# Modal stability and Squire's theorem for an inhomogeneous viscoelastic suspension

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**Abstract**

We study the linear stability of the Poiseuille flow of a viscoelastic upper convected Maxwell fluid in which the rheological parameters depend on the concentration of particles suspended in the fluid (dense suspension with negligible diffusion). After determining the basic flow and basic concentration profile we consider a temporal three dimensional perturbation in the form of a stream-wise and cross-wise wave. We derive the linearized perturbed equation and prove the validity of Squire's theorem, extending the result of [13] in which the theorem was proved for constant rheological parameters. We discuss the relation between the Weissenberg and the Reynolds numbers. We finally study the 2D eigenvalue problem for the case of constant coefficients and for non-constant coefficients with low Weissenberg number. We solve the problem numerically by means of a spectral collocation method and we plot the marginal stability curves discussing how stability depends on the fluid rheology.

*Keywords:* Dense suspension, Viscoelastic fluid, Linear Stability, Neutral stability curves

## 1. Introduction

One of the first investigation on the rheology of suspensions is probably the pioneering work of Einstein [3] on the viscosity of a hard sphere suspension at low concentration. Since then the rheological behavior of suspensions has attracted

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the interest of a considerable number of scientists and many studies have been carried out to investigate the characteristics of these materials. Many papers have focussed on highly-concentrated (or dense) suspensions, where the mean distance between particles is less than the average particle size, [12]. Others are concerned with the so-called dilute or semi-dilute suspensions, where the attraction between particles is negligible and the system is essentially governed by hydrodynamic forces and Brownian diffusion, [1].

Suspensions are of great importance in many applications such as biochemical systems (blood), industrial products (paints, foams) and industrial processing (oils, slurries). Dense suspensions are characterized by strong effects of particle interactions occurring at the microscopic level. The macroscopic response is determined by the microstructure dynamics of the particles and by the suspending fluid around them. Dense suspensions have negligible diffusive effects and multiple-body interactions contribute significantly to the rheology of the system. In particular, in many cases one can also observe non-Newtonian behaviors such as shear thinning, memory and first and second normal stress differences.

The rheology of the suspension in a non-Newtonian medium must be taken into account when one models applications as filled polymer processing, fiber extrusion, ceramic processing, drilling muds in oil recovery, slurry and oil transportation in pipelines. Studies on suspensions in non-Newtonian media are available in the literature, but the vast majority consider only homogeneous dilute/dense suspensions, [9]. In [10] the authors obtain a reliable prediction for the stress in a suspension of spherical particles in a viscoelastic fluid. Unfortunately, the result is confined to a homogeneous suspension and to a linear velocity field. The extension of theoretical models to non-homogeneous suspensions is extremely important when modeling processes as the ones we have mentioned previously. The heterogeneity of the particles distribution may play a significant role in the dynamics of the system and transport mechanisms such as advection or diffusion of the solid particles must be taken into account.

In this paper we are interested in studying the linear stability of an isothermal dense suspension modeled as a upper convected Maxwell fluid flowing between parallel plates (Poiseuille flow). In particular, we investigate the stability assuming that the suspension is not homogeneous, i.e. the particle concentration may change in space and time. The rheological parameters in the constitutive equations are supposed to depend on the particles distribution, so that they are no longer constant. Since we are considering a dense suspension, we neglect diffusion and assume that the evolution of the particle density is governed by a simple transport equation. The linear stability of the Poiseuille flow of a dense Newtonian and generalized Newtonian suspension has been studied in [6], [7]. The present work can be seen as a follow up of the paper [13] in which the linear stability of an upper convected Maxwell fluid with constant rheology is investigated.

## 2. The mathematical model

We consider an inhomogeneous incompressible upper convected Maxwell fluid with constitutive equation given by (the starred quantities are dimensional)

$$\mathbf{T}^* + \lambda^*(\varphi)\overset{\triangledown}{\mathbf{T}}{}^* = 2\mu^*(\varphi)\mathbf{D}^*, \tag{1}$$

where $\varphi \in [0,1]$ is the concentration (volume fraction) of the suspended particles in the fluid that does not affect the density, but whose variation strongly affects the rheology of the fluid (e.g. RBCs in blood flow). The quantities $\lambda^*$, $\mu^*$ are the relaxation time and the viscosity of the fluid respectively, both depending on $\varphi$. The tensor $\mathbf{T}^*$ represents the deviatoric part of the Cauchy stress tensor $\boldsymbol{\sigma}^* = -p^*\boldsymbol{I}+\mathbf{T}^*$, where $p^*$ is the Lagrange multiplier due to the incompressibility constraint (pressure). The upper convected time derivative is given by

