BOUNDARY VALUE PROBLEMS FOR FRACTIONAL-ORDER DIFFERENTIAL INCLUSIONS IN BANACH SPACES WITH NONDENSELY

(Article begins on next page)

03 May 2024

BOUNDARY VALUE PROBLEMS FOR FRACTIONAL-ORDER DIFFERENTIAL INCLUSIONS IN BANACH SPACES WITH NONDENSELY DEFINED OPERATORS

VALERI OBUKHOVSKII*, PIETRO ZECCA** AND MARIA AFANASOVA***

*Faculty of Physics and Mathematics, Voronezh State Pedagogical University,
394043 Voronezh, Russia
E-mail: valerio-ob2000@mail.ru
(Corresponding author)

**Università degli Studi di Firenze, Via Santa Marta 3, 50139 Firenze, Italy
E-mail:zecca@unifi.it

***Faculty of Physics and Mathematics,
Voronezh State Pedagogical University, 394043 Voronezh, Russia
E-mail: marya.afanasowa@yandex.ru

Abstract. We consider a nonlocal boundary value problem for a semilinear differential inclusion of a fractional order in a Banach space assuming that its linear part is a non-densely defined Hille-Yosida operator. We apply the theory of integrated semigroups, fractional calculus and the fixed point theory of condensing multivalued maps to obtain a general existence principle (Theorem 3.2). Theorem 3.3 gives an example of a concrete realization of this result. Some important particular cases including a nonlocal Cauchy problem, periodic and anti-periodic boundary value problems are presented.
Key Words and Phrases: Fractional differential inclusion, boundary value problem, nonlocal Cauchy problem, periodic problem, Hille-Yosida operator, integrated semigroup, measure of noncompactness, fixed point, topological degree, multivalued map, condensing map.
2020 Mathematics Subject Classification: 34B10, 34A08, 34A60, 34C25, 34G25, 47D62, 47H04, 47H08, 47H10, 47H11.

1. INTRODUCTION

Starting from the paper [31], nonlocal boundary value problems for differential inclusions in a Banach space are the subject of investigation for many researchers, see for example the works of Ding and Kartsatos [12], Kravvaritis and Papageorgiou [19], Marino [20], Obukhovskii and Zecca [25], Papageorgiou [27, 28]. Some nonlocal problems for fractional differential equations and inclusions were considered recently in the works of Agarwal, Baleanu, Nieto, Torres and Zhou [2], Ahmad, Nieto, Alsaedi and Aqlan [3], Anh and Ke [4], Benedetti, Obukhovskii and Taddei [9] and others.

In the last decades the interest to the theory of differential equations and inclusions of fractional order essentially strengthened due to important applications to control theory, variational principles, Lagrangian and Hamiltonian dynamics, physics, engineering, biology, economics and other branches of natural sciences (among others see, e.g., monographs of Kilbas, Srivastava and Trujillo [18], Podlubny [29], Zhou [33] and references therein). It should be mentioned that one of the most valuable advantages of fractional order models in comparison with integer order ones is that fractional order derivative of a function depends on the past values of such function and hence it becomes a powerful tool for the description of memory and hereditary properties of some systems. From the other side, it is well known that the study of systems with distributed parameters leads to the corresponding problems in infinite dimensional Banach spaces. The analysis of such problems (for the case of an ordinary semilinear differential inclusion) by the methods of the theory of condensing operators was developed in the monograph of Kamenskii, Obukhovskii and Zecca [15]. The use of this technique allows to overcome the difficulty lying in the fact that in these problems compactness assumptions are not posed neither on a semigroup generated by the linear part of the inclusion nor on a multivalued nonlinearity.

At the same time it is worth noting that in most publications concerning semilinear inclusions it is assumed that their linear part is densely defined on the whole phase space. Meantime in many situations this assumption looks rather onerous: for example, from restrictions on the space on which the linear part is defined (periodic continuous functions, Hölder continuous functions, etc.) or from boundary conditions (e.g., the space of smooth functions vanishing on the boundary of a domain is not dense in the space of continuous functions) (see [11, 22] et al.) In this case the notions of an integrated semigroup introduced by Arendt [5] and a mild (or integral) solution of an equation with a non-densely defined operator considered by Da Prato and Sinestrari [11] turned out to be very useful. In particular, in the paper [26] for a semilinear differential inclusion with a linear part satisfying the Hille-Yosida condition some existence and continuous dependence results were obtained. Existence results for nondensely defined fractional differential inclusions in a Banach space were proved in [32, 33] and some other works.

In the present paper we study a general nonlocal boundary value problem for a fractional-order semilinear differential inclusion in a Banach space. It is organized as follows. In the next section we give necessary preliminaries from the theory of integrated semigroups and their generators as well as from the theory of measures of noncompactness and multivalued analysis. In Section 3 we describe the class of considered differential inclusions in a Banach space and present the general boundary value problem which we are going to study. We introduce a multivalued operator in a functional space whose fixed points are mild solutions of the considered problem and investigate its properties. In particular, it is shown that it is so called quasi-R_δ multioperator that allows to apply to it the corresponding topological degree theory (see [15]). This application yields a general existence principle (Theorem 3.2) and an example of a concrete realization of this principle which gives Theorem 3.3. In the last section we consider some particular cases of the considered boundary value problem including a nonlocal Cauchy problem and a periodic problem.

2. Preliminaries

2.1. **Integrated Semigroups and Nondensely Defined Operators.** In this section we recall some necessary facts about integrated semigroups and their generators. For more details, we refer the reader to [5, 6, 7, 17, 21, 22, 30]. We start with the following notions (see [5, 7]).

Let E be a Banach space.

Definition 2.1. A family $\{V(t)\}_{t\geq 0}$ of bounded linear operators on E is called an integrated semigroup if:

 $(V1)$ $V(0) = 0$;
 $(V2)$ $V(\cdot)$ is strongly continuous, i.e., the function $t \to V(t)x$, $t \geq 0$ is continuous for each $x \in E$;
 $(V3)$ $V(s)V(t) = \int_0^s (V(t + \tau) - V(\tau))d\tau$ for each $t, s \geq 0$.

Definition 2.2. A family of linear operators $\{V(t)\}_{t\geq 0}$ is called exponentially bounded if there exist constants $\mathcal{C} \geq 0$ and $\omega \geq 0$ such that

$$\|V(t)\| \leq \mathcal{C}e^{\omega t} \quad \text{for} \quad t \geq 0.$$

In this case we will denote $V(\cdot) \in G(\mathcal{C}, \omega)$.

Definition 2.3. A linear operator A on E is called the generator of an integrated semigroup if there exists $\omega \geq 0$ such that $(\omega, +\infty) \subset \rho(A)$ (the resolvent set of A) and there exists a strongly continuous exponentially bounded family $\{V(t)\}_{t\geq 0}$ of bounded linear operators such that $V(0) = 0$, $V \in G(M, \omega)$, and the following representation for the resolvent of A holds:

$$R(\lambda, A) := (\lambda I - A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t}V(t)dt \quad (\lambda > \omega).$$

In this case, $V(\cdot)$ is said to be the integrated semigroup generated by A.

Notice the following relations between an integrated semigroup and its generator.

