseteq \alpha Q(D, Y, B)$, observe that the argument is basically symmetric to the above one.
8. $\subseteq$ For any $\alpha q \in \alpha \mathrm{Cl}\,(Q\,(D, Y, B))$, there exists $\{q_n\}_{n\in\mathbb{N}} \subseteq Q\,(D, Y, B)$ such that $\lim_{n\to+\infty} q_n = q$. From item 7, $\{\alpha q_n\}_{n\in\mathbb{N}} \subseteq Q\left(D, \alpha Y, \frac{B}{\alpha}\right)$; then $\alpha q \in \mathrm{Cl}\, Q\left(D, \alpha Y, \frac{B}{\alpha}\right)$.

   $\supseteq$ From 7. above, it suffices to show that

$$\mathrm{Cl}\,(\alpha Q\,(D, Y, B)) \supseteq \alpha \mathrm{Cl}\,(Q\,(D, Y, B))\ .$$

   Indeed, $f : \mathbb{R}^A \to \mathbb{R}^A$, $f\,(x) = \alpha x$ is continuous and, from basic general topology, for any set $X \subseteq \mathbb{R}^n$, $f\,(\mathrm{Cl}\,(X)) \subseteq \mathrm{Cl}\,(f\,(X))$ and therefore the desired result follows. □

Proposition 3.2 *Let $B \subseteq \mathbb{R}^A$ be given.*

1. *If B is a convex set such that $0_A \in B$, then for any $\lambda \in [0, 1]$, one has $\lambda B \subseteq B$.*
2. *If B is nonempty, closed, convex and $0_A \in B$, then for any $\alpha \in \mathbb{R} \setminus \{0\}$ one has that $\alpha\, B$ is nonempty, closed, convex and $0_A \in \alpha\, B$.*
3. *If $Ker\, Y \cap rec\, B = \{0_A\}$, then for any $\alpha,\ \beta \in \mathbb{R} \setminus \{0\}$ one has $Ker\,(\beta\, Y) \cap rec\,(\alpha\, B) = \{0_A\}$.*

Proof 1. For any $\lambda \in [0, 1]$ and $b \in B$ one has $\lambda b = (1 - \lambda)0_A + \lambda b \in B$.
2. The thesis follows from properties of B.
3. Since $\mathrm{Ker}\,(\beta Y) := \left\{b \in \mathbb{R}^A : \beta Y b = 0\right\} = \mathrm{Ker}\, Y$ and $\mathrm{rec}(\alpha B) = \mathrm{sign}(\alpha)\mathrm{rec}\, B$ (see Proposition 6.1 in the Appendix), one has

$$\mathrm{Ker}\,(\beta\, Y) \cap \mathrm{rec}\,(\alpha\, B) = \mathrm{Ker}\, Y \cap (\mathrm{sign}\,(\alpha)\, \mathrm{rec}\, B)\, .$$

   If there exists $b \neq 0_A$ such that $b \in \mathrm{Ker}\, Y$ and $b \in \mathrm{sign}\,(\alpha)\, \mathrm{rec}\, B$ one has that $b^* = (\mathrm{sign}\,\alpha)\, b \neq 0_A$ and $b^* \in \mathrm{Ker}\, Y \cap \mathrm{rec}\, B$, a contradiction. □

Proposition 3.3 *Let a set B satisfying Assumptions 4.1 and 4.2 be given. Let also a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\alpha_n = \frac{x_n}{\|y_n\|}$ with $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ and $\lim_{n\to+\infty} \|y_n\| = +\infty$ and $\lim_{n\to+\infty} \alpha_n = \alpha$ be given. Then,*

1. *for any $n \in \mathbb{N}$, $\alpha_n \in rec\ B \subseteq B$;*
2. *$\alpha \in rec\ B \subseteq B$.*

Proof 1. Thanks to Proposition 6.1 in Appendix, it is sufficient to prove that for any $\lambda \geq 0$, $\lambda\alpha_n \in B$. For sufficiently large n, one has $\frac{\lambda}{\|y_n\|} \in (0, 1)$; then, since B is convex and $0_A \in B$

$$\lambda\alpha_n = \left(1 - \frac{\lambda}{\|y_n\|}\right) \cdot 0_A + \frac{\lambda}{\|y_n\|} x_n \in B$$

that is $\alpha_n \in \text{rec } B \subseteq B$.

2. Let an arbitrary $\lambda \in \mathbb{R}_+$ be given. Since $\lambda\alpha_n \in B$ and $\lim_{n\to+\infty} \lambda\alpha_n = \lambda\alpha$, being B a closed set, one has $\lambda\alpha \in B$. Hence $\alpha \in \text{rec } B$.

□

Proposition 3.4 *Let a set B satisfying Assumptions 4.1 and 4.2 be given. If $\{b_n\}_{n\in\mathbb{N}} \subseteq B$ and there exists $\widehat{y} \in \mathbb{R}^S$ such that $\lim_{n\to+\infty} Yb_n = \widehat{y}$, then there exists a subsequence $\{b_{n_k}\}_{k\in\mathbb{N}}$ of $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim_{k\to+\infty} b_{n_k} = \widehat{b} \in B$.*

Proof Given a sequence $\{b_n\}_{n\in\mathbb{N}} \subseteq B \subseteq \mathbb{R}^A$, then either the sequence $\{b_n\}_{n\in\mathbb{N}}$ is bounded or it is unbounded. If it is bounded, taking into account that B is closed, then $\{b_n\}$ admits a convergent subsequence $\{b_{n_k}\}$, that is $\lim_{k\to+\infty} b_{n_k} = \widehat{b} \in B$. Now, we want to show that the sequence $\{b_n\}_{n\in\mathbb{N}}$ cannot be unbounded. Suppose otherwise $\lim_{n\to+\infty} ||b_n|| = +\infty$. Define $\overline{b}_n = \frac{b_n}{||b_n||} \in \partial\text{B}(0_A, 1)$, then $\{\overline{b}_n\}$ admits a convergent subsequence $\{\overline{b}_{n_k}\}$ such that $\lim_{k\to+\infty} \overline{b}_{n_k} = \overline{b} \neq 0_A$. From Proposition 3.3.2, one has that $\overline{b}$ $\in$rec $B \subseteq B$. Moreover

$$\lim_{k\to+\infty} Y\overline{b}_{n_k} = Y\overline{b} \qquad \textit{and} \qquad \lim_{k\to+\infty} Y\overline{b}_{n_k} = \lim_{k\to+\infty} Yb_{n_k} \frac{1}{||b_{n_k}||} = \widehat{y} \cdot 0 = 0_S$$

Hence $Y\overline{b} = 0_S$. Summarizing, we have that $\overline{b} \neq 0_A$, $\overline{b} \in \text{rec } B$ and $Y\overline{b} = 0_S$. Then Ker $Y \cap$ rec $B \neq \{0_A\}$, contradicting Assumption 4.2. □

We now define $\rho := \sum_{s\in\mathcal{S}} \sum_{a\in\mathcal{A}} y^{sa}$. The proposition below says that there is no loss of generality in assuming $\rho \geq 0$, a condition which is crucial in the arguments below.

Let $\mathcal{P}^*$ be the set of the preferences set-valued functions satisfying Assumptions 2 and 3; let $\mathcal{F}$[10] be the family of pairs $(Y, B) \in \mathcal{M}_{S,A} \times \mathcal{B}$ satisfying Assumptions 4 and define $\mathcal{E} = \mathbb{R}^{GH}_{++} \times \mathcal{P}^* \times \mathcal{F}$ to be the set of economies satisfying all our maintained assumptions.

[10] $\mathcal{F}$ stays for financial structure.

Proposition 3.5 *For any* $\alpha \in \mathbb{R} \setminus \{0\}$,

1. $\Sigma = (e, P, Y, B) \in \mathcal{E} \Leftrightarrow \Sigma_\alpha := \left(e, P, \alpha Y, \left(\frac{B}{\alpha}\right)\right) \in \mathcal{E}$;
2. $(\widetilde{x}, \widetilde{b}, \widetilde{q}, \widetilde{p})$ *is an equilibrium for* Σ *if and only if* $\left(\widetilde{x}, \frac{\widetilde{b}}{\alpha}, \alpha \widetilde{q}, \widetilde{p}\right)$ *is an equilibrium for* Σ_α.

Proof 1. Thanks to Propositions 3.1, 3.2, economy Σ satisfies Assumptions 1, 2, 3 and 4 .1 and 4.2 if and only if Σ_α satisfies Assumptions 1, 2, 3 and 4.1 and 4.2. About Assumption 4.3, observe what follows. We assume that economy Σ satisfies 4.3. Taken $\widehat{q} \in \mathrm{Cl}\left(Q_h(D, \alpha Y, \frac{B_h}{\alpha})\right) \setminus \{0_A\}$, from Proposition 3.1.8, $\frac{\widehat{q}}{\alpha} \in \mathrm{Cl}\,(Q_h(D, Y, B_h)) \setminus \{0_A\}$. Then, by Assumption 4.3, there exists $b_h \in B_h$ such that $\langle \frac{\widehat{q}}{\alpha}, b_h \rangle_A < 0$, that is equivalent to have there exists $\widehat{b_h} = \frac{b_h}{\alpha} \in \frac{B_h}{\alpha}$ such that $\langle \widehat{q}, \widehat{b_h} \rangle_A < 0$, as desired. The proof of the opposite implication is symmetric to the above one.

2. Since $(\widetilde{x}_h, \widetilde{b}_h) \in \Gamma_h(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^{\mathbf{1}})$ if and only if $\left(\widetilde{x}_h, \frac{\widetilde{b}_h}{\alpha}\right) \in \Gamma_h(\alpha \widetilde{q}, \widetilde{p}^0, \widetilde{p}^{\mathbf{1}})$, then the desired result holds true. □

Remark 3.1 Thanks to Proposition 3.5, in order to prove existence of equilibria, we can assume that $\rho \geq 0$. Indeed, let $\Sigma = (e, Y, B, P)$ be an economy with associated ρ being strictly negative and consider the economy $\Sigma_{-1} := (e, -Y, -B, P)$, whose associated ρ is strictly positive. Then, from Proposition 3.5, if $(\widetilde{x}, \widetilde{b}, \widetilde{q}, \widetilde{p})$ is an equilibrium for Σ_{-1}, then $(\widetilde{x}, -\widetilde{b}, -\widetilde{q}, \widetilde{p})$ is an equilibrium for the original economy Σ.

The following proposition gives some preliminary properties.

Proposition 3.6 *Let Assumptions 2 be satisfied. If for any* $h \in \mathcal{H}$, $(\widetilde{x}_h, \widetilde{b}_h)$ *is optimal in the budget constraints set* $\Gamma_h(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^1)$, *then*

1. $\widetilde{p}^0 > 0, \quad (\widetilde{p}^{sC})_{s \in \mathcal{S}} >> 0$;
2. *for any* $h \in \mathcal{H}$,

$$\langle \widetilde{p}^0, \widetilde{x}_h^0 - e_h^0 \rangle_C + \langle \widetilde{q}, \widetilde{b}_h \rangle_A = 0\,,$$

$$\langle \widetilde{p}^s, \widetilde{x}_h^s - e_h^s \rangle_C - \widetilde{p}^{sC} \langle y^s, \widetilde{b}_h \rangle_A = 0, \qquad \forall s \in \mathcal{S}\,;$$

3. *the following so-called* $S + 1$ *Walras laws hold true*

$$\langle \widetilde{p}^0, \sum_{h \in \mathcal{H}} (\widetilde{x}_h^0 - e_h^0) \rangle_C + \langle \widetilde{q}, \sum_{h \in \mathcal{H}} \widetilde{b}_h \rangle_A = 0\,,$$

$$\langle \widetilde{p}^s, \sum_{h \in \mathcal{H}} (\widetilde{x}_h^s - e_h^s) \rangle_C - \widetilde{p}^{sC} \langle y^s, \sum_{h \in \mathcal{H}} \widetilde{b}_h \rangle_A = 0, \qquad \forall s \in \mathcal{S}\,;$$

4. $\widetilde{q} \in Q(D, Y, B) = Q(Y, B)$.

Proof 1. We first prove that $\widetilde{p}^0 > 0$. Suppose $\widetilde{p}^0 = 0_C$. By Assumption 2.3, we have that for any $\varepsilon > 0$, there exists $\widehat{x}_h = (\widehat{x}_h^0, \widetilde{x}_h^1, \dots, \widetilde{x}_h^S) \in \mathbb{R}_+^G \cap \mathcal{B}(\widetilde{x}_h, \varepsilon)$ such that $\widehat{x}_h \in P_h(\widetilde{x}_h)$. Moreover $(\widehat{x}_h, \widetilde{b}_h) \in \Gamma_h(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^{\mathbf{1}})$, contradicting the fact that $\widetilde{x}_h$ verifies statement (1).