$$\overset{\triangledown}{\mathbf{T}}{}^* = \frac{\partial \mathbf{T}^*}{\partial t^*} + (\mathbf{v}^* \cdot \nabla^*)\mathbf{T}^* - \mathbf{L}^*\mathbf{T}^* - \mathbf{T}^*{\mathbf{L}^*}^T, \tag{2}$$

where $\mathbf{L}^* = \nabla^* \mathbf{v}^*$ and $\mathbf{v}^*$ is the velocity field. In the absence of body forces the system is governed by

$$\begin{cases} \dfrac{\partial \varphi}{\partial t^*} + \mathbf{v}^* \cdot \nabla^* \varphi = 0, \\ \\ \nabla^* \cdot \mathbf{v}^* = 0, \\ \\ \rho^* \dfrac{D\mathbf{v}^*}{Dt^*} = -\nabla^* p^* + \nabla^* \cdot \mathbf{T}^*, \end{cases} \tag{3}$$

where $\rho^*$ is the constant density of the fluid, $D/Dt^*$ is material differentiation and $\mathbf{T}^*$ is given by (1), (2). Equation $(3)_1$ represents the evolution equation of the particles concentration. Here we assume that the only transport mechanism is advection with no diffusive phenomena (dense suspension). The rheological parameters appearing in (1) depend on $\varphi$ so that we can write

$$\lambda^*(\varphi) = \lambda_p^* \lambda(\varphi), \qquad \mu^*(\varphi) = \mu_p^* \mu(\varphi), \tag{4}$$

where $\lambda$, $\mu$ are positive smooth functions of $\varphi$ such that $\lambda(0) = \mu(0) = 1$ and where $\lambda_p^*$, $\mu_p^*$ are the constant relaxation time and viscosity when $\varphi = 0$ (zero particle concentration).

The system (3) is put in a dimensionless form scaling $\mathbf{x}^*$ with $L^*$, $\mathbf{v}^*$ with $U^*$, $t^*$ with $L^* {U^*}^{-1}$, $p^*$ with $\rho^* {U^*}^2$ and $\mathbf{T}^*$ with $\mu_p^* U^* {L^*}^{-1}$, $\mathbf{D}^*$ and $\mathbf{L}^*$ with $U^* {L^*}^{-1}$. The selection of the reference pressure is made to follow the one adopted in [13]. We find

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + \mathbf{v} \cdot \nabla \varphi = 0, \\ \\ \nabla \cdot \mathbf{v} = 0, \\ \\ \mathcal{R} \, \dfrac{D\mathbf{v}}{Dt} = -\mathcal{R} \, \nabla p + \nabla \cdot \mathbf{T}, \end{cases} \tag{5}$$

where $\mathcal{R} = (\rho^* U^* L^* {\mu_p^*}^{-1})$ is the Reynolds number. The dimensionless constitutive equation becomes

$$
\begin{cases}
\mathbf{T} + \mathcal{W}\,\lambda(\varphi)\overset{\triangledown}{\mathbf{T}} = 2\mu(\varphi)\mathbf{D}, \\[2ex]
\overset{\triangledown}{\mathbf{T}} = \dfrac{\partial \mathbf{T}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{T} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T,
\end{cases}
\tag{6}
$$

where $\mathcal{W} = (\lambda_p^* U^* {L^*}^{-1})$ is the Weissenberg number. We observe that

$$
\frac{\mathcal{W}}{\mathcal{R}} = \frac{\lambda_p^* \mu_p^*}{\rho^* {L^*}^2} = \frac{\lambda_p^* \nu_p^*}{{L^*}^2} = \frac{\lambda_p^*}{\lambda_k^*} = \mathcal{E}\,,
\tag{7}
$$