Proposition 2.1. (see [5, 7]) *Let A be the generator of an integrated semigroup* $\{V(t)\}_{t\geq 0}$. *Then,*
 (i) for all $x \in E$ and $t \geq 0$:

$$\int_0^t V(\tau)x d\tau \in D(A) \quad \text{and} \quad V(t)x = A(\int_0^t V(\tau)d\tau) + tx;$$

 (ii) for all $x \in D(A)$ and $t \geq 0$:

$$V(t)x \in D(A), \qquad AV(t)x = V(t)Ax,$$

 and

$$V(t)x = \int_0^t V(\tau)Ax d\tau + tx;$$

 (iii) $R(\lambda, A)V(t) = V(t)R(\lambda, A)$ for all $t \geq 0, \lambda > \omega$.

Definition 2.4. ([17]) An integrated semigroup $\{V(t)\}_{t\geq 0}$ is called locally Lipschitz continuous if for each $r > 0$ there exists a constant $L_r > 0$ such that

$$\|V(t) - V(s)\| \leq L_r|t - s| \quad \text{for all} \quad t, s \in [0, r].$$

It is known (see [17]) that a Lipschitz continuous integrated semigroup is exponentially bounded.

Definition 2.5. A linear (not necessarily densely defined) operator $A : D(A) \subset E \to E$ is said to be the Hille-Yosida operator if there exist $M \geq 0$ and $\omega \geq 0$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$(\lambda - \omega)^n \|R(\lambda, A)^n\| \leq M \quad \text{for all} \quad n = 1, 2, \dots \quad \text{and} \quad \lambda > \omega. \tag{2.1}$$

Proposition 2.2. ([17]) *The following assertions are equivalent:*

(i) *A is a Hille-Yosida operator;*
(ii) *A is the generator of a locally Lipschitz continuous integrated semigroup.*

It is known (see [17, 30]) that if A is a Hille-Yosida operator and $\{V(t)\}_{t\geq 0}$ is the locally Lipschitz integrated semigroup generated by A, then the function $t \to V(t)x$, $t \geq 0$ is differentiable for each $x \in \overline{D(A)}$ and, moreover, the derivative $\{V'(t)\}_{t\geq 0}$ is a C_0-semigroup on $\overline{D(A)}$ generated by the part A_0 of the operator A which is defined by

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\},$$

$$A_0 x = Ax \quad \text{for} \quad x \in D(A_0).$$

In the what follows we will denote

$$E_0 = \overline{D(A)}.$$

Proposition 2.3. ([17, 30]) *Let $\{V(t)\}_{t\geq 0}$ be a locally Lipschitz continuous integrated semigroup on E and $f : [0, T] \to E$ be a Bochner integrable function. Then the function $B : [0, T] \to E$,*

$$B(t) = \int_0^t V(t - s)f(s)ds$$

is continuously differentiable and, moreover,

$$\|\frac{d}{dt}B(t)\| \leq 2L_T \int_0^t \|f(s)\|ds \quad \text{for all} \quad t \in [0, T], \tag{2.2}$$

where L_T is the Lipschitz constant of V on $[0, T]$.

Let A be a Hille-Yosida operator generating a locally Lipschitz continuous integrated semigroup $\{V(t)\}_{t\geq 0}$, function $B : [0, t] \to E$ be defined as in Proposition 2.3. Then, by applying Proposition 2.1 (iii) and using the Lipschitz continuity of $V(\cdot)$, one may verify the following relation (see [1]):

$$R(\lambda, A)\frac{d}{dt}B(t) = \int_0^t V'(t - s)R(\lambda, A)f(s)ds. \tag{2.3}$$

Moreover, taking into account that $\lim_{\lambda \to +\infty} \lambda R(\lambda, A)x = x$ for each $x \in E_0$ (see, e.g. [7]), we come to the following equality:

$$\frac{d}{dt}B(t) = \lim_{\lambda \to +\infty} \int_0^t V'(t-s)\lambda R(\lambda, A)f(s)ds. \qquad (2.4)$$

2.2. **Measures of Noncompactness and Multivalued Maps.** We will need some notions from the multivalued analysis and topological degree theory for condensing maps (see, e.g., [8, 10, 13, 15, 24]). Let X be a metric space; $\mathcal{E}$ a normed space; $P(\mathcal{E})$ denote the collection of all nonempty subsets of $\mathcal{E}$. By symbols $K(\mathcal{E})$ and $Kv(\mathcal{E})$ we denote the collections of all nonempty compact and, respectively, compact convex subsets of $\mathcal{E}$.

For $\Omega \in K(\mathcal{E})$ we denote

$$\|\Omega\| = \max\{\|\omega\| : \omega \in \Omega\}.$$

Definition 2.6. A multivalued map (multimap) $\mathcal{F} : X \to P(\mathcal{E})$ is said to be:

(i) upper semicontinuous (u.s.c. for short) if $\mathcal{F}^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$ is an open subset of X for every open $V \subset \mathcal{E}$;

(ii) closed if its graph $Gr_{\mathcal{F}} = \{(x, y) \in X \times \mathcal{E} : y \in \mathcal{F}(x)\}$ is a closed subset of $X \times \mathcal{E}$.

Definition 2.7. A multivalued map (multimap) $\mathcal{F} : X \to K(\mathcal{E})$ is said to be:

(i) compact if its range $\mathcal{F}(X)$ is a relatively compact subset of $\mathcal{E}$,

(ii) quasicompact if its restriction to each compact set is compact.

If a u.s.c. multimap $\mathcal{F}$ is compact on bounded subsets of X it is called completely u.s.c.

Proposition 2.4. ([15], Theorem 1.1.12). *A closed and quasicompact multimap* $\mathcal{F} : X \to K(\mathcal{E})$ *is u.s.c.*

Definition 2.8. Let $\mathcal{E}$ be a normed space; $(\mathcal{A}, \geq 0)$ a (partially) ordered set. A function $\beta : P(\mathcal{E}) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in $\mathcal{E}$ if

$$\beta(\overline{co}\Omega) = \beta(\Omega)$$

for every $\Omega \in P(\mathcal{E})$.

A MNC β is called:

(a) *monotone* if $\Omega_1 \subseteq \Omega_2$ implies $\beta(\Omega_1) \leq \beta(\Omega_2)$;

(b) *nonsingular* if $\beta(\Omega \cup \{a\}) = \beta(\Omega)$ for every $a \in \mathcal{E}$, $\Omega \in P(\mathcal{E})$;

(c) *invariant with respect to union with compact sets* if $\beta(\Omega \cup K) = \beta(\Omega)$ for every $\Omega \in P(\mathcal{E})$, K is relatively compact in $\mathcal{E}$;

(d) *invariant with respect to reflection through the origin* if $\beta(-\Omega) = \beta(\Omega)$ for every $\Omega \in P(\mathcal{E})$;

(e) *real* if $\mathcal{A} = [0, +\infty]$ with natural ordering.