We now prove that $(\widetilde{p}^{sC})_{s \in \mathcal{S}} >> 0$. Suppose that there exists $s^* \in \mathcal{S}^0$ such that $\widetilde{p}^{s^*C} = 0$. Then, define $\widehat{x}_h$ such that

$$\widehat{x}_h^{sc} := \begin{cases} \widetilde{x}_h^{s^*C} + 1 & \text{if} \quad sc = s^*C, \\ \widetilde{x}_h^{sc} & \text{otherwise.} \end{cases}$$

Then, for any $h \in \mathcal{H}$, $\left(\widehat{x}_h, \widetilde{b}_h\right) \in \Gamma_h(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^{\mathbf{1}})$ and, since $\widehat{x}_h \geq \widetilde{x}_h$ and $\widehat{x}_h^{s^*C} > \widetilde{x}_h^{s^*C}$, from Assumption 2.2 one has $\widehat{x}_h \in P_h(\widetilde{x}_h)$, contradicting the fact that $\widetilde{x}_h$ verifies statement (1).

2. Suppose otherwise; then there exists $h \in \mathcal{H}$ such that

$$\text{either} \quad \langle \widetilde{p}^0, \widetilde{x}_h^0 - e_h^0 \rangle_C + \langle \widetilde{q}, \widetilde{b}_h \rangle_A < 0,$$

$$\text{or } \exists s \in \mathcal{S} \text{ such that } \quad \langle \widetilde{p}^s, \widetilde{x}_h^s - e_h^s \rangle_C - \widetilde{p}^{sC} \langle y^s, \widetilde{b}_h \rangle_A < 0.$$

Suppose $\langle \widetilde{p}, \widetilde{x}_h^0 - e_h^0 \rangle_C + \langle \widetilde{q}, \widetilde{b}_h \rangle_A < 0$. There exists $\varepsilon > 0$ such that

$$\forall x_h^0 \in \mathbb{R}_+^C \; s.t. \; \|x_h^0 - \widetilde{x}_h^0\| < \varepsilon \quad \Rightarrow \quad \langle \widetilde{p}^0, x_h^0 - e_h^0 \rangle_C + \langle \widetilde{q}, \widetilde{b}_h \rangle_A < 0 \,. \qquad (2)$$

From Assumption 2.3, there exists $\widehat{x}_h = (\widehat{x}_h^0, \widetilde{x}_h^1, \dots, \widetilde{x}_h^S) \in \mathbb{R}_+^C$ such that $\|\widehat{x}_h^0 - \widetilde{x}_h^0\| < \varepsilon$ and $\widehat{x}_h \in P_h(\widetilde{x}_h)$. Moreover, from Eq. 2 it follows $(\widehat{x}_h, \widetilde{b}_h) \in \Gamma_h(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^{\mathbf{1}})$, contradicting the fact that $\widetilde{x}_h$ verifies (1).

Assume now that $\langle \widetilde{p}^s, \widetilde{x}_h^s - e_h^s \rangle_C - \widetilde{p}^{sC} \langle y^s, \widetilde{b}_h \rangle_A < 0$, for some $s \in \mathcal{S}$. Since $(\widetilde{p}^{sC})_{s \in \mathcal{S}} >> 0$, define

$$\widehat{x}_h^{sc} := \begin{cases} \widetilde{x}_h^{sC} - \frac{1}{\widetilde{p}^{sC}} \left(\langle \widetilde{p}^s, \widetilde{x}_h{}^s - e_h^s \rangle_C - \widetilde{p}^{sC} \langle y^s, \widetilde{b}_h \rangle_A \right) & \text{if } sc = sC \\ \widetilde{x}_h^{sc} & \text{otherwise.} \end{cases}$$

Then $\left(\widehat{x}_h, \widetilde{b}_h\right) \in \Gamma_h(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^{\mathbf{1}})$ and from Assumption 2.2 one has $\widehat{x}_h \in P_h(\widetilde{x}_h)$, contradicting the fact that $\widetilde{x}_h$ verifies (1).

3. It follows from summing up with respect to h the equalities obtained above.
4. Suppose that $\widetilde{q} \notin Q(D, Y, B)$, i.e., there exists $h' \in \mathcal{H}$ and $b_{h'}^* \in \mathrm{rec} B_{h'}$ such that $\begin{bmatrix} -\widetilde{q} \\ \widetilde{D}Y \end{bmatrix} b_{h'}^* > 0$. First, suppose that $\langle \widetilde{q}, b_{h'}^* \rangle_A < 0$ and $\widetilde{D} Y b_{h'}^* \geq 0$. Then, one has

$$\langle \widetilde{p}^0, \widetilde{x}_{h'}^0 - e_{h'}^0 \rangle_C + \langle \widetilde{q}, (\widetilde{b}_{h'} + b_{h'}^*) \rangle_A < 0 \,.$$

Then, there exists $\varepsilon > 0$ such that

$$\forall x_{h'}^0 \in \mathbb{R}_+^C \; s.t. \; \|x_{h'}^0 - \widetilde{x}_{h'}^0\| < \varepsilon \quad \Rightarrow \quad \langle \widetilde{p}^0, x_{h'}^0 - e_{h'}^0 \rangle_C + \langle \widetilde{q}, (\widetilde{b}_{h'} - b_{h'}^*) \rangle_A < 0 \,. \qquad (3)$$

From Assumption 2.3, there exists $\widehat{x}_{h'} = (\widehat{x}_{h'}^0, \widetilde{x}_{h'}^1, \dots, \widetilde{x}_{h'}^S) \in \mathbb{R}_+^G$ such that $\|\widehat{x}_{h'}^0 - \widetilde{x}_{h'}^0\| < \varepsilon$ and $\widehat{x}_{h'} \in P_h(\widetilde{x}_{h'})$. Moreover, from Eq. 3 it follows $(\widehat{x}_{h'}, (\widetilde{b}_{h'} + b_{h'}^*)) \in \Gamma_{h'}(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^{\mathbf{1}})$, contradicting the fact that $\widetilde{x}_{h'}$ satisfies (1).

Assume now that $\langle \widetilde{q}, b^*_{h'} \rangle_A \leq 0$ and $\widetilde{D} Y b^*_{h'} > 0$. Let $s^* \in \mathcal{S}$ be a state corresponding to a strictly positive component of the vector $\widetilde{D} Y b^*_{h'}$. Define $\left(\widehat{x}_{h'}, \widehat{b}_{h'}\right)$ as follows:

$$\widehat{x}^{sc}_{h'} := \begin{cases} \widetilde{x}^{s^*C}_{h'} + \langle y^{s^*}, b^*_{h'} \rangle_A & \text{if} \quad sc = s^*C, \\ \widetilde{x}^{sc}_{h'} & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \widehat{b}_{h'} = \widetilde{b}_{h'} + b^*_{h'}.$$

This vector satisfies all budget constraints and from Assumption 2.3, we obtain a contradiction. Hence $\widetilde{q} \in Q(D, Y, B)$. □

We consider the following price sets.

$$\Delta_0 := \left\{ (q, p^0) \in \mathrm{Cl}\,(Q\,(Y, B)) \times \mathbb{R}^C_+ : \sum_{c \in \mathcal{C}} p^{0c} + \sum_{a \in \mathcal{A}} q^a = 1 + \rho \right\},$$

$$\Delta_s := \left\{ p^s \in \mathbb{R}^C_+ : \sum_{c \in \mathcal{C}} p^{sc} = 1 \right\}, \qquad \Delta_{\mathbf{1}} := \prod_{s \in \mathcal{S}} \Delta_s, \qquad \Delta := \prod_{s \in \mathcal{S}^0} \Delta_s \quad .$$

For any $h \in \mathcal{H}$, let $\Gamma_h : \Delta \rightrightarrows \mathbb{R}^G_+ \times B_h$ be the budget set-valued function,

$$\Gamma_h(q, p^0, p^{\mathbf{1}}) : = \{(x_h, b_h) \in \mathbb{R}^G_+ \times B_h : \langle p^0, x^0_h - e^0_h \rangle_C + \langle q, b_h \rangle_A \leq 0,$$

$$\langle p^s, x^s_h - e^s_h \rangle_C - p^{sC} \langle y^s, b_h \rangle_A \leq 0, \quad \forall s \in \mathcal{S}\},$$

for any $\left(q, p^0, p^{\mathbf{1}}\right) \in \Delta$. We define $\Gamma\left(q, p^0, p^{\mathbf{1}}\right) := \prod_{h \in \mathcal{H}} \Gamma_h \left(q, p^0, p^{\mathbf{1}}\right)$.

The following proposition describes useful properties of Γ_h.

Proposition 3.7 *Let Assumptions 4.1, 4.2 be satisfied. Then, for any $h \in \mathcal{H}$, the set-valued function Γ_h is*

1. *nonempty and convex valued;*
2. *closed;*
3. *compact valued for any $\left(q, p^0, p^{\mathbf{1}}\right) \in \Delta$ such that $p \in \mathbb{R}^G_{++}$ and $q \in Q(Y, B)$;*
4. *lower semicontinuous for any $\left(q, p^0, p^{\mathbf{1}}\right) \in \Delta$ such that $\left(p^{sC}\right)_{s \in \mathcal{S}^0} >> 0$;*
5. *upper semicontinuous for any $\left(q, p^0, p^{\mathbf{1}}\right) \in \Delta$ such that $p \in \mathbb{R}^G_{++}$ and $q \in Q(Y, B)$.*[11]

Proof 1. Since $0_A \in B_h$ (from Assumption 4.1), one has that $(e_h, 0_A) \in \mathbb{R}^G_+ \times B_h$, then $(e_h, 0) \in \Gamma_h\left(q, p^0, p^{\mathbf{1}}\right)$ for any $\left(q, p^0, p^{\mathbf{1}}\right) \in \Delta$. Convexity is obvious.

2. Let $\{\left(q_n, p^0_n, p^{\mathbf{1}}_n\right)\}_{n \in \mathbb{N}} \subseteq \Delta$ be a sequence such that $\lim\limits_{n \to +\infty} \left(q_n, p^0_n, p^{\mathbf{1}}_n\right) = \left(q, p^0, p^{\mathbf{1}}\right)$ and let $\{\left(x_{h,n}, b_{h,n}\right)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^G_+ \times \mathbb{R}^A$ be such that $\left(x_{h,n}, b_{h,n}\right) \in \Gamma_h\left(q_n, p^0_n, p^{\mathbf{1}}_n\right)$ and $\lim\limits_{n \to +\infty} \left(x_{h,n}, b_{h,n}\right) = (x_h, b_h)$. Since B_h is a closed set, one

[11] Recall that, from Remark 2.1, if $\left(p^{sC}\right)_{s \in \mathcal{S}} \in \mathbb{R}^S_{++}$, then $Q_h(Y, B_h) = Q_h(D, Y, B_h)$.

has $(x_h, b_h) \in \mathbb{R}^G_+ \times B_h$ and from assumptions

$$\langle p^0_n, x^0_{h,n} - e^0_h\rangle_C + \langle q_n, b_{h,n}\rangle_A \leq 0,$$
$$\langle p^s_n, x^s_{h,n} - e^s_h\rangle_C - p^{sC}_n \langle y^s, b_{h,n}\rangle_A \leq 0 \qquad \forall s \in \mathcal{S},$$

taking limits we get that $(x_h, b_h) \in \Gamma_h(q, p^0, p^{\mathbf{1}})$. Then Γ_h is closed.

3. Since B_h is a closed set and Γ_h is defined in terms of weak inequalities via continuous function, it is closed valued. We are left with showing that Γ_h is bounded valued. Suppose otherwise. Then there exists a sequence $\{(x_{h,n}, b_{h,n})\}_{n\in\mathbb{N}} \subseteq \Gamma_h \left(q, p^0, p^{\mathbf{1}}\right)$ such that $\lim_{n\to+\infty} \left\|(x_{h,n}, b_{h,n})\right\| = +\infty$. Consider

$$\alpha_n := \left(\alpha_{0,n}, (\alpha_{s,n})_{s\in\mathcal{S}}, \alpha_{b,n}\right) = \frac{1}{\left\|(x_{h,n}, b_{h,n})\right\|} \left(x^0_{h,n}, (x^s_{h,n})_{s\in\mathcal{S}}, b_{h,n}\right).$$

Since, for every $n \in \mathbb{N}$, $\|\alpha_n\| = 1$, the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ lies in the boundary of the closed ball of radius 1, $\partial \mathrm{B}(0, 1)$; then, without loss of generality, one has

$$\lim_{n\to\infty} \alpha_n = \alpha := (\alpha_0, (\alpha_s)_{s\in\mathcal{S}}, \alpha_b) \ ,$$

and α is such that $\alpha \in \partial\mathrm{B}(0, 1)$, $\alpha_0 \geq 0$, $\alpha_s \geq 0$ for all $s \in \mathcal{S}$, $\alpha \neq 0$ and from Proposition 3.3.2, $\alpha_b \in \operatorname{rec} B_h$. Moreover, since $(x_{h,n}, b_{h,n}) \in \Gamma_h(q, p^0, p^{\mathbf{1}})$, for all $n \in \mathbb{N}$, one has:

$$\langle p^0, \alpha_{0,n}\rangle_C - \frac{1}{\left\|(x_{h,n}, \alpha_{b,n})\right\|} \langle p^0, e^0_h\rangle_C + \langle q, \alpha_{b,n}\rangle_A \leq 0,$$

$$\langle p^s, \alpha_{s,n}\rangle_C - \frac{1}{\left\|(x_{h,n}, \alpha_{b,n})\right\|} \langle p^s, e^s_h\rangle_C - p^{sC} \langle y^s, \alpha_{b,n}\rangle_A \leq 0 \qquad \forall s \in \mathcal{S}.$$

and, taking limits for $n \to +\infty$, we get

$$\langle p^0, \alpha_0\rangle_C + \langle q, \alpha_b\rangle_A \leq 0, \qquad \langle p^s, \alpha_s\rangle_C - p^{sC} \langle y^s, \alpha_b\rangle_A \leq 0 \qquad \forall s \in \mathcal{S}. \tag{4}$$

We, now, distinguish two cases:

- If$(\alpha_s)_{s\in\mathcal{S}^0} \neq 0$, that is there exists $s \in \mathcal{S}^0$ such that $\alpha_s > 0$. Then, from Eq. 4, one has, for some $s \in \mathcal{S}$,

$$-\langle q, \alpha_b\rangle_A \geq \langle p^0, \alpha_0\rangle_C > 0 \quad \text{or} \quad p^{sC} \langle y^s, \alpha_b\rangle_A \geq \langle p^s, \alpha_s\rangle_C > 0,$$

  with $\alpha_b \in \operatorname{rec} B_h$ and $\langle p^s, \alpha_s\rangle_C \geq 0$ for any $s \in \mathcal{S}^0$, which contradicts the fact that $q \in Q(Y, B)$.
- If $\alpha_b \neq 0$. We suppose that $\alpha_s = 0$ for any $s \in \mathcal{S}^0$. From Eq. 4, it follows

$$-\langle q, \alpha_b\rangle_A \geq 0 \qquad \text{and} \qquad p^{sC} \langle y^s, \alpha_b\rangle_A \geq 0 \qquad \text{for all } s \in \mathcal{S}.$$

  If $\langle y^s, \alpha_b\rangle_A > 0$ for some $s \in \mathcal{S}$, since $p \in \mathbb{R}^G_{++}$, $p^{sC} \langle y^s, \alpha_b\rangle_A > 0$ with $\alpha_b \in \operatorname{rec} B_h$, this contradicts the fact that $q \in Q(Y, B)$.