where $\nu_p^*$ is the kinematic viscosity, $\lambda_k^* = {L^*}^2/\nu_p^*$ is the kinematical diffusion characteristic time and $\mathcal{E}$ is the so-called "elasticity number". From (7) we realize that $\mathcal{E}$ is a non dimensional parameter that depends only on the material characteristics of the fluid and on the geometrical setting (not on the kinematics, i.e. not on $U^*$). The Weissenberg number $\mathcal{W} = (\lambda_p^* U^* {L^*}^{-1})$, on the other hand, depends on $U^*$ and hence on the kinematics. This means that, when studying linear stability of shear flows where $\mathcal{R} = (U^* L^*/\nu_p^*)$ is progressively increased to detect the critical Reynolds number that marks the appearance of unstable modes, the number $\mathcal{W}$ cannot be treated as a fixed constant, unless we assume that the variation of $\mathcal{R}$ is due to a change in the kinematic viscosity $\nu_p^*$ and not in the characteristic velocity $U^*$. There are some works in which the marginal stability curves are computed assuming that $\mathcal{W}$ and $\mathcal{R}$ are independent, see [2], [8], [13], [14].

To better explain the relation between $\mathcal{W}$ and $\mathcal{R}$ let us consider the classical way in which marginal stability curves are obtained in 2D modal stability analysis for confined flows. The shear flow is perturbed by a disturbance in the form of a streamwise wave with velocity $c \in \mathbb{C}$, wave number $\alpha$ and wave amplitude that depends on the transversal coordinate. The balance equations are then linearized and the system is reduced to an eigenvalue problem whose numerical solution provides a discrete set of complex eigenvalues for the selected pair $(\mathcal{R}\,,\alpha)$. The maximum of the imaginary part of the complex eigenvalues $c_i(\mathcal{R}\,,\alpha)$, which is clearly a function or $\mathcal{R}$ and $\alpha$, is finally computed. If this value is negative the relative mode is stable, if it is positive the mode is unstable. The marginal stability curve is therefore the zero level set of the function $c_i(\mathcal{R}, \alpha)$, a curve that can be computed varying the Reynolds number $\mathcal{R}$ and the wave number $\alpha$ in the eigenvalue problem. If we take $\mathcal{W}$ as a fixed constant in this process, then for each different $\mathcal{R}$ we must have a different $\mathcal{E}$ in order to ensure that $\mathcal{W} = \mathcal{E}\mathcal{R} = const$. This means that changing $\mathcal{R}$ implies a change in $\mathcal{E}$, i.e. a change in the material properties of the fluid. In this paper we consider the case in which $\mathcal{W}$ and $\mathcal{R}$ are independent (and hence $\mathcal{E}$ is not a fixed constant) and the case in which $\mathcal{W}$ varies with $\mathcal{R}$ (and hence $\mathcal{E}$ is a fixed constant).

## 3. Basic solution

Let us consider the flow taking place in the layer $y \in [-1, 1]$ (in dimensional variables $[-L^*, L^*]$) driven by a known constant pressure gradient $-G$ (the dimensional pressure gradient is thus $G^* = \rho^* {U^*}^2 {L^*}^{-1} G$). We impose no-slip boundary conditions $\mathbf{v}|_{y=\pm 1} = 0$ on the plates and we look for a basic solution of the form

$$\mathbf{v}_o = u_o(y)\mathbf{i}, \qquad \mathbf{T}_o = \mathbf{T}_o(y), \qquad \varphi_o = \varphi_o(y), \qquad p_o = p_o(x). \tag{8}$$

We note that a solution like (8) automatically satisfies equation $(5)_1$ and hence the quantity $\varphi_o(y)$ is now a datum of the problem. For simplicity we assume that such a function is symmetric with respect to $y = 0$ but minor changes allow to treat the non-symmetric case. We write

$$\lambda_o(y) = \lambda(\varphi_o(y)), \qquad \mu_o(y) = \mu(\varphi_o(y)). \tag{9}$$

The only non-zero component of the tensor $\mathbf{L}_o$ is $(\mathbf{L}_o)_{12} = u_o'(y)$ (the prime denotes differentiation with respect to $y$) and

$$\frac{\partial \mathbf{T}_o}{\partial t} = (\mathbf{v}_o \cdot \nabla)\mathbf{T}_o = 0. \tag{10}$$

Equations $(5)_{1,2}$ are automatically satisfied. On substituting (8)-(10) into (6) we find $T_{13} = T_{23} = T_{22} = T_{33} = 0$ and

$$T_{11} = 2\mathcal{W}\lambda_o \mu_o {u_o'}^2, \qquad T_{12} = \mu_o u_o'. \tag{11}$$