If $\mathcal{A}$ is a cone in a Banach space, we say that the MNC β is:

(f) *algebraically semiadditive* if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$
for every $\Omega_0, \Omega_1 \in P(\mathcal{E})$;

(g) *regular* if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As the example of the MNC possessing all these properties, we may consider the *Hausdorff MNC*:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon \text{ - net }\}.$$

Definition 2.9. A multimap $\mathcal{F} : X \subseteq \mathcal{E} \to K(\mathcal{E})$ is called condensing w.r.t. a MNC β in $\mathcal{E}$ (or β-condensing), if for every $\Omega \subseteq X$, that is not relatively compact, we have

$$\beta(\mathcal{F}(\Omega)) \not\geq \beta(\Omega).$$

As examples of real MNCs defined on the space of continuous functions $C([a,b]; E)$ with values in a Banach space E we may consider:

(*i*) *the module of fiber noncompactness*

$$\varphi_C(\mathcal{D}) = \sup_{t \in [a,b]} \chi(\mathcal{D}(t))$$

and

(*ii*) *the module of equicontinuity*

$$mod_C(\mathcal{D}) = \lim_{\delta \to 0} \sup_{x \in \mathcal{D}} \max_{|t_1 - t_2| \leq \delta} \|x(t_1) - x(t_2)\|$$

for each bounded $\mathcal{D} \subset C([a,b]; E)$.

These MNCs possess all the above properties except regularity. Moreover if we denote by χ_C the Hausdorff MNC in the space $C([a,b]; E)$, we have the following relation (see [15], Example 2.1.3):

$$\varphi_C(\mathcal{D}) \leq \chi_C(\mathcal{D}). \tag{2.5}$$

Let $\mathcal{E}$, $\mathcal{E}'$ be normed spaces with the Hausdorff MNCs $\chi_\mathcal{E}$ and $\chi_{\mathcal{E}'}$ respectively and $\mathcal{L} : \mathcal{E} \to \mathcal{E}'$ a bounded linear operator. The number

$$\|\mathcal{L}\|^{(\chi)} := \chi_{\mathcal{E}'}(\mathcal{L}S),$$

where $S \subset \mathcal{E}$ is a unit sphere, is called *the* (χ)*-norm of the operator* $\mathcal{L}$.

It is easy to verify the following properties:

$$\|\mathcal{L}\|^{(\chi)} \leq \|\mathcal{L}\| \tag{2.6}$$

and

$$\chi_{\mathcal{E}'}(\mathcal{L}\Omega) \leq \|\mathcal{L}\|^{(\chi)} \chi_\mathcal{E}(\Omega) \tag{2.7}$$

for each bounded $\Omega \subset \mathcal{E}$.

Let E be a Banach space, $\mathcal{L} : E \to C([a,b]; E)$ a bounded linear operator and φ_C the module of fiber noncompactness in $C([a,b]; E)$.

Definition 2.10. The value $\|\mathcal{L}\|^{(\chi,\varphi)}$ equal to the infimum of all such $C \geq 0$ for which

$$\varphi_C(\mathcal{L}\Omega) \leq C\chi_E(\Omega)$$

for all bounded $\Omega \subset E$ is called the (χ, φ)-norm of $\mathcal{L}$.

From (2.5) it follows that

$$\|\mathcal{L}\|^{(\chi,\varphi)} \le \|\mathcal{L}\|^{(\chi)}. \tag{2.8}$$

Recall (see, e.g., [13]) that a metric space X is an absolute neighborhood retract (or an ANR-space) if for every homeomorphism h that maps it onto a closed subset $h(X)$ of a metric space Y, the set $h(X)$ is a retract of its certain neighborhood. The class of ANR-spaces is sufficiently large: in particular, any union of a finite number of convex closed subsets of a normed space is an ANR-space.

Definition 2.11. (See [13, 14]) A nonempty compact subset $\mathcal{A}$ of an ANR-space is called an R_δ-set if there exists a decreasing sequence $\{\mathcal{A}_n\}$ of compact contractible sets such that

$$\mathcal{A} = \bigcap_{n \ge 1} \mathcal{A}_n.$$

It is clear that compact convex or, more generally, contractible sets present examples of R_δ-sets. At the same time, an R_δ-set needs not be contractible (see an example in [13]).

Let $\mathfrak{X}$ be a subset of $\mathcal{E}$.

Definition 2.12. A u.s.c. multimap $\mathcal{F} : \mathfrak{X} \to K(\mathcal{E})$ is called: (i) an R_δ-multimap (or J-multimap) if every value $\mathcal{F}(x)$, $x \in \mathfrak{X}$ is an R_δ-set; (ii) quasi-R_δ-multimap (or CJ-multimap) if there exists a normed space $\mathcal{E}_1$, an R_δ-multimap $\mathcal{F}_1 : \mathfrak{X} \to K(\mathcal{E}_1)$ and a continuous map $\mathfrak{g} : \mathcal{E}_1 \to \mathcal{E}$ such that $\mathcal{F} = \mathfrak{g} \circ \mathcal{F}_1$.

This definition and continuity properties of multimaps (see, e.g., [15]) imply the following assertion.

Proposition 2.5. *If $\mathcal{F}, \mathcal{G} : \mathfrak{X} \to K(\mathcal{E})$ be quasi-R_δ-multimaps then their sum $\mathcal{F} + \mathcal{G} : \mathfrak{X} \to K(\mathcal{E})$,*

$$\big(\mathcal{F} + \mathcal{G}\big)(x) = \mathcal{F}(x) + \mathcal{G}(x)$$

is also quasi-R_δ.

Let β be a monotone nonsingular MNC in $\mathcal{E}$, $\mathfrak{U}$ an open bounded subset of $\mathcal{E}$ and $\mathcal{F} : \overline{\mathfrak{U}} \to K(\mathcal{E})$ be a β-condensing quasi-R_δ-multimap, moreover, let $x \notin \mathcal{F}(x)$ for all $x \in \partial\mathfrak{U}$, where $\partial\mathfrak{U}$ denote the boundary of the set $\mathfrak{U}$. In this situation for the corresponding multifield $i - \mathcal{F}$ the characteristic

$$deg\,(i - \mathcal{F}, \overline{\mathfrak{U}}),$$

called the *topological degree*, having all standard properties, is defined (see [15], Chapter 3.4). In particular, the difference of this characteristic from zero implies the existence of at least one fixed point $x \in \mathfrak{U}$, $x \in \mathcal{F}(x)$.

As the consequence of this topological degree theory we get the following fixed point principle (see [15], Corollary 3.4.2).

Proposition 2.6. *Let $\mathfrak{M}$ be a closed bounded subset of $\mathcal{E}$ and $\mathcal{F} : \mathfrak{M} \to K(\mathfrak{M})$ a β-condensing quasi-R_δ-multimap. Then $\mathcal{F}$ has a fixed point $x_\star \in \mathfrak{M}, x_\star \in \mathcal{F}(x_\star)$.*

Recall some notions (see, e.g., [15, 24]). Let E be a separable Banach space.

Definition 2.13. *For a given $p \ge 1$, a multifunction $\mathfrak{G} : [0, T] \to K(E)$ is called:*

- L^p–integrable *if it admits an L^p–Bochner integrable selection, i.e., there exists a function $g \in L^p([0,T]; E)$ such that $\quad g(t) \in \mathfrak{G}(t)$ for a.e. $t \in [0,T]$;*
- L^p–integrably bounded *if there exists a function $\xi \in L^p([0,T])$ such that*

$$\|\mathfrak{G}(t)\| \le \xi(t)$$

for a.e. $t \in [0,T]$.

The set of all L^p–Bohner integrable selections of a multifunction $\mathfrak{G} \colon [0,T] \to K(E)$ is denoted by $\mathbb{S}^p(\mathfrak{G})$.