  If $\langle y^s, \alpha_b\rangle_A = 0$ for all $s \in \mathcal{S}$, one has $\alpha_b \neq 0_A$ and $\alpha_b \in \operatorname{Ker} Y \cap \operatorname{rec} B_h$, and this contradicts Assumption 4.2.

Hence, we can conclude that, for any $\left(q, p^0, p^{\mathbf{1}}\right) \in \Delta$ such that $p \in \mathbb{R}^G_{++}$ and $q \in Q(Y, B)$, $\Gamma_h(q, p^0, p^{\mathbf{1}})$ is a compact set.

4. To prove the lower semicontinuity of Γ_h, we define the set-valued function $\widetilde{\Gamma}_h : \Delta \rightrightarrows \mathbb{R}^G_+ \times B_h$ such that for any $\left(q, p^0, p^{\mathbf{1}}\right) \in \Delta$

$$\widetilde{\Gamma}(q, p^0, p^{\mathbf{1}}) := \{(x_h, b_h) \in \mathbb{R}^G_+ \times B_h : \langle p^0, x^0_h - e^0_h \rangle_C + \langle q, b_h \rangle_A < 0,$$

$$\langle p^s, x^s_h - e^s_h \rangle_C - p^{sC} \langle y^s, b_h \rangle_A < 0 \quad \forall s \in \mathcal{S}\}.$$

Since Cl $(\widetilde{\Gamma}_h) = \Gamma_h$ and since a set-valued function is lower semicontinuous if and only if its closure is lower semicontinuous, it is sufficient to prove that $\widetilde{\Gamma}_h$ is lower semicontinuous. Firstly, we have $(x_h, b_h) = (0_G, 0_A) \in \widetilde{\Gamma}_h(q, p^0, p^{\mathbf{1}})$, that is $\widetilde{\Gamma}_h$ is nonempty valued. Observe that, from Proposition 4 in Ok (2007) p. 229, to show the lower semicontinuity of $\widetilde{\Gamma}_h$ it is sufficient to prove that for every sequence $\{(q_n, p^0_n, p^{\mathbf{1}}_n)\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}(q_n, p^0_n, p^{\mathbf{1}}_n) = (q, p^0, p^{\mathbf{1}})$ and any $(x_h, b_h) \in \widetilde{\Gamma}_h(q, p^0, p^{\mathbf{1}})$, there exists $\{(x_{h,n}, b_{h,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^G_+ \times B_h$ such that $\lim\limits_{n\to+\infty}(x_{h,n}, b_{h,n}) = (x_h, b_h)$ and $(x_{h,n}, b_{h,n}) \in \widetilde{\Gamma}_h(q_n, p^0_n, p^{\mathbf{1}}_n)$ for each n. Hence, for every sequence $\{(q_n, p^0_n, p^{\mathbf{1}}_n)\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty}(q_n, p^0_n, p^{\mathbf{1}}_n) = (q, p^0, p^{\mathbf{1}})$, let $(x_h, b_h) \in \widetilde{\Gamma}_h(q, p^0, p^{\mathbf{1}})$. Since

$$\lim_{n\to+\infty}\left(\langle p^0_n, x^0_h - e^0_h \rangle_C + \langle q_n, b_h \rangle_A\right) = \left(\langle p^0, x^0_h - e^0_h \rangle_C + \langle q, b_h \rangle_A\right) < 0,$$

and, for any $s \in \mathcal{S}$,

$$\lim_{n\to+\infty}\left(\langle p^s_n, x^s_h - e^s_h \rangle_C - p^{sC}_n \langle y^s, b_h \rangle_A\right) = \left(\langle p^s, x^s_h - e^s_h \rangle_C - p^{sC} \langle y^s, b_h \rangle_A\right) < 0,$$

there exists $\nu \in \mathbb{N}$ such that for all $n > \nu$ one has $(x_h, b_h) \in \widetilde{\Gamma}_h(q_n, p^0_n, p^{\mathbf{1}}_n)$. Then, we can choose the sequence $\{(x_{h,n}, b_{h,n})\}_{n\in\mathbb{N}}$ as $(x_{h,n}, b_{h,n}) = (x_h, b_h)$ and we can conclude that $\widetilde{\Gamma}_h$ is lower semicontinuous.
5. It follows from the four results above and Lemma 1, page 33, in Hildebrand (1974). □

4 The Sequence of Variational Inequality Problems

First of all we observe that, given our assumptions on the utility functions, the no-free lunch good price set is $\mathbb{R}^G_{++}$. Now, we consider a non-zero lower bound on prices and to this aim we define the following prices sets

$$\Delta^n_0 := \left\{(q, p^0) \in \Delta_0 : p^0 \geq \frac{1}{n}\mathbf{1}_C, \;\; q \in \left\{\frac{1}{n}\mathbf{1}_S Y\right\} + \mathrm{Cl}\,(\in Q(Y, B)), \; q \geq -n\mathbf{1}_A\right\},$$

$$\Delta^n_s := \left\{p^s \in \Delta_s : p^s \geq \frac{1}{n}\mathbf{1}_C\right\}, \qquad \Delta^n_{\mathbf{1}} := \prod_{s\in\mathcal{S}} \Delta^n_s, \qquad \text{and} \qquad \Delta^n := \prod_{s\in\mathcal{S}^0} \Delta_s.$$

Proposition 4.1 *Let $\left(q, p^0, p^1\right) \in \Delta^n$ be given; the following properties hold true.*

1. $Q(D, Y, B) = Q(Y, B) := Q$ *and* $Q^u(D, Y) = Q^u(Y) := Q^u$;
2. *for any* $n \in \mathbb{N}$, *one has*
$$\frac{1}{n}\mathbf{1}_S Y \in Q^u \subseteq Q \subseteq \mathrm{Cl}(Q), \quad \left\{\frac{1}{n}\mathbf{1}_S Y\right\} + \mathrm{Cl}(Q) \subseteq \mathrm{Cl}(Q)$$
and $\mathbf{1}_S Y \in \left\{\frac{1}{n}\mathbf{1}_S Y\right\} + \mathrm{Cl}(Q)$;
3. *let* $\{q_n\}_{n\in\mathbb{N}}$ *be a sequence such that* $\lim_{n\to+\infty} q_n = q$ *and* $q \in Q$. *Then, there exists* $\nu \in \mathbb{N}$ *such that for all* $n > \nu$ *one has* $q_n \in \left\{\frac{1}{n}\mathbf{1}_S Y\right\} + \mathrm{Cl}(Q)$;
4. $q \in Q$ *and* $p^0 \in \mathbb{R}^S_{++}$.

Proof 1. It follows from Remark 2.1.

2. By definition of Q^u, point 1. above and from Proposition 3.1.1, $\frac{1}{n}\mathbf{1}_S Y \in Q^u \subseteq Q \subseteq \mathrm{Cl}(Q)$.

   Since from Proposition 3.1, $\mathrm{Cl}(Q)$ is a convex cone,[12] we do have that for any $q \in \mathrm{Cl}(Q)$, $\frac{1}{n}\mathbf{1}_S Y + q \in \mathrm{Cl}(Q)$.

   Finally, since $\left(1 - \frac{1}{n}\right)\mathbf{1}_S Y \in Q^u \subseteq Q \subseteq \mathrm{Cl}(Q)$, then $\mathbf{1}_S Y = \frac{1}{n}\mathbf{1}_S Y + \left(1 - \frac{1}{n}\right)\mathbf{1}_S Y \in \left\{\frac{1}{n}\mathbf{1}_S Y\right\} + \mathrm{Cl}(Q)$.

3. Since $q \in Q = \mathrm{int}\, Q$, there exists $\delta > 0$ and $\nu_1 \in \mathbb{N}$ such that for all $n > \nu_1$ it results $q_n \in \overline{\mathrm{B}}\left(q, \frac{\delta}{2}\right) \subseteq \mathrm{B}(q, \delta) \subseteq Q$. Moreover, for any $n > \max\left\{\nu_1, \frac{2\|\mathbf{1}_S Y\|}{\delta}\right\}$, we have
$$q_n = \frac{1}{n}\mathbf{1}_S Y + \left(q_n - \frac{1}{n}\mathbf{1}_S Y\right) \in \left\{\frac{1}{n}\mathbf{1}_S Y\right\} + \mathrm{Cl}(Q).$$
   Indeed, since
$$\|\left(q_n - \frac{1}{n}\mathbf{1}_S Y\right) - q\| \le \frac{1}{n}\|\mathbf{1}_S Y\| + \|q_n - q\| \le \frac{1}{n}\|\mathbf{1}_S Y\| + \frac{\delta}{2} < \delta,$$
   one has $\left(q_n - \frac{1}{n}\mathbf{1}_S Y\right) \in \mathrm{B}(q, \delta) \subseteq Q$.

4. It is enough to prove that $q \in Q$. Since $q \in \left\{\frac{1}{n}\mathbf{1}_S Y\right\} + \mathrm{Cl}(Q)$, then there exists $q' \in \mathrm{Cl}(Q)$ such that
$$q = \frac{1}{n}\mathbf{1}_S Y + q' \quad \Rightarrow \quad q = \left(1 - \frac{1}{n}\right)\frac{1}{n}\mathbf{1}_S Y + \frac{1}{n}\left(\frac{1}{n}\mathbf{1}_S Y + nq'\right).$$
   Since $\frac{1}{n}\mathbf{1}_S Y \in Q$, where Q is open, $\frac{1}{n}\mathbf{1}_S Y \in \mathrm{Int}(Q)$. Moreover, since $\mathrm{Cl}(Q)$ is a cone $nq' \in \mathrm{Cl}(Q)$ and, from item 1, $\left(\frac{1}{n}\mathbf{1}_S Y + nq'\right) \in \mathrm{Cl}(Q)$. Then, since q can be written as a convex combination of an element in the interior of Q and an element in the closure of Q, one has $q \in \mathrm{Int}(Q) = Q$. □

[12] Recall that $A \subseteq \mathbb{R}^n$ is a convex cone if and only if $\forall a, b \in A, \lambda, \mu \ge 0, \lambda a + \mu b \in A$.

Proposition 4.2 *For any $n \geq C$ and $n^2 > \max_{a \in \mathcal{A}} \left\{ -\sum_{s \in \mathcal{S}} y^{sa} \right\}$, one has:*

1. *Δ_0^n is nonempty, convex, compact;*
2. *Δ_s^n is nonempty, convex, compact for any $s \in \mathcal{S}$;*
3. *Δ^n is nonempty, convex, compact.*

Proof 1. *Δ_0^n is nonempty*. Take $(\widehat{q}, \widehat{p}^0)$ such that

$$\widehat{p}^{0c} := \begin{cases} \frac{1}{n} & \text{if } c \neq C, \\ 1 - \frac{C-1}{n} + \left(1 - \frac{1}{n}\right)\rho & \text{if } c = C, \end{cases} \qquad \text{and} \qquad \widehat{q} := \frac{1}{n} \mathbf{1}_S Y\,,$$

One has $(\widehat{q}, \widehat{p}^0) \in \Delta_0^n$. Indeed, $\widehat{p}^0 \geq \frac{1}{n}\mathbf{1}_C$. For $c \neq C$, the result is obvious; for $c = C$, we have

$$1 - \frac{C-1}{n} + \left(1 - \frac{1}{n}\right)\rho \geq \frac{1}{n} \qquad \Leftrightarrow \qquad n \geq \frac{C+\rho}{1+\rho}$$

which is true since, from Proposition 3.5, $1 + \rho > 0$ and since we assumed $n \geq C$.

From Proposition 3.1.5, it follows that $0_A \in \mathrm{Cl}\,(Q)$, then $\widehat{q} \in \left\{\frac{1}{n}\mathbf{1}_S Y\right\} + \mathrm{Cl}\,(Q)$ and, from Proposition 4.1.2, $\widehat{q} \in \mathrm{Cl}\,(Q)$. Moreover, $\widehat{q} \geq -n\mathbf{1}_A$ if and only if $n^2 \geq -\sum_{s \in \mathcal{S}} y^{sa}$ for any $a \in \mathcal{A}$, as we assumed.

Finally, since $\sum_{a \in \mathcal{A}} \sum_{s \in \mathcal{S}} y^{sa} = \rho$, we have

$$\sum_{c \in \mathcal{C}} \widehat{p}^{0c} + \sum_{a \in \mathcal{A}} \widehat{q}^a = (C-1)\frac{1}{n} + 1 - \frac{C-1}{n} + \left(1 - \frac{1}{n}\right)\rho + \frac{1}{n}\sum_{a \in \mathcal{A}}\sum_{s \in \mathcal{S}} y^{sa} = 1 + \rho\,.$$

Then $(\widehat{q}, \widehat{p}^0) \in \Delta_0^n$.

Δ_0^n is closed.

By definition of Δ_0^n, we have that it is a closed set because defined in terms of weak inequalities via continuous functions on the closed set Δ_0 and because $\left\{\frac{1}{n}\mathbf{1}_S Y\right\} + \mathrm{Cl}\,(Q)$ is the sum of a compact set and a closed set and therefore it is closed.

Δ_0^n is bounded.

By definition, Δ_0^n is bounded below. In order to prove that it is bounded above, observe that, by definition $q \geq -n\mathbf{1}_A$, i.e., for any $a \in \mathcal{A}$, $q^a \geq -n$ and then we also have $\sum_{a \in \mathcal{A}} q^a \geq -nA$ and $-\sum_{a \in \mathcal{A}} q^a \leq nA$. Then, for any $c' \in \mathcal{C}$, $p^{0c'} = 1 + \rho - \sum_{c \neq c'} p^{0c} - \sum_{a \in \mathcal{A}} q^a \leq 1 + \rho + nA$; for any $a' \in \mathcal{A}$, $q^{a'} = 1 + \rho - \sum_{c \in \mathcal{C}} p^{0c} - \sum_{a \neq a'} q^a \leq 1 + \rho + n(A-1)$.

Δ_0^n is convex.

It follows from the convexity of $\mathrm{Cl}\,(Q)$ given in Proposition 3.1.6.

Δ_s^n is nonempty.

Take $\widehat{p}^s \in \mathbb{R}^C_+$ such that

$$\widehat{p}^{sc} := \begin{cases} \frac{1}{n} & \text{if } c \neq C, \\ 1 - \frac{C-1}{n} & \text{if } c = C. \end{cases}$$

One has $\widehat{p}^s \in \Delta^n_s$. Indeed, clearly for $c \neq C$, $\widehat{p}^s \geq \frac{1}{n}\mathbf{1}_C$; for $c = C$, $\widehat{p}^s \geq \frac{1}{n}\mathbf{1}_C$ if and only if $n \geq C$. Moreover $\sum_{c \in \mathcal{C}} \widehat{p}^{sc} = (C-1)\frac{1}{n} + 1 - \frac{C-1}{n} = 1$.

Δ^n_s is compact.

Δ^n_s is a closed subset of the simplex, which is a compact set, and therefore Δ^n_s is a compact set, as well.

Δ^n_s is convex.

It follows from the convexity of Δ_s.

Δ^n is nonempty, compact and convex.

It follows from all properties of sets Δ^n_s, with $s \in \mathcal{S}^0$. □

Now, in order to relate the consumer h's maximization problem with a variational problem, we adapt the definition of the operator given in Aussel and Dutta (2008) (see also Eq. 37 of the Appendix). To this aim, for any $h \in \mathcal{H}$, we consider the set-valued function

$$\mathcal{G}_h : \mathbb{R}^G \rightrightarrows \mathbb{R}^G \text{ such that}$$

$$\mathcal{G}_h(x_h) := conv\left(N_h^{>}(x_h) \cap S(0,1)\right) \qquad \forall x_h \in \mathbb{R}^G,$$

where $N_h^{>}(x_h)$ is the normal cone to the convex set $P_h(x_h)$ defined in the Appendix.

Let f_C^C be the element of the canonical base of $\mathbb{R}^C$ with 1 in the component C. Now, we introduce the following GQVI$_n$:

Find $\left((\widetilde{x}_n, \widetilde{b}_n), (\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n)\right) \in \Gamma(\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n) \times \Delta^n$ *such that there exists* $g_n = (g_{h,n})_{h\in\mathcal{H}} \in \prod_{h\in\mathcal{H}} \mathcal{G}_h(\widetilde{x}_{h,n})$ *with*

$$\langle -g_n, x_n - \widetilde{x}_n\rangle_{GH} + \langle \sum_{h\in\mathcal{H}} \widetilde{b}_{h,n}, q_n - \widetilde{q}_n\rangle_A + \langle \sum_{h\in\mathcal{H}} (\widetilde{x}^{\,0}_{h,n} - e^0_h), p^0_n - \widetilde{p}^{\,0}_n\rangle_C$$
$$+ \sum_{s\in\mathcal{S}} \langle \sum_{h\in\mathcal{H}} (\widetilde{x}^s_{h,n} - e^s_h - f^C_C \langle y^s, \widetilde{b}_h\rangle_A), p^s_n - \widetilde{p}^{\,s}_n\rangle_C \leq 0,$$
$$\forall \left((x_n, b_n), (q_n, p^0_n, p^{\mathbf{1}}_n)\right) \in \Gamma(\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\mathbf{1}}_n) \times \Delta^n. \tag{5}$$

Remark 4.1 $\left((\widetilde{x}_n, \widetilde{b}_n), (q_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n)\right)$ is a solution to GQVI$_n$ (5) if and only if, simultaneously we have,

for any $h \in \mathcal{H}$,

$$\langle -g_{h,n}, x_{h,n} - \widetilde{x}_{h,n}\rangle_G \leq 0, \quad \forall (x_{h,n}, b_{h,n}) \in \Gamma_h(\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\mathbf{1}}_n); \tag{6}$$

$$\langle \sum_{h\in\mathcal{H}} \widetilde{b}_{h,n}, q_n - \widetilde{q}_n\rangle_A + \langle \sum_{h\in\mathcal{H}} (\widetilde{x}^{\,0}_{h,n} - e^0_h), p^0_n - \widetilde{p}^{\,0}_n\rangle_C \leq 0 \quad \forall (q_n, p^0_n) \in \Delta^n_0; \tag{7}$$

for any $s \in \mathcal{S}$,

$$\langle \sum_{h\in\mathcal{H}} (\widetilde{x}^s_{h,n} - e^s_h - f^C_C \langle y^s, \widetilde{b}_h\rangle_A), p^{\,s}_n - \widetilde{p}^{\,s}_n\rangle_C \leq 0, \quad \forall p^s_n \in \Delta^n_s. \tag{8}$$

The above equivalence follows by replacing in Eq. 5, respectively,

$$\left((x_n, b_n), (q_n, p_n^0, p_n^{\mathbf{1}})\right) = \left((x_n, b_n), (\widetilde{q}_n, \widetilde{p}_n^0, \widetilde{p}_n^{\mathbf{1}})\right),$$

$$((x_n, b_n), (q_n, p_n^0, p_n^{\mathbf{1}})) = (\widetilde{x}_n, \widetilde{b}_n), (q_n, p_n^0, \widetilde{p}_n^{\mathbf{1}}))$$

and

$$\left((x_n, b_n), (q_n, p_n^0, (p_n^{s'})_{s' \in \mathcal{S}})\right) = \left((\widetilde{x}_n, \widetilde{b}_n), (\widetilde{q}_n, \widetilde{p}_n^0, (\widetilde{p}_n^{s'})_{s' \in \mathcal{S} \setminus s}, p_n^s)\right).$$

Theorem 1 *Let Assumptions 1, 2 and 4 hold true. For any $n \in \mathbb{N}$, with $n \geq C$ and $n^2 > \max_{a \in \mathcal{A}} \{-\sum_{s \in \mathcal{S}} y^{sa}\}$, GQVI$_n$ (5) admits at least one solution.*

Proof To get the desired result, we apply Theorem 4 in the Appendix. Consistently with Definition A.2 in the Appendix, the variational problem (5) represents a generalized quasi-variational inequality associated with

$$C := conv\left(\Gamma\left(\Delta^n\right)\right) \times \Delta^n$$

and, for any $\left((x_n, b_n), (q_n, p_n^0, p_n^1)\right) \in \Gamma(q_n, p_n^0, p_n^1) \times \Delta^n$,

$$S\left((x_n, b_n), (q_n, p_n^0, p_n^1)\right) := \Gamma\left(q_n, p_n^0, p^{\mathbf{1}}\right) \times \Delta^n$$

$$\Phi\left((x_n, b_n), (q_n, p_n^0, p_n^1)\right) :=$$

$$-\left(-\prod_{h \in \mathcal{H}} \mathcal{G}_h(x_{h,n}), \sum_{h \in \mathcal{H}} b_{h,n}, \sum_{h \in \mathcal{H}} (x_{h,n}^0 - e_h^0), \left(\sum_{h \in \mathcal{H}} (x_{h,n}^s - e_h^s - f_C^C \langle y^s, \widetilde{b}_h \rangle_A)\right)_{s \in \mathcal{S}}\right).$$

Now, we have to check that Assumptions of Theorem 4 are satisfied.

- ◇ *C is nonempty, convex, compact*. From Proposition 4.2, Δ^n is nonempty, convex and compact and, from Proposition 3.7, for any $h \in \mathcal{H}$, Γ_h is nonempty, convex, compact valued, closed, lower semicontinuous and upper semicontinuous on Δ^n. Then, $\Gamma(\Delta^n) := \prod_{h \in \mathcal{H}} \Gamma_h(\Delta^n)$ is nonempty, convex, compact valued, closed, lower semicontinuous and upper semicontinuous as well (see Propositions 4 and 8 of Hildebrand (1974)). Hence, $\Gamma(\Delta^n)$ is compact (see Proposition 3 of Hildebrand (1974)) and then $conv(\Gamma(\Delta^n))$ is convex and compact.
- ◇ *The set-valued function $S(\cdot)$ is nonempty, convex, compact valued, closed, lower semicontinuous and upper semicontinuous* . It follows from what said above.
- ◇ *The set-valued function $\Phi(\cdot)$ is nonempty, convex, compact valued, closed and upper semicontinuous*. From Assumptions 2.2, 2.3, one has $P_h(x_h) \neq \emptyset$ for all $x_h \in \mathbb{R}^G$; then the set-valued function $\mathcal{G}_h$ is equal to the set-valued function $\widetilde{\mathcal{G}}$, presented in Eq. 37 in the Appendix. Then, for any $h \in \mathcal{H}$, we have that $\mathcal{G}_h$ is nonempty, convex, compact valued, closed and upper semicontinuous (see Proposition 5 in the Appendix) and furthermore the other components of Φ are continuous functions (see Proposition 4 of Hildebrand (1974)).

  All assumptions of Theorem 4 are satisfied, then GQVI$_n$ (5) associated with C, S and Φ admits at least a solution. □

Theorem 2 *Let Assumptions 2 be satisfied. For any* $n \in \mathbb{N}$ *such that* $n \geq C$, $n > \max_{a\in\mathcal{A}}\left\{-\sum_{s\in\mathcal{S}} y^{sa}\right\}$ *and* $n^2 > \max_{a\in\mathcal{A}}\left\{-\sum_{s\in\mathcal{S}} y^{sa}\right\}$*,let* $\left((\widetilde{x}_n, \widetilde{b}_n), (\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n)\right)$ *be a solution to GQVI*$_n$ *(5). Then, one has*

- (i) *for any* $h \in \mathcal{H}$, $(\widetilde{x}_{h,n}, \widetilde{b}_{h,n})$ *is optimal in the constraints set* $\Gamma_h((\widetilde{q}_n, \widetilde{p}^{\,0}_n), \widetilde{p}^{\mathbf{1}}_n)$.
- (ii) *for any* $h \in \mathcal{H}$ *and* $s \in \mathcal{S}$,

$$\langle \widetilde{p}^{\,0}_n, \widetilde{x}^0_{h,n} - e^0_h\rangle_C + \langle \widetilde{q}_n, \widetilde{b}_{h,n}\rangle_A = 0, \tag{9}$$

$$\langle \widetilde{p}^{\,s}_n, \widetilde{x}^s_{h,n} - e^s_h\rangle_C = \widetilde{p}^{\,sC}_n \langle y^s, \widetilde{b}_{h,n}\rangle; \tag{10}$$

- (iii) *for any* $s \in \mathcal{S}$,

$$\langle \widetilde{p}^{\,0}_n, \sum_{h\in\mathcal{H}} (\widetilde{x}^0_{h,n} - e^0_h)\rangle_C + \langle \widetilde{q}_n, \sum_{h\in\mathcal{H}} \widetilde{b}_{h,n}\rangle_A = 0, \tag{11}$$

$$\langle \widetilde{p}^{\,s}_n, \sum_{h\in\mathcal{H}} (\widetilde{x}^s_{h,n} - e^s_h)\rangle_C = \widetilde{p}^{\,sC}_n \langle y^s, \sum_{h\in\mathcal{H}} \widetilde{b}_{h,n}\rangle; \tag{12}$$

- (iv) *for any* $h \in \mathcal{H}$, $c \in \mathcal{C}$ *and* $s \in \mathcal{S}^0$,

$$0 \leq \widetilde{x}^{sc}_{h,n} \leq \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} e^{0c}_h + C \sum_{s\in\mathcal{S}} \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} e^{sc}_h . \tag{13}$$

Proof Thanks to Remark 4.11, for any $h \in \mathcal{H}$, $(\widetilde{x}_{h,n}, \widetilde{b}_{h,n})$ is a solution to GVI$_n$ (6), $(\widetilde{q}_n, \widetilde{p}^0_n)$ is a solution to Eq. 7 and for all $s \in S$, $\widetilde{p}^s_n$ is a solution to Eq. 8.