Proposition 2.7. (see [23, 15]). *Let $\mathfrak{G} \colon [0,T] \to P(E)$ be an L^1-integrable and L^1-integrably bounded multifunction. If there exists an L^1-function $\upsilon \colon [0,T] \to \mathbb{R}_+$ such that*

$$\chi_E(\mathfrak{G}(t)) \le \upsilon(t)$$

for a.e. $t \in [0,T]$ then

$$\chi_E\left(\int_a^t \mathfrak{G}(s)ds\right) \le \int_a^t \upsilon(s)ds, \quad t \in [0,T],$$

where

$$\int_a^t \mathfrak{G}(s)ds = \left\{\int_a^t \mathfrak{g}(s)ds : \mathfrak{g} \in \mathbb{S}^1(\mathfrak{G})\right\}.$$

3. Generalized Boundary Value Problems

Recall some notions from the fractional analysis (details can be found, e.g., in [18, 29, 33]).

Let E be a real Banach space.

Definition 3.1. The Riemann–Liouville fractional derivative of the order $q \in (0,1)$ of a continuous function $g : [0,T] \to E$ is the function $D^q g$ of the following form:

$$D^q g(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} g(s)\, ds$$

provided the right-hand side of this equality is well defined.

Here Γ is the Euler gamma-function

$$\Gamma(r) = \int_0^\infty s^{r-1} e^{-s} ds.$$

Definition 3.2. The Caputo fractional derivative of the order $q \in (0,1)$ of a continuous function $g : [0,T] \to E$ is the function $^C D^q g$ defined in the following way:

$$^C D^q g(t) = \Big(D^q(g(\cdot) - g(0))\Big)(t)$$

provided the right-hand side of this equality is well defined.

We will consider some boundary value problems for a semilinear differential inclusion in a separable Banach space E of a fractional order $0 < q < 1$:

$$^C D^q x(t) \in Ax(t) + F(t, x(t)),\ t \in [0,T]. \tag{3.1}$$

It will be supposed that:

(A1) $A : D(A) \subset E \to E$ is a Hille-Yosida operator generating a locally Lipschitz integrated semigroup $\{V(t)\}_{t \geq 0}$;

(A2) the semigroup $\{V'(t)\}_{t \geq 0}$ is uniformly bounded, i.e.,

$$\sup_{t \geq 0} \|V'(t)\| \leq D$$

for some $D > 0$.

We assume that the multimap $F : [0,T] \times E \to Kv(E)$ obeys, for a given $p > \frac{1}{q}$, the following conditions:

(F1) for each $x \in E$, the multifunction $F(\cdot, x) : [0;T] \to Kv(E)$ admits a measurable selection;

(F2) for a.e. $t \in [0;T]$, the multimap $F(t, \cdot) : E \to Kv(E)$ is u.s.c.;

(F3) for each $r > 0$, there exists a function $\alpha_r \in L^p_+[0,T]$ such that for each $x \in E, \|x\| \leq r$ we have

$$\|F(t,x)\|_E \leq \alpha_r(t)$$

for a.e. $t \in [0,T]$;

(F4) there exists a function $\mu \in L^p_+[0,T]$ such that for each nonempty bounded set $\Omega \subset E$

$$\chi(F(t,\Omega)) \leq \mu(t)\chi(\Omega)$$

for a.e. $t \in [0,T]$, where χ is the Hausdorff MNC in E.

Remark 3.1. From the Kuratowski – Ryll-Nardzewski selection theorem (see, e.g., [8, 13, 15, 24]) it is known that condition $(F1)$ is fulfilled if the multifunction $F(\cdot, x)$ is measurable for each $x \in E$.

Remark 3.2. Condition $(F4)$ is fulfilled in case when the multimap F may be represented as

$$F(t,x) = F_1(t,x) + F_2(t,x),$$

where multimaps $F_1, F_2 : [0,T] \times E \to K(E)$ are such that F_1 is completely u.s.c. in the second argument and F_2 is $\mu(t)$-Lipschitz in the second argument in the sense that

$$h\left(F_2\left(t,x'\right), F_2\left(t,x''\right)\right) \leq \mu\left(t\right)\left\|x' - x''\right\| \ \ for \ a.e. \ t \in [0,T]$$

for each $x', x'' \in E$, where h is the Hausdorff metric on $K(E)$ (see [24], Theorem 2.3.4).

From conditions $(F1) - (F3)$ it follows (see, e.g. [15], Theorem 1.3.5) that the superposition multioperator $\mathcal{P}_F : C([0,T]; E) \to P(L^p((0,T); E))$, given by

$$\mathcal{P}_F(x) = \{f \in L^p((0,T); E) : f(t) \in F(t,x(t)) \ \text{a.e.} \ t \in [0,T]\}$$

is well defined. Moreover, $\mathcal{P}_F$ is weakly closed in the sense that $\{x_n\} \subset C([0,T]; E)$, $x_n \to x_0$, $\{f_n\} \subset L^p((0,T); E)$, $f_n \rightharpoonup f_0$ in $L^1((0,T); E)$ imply $f_0 \in \mathcal{P}_F(x_0)$ (see [15], Lemma 5.1.1).

Definition 3.3. (see, e.g., [32, 33]). A mild solution of inclusion (3.1) with an initial value

$$x(0) = x_0 \in E_0 \tag{3.2}$$

is a function $x \in C([0, T]; E)$ such that:

(i) $\int_0^t (t-s)^{q-1} x(s) \, ds \in D(A)$ for $t \in [0, T]$;

(ii) $x(t) = x_0 + \frac{1}{\Gamma(q)} A \int_0^t (t-s)^{q-1} x(s) \, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) \, ds$,
    where $f \in \mathcal{P}_F(x)$.

Remark 3.3. From condition (i) it follows that $x(t) \in E_0 = \overline{D(A)}$, $t \in [0, T]$ (see [32, 33]).

Proposition 3.1. (see [32, 33]). *A mild solution to problem* (3.1) - (3.2) *may be represented in the following form:*

$$x(t) = \mathcal{G}(t)x_0 + \lim_{\lambda \to \infty} \int_0^t (t-s)^{q-1} \mathcal{T}(t-s)\lambda R(\lambda, A) f(s) ds, \qquad t \in [0, T], \tag{3.3}$$

where $f \in \mathcal{P}_F(x)$ *and*

$$\mathcal{G}(t) = \int_0^\infty \xi_q(\theta) V'(t^q\theta) d\theta, \qquad \mathcal{T}(t) = q \int_0^\infty \theta \xi_q(\theta) V'(t^q\theta) d\theta,$$

where ξ_q *is the Wright function defined as*

$$\xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \Psi_q(\theta^{-1/q}),$$

$$\Psi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \theta \in \mathbb{R}^+.$$

Lemma 3.1. (see, e.g., [33]) *The operators* $\mathcal{G}$ *and* $\mathcal{T}$ *possess the following properties:*

1) *For each* $t \in [0, T]$, $\mathcal{G}(t)$ *and* $\mathcal{T}(t)$ *are linear bounded operators, more precisely, for each* $x \in E_0$ *we have*

$$\|\mathcal{G}(t)x\|_E \leq D \|x\|_E; \tag{3.4}$$

$$\|\mathcal{T}(t)x\|_E \leq \frac{qD}{\Gamma(1+q)} \|x\|_E; \tag{3.5}$$

2) *the operator functions* $\mathcal{G}(\cdot)$ *and* $\mathcal{T}(\cdot)$ *are strongly continuous, i.e., functions* $t \in [0, T] \to \mathcal{G}(t)x$ *and* $t \in [0, T] \to \mathcal{T}(t)x$ *are continuous for each* $x \in E_0$.