- (i). We suppose that, for some $h \in \mathcal{H}$, $(\widetilde{x}_{h,n}, \widetilde{b}_{h,n})$ is not optimal in $\Gamma_h((\widetilde{q}_n, \widetilde{p}^{\,0}_n), \widetilde{p}^{\mathbf{1}}_n)$: there exists $\widetilde{z}_h \in P_h(\widetilde{x}_{h,n}) \cap \Gamma_h((\widetilde{q}_n, \widetilde{p}^{\,0}_n), \widetilde{p}^{\mathbf{1}}_n)$. Since $P_h(\widetilde{x}_{h,n})$ is an open set, there exists $\alpha > 0$ such that $z := \widetilde{z}_h + \alpha g_{h,n} \in P_h(\widetilde{x}_{h,n})$. Hence, since $g_{h,n} \in N^>(\tilde{x}_{h,n})$, one has $\langle g_{h,n}, z - \widetilde{x}_{h,n}\rangle = \alpha\|g_{h,n}\|^2 \leq 0$, that is $\|g_{h,n}\| = 0$. A contradiction with $g_{h,n} \neq 0$.[13]
- (ii) and (iii). Both statements follow from the fact $(\widetilde{x}_{h,n}, \widetilde{b}_{h,n})$ is optimal in $\Gamma_h((\widetilde{q}_n, \widetilde{p}^{\,0}_n), \widetilde{p}^{\mathbf{1}}_n)$ and from Proposition 3.6.
- (iv). From Eq. 11, inequality (7) becomes

$$\langle \sum_{h\in\mathcal{H}} \widetilde{b}_{h,n}, q_n\rangle_A + \langle \sum_{h\in\mathcal{H}} (\widetilde{x}^{\,0}_{h,n} - e^0_h), p^0_n\rangle_C \leq 0 \qquad \forall (q_n, p^0_n) \in \Delta^n_0; \tag{14}$$

and, from Eq. 10, for any $s \in \mathcal{S}$, inequality (8) becomes

$$\langle \sum_{h\in\mathcal{H}} (\widetilde{x}^s_{h,n} - e^s_h), p^{\,s}_n\rangle_C - p^{sC}_n \langle y^s, \sum_{h\in\mathcal{H}} \widetilde{b}_{h,n}\rangle_A \leq 0 \qquad \forall p^s_n \in \Delta^n_s$$

that is

$$\langle \sum_{h\in\mathcal{H}} (\widetilde{x}^s_{h,n} - e^s_h), \frac{p^{\,s}_n}{p^{sC}_n}\rangle_C - \langle y^s, \sum_{h\in\mathcal{H}} \widetilde{b}_{h,n}\rangle_A \leq 0, \quad \forall p^s_n \in \Delta^n_s . \tag{15}$$

[13] Also in the case of non-complete and non-transitive preferences, we can adapt the results obtained in Aussel (2014) and Aussel and Dutta (2008) and Milasi et al. (2019)

Summing up (14) and (15), for any $(q_n, p_n^0, p_n^{\mathbf{1}}) \in \Delta^n$, one has:

$$\langle \sum_{h\in\mathcal{H}} \widetilde{b}_{h,n}, q_n\rangle_A + \langle \sum_{h\in\mathcal{H}} (\widetilde{x}^0_{h,n} - e^0_h), p^0_n\rangle_C + \sum_{s\in\mathcal{S}} \langle \sum_{h\in\mathcal{H}} (\widetilde{x}^s_{h,n} - e^s_h), \frac{p^s_n}{p^{sC}_n}\rangle_C$$
$$- \langle \sum_{s\in\mathcal{S}} y^s, \sum_{h\in\mathcal{H}} \widetilde{b}_{h,n}\rangle_A \leq 0 . \tag{16}$$

Now, choose $q_n = \mathbf{1}_S Y$ and $\widehat{p}^{sc} = \frac{1}{C}$[14] for any $c \in \mathcal{C}$ and $s \in \mathcal{S}^0$. From Proposition 4.1.2, $\mathbf{1}_S Y \in \{\frac{1}{n}\mathbf{1}_S Y\} + \mathrm{Cl}(Q)$; since $n > \max_{a\in\mathcal{A}} \left\{-\sum_{s\in\mathcal{S}} y^{sa}\right\}$, $q_n = \sum_{s\in\mathcal{S}} y^s \geq -n\mathbf{1}_A$, and $\sum_{a\in\mathcal{A}} q^a_n = \rho$. Hence $(q_n, \widehat{p}^0_n, \widehat{p}^{\mathbf{1}}_n) \in \Delta^n$ and replacing $(q_n, p^0_n, p^{\mathbf{1}}_n)$ with $(\mathbf{1}_S Y, \widehat{p}^0_n, \widehat{p}^{\mathbf{1}}_n)$ in Eq. 16,

$$\frac{1}{C} \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} (\widetilde{x}^{0c}_{h,n} - e^{0c}_h) + \sum_{s\in\mathcal{S}} \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} (\widetilde{x}^{sc}_{h,n} - e^{sc}_h) \leq 0.$$

Then

$$\frac{1}{C} \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} \widetilde{x}^{0c}_{h,n} + \sum_{s\in\mathcal{S}} \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} \widetilde{x}^{sc}_{h,n} \leq \frac{1}{C} \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} e^{0c}_h + \sum_{s\in\mathcal{S}} \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} e^{sc}_h .$$

Hence, being $C > 1$, for any $s \in \mathcal{S}^0$, $c \in \mathcal{C}$ and $h \in \mathcal{H}$, we have:

$$0 \leq \widetilde{x}^{sc}_{h,n} \leq \sum_{s\in\mathcal{S}^0} \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} \widetilde{x}^{sc}_{h,n} < \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} \widetilde{x}^{0c}_{h,n} + C \sum_{s\in\mathcal{S}} \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} \widetilde{x}^{sc}_{h,n} \leq \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} e^{0c}_h$$
$$+ C \sum_{s\in\mathcal{S}} \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} e^{sc}_h .$$

□

Proposition 4.3 *Let Assumptions 1, 2, 3 and 4.3 be satisfied.*
Let $\left\{\left((\widetilde{x}_n, \widetilde{b}_n), (\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n)\right)\right\}_{n\in\mathbb{N}}$ *be the sequence such that, for any* $n \in \mathbb{N}$ *with* $n \geq C$ *and* $n^2 > \max_{a\in\mathcal{A}} \left\{-\sum_{s\in\mathcal{S}} y^{sa}\right\}$, $\left((\widetilde{x}_n, \widetilde{b}_n), (\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n)\right)$ *is a solution to* $GQVI_n$ *(5). Then there exists a subsequence converging to* $((\widetilde{x}, \widetilde{b}), (\widetilde{q}, \widetilde{p}^{\,0}, \widetilde{p}^{\,\mathbf{1}}))$ *such that* $(\widetilde{q}, \widetilde{p}^{\,0}, \widetilde{p}^{\,\mathbf{1}}) \in \Delta$ *with* $\widetilde{p} >> 0$, $\widetilde{q} \in Q$ *and* $(\widetilde{x}, \widetilde{b}) \in \Gamma(\widetilde{q}, \widetilde{p}^{\,0}, \widetilde{p}^{\,\mathbf{1}})$.

Proof One has:

◁ $\lim_{n\to+\infty} (\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n) = (\widetilde{q}, \widetilde{p}^{\,0}, \widetilde{p}^{\,\mathbf{1}}) \in \Delta$.

Since $\{(\widetilde{q}_n, \widetilde{p}^{\,0}_n)\}_{n\in\mathbb{N}} \subseteq \Delta_0$ and for any $s \in \mathcal{S}$, $\{\widetilde{p}^{\,s}_n\}_{n\in\mathbb{N}} \subseteq \Delta_s$, with Δ_0 and Δ_s compact sets, without loss of generality, for any $s \in \mathcal{S}$, it follows that

$$\lim_{n\to+\infty} (\widetilde{q}_n, \widetilde{p}^{\,0}_n) = (\widetilde{q}, \widetilde{p}^{\,0}) \in \Delta_0, \qquad \lim_{n\to+\infty} \widetilde{p}^{\,s}_n = \widetilde{p}^{\,s} \in \Delta_s .$$

Hence $(\widetilde{q}, \widetilde{p}^{\,0}, \widetilde{p}^{\,\mathbf{1}}) \in \Delta$.

$$\lhd\ For any h \in \mathcal{H}, \lim_{n\to+\infty} \widetilde{x}_{h,n} = \widetilde{x}_h \in \mathbb{R}^G .$$

[14] Observe that $\frac{1}{C} \geq \frac{1}{n}$ since by assumption $n \geq C$.

From inequality (13), for any $h \in \mathcal{H}$, $s \in \mathcal{S}^0$ and $c \in \mathcal{C}$, the sequence $\{\widetilde{x}_{h,n}^{sc}\}_{n\in\mathbb{N}}$ is bounded, then claim holds.

$$\triangleleft\ For any s \in \mathcal{S},\ \widetilde{p}^{\,s} >> 0.$$

Suppose that there exist $s' \in \mathcal{S}$ and $c' \in \mathcal{C}$ such that $\widetilde{p}^{s'c'} = 0$. From Assumption 3 there exists $\varepsilon > 0$ and $h' \in \mathcal{H}$ such that $P_{h'|\mathcal{B}(x_{h'},\varepsilon)}$ is strictly increasing in $s'c'$. Define $(\widehat{x}_{h',n}, \widehat{b}_{h',n})$ such that

$$\widehat{x}_{h',n}^{sc} := \begin{cases} (1-\widetilde{p}_n^{s'c'})\widetilde{x}_{h',n}^{s'c'} + \frac{1}{m}\langle \widetilde{p}_n^{s'}, e_{h'}^{s'}\rangle_C & \text{if} \quad sc = s'c', \\ (1-\widetilde{p}_n^{s'c'})\widetilde{x}_{h',n}^{sc} & \text{otherwise,} \end{cases} \qquad \widehat{b}_{h',n} := (1-\widetilde{p}_n^{s'c'})\widetilde{b}_{h',n}$$

with $m \in \mathbb{N}$. Firstly we observe that, since for n sufficiently large $(1-\widetilde{p}_n^{s'c'}) \in (0, 1]$, from Proposition 3.2.1, we have that $\widehat{b}_{h',n} \in B_{h'}$. One has that $\lim_{n\to+\infty} \widehat{x}_{h',n} = \widehat{x}_{h'}$ where

$$\widehat{x}_{h'}^{sc} := \begin{cases} \widetilde{x}_{h'}^{s'c'} + \frac{1}{m}\langle \widetilde{p}^{s'}, e_{h'}^{s'}\rangle_C & \text{if} \quad sc = s'c', \\ \widetilde{x}_{h'}^{sc} & \text{otherwise.} \end{cases}$$

Since $\widetilde{p}^{s'} \in \Delta^{s'}$ and from Assumption 1, we have that $\langle \widetilde{p}^{s'}, e_{h'}^{s'}\rangle_C > 0$. Since $\widehat{x}_{h'} \geq \widetilde{x}_{h'}$ and $\widehat{x}_{h'}^{s'c'} > \widetilde{x}_{h'}^{s'c'}$, and for m sufficiently large $\widehat{x}_{h'} \in \mathcal{B}(x_{h'}, \varepsilon)$, from Assumption 3, it follows that $\widehat{x}_{h'} \in P_h(\widetilde{x}_{h'})$.

Moreover, from Assumption on preference set-valued map ($P_h(\widetilde{x}_{h'})$ is an open set) there exists $n_1 \in \mathbb{N}$ such that $\forall n > n_1$,

$$\widehat{x}_{h',n} \in P_{h'}\left(\widetilde{x}_{h',n}\right) \tag{17}$$

Now, we show that

$$(\widehat{x}_{h',n}, \widehat{b}_{h',n}) \in \Gamma_{h'}(\widetilde{q}_n, \widetilde{p}_n^0, \widetilde{p}_n^{\mathbf{1}}). \tag{18}$$

Taking into account that $(\widetilde{x}_{h',n}, \widetilde{b}_{h',n}) \in \Gamma_{h'}(\widetilde{q}_n, \widetilde{p}_n^0, \widetilde{p}_n^{\mathbf{1}})$, to check (18), we go through 3 steps.

Step 1. At $s = 0$. We have that

$$\begin{aligned} &\langle \widetilde{p}_n^0, \widehat{x}_{h',n}^0 - e_{h'}^0\rangle_C + \langle \widetilde{q}_n, \widehat{b}_{h',n}\rangle_A \\ &= (1-\widetilde{p}_n^{s'c'})\Big[\langle \widetilde{p}_n^0, \widetilde{x}_{h',n}^0 - e_{h'}^0\rangle_C + \langle q_n, \widetilde{b}_{h',n}\rangle_A\Big] - \widetilde{p}_n^{s'c'}\langle \widetilde{p}_n^0, e_{h'}^0\rangle_C < 0. \end{aligned} \tag{19}$$

Step 2. At $s \in \mathcal{S}\setminus\{s'\}$.

We have that

$$\begin{aligned} &\langle \widetilde{p}_n^s, \widehat{x}_{h',n}^s - e_{h'}^s\rangle_C - \widetilde{p}_n^{sC}\langle y^s, \widehat{b}_{h',n}\rangle_A \\ &= (1-\widetilde{p}_n^{s'c'})\Big[\langle \widetilde{p}_n^s, \widetilde{x}_{h',n}^s - e_{h'}^s\rangle_C - \widetilde{p}_n^{sC}\langle y^s, \widetilde{b}_{h',n}\rangle_A\Big] - \widetilde{p}_n^{s'c'}\langle \widetilde{p}_n^s, e_{h'}^s\rangle_C < 0. \end{aligned} \tag{20}$$

Step 3. At $s = s'$.

We have that

$$\langle \widetilde{p}^{s'}_n, \widehat{x}^{s'}_{h',n} - e^{s'}_{h'}\rangle_C - \widetilde{p}^{s'C}_n \langle y^{s'}, \widehat{b}_{h',n}\rangle_A = (1-\widetilde{p}^{s'c'}_n)\Big[\langle \widetilde{p}^{s'}_n, \widetilde{x}^{s'}_{h',n} - e^{s'}_{h'}\rangle_C$$
$$-\widetilde{p}^{s'C}_n \langle y^{s'}, \widetilde{b}_{h',n}\rangle_A\Big] + \left(\frac{1}{m} - 1\right)\widetilde{p}^{s'c'}_n \langle \widetilde{p}^{s'}_n, e^{s'}_{h'}\rangle_C < 0. \quad (21)$$

Then, from Eqs. 19, 20 and 21 claim (18) is proved. Hence (17) contradicts the fact that $(\widetilde{x}_{h',n}, \widetilde{b}_{h',n})$ is optimal in $\Gamma_h(((\tilde{q}_n, \tilde{p}_n), \tilde{p}^{\mathbf{1}}_n))$. Then, for any $s \in \mathcal{S}$ it results that $\widetilde{p}^{sc} > 0$ for any $c \in \mathcal{C}$.

$$\lhd \ \widetilde{p}^{\,0} >> 0.$$

Suppose our claim is false, i.e., there exists $c' \in \mathcal{C}$ such that $\widetilde{p}^{\,0c} = 0$. Then either at least a component in $\widetilde{p}^{\,0}$ is zero or all components are zero. Both cases are analyzed below.

Case 1. $\widetilde{p}^0 > 0$.

We suppose that there exists $c' \in \mathcal{C}$ such that $\widetilde{p}^{0c'} = 0$. From Assumption 3 there exists $\varepsilon > 0$ and $h' \in \mathcal{H}$ such that $P_{h'|\mathcal{B}(x_{h'},\varepsilon)}$ is strictly increasing in $0c'$. Define $(\widehat{x}_{h',n}, \widehat{b}_{h',n})$ such that

$$\widehat{x}^{sc}_{h',n} := \begin{cases} (1-\widetilde{p}^{0c'}_n)\widetilde{x}^{0c'}_{h',n} + \frac{1}{m}\langle \widetilde{p}^0_n, e^0_{h'}\rangle_C & \text{if} \quad sc = 0c', \\ \\ (1-\widetilde{p}^{0c'}_n)\widetilde{x}^{sc}_{h',n} & \text{otherwise,} \end{cases}$$

with $m \in \mathbb{N}$, and $\widehat{b}_{h',n} := (1-\widetilde{p}^{0c'}_n)\widetilde{b}_{h',n} \in B_{h'}$. One has that $\lim_{n\to+\infty}\widehat{x}_{h',n} = \widehat{x}_{h'}$ where

$$\widehat{x}^{sc}_{h'} := \begin{cases} \widetilde{x}^{0c'}_{h'} + \frac{1}{m}\langle \widetilde{p}^0, e^0_{h'}\rangle_C & \text{if} \quad sc = 0c', \\ \\ \widetilde{x}^{sc}_{h'} & \text{otherwise.} \end{cases}$$

Since $\widetilde{p}^{0c} > 0$ for some $c \in \mathcal{C}$ and, from Assumption 1, we have that $\langle \widetilde{p}^0, e^0_{h'}\rangle_C > 0$. Since $\widehat{x}_{h'} \geq \widetilde{x}_{h'}$ and $\widehat{x}^{0c'}_{h'} > \widetilde{x}^{0c'}_{h'}$, and for m sufficiently large $\widehat{x}_{h'} \in \mathcal{B}(x_{h'}, \varepsilon)$, from Assumption 3, it follows that $\widehat{x}_{h'} \in P'_h(\widetilde{x}_{h'})$.

Moreover, since P_h has open values, there exists $n_1 \in \mathbb{N}$ such that $\forall n > n_1$,

$$\widehat{x}_{h',n} \in P_{h'}\left(\widetilde{x}_{h',n}\right) \quad (22)$$

Now, we show that

$$(\widehat{x}_{h',n}, \widehat{b}_{h',n}) \in \Gamma_{h'}(\widetilde{q}_n, \widetilde{p}^0_n, \widetilde{p}^{\mathbf{1}}_n). \quad (23)$$

Taking into account that $(\widetilde{x}_{h',n}, \widetilde{b}_{h',n}) \in \Gamma_{h'}(\widetilde{q}_n, \widetilde{p}^0_n, \widetilde{p}^{\mathbf{1}}_n)$, to check (23), we go through 2 steps.

Step 1. At $s = 0$.

We have that

$$\langle \widetilde{p}^0_n, \widehat{x}^0_{h',n} - e^0_{h'}\rangle_C + \langle \widetilde{q}_n, \widehat{b}_{h',n}\rangle_A = (1-\widetilde{p}^{0c'}_n)\Big[\langle \widetilde{p}^0_n, \widetilde{x}^0_{h',n} - e^0_{h'}\rangle_C + \langle \widetilde{q}_n, \widetilde{b}_{h',n}\rangle_A\Big] < 0 \quad (24)$$

Step 2. At $s \in \mathcal{S}$.

We have that

$$\langle \widetilde{p}^s_n, \widehat{x}^s_{h',n} - e^s_{h'} \rangle_C - \widetilde{p}^{sC}_n \langle y^s, \widehat{b}_{h',n} \rangle_A$$
$$= (1-\widetilde{p}^{0c'}_n)\Big[\langle \widetilde{p}^s_n, \widetilde{x}^s_{h',n} - e^s_{h'} \rangle_C - \widetilde{p}^{sC}_n \langle y^s, \widetilde{b}_{h',n} \rangle_A\Big] - \widetilde{p}^{0c'}_n \langle \widetilde{p}^s_n, e^s_{h'} \rangle_C < 0. \quad (25)$$

Then, from Eqs. 24 and 25 claim (23) is proved. Hence (22) contradicts the fact that $(\widetilde{x}_{h',n}, \widetilde{b}_{h',n})$ is optimal in $\Gamma_h((\tilde{q}_n, \tilde{p}_n), \tilde{p}^{\mathbf{1}}_n)$. Then, it results that $\widetilde{p}^{0c} > 0$ for any $c \in \mathcal{C}$.

Case 2. $\widetilde{p}^0 = 0_C$ and therefore $\widetilde{q} \neq 0_A$.

From Assumption 3 there exists $\varepsilon > 0$ and $h' \in \mathcal{H}$ such that $P_{h'|\mathcal{B}(x_{h'},\varepsilon)}$ is strictly increasing in $0c'$. From Assumption 4.3, there exists $b^*_{h'} \in B_{h'}$ such that $\langle -\widetilde{q}, b^*_{h'} \rangle_A > 0$. Define $(\widehat{x}_{h',n}, \widehat{b}_{h',n})$ as follows

$$\widehat{x}^{sc}_{h',n} := \begin{cases} (1-\widetilde{p}^{0c'}_n)\widetilde{x}^{0c'}_{h',n} - \frac{1}{m}\langle \widetilde{q}_n, \frac{b^*_{h'}}{\alpha_{h'}} \rangle_A & \text{if} \quad sc = 0c', \\ (1-\widetilde{p}^{0c'}_n)\widetilde{x}^{sc}_{h',n} & \text{otherwise.} \end{cases}$$

with $m \in \mathbb{N}$, and $\widehat{b}_{h',n} := (1-\widetilde{p}^{0c'}_n)\widetilde{b}_{h',n} + \frac{\widetilde{p}^{0c'}_n}{\alpha_{h'}} b^*_{h'}$, where

$$\alpha_{h'} := \max\left(\left\{ \frac{\widetilde{p}^{sC}_n \langle -y^s, b^*_{h'} \rangle_A}{\langle \widetilde{p}^s_n, e^s_{h'} \rangle_C} : s \in \mathcal{S}, n \in \mathbb{N} \right\}, 1\right). \quad (26)$$

Observe that, since the sequence $\{\widetilde{p}^{\mathbf{1}}_n\}_{n\in\mathbb{N}}$ converges to $\widetilde{p}^{\mathbf{1}} >> 0$, the maximum introduced in Eq. 26 exists. Moreover, by definition $\alpha_{h'} \geq 1$ and then $0 < \frac{1}{\alpha_{h'}} \leq 1$, hence $\frac{b^*_{h'}}{\alpha_{h'}} \in B_{h'}$. Being $\widetilde{p}^0 = 0_C$, for $n \in \mathbb{N}$ sufficiently large, $\widetilde{p}^{0c'}_n \in (0, 1)$ and then, we have $\widehat{b}_{h',n} \in B_{h'}$.

One has that $\lim_{n\to+\infty} \widehat{x}_{h',n} = \widehat{x}_{h'}$ where

$$\widehat{x}^{sc}_{h'} := \begin{cases} \widetilde{x}^{0c'}_{h'} - \frac{1}{m}\langle \widetilde{q}, \frac{b^*_{h'}}{\alpha_{h'}} \rangle_A & \text{if} \quad sc = 0c', \\ \widetilde{x}^{sc}_{h'} & \text{otherwise.} \end{cases}$$

Since $\widehat{x}_{h'} \geq \widetilde{x}_{h'}$ and $\widehat{x}^{0c'}_{h'} > \widetilde{x}^{0c'}_{h'}$, and for m sufficiently large $\widehat{x}_{h'} \in \mathcal{B}(x_{h'}, \varepsilon)$, from Assumption 3, it follows that $\widehat{x}_{h'} \in P_{h'}(\widetilde{x}_{h'})$. Moreover, since P_h has open values, there exists $n_1 \in \mathbb{N}$ such that $\forall n > n_1$,

$$\widehat{x}_{h',n} \in P_{h'}\left(\widetilde{x}_{h',n}\right). \quad (27)$$

Now, we show that

$$(\widehat{x}_{h',n}, \widehat{b}_{h',n}) \in \Gamma_{h'}(\widetilde{q}_n, \widetilde{p}^0_n, \widetilde{p}^{\mathbf{1}}_n). \quad (28)$$

Taking into account that $(\widetilde{x}_{h',n}, \widetilde{b}_{h',n}) \in \Gamma_{h'}\left(\widetilde{q}_n, \widetilde{p}^0_n, \widetilde{p}^{\mathbf{1}}_n\right)$, to check (28), we go through 2 steps.

Step 1. At $s = 0$.

We have that

$$\langle \widetilde{p}^0_n, \widehat{x}^0_{h',n} - e^0_{h'}\rangle_C + \langle \widetilde{q}_n, \widehat{b}_{h',n}\rangle_A = (1-\widetilde{p}^{0c'}_n)\Big[\langle \widetilde{p}^0_n, \widetilde{x}^0_{h',n} - e^0_{h'}\rangle_C + \langle \widetilde{q}_n, \widetilde{b}_{h',n}\rangle_A\Big]$$

$$-\widetilde{p}^{0c'}_n\langle \widetilde{p}^0_n, e^0_{h'}\rangle_C + \widetilde{p}^{0c'}_n\left(1-\frac{1}{m}\right)\langle \widetilde{q}_n, \frac{b^*_{h'}}{\alpha_{h'}}\rangle_A = -\widetilde{p}^{0c'}_n\left[\langle \widetilde{p}^0_n, e^0_{h'}\rangle_C + \left(\frac{1}{m}-1\right)\langle \widetilde{q}_n, \frac{b^*_{h'}}{\alpha_{h'}}\rangle_A\right] \le 0.$$

Step 2. At $s \in \mathcal{S}$.