We will study the problem of existence of mild solutions of the above differential inclusion satisfying the following general boundary value condition:

$$\mathcal{Q}(x) \in \mathcal{S}(x), \tag{3.6}$$

where $\mathcal{Q} : C([0, T]; E_0) \to E_0$ is a bounded linear operator, $\mathcal{S} : C([0, T]; E_0) \to K(E_0)$ is a completely u.s.c. quasi-R_δ-multioperator.

To search our problem, consider the linear operator

$$G : L^p([0, T]; E) \to C([0, T]; E_0),$$

defined as

$$(Gf)(t) = \lim_{\lambda \to \infty} \int_0^t (t-s)^{q-1} \mathcal{T}(t-s)\,\lambda R(\lambda, A) f(s)\,ds. \tag{3.7}$$

Denote by $\mathcal{C}_0$ the subspace of $C([0,T]; E_0)$, consisting of functions of the form

$$x(t) = \mathcal{G}(t)\,x(0), \quad t \in [0,T]$$

and denote by $\mathcal{Q}_0$ the restriction of $\mathcal{Q}$ to $\mathcal{C}_0$.

Our main assumption on boundary operators $\mathcal{Q}$ and $\mathcal{S}$ is the following:

$(\mathcal{QS})$ There exists a continuous linear operator $\Lambda : E_0 \to \mathcal{C}_0$ such that

$$(I - \mathcal{Q}_0\Lambda)(z - \mathcal{Q}Gf) = 0$$

for each $x \in C([0,T]; E_0)$, $z \in \mathcal{S}(x)$, $f \in \mathcal{P}_F(x)$.

To present an example of the realization of condition $(\mathcal{QS})$, consider the continuous linear operator $r : E_0 \to \mathcal{C}_0$ defined in the following way:

$$r(\varsigma)(t) = \mathcal{G}(t)\,\varsigma, \quad t \in [0;T].$$

Assume that

$(\widetilde{\mathcal{Q}})$ The linear continuous operator $\widetilde{\mathcal{Q}} : E_0 \to E_0$ defined as $\widetilde{\mathcal{Q}}\varsigma = \mathcal{Q}(r(\varsigma))$ has the continuous inverse $\widetilde{\mathcal{Q}}^{-1}$.

It is easy to see that under condition $(\widetilde{\mathcal{Q}})$ the operator Λ may be presented in the following explicit form:

$$\Lambda\varsigma = r[\widetilde{\mathcal{Q}}^{-1}(\varsigma)]. \tag{3.8}$$

Supposing that condition $(\mathcal{QS})$ is fulfilled, consider the multioperator

$$\Theta : C([0,T]; E_0) \to K(C([0,T]; E_0))$$

defined in the following way:

$$\Theta(x) = \Lambda\mathcal{S}(x) + (I - \Lambda\mathcal{Q})G\mathcal{P}_F(x).$$

The main property of the multioperator Θ is described by the following assertion.

Theorem 3.1. *Every fixed point of Θ, i.e., a function $x(\cdot)$ satisfying*

$$x = \Lambda z + (I - \Lambda\mathcal{Q})Gf \tag{3.9}$$

for some $z \in \mathcal{S}(x)$, $f \in \mathcal{P}_F(x)$ is a mild solution of problem (3.1), (3.6).

Conversely, under condition $(\widetilde{\mathcal{Q}})$, if x is a mild solution of (3.1), (3.6) then it is a fixed point of Θ.

Proof. (i) Since the function x may be represented in the form

$$x = \Lambda(z - \mathcal{Q}Gf) + Gf$$

we obtain that x satisfies integral equation (3.3).

Let us verify the fulfilment of the boundary condition. Using condition $(\mathcal{QS})$ we get

$$\mathcal{Q}x = \mathcal{Q}_0\Lambda z + \mathcal{Q}(I - \Lambda\mathcal{Q})Gf = z - (z - \mathcal{Q}_0\Lambda z) + \mathcal{Q}Gf + \mathcal{Q}_0\Lambda\mathcal{Q}Gf$$
$$= z - (I - \mathcal{Q}_0\Lambda)(z - \mathcal{Q}Gf) = z \in \mathcal{S}x.$$

(ii) Now, let x be a mild solution of (3.1), (3.6). Then it satisfies the relation

$$x = r(x(0)) + Gf$$

for some $f \in \mathcal{P}_F(x)$. Then

$$\mathcal{Q}x = \widetilde{\mathcal{Q}}(x(0)) + \mathcal{Q}Gf$$

from where we get

$$x(0) = \widetilde{\mathcal{Q}}^{-1}(\mathcal{Q}x - \mathcal{Q}Gf)$$

implying

$$r(x(0)) = \Lambda(\mathcal{Q}x - \mathcal{Q}Gf).$$

Therefore,

$$x = \Lambda(\mathcal{Q}x - \mathcal{Q}Gf) + Gf = \Lambda\mathcal{Q}x + (I - \Lambda\mathcal{Q})Gf \in \Theta(x).$$

$\square$

To describe subsequent properties of the multioperator Θ we need the following notion.

Definition 3.4. (cf. [15], Definition 4.2.1). For $1 \le p \le \infty$, a sequence of functions $\{\xi_n\} \subset L^p((0,t); E)$ is called L^p-semicompact if it is L^p-integrably bounded, i.e.,

$$\|\xi_n(t)\| \le \zeta(t) \text{ a.e. } t \in [0,T], \ n \ge 1,$$

where $\zeta \in L^p(0,T)$ and the set $\{\xi_n(t)\}$ is relatively compact in E for a.e. $t \in [0,T]$.

Proposition 3.2. (cf. [15], Proposition 4.2.1). *Every L^p-semicompact sequence $\{\xi_n\}$ is weakly compact in $L^1((0,T); E)$.*

Following the lines of [16] one can verify the following property of the operator G defined in (3.7).

Proposition 3.3. (i) *If $\frac{1}{q} < p < \infty$, then there exists a constant $C_p > 0$ such that*

$$\|G(\xi)(t) - G(\eta)(t)\|_E^p \le C_p \int_0^t \|\xi(s) - \eta(s)\|_E^p ds, \quad \xi, \eta \in L^p((0,T); E);$$

(ii) *Let $\{\xi_n\}$ be an L^p-semicompact sequence in $L^p((0,T); E)$. Then the sequence $\{G\xi_n\}$ is relaively compact in $C([0,T]; E_0)$ and moreover, the weak convergence $\xi_n \rightharpoonup \xi_0$ in $L^1((0,T); E)$ implies $G\xi_n \to G\xi_0$ in $C([0,T]; E_0)$.*

Lemma 3.2. *The multioperator Θ is quasi-R_δ.*

Proof. Decompose Θ in the sum

$$\Theta = \Theta_1 + \Theta_2,$$

where $\Theta_1 = \Lambda\mathcal{S}$ and $\Theta_2 = (I - \Lambda\mathcal{Q})G\mathcal{P}_F$.

From conditions imposed on the multimap $\mathcal{S}$ it follows that Θ_1 is quasi-R_δ.

Consider the multimap Θ_2. It is clear that it is convex-valued. Let sequences $\{x_n\}, \{z_n\} \subset C([0,T]; E_0)$ be such that $x_n \to x_0$, $z_n \in \Theta_2(x_n)$, $n \ge 1$. Then we have

$$z_n = (I - \Lambda\mathcal{Q})G(f_n), \quad n \ge 1,$$

where $f_n \in \mathcal{P}_F(x_n)$, $n \geq 1$. According to conditions (F3) and (F4), the sequence $\{f_n\}$ is L^p-semicompact and, due to Proposition 3.2 it is weakly compact in $L^1((0,T);E)$ and hence we may assume, w.l.o.g., that $f_n \rightharpoonup f_0 \in \mathcal{P}_F(x_0)$ in $L^1((0,T);E)$. Applying Proposition 3.3 (ii) we get $G(f_n) \to G(f_0)$ and hence, by the continuity of the linear operator $I - \Lambda\mathcal{Q}$ we have that

$$z_n \to z_0 = (I - \Lambda\mathcal{Q})G(f_0) \in \Theta_2(x_0).$$

This reasoning shows that the multimap Θ_2 is closed and quasicompact and hence u.s.c. (Proposition 2.4). This means that Θ_2 is quasi-R_δ.

It remains to apply Proposition 2.5. $\qquad\square$

Now our goal is to show that the multioperator Θ is condensing with respect to an appropriate measure of noncompactness. To do so, we need some additional assumptions.

Let χ be the Hausdorff MNC in E_0. Denote

$$d = \sup_{0 \leq t \leq T} \|\mathcal{T}(t)\|^{(\chi)}.$$

Notice that from (3.5) and (2.6) it follows that

$$0 \leq d \leq \frac{qD}{\Gamma(1+q)}.$$

Let φ be the module of fiber noncompactness in the space $C([0,T];E_0)$ (see Section 2.2). Assume that

(H1) there exists $\rho \geq 0$ such that for each bounded $\Omega \subset C([0,T];E_0)$ we have

$$\chi(\mathcal{Q}\Omega) \leq \rho\varphi(\Omega)$$

(H2) $d \sup\limits_{0 \leq t \leq T} \int_0^t (t-s)^{q-1}\mu(s)\,ds < \dfrac{1}{1 + \|\Lambda\|^{(\chi,\varphi)}\rho}$,

where μ is the function from condition (F4).

Remark 3.4. Condition (H2) is fulfilled if, in particular, the semigroup V' is compact (in this case $d = 0$, see [33], Lemma 2.9) or the multimap F is completely u.s.c. in the second argument ($\mu = 0$).

Define the vector measure of noncompactness in the space $C([0,T];E_0)$

$$\nu : P(C([0,T];E_0)) \to \mathbb{R}_+^2$$

with the values in the cone $\mathbb{R}_+^2$ given as

$$\nu(\Omega) = (\varphi(\Omega), mod_C(\Omega)),$$

where $mod_C(\Omega)$ is the module of equicontinuity in the space $C([0,T];E_0)$ (see Section 2.2).

Lemma 3.3. *The multioperator Θ is ν-condensing.*

Proof. Since the multioperator Θ_1 is completely u.s.c. and the MNC ν is monotone, algebraically semiadditive and invariant with respect to union with a compact set (see [15]) it is sufficient to prove the assertion for the multioperator Θ_2, i.e., to demonstrate that for any bounded set $\Omega \subset C([0,T]; E_0)$ the relation

$$\nu(\Theta_2(\Omega)) \geq \nu(\Omega) \tag{3.10}$$

(taken in the sense of the partial order in $\mathbb{R}^2$ induced by the cone $\mathbb{R}_+^2$) implies the relative compactness of Ω. This can be done by following the reasonings.

From (3.10) it follows that

$$\varphi(\Theta_2(\Omega)) \geq \varphi(\Omega). \tag{3.11}$$

Take an arbitrary $t \in [0,T]$ and estimate $\chi(\Theta_2(\Omega)(t))$. We have

$$\chi\big(\Lambda\mathcal{Q}G\mathcal{P}_F(\Omega)(t)\big) \leq \varphi\big(\Lambda\mathcal{Q}G\mathcal{P}_F(\Omega)\big) \leq \|\Lambda\|^{(\chi,\varphi)}\chi\big(\mathcal{Q}G\mathcal{P}_F(\Omega)\big)$$

$$\leq \|\Lambda\|^{(\chi,\varphi)}\rho\,\varphi\big(G\mathcal{P}_F(\Omega)\big)\|\Lambda\|^{(\chi,\varphi)}\rho \sup_{\tau\in[0,T]} \chi\big(G\mathcal{P}_F(\Omega)(\tau)\big).$$

To estimate $\chi\big(G\mathcal{P}_F(\Omega)(t)\big)$ introduce the family of operators

$$G_\lambda \colon L^p((0,T); E) \to C([0,T]; E_0), \ \lambda > \omega$$

defined as

$$(G_\lambda f)(t) = \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)\,\lambda R(\lambda, A)f(s)\,ds.$$

Then for any $\lambda > \omega$ and $0 \leq s \leq t$ by using the Hille-Yosida condition we get

$$\chi\big((t-s)^{q-1}\mathcal{T}(t-s)\lambda R(\lambda, A)F(s,\Omega(s))\big)$$

$$\leq (t-s)^{q-1}\|\mathcal{T}(t-s)\|^{(\chi)}\lambda\|R(\lambda, A)\|\chi\big(F(s,\Omega(s))\big)$$

$$\leq d\frac{\lambda}{\lambda-\omega}(t-s)^{q-1}\mu(s)\chi(\Omega(s)) \leq d\frac{\lambda}{\lambda-\omega}(t-s)^{q-1}\mu(s)\varphi(\Omega).$$

Then according to Proposition 2.7 we have

$$\chi\big(G_\lambda\mathcal{P}_F(\Omega)(t)\big) \leq d\frac{\lambda}{\lambda-\omega}\int_0^t (t-s)^{q-1}\mu(s)ds \cdot \varphi(\Omega).$$

Passing to the limit as $\lambda \to \infty$ yields

$$\chi\big(G\mathcal{P}_F(\Omega)(t)\big) \leq d\int_0^t (t-s)^{q-1}\mu(s)ds \cdot \varphi(\Omega).$$

and, by using the property of algebraic subadditivity of χ we obtain

$$\chi\big((I - \Lambda\mathcal{Q})G\mathcal{P}_F(\Omega)(t)\big) \leq d\left(1 + \|\Lambda\|^{(\chi,\varphi)}\rho\right) \sup_{t\in[0,T]}\int_0^t (t-s)^{q-1}\mu(s)ds \cdot \varphi(\Omega)$$

$$= \kappa \cdot \varphi(\Omega),$$

where

$$\kappa = d\left(1 + \|\Lambda\|^{(\chi,\varphi)}\rho\right) \sup_{t\in[0,T]}\int_0^t (t-s)^{q-1}\mu(s)ds < 1$$

by assumption $(H2)$.