We have that

$$\langle \widetilde{p}^s_n, \widehat{x}^s_{h',n} - e^s_{h'}\rangle_C - \widetilde{p}^{sC}_n\langle y^s, \widehat{b}_{h',n}\rangle_A = (1-\widetilde{p}^{0c'}_n)\Big[\langle \widetilde{p}^s_n, \widetilde{x}^s_{h',n} - e^s_{h'}\rangle_C - \widetilde{p}^{sC}_n\langle y^s, \widetilde{b}_{h',n}\rangle_A\Big]$$

$$-\widetilde{p}^{0c'}_n\left[\langle \widetilde{p}^s_n, e^s_{h'}\rangle_C + \widetilde{p}^{sC}_n\langle y^s, \frac{b^*_{h'}}{\alpha_{h'}}\rangle_A\right] \le 0,$$

where the inequality holds true for the definition of $\alpha_{h'}$. This fact contradicts that $(\widetilde{x}_{h',n}, \widetilde{b}_{h',n})$ is optimal in $\Gamma_h(((\tilde{q}_n, \tilde{p}_n), \tilde{p}^{\mathbf{1}}_n))$. Then, it results that $\widetilde{p}^{0c} > 0$ for any $c \in \mathcal{C}$.

$$\lhd \lim_{n\to+\infty} \widetilde{b}_n = \widetilde{b} \in B.$$

From Eq. 10, taking limits, for all $s \in \mathcal{S}$, we get

$$\widetilde{p}^{sC}\langle y^s, \lim_{n\to+\infty} \widetilde{b}_{h,n}\rangle_C = \langle \widetilde{p}^{\,s}, \widetilde{x}^s_h - e^s_h\rangle_C.$$

Then, from Proposition 3.4, we can conclude that for any $h \in \mathcal{H}$, $\lim_{n\to+\infty} \widetilde{b}_{h,n} = \widetilde{b}_h \in B_h$.

$$\lhd\ (\widetilde{x}, \widetilde{b}) \in \Gamma(\widetilde{q}, \widetilde{p}^{\,0}, \widetilde{p}^{\,\mathbf{1}}).$$

It follows from the fact that $(\widetilde{x}, \widetilde{b}) \in \mathbb{R}^{GH}_+ \times B$ and from Proposition 3.7, Γ is closed.

$$\lhd\ \widetilde{q} \in Q.$$

The proof of this claim is similar to the one given in Proposition 3.6. □

Theorem 3 *Let Assumptions 1, 2, 3 and 4 be satisfied. Then, for any financial economy $\Sigma \in \mathcal{E}$, there exists an equilibrium vector with restricted participation and numeraire assets.*

Proof From Theorem 1, for any $n \in \mathbb{N}$, with $n \ge C$ and $n^2 > max_{a\in\mathcal{A}}\{-\sum_{s\in S} y^{sa}\}$ there exists $\big((\widetilde{x}_n, \widetilde{b}_n), (\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n)\big) \in \Gamma(\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n) \times \Delta^n$ solution to GQVI$_n$ (5). Let $\big\{\big((\widetilde{x}_n, \widetilde{b}_n), (\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n)\big)\big\}_{n\in\mathbb{N}}$ be the sequence of solutions; from Proposition 4.3 one has

$$\lim_{n\to+\infty}\Big((\widetilde{x}_n, \widetilde{b}_n), (\widetilde{q}_n, \widetilde{p}^{\,0}_n, \widetilde{p}^{\,\mathbf{1}}_n)\Big) = \Big((\widetilde{x}, \widetilde{b}), (\widetilde{q}, \widetilde{p}^{\,0}, \widetilde{p}^{\,\mathbf{1}})\Big)$$

with $(\widetilde{x}, \widetilde{b}) \in \Gamma(\widetilde{q}, \widetilde{p}^{\,0}, \widetilde{p}^{\,\mathbf{1}})$, $\widetilde{p}^{\,0} >> 0$, $\widetilde{p}^{\,\mathbf{1}} >> 0$ and $q \in Q$. Moreover, we observe that for any sequence $\{l_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} l_n = l > 0$ there exists $N \in \mathbb{N}$ such that $l > \frac{1}{N}$ and, from the theorem of permanence of sign for sequences, there exists $\nu \in \mathbb{N}$ such for any $n > \nu$, $l_n > \frac{1}{N}$. Then defined $\overline{\nu} = \max\{\nu, N\}$, one has, for any $n > \overline{\nu} \ge N$, $l_n > \frac{1}{N} > \frac{1}{n}$. Then, since for all $s \in \mathcal{S}^0$ $\lim_{n\to+\infty} \widetilde{p}^{\,s}_n = \widetilde{p}^{\,s} >> 0$, then there exist ν such that, for all $n > \nu$, one has $\widetilde{p}^s_n > \frac{1}{n}\mathbf{1}_C$. We now prove that the vector $(\widetilde{x}, \widetilde{b}, \widetilde{q}, \widetilde{p}^{\,0}, \widetilde{p}^{\,\mathbf{1}})$ is an equilibrium vector according to Definition 2.1.

$\lhd$ *Condition* (1). Fixed $h \in \mathcal{H}$, we suppose that there exists $(x_h, b_h) \in (P_h(\widetilde{x}_h) \times B_h) \cap \Gamma_h(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^1)$. From Proposition 3.7 and the fact that $\widetilde{p} >> 0$, we have that Γ_h is lower semicontinuous at $(\widetilde{q}, \widetilde{p}^0, \widetilde{p}^{\mathbf{1}})$, that is there exists a sequence $\{(x_{h,n}, b_{h,n})\}_{n\in\mathbb{N}}$ such that $(x_{h,n}, b_{h,n}) \in \Gamma_h(\widetilde{q}_n, \widetilde{p}^0_n, \widetilde{p}^{\mathbf{1}}_n)$ and $\lim_{n+\infty}(x_{h,n}, b_{h,n}) = (x_h, b_h)$. Since $P_h(\tilde{x}_h)$ is an open set, it follows that there exists n_1 such that $x_{h,n} \in P_h(\tilde{x}_h)$, for all $n > n_1$. But this contradicts the fact that $(\tilde{x}_{h,n}, \tilde{b}_{h,n})$ is optimal on $\Gamma_h(\widetilde{q}_n, \widetilde{p}^0_n, \widetilde{p}^{\mathbf{1}}_n)$.

$\lhd$ *Condition* 3. Since $\widetilde{q} \in Q = \operatorname{int} Q$, there exists $\varepsilon > 0$ such that $B(\widetilde{q}, \varepsilon) \subseteq Q$. For all $a' \in \mathcal{A}$, let $f_A^{a'}$ be the element of the canonical basis of $\mathbb{R}^A$ with component a' equal to 1 . For any $\delta \in \left(0, \min\left\{\frac{1+\rho}{2}, \frac{\varepsilon}{(1+\rho)+\|\widetilde{q}\|}\right\}\right)$,

take vectors:

$$\widehat{q}_n := (1-\widehat{\lambda})(\widetilde{q}_n+\delta f_A^{a'}) \quad \text{and} \quad \widehat{p}^{\,0}_n := (1-\widehat{\lambda})\widetilde{p}^{\,0}_n \quad \text{with} \quad \widehat{\lambda} := \frac{\delta}{1+\rho+\delta} \in (0,1)$$

and

$$\widehat{\widehat{q}}_n := (1+\widehat{\widehat{\lambda}})(\widetilde{q}_n-\delta f_A^{a'}) \quad \text{and} \quad \widehat{\widehat{p}}^{\,0}_n := (1+\widehat{\widehat{\lambda}})\widetilde{p}^{\,0}_n \quad \text{with} \quad \widehat{\widehat{\lambda}} := \frac{\delta}{1+\rho-\delta} \in (0,1)\,.$$

We have $(\widehat{q}_n, \widehat{p}^0_n)$, $(\widehat{\widehat{q}}_n, \widehat{\widehat{p}}^0_n) \in \Delta^0_n$. Indeed:

(i) $\sum_{c\in\mathcal{C}} \widehat{p}^{\,0c}_n + \sum_{a\in\mathcal{A}} \widehat{q}^{\,a}_n = (1-\widehat{\lambda})\left(\sum_{c\in\mathcal{C}} \widetilde{p}^{\,0c}_n + \sum_{a\in\mathcal{A}} \widetilde{q}^{\,a}_n\right) + (1-\widehat{\lambda})\delta = (1-\widehat{\lambda})(1+\rho+\delta) = 1+\rho$

$$\sum_{c\in\mathcal{C}} \widehat{\widehat{p}}^{\,0c}_n + \sum_{a\in\mathcal{A}} \widehat{\widehat{q}}^{\,a}_n = (1+\widehat{\widehat{\lambda}})\left(\sum_{c\in\mathcal{C}} \widetilde{p}^{\,0c}_n + \sum_{a\in\mathcal{A}} \widetilde{q}^{\,a}_n\right) - (1+\widehat{\widehat{\lambda}})\delta = (1+\widehat{\widehat{\lambda}})(1+\rho-\delta) = 1+\rho$$

(ii) As observed above, since

$$\lim_{n\to+\infty} \widehat{p}^{\,0}_n = (1-\widehat{\lambda})\widetilde{p}^{\,0} >> 0 \qquad \lim_{n\to+\infty} \widehat{\widehat{p}}^{\,0}_n = (1+\widehat{\widehat{\lambda}})\widetilde{p}^{\,0} >> 0$$

there exist $\widehat{\nu}$ and $\widehat{\widehat{\nu}}$ such that, for any $n > \max\{\widehat{\nu}, \widehat{\widehat{\nu}}\}$, one has $\widehat{p}^0_n > \frac{1}{n}\mathbf{1}_C$ and $\widehat{\widehat{p}}^0_n > \frac{1}{n}\mathbf{1}_C$.

(iii) We have

$$\lim_{n\to+\infty} \widehat{q}_n = (1-\widehat{\lambda})(\widetilde{q}+\delta f_A^{a'}) \quad \text{and} \quad \lim_{n\to+\infty} \widehat{\widehat{q}}_n = (1+\widehat{\widehat{\lambda}})(\widetilde{q}-\delta f_A^{a'}). \tag{29}$$

One has that

$$(1-\widehat{\lambda})(\widetilde{q}+\delta f_A^{a'}) \in Q \quad \text{and} \quad (1+\widehat{\widehat{\lambda}})(\widetilde{q}-\delta f_A^{a'}) \in Q\,.$$

Indeed $\quad \|(1-\widehat{\lambda})(\widetilde{q}+\delta f_A^{a'}) - \widetilde{q}\| = \|\frac{1+\rho}{1+\rho+\delta}(\widetilde{q}+\delta f_A^{a'}) - \widetilde{q}\| \leq$

$$\leq \frac{1}{1+\rho+\delta}\|(1+\rho)(\widetilde{q}+\delta f_A^{a'}) - (1+\rho)\widetilde{q} - \delta\widetilde{q}\| < \|(1+\rho)\delta f_A^{a'} + \delta\widetilde{q}\|$$

$$< \delta\left((1+\rho)+\|\widetilde{q}\|\right) < \varepsilon \quad \Rightarrow \quad (1-\widehat{\lambda})(\widetilde{q}+\delta f_A^{a'}) \in B(\widetilde{q}, \varepsilon) \subseteq Q\,.$$

and $\quad \|(1+\widehat{\widehat{\lambda}})(\widetilde{q}-\delta f_A^{a'})-\widetilde{q}\| = \|\frac{1+\rho}{1+\rho-\delta}(\widetilde{q}-\delta f_A^{a'})-\widetilde{q}\| \leq$

$$\leq \frac{1}{1+\rho-\delta}\|(1+\rho)(\widetilde{q}-\delta f_A^{a'})-(1+\rho)\widetilde{q}+\delta\widetilde{q}\| < \|-(1+\rho)\delta f_A^{a'}+\delta\widetilde{q}\|$$

$$\leq \delta\,((1+\rho)+\|\widetilde{q}\|) < \varepsilon \quad \Rightarrow \quad (1+\widehat{\widehat{\lambda}})(\widetilde{q}-\delta f_A^{a'}) \in B(\widetilde{q},\varepsilon) \subseteq Q\,.$$

Then, from Eq. 29 and Proposition 4.1.3, one has $\widehat{q}_n,\ \widehat{\widehat{q}}_n \in \left\{\frac{1}{n}\mathbf{1}_S Y\right\} + \mathrm{Cl}(Q)$.

(iv) From Eq. 29, it follows that $\widehat{q}_n \geq -n\mathbf{1}_A$ and $\widehat{\widehat{q}}_n \geq -n\mathbf{1}_A$.