But then also

$$\varphi(\Theta_2(\Omega)) \le \kappa \cdot \varphi(\Omega)$$

implying

$$\varphi(\Omega) = 0. \tag{3.12}$$

Now we will show that the set Ω is equicontinuous. Notice that from the relation

$$mod_C(\Theta_2(\Omega)) \ge mod_C(\Omega)$$

it follows that it is sufficient to prove the equicontinuity of the set $\Theta_2(\Omega)$. This is equivalent to the fact that this property holds for any sequence

$$\{z_n\} \subset (I - \Lambda\mathcal{Q})G\mathcal{P}_F(\Omega).$$

Take sequences $\{x_n\} \subset \Omega$ and $\{f_n\}, f_n \in \mathcal{P}_F(x_n)$ such that

$$z_n = (I - \Lambda\mathcal{Q})Gf_n, \quad n = 1, 2, \dots$$

From condition $(F3)$ it follows that the sequence $\{f_n\}$ is L^p-integrably bounded. Relation (3.12) yields the equality

$$\chi(\{x_n(t)\}) = 0, \quad \forall t \in [0, T]$$

and hence by condition $(F4)$ we get

$$\chi(\{f_n(t)\}) = 0, \quad \text{a.e. } t \in [0, T].$$

From Proposition 3.3(ii) it follows that the sequence $\{Gf_n\}$ and hence z_n is relatively compact and hence equicontinuous. Now the relative compactness of the set Ω follows from the Arzelà–Ascoli theorem. $\square$

We see that the properties of the multioperator Θ open the possibility to apply the topological degree theory described in Section 2.2 for its investigation. We can formulate the following general principle for the existence of mild solutions to problem (3.1), (3.6).

Theorem 3.2. *Under above conditions, let an open bounded set $\Omega \subset C([0, T]; E_0)$ does not have mild solutions of problem (3.1), (3.6) on its boundary $\partial\Omega$ and let*

$$deg(i - \Theta, \overline{\Omega}) \ne 0.$$

Then the set of mild solutions of problem (3.1), (3.6) is non empty.

As an example of application of this principle consider the following assertion.

Theorem 3.3. *Under above conditions, let us assume, in addition, that*

(H3) *there exists a sequence of functions $\omega_n \in L_+^p(0; T)$, $n = 1, 2, \dots$ such that:*

$$\liminf_{n \to \infty} \frac{1}{n}\|\omega_n\|_p = 0;$$

and

$$\sup_{\|x\| \le n} \|F(t, x)\| \le \omega_n(t) \qquad \textit{for a.e. } t \in (0; T),$$

(H4) *the following asymptotic condition holds:*

$$\liminf_{\|x\|\to\infty} \frac{\|\mathcal{S}(x)\|}{\|x\|} = 0.$$

Then the set of mild solutions to problem (3.1), (3.6) is non empty.

Proof. Let us show that there exists a closed ball $B_R \subset C([0,T];E_0)$ such that $\Theta(B_R) \subseteq B_R$.

Supposing the contrary we have sequences $\{x_n\}, \{z_n\} \subset C([0,T];E_0)$ such that $z_n \in \Theta(x_n)$, $\|x_n\| \le n$, $\|z_n\| > n$. From conditions posed on operators $\mathcal{Q}, \mathcal{S}$, condition $(F3)$ and Proposition 3.3 (i) it follows that the multimap Θ transforms bounded sets into bounded ones. This means that, passing to a subsequence if necessary, we may assume, w.l.o.g. that $\|x_n\| \to \infty$. Then we obtain

$$\|z_n\| \le \|\Lambda \mathcal{S} x_n\| + \|I - \Lambda \mathcal{Q}\| \cdot \|G f_n\|$$

for some $f_n \in \mathcal{P}_F(x_n)$. Applying Proposition 3.3 (i) we get

$$\|z_n\| \le \|\Lambda\| \|\dot{\mathcal{S}} x_n\| + \|I - \Lambda \mathcal{Q}\| \sqrt[p]{C_p} \|f_n\|_p.$$

Then we have

$$1 < \frac{\|z_n\|}{n} \le \|\Lambda\| \frac{\|\mathcal{S} x_n\|}{n} + \|I - \Lambda \mathcal{Q}\| \sqrt[p]{C_p} \frac{1}{n} \|f_n\|_p$$
$$\le \|\Lambda\| \frac{\|\mathcal{S} x_n\|}{\|x_n\|} + \|I - \Lambda \mathcal{Q}\| \sqrt[p]{C_p T} \frac{1}{n} \|\omega_n\|_p$$

contrary to assumptions $(H3)$ and $(H4)$.

It remains to apply Proposition 2.6 to the restriction of Θ to B_R. $\square$

4. Some particular cases

4.1. **A nonlocal Cauchy problem.** Consider differential inclusion (3.1) with the following boundary condition

$$x(0) \in x_0 + \mathfrak{S}(x), \tag{4.1}$$

where $x_0 \in E_0$ and $\mathfrak{S} : C([0,T];E_0) \to K(E_0)$ is a completely u.s.c. quasi-R_δ-multi-operator.

Notice that in this case the operator $\mathcal{Q}$ has the form $\mathcal{Q}(x) = x(0)$ and hence condition $(H1)$ is fulfilled with $\rho = 1$. Further, the operator $\widetilde{\mathcal{Q}}$ is the identity, so condition $(\widetilde{\mathcal{Q}})$ is fulfilled and moreover, the operator Λ coincides with r. Denoting

$$d' = \sup_{0 \le t \le T} \|\mathcal{G}(t)\|^{(\chi)}$$

we get the estimate

$$\|\Lambda\|^{(\chi,\varphi)} \le d' \le D$$

(the last inequality follows from (3.4) and (2.6)).

As a result, we get the following corollary of Theorem 3.3.

Proposition 4.1. *Under conditions* $(A1), (A2), (F1), (F2), (F4)$,

$(H2')$ d $\displaystyle\sup_{0 \le t \le T} \int_0^t (t-s)^{q-1}\mu(s)\,ds < \frac{1}{1+d'}$,

$(H3)$ *and* $(H4)$ *(with the replacement of* $\mathcal{S}(x)$ *with* $\mathfrak{S}(x)$*) problem (3.1), (3.6) has a mild solution.*

4.2. **A periodic boundary value problem.** A periodic boundary value problem for differential inclusion (3.1)

$$x(0) = x(T) \tag{4.2}$$

may be written in the form of a boundary condition

$$\mathcal{Q}x = 0, \tag{4.3}$$

where $\mathcal{Q}x = x(T) - x(0)$.

Notice that for this operator condition $(H1)$ is fulfilled with the constant $\rho = 2$.

In fact, for every bounded $\Omega \subset C([0,T]; E_0)$ we have:

$$\chi(\mathcal{Q}(\Omega)) \le \chi\big(\Omega(T) - \Omega(0)\big) \le \chi\big(\Omega(T)\big) + \chi\big(\Omega(0)\big) \le 2\varphi(\Omega).$$

We will assume that the following condition holds:

(A_T) the linear operator $\mathcal{G}(T) - I$ has a continuous inverse on E_0.

Under condition (A_T) condition $(\widetilde{\mathcal{Q}})$ is fulfilled. In fact, for any $\varsigma \in E_0$

$$\widetilde{\mathcal{Q}}\varsigma = \mathcal{G}(T)\varsigma - \mathcal{G}(0)\varsigma = (\mathcal{G}(T) - I)\varsigma$$

and hence

$$\widetilde{\mathcal{Q}}^{-1} = (\mathcal{G}(T) - I)^{-1}$$

and

$$(\Lambda\varsigma)(t) = \mathcal{G}(t)(\mathcal{G}(T) - I)^{-1}\varsigma.$$

So, we have the following estimate

$$\|\Lambda\|^{(\chi,\varphi)} \le d' \|(\mathcal{G}(T) - I)^{-1}\|^{(\chi)}.$$

The multioperator Θ for the periodic problem has the form

$$\Theta(x) = (I - \Lambda\mathcal{Q})G\mathcal{P}_F(x).$$

To present it in a more explicit form, take $x \in C([0,T]; E_0)$. Then $\Theta(x)$ consists of all functions $z \in \mathcal{C}([0,T]; E_0)$ which, for any $f \in \mathcal{P}_F(x)$ have the form

$$z = (I - \Lambda\mathcal{Q})Gf,$$

i.e.,

$$z(t) = \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f(s)ds + \Lambda\left(\int_0^T (T-s)^{q-1}\mathcal{T}(T-s)f(s)ds\right)$$

$$= \mathcal{G}(t)(I - \mathcal{G}(T))^{-1}\int_0^T (T-s)^{q-1}\mathcal{T}(T-s)f(s)ds + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f(s)ds$$

(cf. [15], Section 6.1).

Again, by applying Theorem 3.3 we obtain the following assertion.

Proposition 4.2. *Under conditions* $(A1), (A2), (A_T), (F1), (F2), (F4), (H3)$ *and*

$(H2'')$ $d \sup\limits_{0 \le t \le T} \int_0^t (t-s)^{q-1} \mu(s)\, ds < \dfrac{1}{1 + 2d' \|(\mathcal{G}(T) - I)^{-1}\|^{(\chi)}}$

problem (3.1), (4.2) has a mild solution.

Notice in conclusion that by using the same methods we can solve the anti-periodic boundary value problem

$$x(0) = -x(T). \tag{4.4}$$

References

[1] M. Adimy, H. Bouzahir, K. Ezzinbi, *Existence for a class of partial functional differential equations with infinite delay*, Nonlinear Anal. - TMA, **46**(2001), no. 1, 91-112.

[2] R.P. Agarwal, D. Baleanu, J.J. Nieto, D.F.M. Torres, Y. Zhou, *A survey on fuzzy fractional differential and optimal control nonlocal evolution equations*, J. Comput. Appl. Math., **339**(2018), 3-29.

[3] B. Ahmad, J.J. Nieto, A. Alsaedi, M.H. Aqlan, *A coupled system of Caputo-type sequential fractional differential equations with coupled (periodic/anti-periodic type) boundary conditions*, Mediterr. J. Math., **14**(2017), art. 227, 15 pp.

[4] C.T. Anh, T.D. Ke, *On nonlocal problems for retarded fractional differential equations in Banach spaces*, Fixed Point Theory, **15**(2014), no. 2, 373-392.

[5] W. Arendt, *Resolvent positive operators*, Proc. London Math. Soc., **54**(3)(1987), no. 2, 321-349.

[6] W. Arendt, *Vector-valued Laplace transforms and Cauchy problems*, Israel J. Math., **59**(1987), no.3, 327-352.

[7] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, 96. Birkhauser Verlag, Basel, 2001.

[8] A.V. Arutyunov, V. Obukhovskii, *Convex and Set-Valued Analysis. Selected Topics*, De Gruyter Graduate, De Gruyter, Berlin, 2017.

[9] I. Benedetti, V. Obukhovskii, V. Taddei, *On noncompact fractional order differential inclusions with generalized boundary condition and impulses in a Banach space*, J. Funct. Spaces, **2015**, art. ID 651359, 10 pp.

[10] Yu.G. Borisovich, B.D. Gelman, A.D. Myshkis, V.V. Obukhovskii, *Topological methods in the theory of fixed points of multivalued mappings*, (Russian), Uspekhi Mat. Nauk, **35**(1980), no. 1, 59-126; English translation: Russian Math. Surveys, **35**(1980), no.1, 65-143.

[11] G. Da Prato, E. Sinestrari, it Differential operators with nondense domain, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **14**(4)(1988), no. 2, 285-344.

[12] Z. Ding, A.G. Kartsatos, *Nonresonance problems for differential inclusions in separable Banach spaces*, Proc. Amer. Math. Soc., **124**(1996), no. 8, 2357-2365.

[13] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Second ed., Topological Fixed Point Theory and Its Applications, vol. 4. Springer, Dordrecht, 2006.

[14] D.H. Hyman, *On decreasing sequences of compact absolute retracts*, Fund Math., **64**(1969), 91-97.

[15] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, De Gruyter Series in Nonlinear Analysis and Applications, 7. Walter de Gruyter, Berlin - New York, 2001.

[16] T.D. Ke, V. Obukhovskii, N.-C. Wong, J.-C. Yao, *On a class of fractional order differential inclusions with infinite delays*, Appl. Anal., **92**(2013), no. 1, 115-137.

[17] H. Kellerman, M. Hieber, *Integrated semigroups*, J. Funct. Anal., **84**(1989), no.1, 160-180.

[18] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. **204**, Elsevier Science B.V., 2006.

[19] D. Kravvaritis, N.S. Papageorgiou, *A boundary value problem for a class of evolution inclusions*, Comment. Math. Univ. St. Paul., **40**(1991), no. 1, 29-37.

[20] G. Marino, *Nonlinear boundary value problems for multivalued differential equations in Banach spaces*, Nonlinear Anal., **14**(1990), no. 7, 545-558.

[21] I.V. Mel'nikova, A.I. Filinkov, *Integrated semigroups and C-semigroups. Well-posedness and regularization of operator-differential problems*, (Russian), Uspekhi Mat. Nauk **49**(1994), no. 6, 111-150; English translation in Russian Math. Surveys, **49**(1994), no. 6, 115-155.

[22] F. Neubrander, *Integrated semigroups and their applications to the abstract Cauchy problem*, Pacific J. Math., **135**(1988), 111-155.

[23] V.V. Obukhovskii, *Semilinear functional-differential inclusions in a Banach space and controlled parabolic systems*, Soviet J. Automat. Inform. Sci., **24**(1991), no. 3, 71-79.

[24] V. Obukhovskii, B. Gel'man, *Multivalued Maps and Differential Inclusions. Elements of Theory and Applications*, World Scientific Publishing Co., Hackensack, NJ, 2020.

[25] V. Obukhovskii, P. Zecca, *On boundary value problems for degenerate differential inclusions in Banach spaces*, Abstr. Appl. Anal., **2003**, no. 13, 769-784.

[26] V. Obukhovskii, P. Zecca, *On semilinear differential inclusions in Banach spaces with nondensely defined operators*, J. Fixed Point Theory Appl., **9** (2011), no. 1, 85-100.

[27] N.S. Papageorgiou, *Boundary value problems for evolution inclusions*, Comment. Math. Univ. Carolin., **29**(1988), no. 2, 355-363.

[28] N.S. Papageorgiou, *Boundary value problems and periodic solutions for semilinear evolution inclusions*, Comment. Math. Univ. Carolin., **35**(1994), no. 2, 325-336.

[29] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[30] H.R. Thieme, *Integrated semigroups and integrated solutions to abstract Cauchy problems*, J. Math. Anal. Appl., **152**, no.2, 416-447.

[31] P. Zecca, P.L. Zezza, *Nonlinear boundary value problems in Banach space for multivalued differential equations on noncompact intervals*, Nonlinear Anal., **3**(1979), no. 3, 347-352.

[32] Z. Zhang, B. Liu, *Existence results on nondensely defined fractional evolution differential inclusions*, J. Appl. Math., **2012**, art. ID 316850, 19 pp.

[33] Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control*, Elsevier Academic Press, London, 2016.