Then, from (i), (ii), (iii) and (iv), we have that $(\widehat{q},\widehat{p}^0)$, $(\widehat{\widehat{q}},\widehat{\widehat{p}}^0) \in \Delta_0^n$. From Eqs. 7 and 11, and since $\delta,\ \widehat{\lambda},\ \widehat{\widehat{\lambda}} \in (0,1)$, one has:

$$\langle\sum_{h\in\mathcal{H}}\widetilde{b}_{h,n},\widehat{q}_n-\widetilde{q}_n\rangle_A + \langle\sum_{h\in\mathcal{H}}(\widetilde{x}^{\,0}_{h,n}-e_h^0),\widehat{p}^{\,0}_{\,n}-\widetilde{p}^{\,0}_{\,n}\rangle_C$$

$$= -\widehat{\lambda}\left(\langle\sum_{h\in\mathcal{H}}\widetilde{b}_{h,n},\widetilde{q}_n\rangle_A + \langle\sum_{h\in\mathcal{H}}(\widetilde{x}^{\,0}_{h,n}-e_h^0),\widetilde{p}^{\,0}_{\,n}\rangle_C\right) + (1-\widehat{\lambda})\delta\sum_{h\in\mathcal{H}}\widetilde{b}^{a'}_{h,n}$$

$$= (1-\widehat{\lambda})\delta\sum_{h\in\mathcal{H}}\widetilde{b}^{a'}_{h,n} \leq 0 \quad \Rightarrow \quad \sum_{h\in\mathcal{H}}\widetilde{b}^{a'}_{h,n} \leq 0 \tag{30}$$

and $\quad \langle\sum_{h\in\mathcal{H}}\widetilde{b}_{h,n},\widehat{\widehat{q}}_n-\widetilde{q}_n\rangle_A + \langle\sum_{h\in\mathcal{H}}(\widetilde{x}^{\,0}_{h,n}-e_h^0),\widehat{\widehat{p}}^{\,0}_n-\widetilde{p}^{\,0}_{\,n}\rangle_C$

$$= \widehat{\widehat{\lambda}}\left(\langle\sum_{h\in\mathcal{H}}\widetilde{b}_{h,n},\widetilde{q}_n\rangle_A + \langle\sum_{h\in\mathcal{H}}(\widetilde{x}^{\,0}_{h,n}-e_h^0),\widetilde{p}^{\,0}_{\,n}\rangle_C\right) - (1+\widehat{\widehat{\lambda}})\delta\sum_{h\in\mathcal{H}}\widetilde{b}^{a'}_{h,n}$$

$$= -(1+\widehat{\widehat{\lambda}})\delta\sum_{h\in\mathcal{H}}\widetilde{b}^{a'}_{h,n} \leq 0 \quad \Rightarrow \quad \sum_{h\in\mathcal{H}}\widetilde{b}^{a'}_{h,n} \geq 0 \tag{31}$$

Hence, from Eqs. 30 and 31, we have that for any $a' \in \mathcal{A}$,

$$\sum_{h\in\mathcal{H}}\widetilde{b}^{a'}_{h,n} = 0. \tag{32}$$

Taking limits, we can conclude that $\sum_{h\in\mathcal{H}}\widetilde{b}^a_h = 0 \qquad \forall a \in \mathcal{A}$.

$\lhd$ *Condition* 2.

Firstly, from condition (32), conditions (11) and (12), for any $s \in \mathcal{S}^0$, become

$$\langle\sum_{h\in\mathcal{H}}(\widetilde{x}^s_{h,n}-e^s_h),\widetilde{p}^{\,s}_{\,n}\rangle_C = 0 \tag{33}$$

and VI_n (7) and (8) become

$$\langle\sum_{h\in\mathcal{H}}(\widetilde{x}^{\,0}_{h,n}-e^0_h),p^0_n\rangle_C \leq 0 \quad \forall (q_n,p^0_n) \in \Delta^n_0; \tag{34}$$

for any $s \in \mathcal{S}$,

$$\langle\sum_{h\in\mathcal{H}}(\widetilde{x}^s_{h,n}-e^s_h),p^{\,s}_{\,n}\rangle_C \leq 0, \quad \forall p^s_n \in \Delta^n_s. \tag{35}$$

For any $n \in \mathbb{N}$ and $s \in \mathcal{S}^0$, define

$$\mathcal{C}_s^- := \{c \in \mathcal{C} : \sum_{h \in \mathcal{H}} (\widetilde{x}_{h,n}^{sc} - e_h^{sc}) \leq 0\}, \qquad \mathcal{C}_s^+ := \{c \in \mathcal{C} : \sum_{h \in \mathcal{H}} (\widetilde{x}_{h,n}^{sc} - e_h^{sc}) > 0\}.$$

We suppose that there exists $s \in \mathcal{S}^0$ such that $\mathcal{C}_s^+ \neq \emptyset$ and we take

$$\widehat{p}_n^{sc} := \begin{cases} \widetilde{p}_n^{sc} + K & \text{if} \quad c \in \mathcal{C}_s^+, \\ \widetilde{p}_n^{sc} - K \frac{|\mathcal{C}_s^+|}{|\mathcal{C}_s^-|} & \text{if} \quad c \in \mathcal{C}_s^-, \end{cases}$$

with $0 < K < \frac{|\mathcal{C}_s^-|}{|\mathcal{C}_s^+|} \left(\widetilde{p}_n^{sc} - \frac{1}{n} \right)$. Clearly, if $s = 0$, $(\widetilde{q}_n, \widehat{p}_n^0) \in \Delta_0^n$ and if $s \in \mathcal{S}$, $\widehat{p}_n^s \in \Delta^n$. Then:

$$\langle \sum_{h \in \mathcal{H}} (\widetilde{x}_{h,n}^s - e_h^s), p_n^s \rangle_C = K \sum_{c \in \mathcal{C}_s^+} \sum_{h \in \mathcal{H}} (\widetilde{x}_{h,n}^{sc} - e_h^{sc}) - K \frac{|\mathcal{C}_s^+|}{|\mathcal{C}_s^-|} \sum_{c \in \mathcal{C}_s^-} \sum_{h \in \mathcal{H}} (\widetilde{x}_{h,n}^{sc} - e_h^{sc}) > 0$$

contradicting (34) if $s = 0$ and (35) otherwise. Then we have that $\mathcal{C}_s^+ = \emptyset$. Hence, being $\widetilde{p} >> 0$, from Eq. 33, for any $s \in \mathcal{S}^0$ and $c \in \mathcal{C}$, it follows that $\sum_{h \in \mathcal{H}} (\widetilde{x}_{h,n}^{sc} - e_h^{sc}) = 0$. Taking limits, we can conclude $\sum_{h \in \mathcal{H}} (\widetilde{x}_h^{sc} - e_h^{sc}) = 0$, for any $s \in \mathcal{S}^0$ and $c \in \mathcal{C}$. □

5 Conclusions

We presented a proof of the existence of equilibria in a model with restricted participation and numeraire asset under relatively general assumptions and by using a variational inequality approach. By using well chosen specifications of the households portfolio sets, we can get existence of equilibria in some extensively studied models of general equilibrium, as an immediate corollary of our result. Indeed, assume that markets are (potentially) complete in our model, i.e., the rank of the yield matrix Y is equal to the the number S of states in the second period. Then, our framework coincides with the standard exchange economy, if each households' portfolio sets is equal to the entire Euclidean space $\mathbb{R}^A$ and with the incomplete market case if restrictions are constant across households and imply that portfolio have to belong to a linear subspace of dimension equal to the number of available assets A smaller than the number S of states.

Finally, since any equilibrium with numeraire asset is an equilibrium with nominal assets (and not vice versa), Villanacci et al. (2002), Proposition 6, page 329, the existence result we provide implies existence in the model by Siconolfi (1988).

We believe that our approach, which uses the variational inequality theory, has the following important characteristics. It does provide a general, relatively simple, few step methodology, which can be applied to show existence of equilibria of any general equilibrium model (see also Donato et al. (2018a, b)). First of all, some crucial properties of the budget set valued functions have to be proven to hold true on a well chosen subset of the price set (Proposition 3.7). That analysis provides simple hints to define a sequence of nonempty, compact, convex sets of prices on which those properties are satisfied. Then, the main step is to find an appropriate sequence of Variational Inequalities problems constructed using those sets (see Theorem 2 and Proposition 4.3). The associated non-empty solution sets are then used to show the existence of a sequence whose limit is an equilibrium. The quite interesting characteristic and probably the main virtue of the above described approach is the following one. The seemingly complicated Variational Inequality (see GQVI_n (5)) is made up by two simple parts. The first one takes care of the individuals maximization problems in a quite standard manner. The second part is constructed exploiting the very nature of the model under analysis: the so-called set of Walras' laws, i.e., the set of equalities obtained summing up budget equations across households.

Our future research work will focus on the analysis of models with real assets, restricted participation, possibility of defaults, multiple periods and possibility of asset trading which are still not completely analyzed in the existing literature. We also plan to apply the Variational Inequality approach to general equilibrium models in which households have noncomplete, nontransitive preferences.

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Appendix

Definition A.1 Let A be a nonempty convex set in $\mathbb{R}^n$. The recession cone of A is denoted and defined as follows

$$\mathrm{rec}A = \left\{ y \in \mathbb{R}^n : \forall x^0 \in A, \forall \lambda \geq 0, x^0 + \lambda y \in A \right\}.$$

Proposition A.1 *(see Soltan (2015)) Let A be a convex set in $\mathbb{R}^n$. Then*

1. *if A is nonempty, closed and $0_n \in A$,*
$$recA = \{z \in \mathbb{R}^n : \ \forall \lambda \geq 0, \ \lambda z \in A\} \subseteq A;$$
2. *$recA$ is a convex cone;*
3. *if A is closed, then $recA$ is closed;*
4. *for any $a \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$,*
$$rec(a + \mu A) = \mu \, recA = sgn(\mu) \, recA \, .$$

Definition A.2 Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set and let $S : C \rightrightarrows \mathbb{R}^n$ and $\Phi : C \rightrightarrows \mathbb{R}^n$ be set-valued maps. A *Generalized Quasi-Variational Inequality* associated with C, S, Φ, denoted by *GQVI*, is the following problem:

$$\text{Find } \overline{x} \in S(\overline{x}) \text{ such that there exists } \varphi \in \Phi(\overline{x}) \text{ with } \langle \varphi, x - \overline{x} \rangle \geq 0, \quad \forall x \in S(\overline{x}) \, . \tag{36}$$

In particular, when $S(x) = C$ for any $x \in C$, (36) is a *Generalized Variational Inequality*, *GVI*; when Φ is single-valued, (36) reduces to the *Quasi-Variational Inequality*, *QVI*. When both $\Phi(x)$ is singleton and $S(x) = C$, for any $x \in C$, we have the classical Stampacchia *Variational Inequality*, *VI*.

Theorem 4 *(see Tan (1985)) Let C be a nonempty, convex and compact subset of $\mathbb{R}^n$. Let $\Phi : C \rightrightarrows \mathbb{R}^n$ and $S : C \rightrightarrows C$ be two set-valued maps satisfying the following properties:*

(i) *S is closed, lower semicontinuous and with nonempty, convex and compact values;*
(ii) *Φ is upper semicontinuous with nonempty, convex and compact values.*

Then the GQVI (36) admits at least a solution.

The above theorem, which establishes the existence of a solution for variational inequalities, is a consequence of a Kakutani's fixed point Theorem.

$$N^{>}(x) := \{h \in \mathbb{R}^n : \langle h, z - x\rangle \leq 0 \quad \forall\, z \in P(x)\}.$$

Let us define $\widetilde{\mathcal{G}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that

$$\widetilde{\mathcal{G}}(x) := \begin{cases} \overline{\mathrm{B}}(0,1) & if \;\; P(x) = \emptyset \\ \\ conv\,(N^{>}(x) \cap S(0,1)) & if \;\; P(x) \neq \emptyset \end{cases} \tag{37}$$

where $\overline{\mathrm{B}}(0,1) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $S(0,1) = \{x \in \mathbb{R}^n : \|x\| = 1\}$ are, respectively, the closed unit ball and the unit sphere of $\mathbb{R}^n$.

Theorem 5 *Let P be lower semicontinuous with open and convex valued. Then the set-valued map $\widetilde{\mathcal{G}}$, defined in* Eq. 37 *is with nonempty and convex and compact values, and upper semicontinuous.*

Proof Firstly, we prove that $N^{>}$ is a closed map. Let $\{x_n\} \subseteq \mathbb{R}^n$ and $\{y_n\} \subseteq \mathbb{R}^n$ be sequences such that $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$ with $y_n \in N^{>}(x_n)$. We have to prove that $y \in N^{>}(x)$, that is $\langle y, z - x_h\rangle \leq 0 \;\; \forall z \in P(x)$. We fix $z \in P(x_h)$ and since P is lower semicontinuous, there exists $\{z_n\}$ such that $z_n \in P(x_n)$ with $\lim_{n\to+\infty} z_n = z$. From $z_n \in P(x_{h,n})$ and $y_n \in N^{>}(x_n)$, it follows $\langle y_n, z_n - x_n\rangle \leq 0$. Passing to the limits $\langle y, z - x\rangle \leq 0$, so $y \in N^{>}(x)$.
All Properties follows from the closedness of $N^{>}$ and the proof of Theorem 3.2 in Milasi et al. (2019). □

Definition A.3 Let C be a convex set of $\mathbb{R}^n$, call (GVI) the following problem:

$$\text{“ Find } \widetilde{x} \in C \text{ such that } \exists\; g \in \mathcal{G}(\widetilde{x}) \;\text{ with }\; \langle g, x - \widetilde{x}\rangle \geq 0 \quad \forall x \in C.\text{”} \tag{38}$$

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