The momentum balance reduces to

$$-\mathcal{R}\,\frac{\partial p_o}{\partial x} = \frac{\partial}{\partial y}\Big[\mu_o u_o'\Big], \tag{12}$$

where $p_o(x) = -Gx + const$. Focusing only on the upper part of the channel we integrate (12) using the no-slip condition obtaining

$$u_o(y) = \int_y^1 \frac{\mathcal{R}\,G\eta}{\mu_o(\eta)} d\eta. \tag{13}$$

Selecting the reference velocity as

$$U^* = G^* \int_0^{L^*} \frac{\eta^*}{\mu_o^*(\eta^*)} d\eta^* \qquad \Big(\longrightarrow \quad 1 = \int_0^1 \frac{\mathcal{R}\,G\eta}{\mu_o(\eta)} d\eta\Big), \tag{14}$$

the basic velocity in the upper layer (in the lower part the solution is symmetric) can be rewritten as

$$u_o(y) = \left( 1 - \frac{\int_0^y \frac{\eta}{\mu_o(\eta)} d\eta}{\int_0^1 \frac{\eta}{\mu_o(\eta)} d\eta} \right) \in [0, 1]. \tag{15}$$

We notice that (15) has a flex point only if there exists a $\bar{y} \in (0, 1)$ such that $\mu_o'(\bar{y})\bar{y} - \mu_o(\bar{y}) = 0$. Therefore, whenever $\mu_o$ is decreasing with $y$ the basic profile is always concave.

**Remark 1.** *The basic solution (15) does not contain the relaxation function $\lambda_o(y)$. This means that when the steady-state is reached the elastic effects are no longer observable and the flow is equivalent to the one of a linear fluid in which the viscosity depends on the volume fraction $\varphi_o$. Moreover, when $\mu_o$ is constant we recover the classical parabolic profile of a viscous fluid.*

## 4. Modal perturbation and Squire's theorem generalization

Let us consider the following 3D perturbation

$$\begin{aligned} &\mathbf{v} = \mathbf{v}_o(y) + \hat{\mathbf{v}}(y)e^{i\alpha x + i\beta z + \sigma t}, \qquad && p = p_o(x) + \hat{p}(y)e^{i\alpha x + i\beta z + \sigma t}, \\ &\varphi = \varphi_o(y) + \hat{\varphi}(y)e^{i\alpha x + i\beta z + \sigma t}, \qquad && \mathbf{T} = \mathbf{T}_o(y) + \hat{\mathbf{T}}(y)e^{i\alpha x + i\beta z + \sigma t}, \end{aligned} \tag{16}$$

where $\hat{(.)}$ are complex functions such that $|\hat{(.)}| \ll 1$, $\alpha$, $\beta$ are the streamwise and the spanwise wave numbers and $\sigma$ is the wave frequency. The velocity field is written component-wise as $\mathbf{v} = (u, v, w)$. Let us insert (16) into (5), (6) and linearize the equations. The equations for the transport of $\varphi$ and for the mass balance become

$$\hat{\varphi}(\sigma + i\alpha u_o) + \varphi_o' \hat{v} = 0, \tag{17}$$

$$i\alpha\hat{u} + i\beta\hat{w} + \hat{v}' = 0. \tag{18}$$

For simplicity of notation we introduce $\Upsilon = e^{i\alpha x + i\beta z + \sigma t}$. To linearize the perturbed equation (6) we expand the functions $\lambda$ and $\mu$ around $\varphi_o$ up to the first order

$$\lambda(\varphi) \cong \lambda(\varphi_o) + \left.\frac{d\lambda}{d\varphi}\right|_{\varphi_o} \hat{\varphi}\Upsilon =: \lambda_o(y) + \lambda_\varphi(y)\hat{\varphi}\Upsilon, \tag{19}$$

$$\mu(\varphi) \cong \mu(\varphi_o) + \left.\frac{d\mu}{d\varphi}\right|_{\varphi_o} \hat{\varphi}\Upsilon =: \mu_o(y) + \mu_\varphi(y)\hat{\varphi}\Upsilon. \tag{20}$$

The constitutive relation (6) becomes

$$\Big(\mathbf{T}_o + \hat{\mathbf{T}}\Upsilon\Big) + \mathcal{W}\Big(\lambda_o + \lambda_\varphi\hat{\varphi}\Upsilon\Big)\Big(\overset{\triangledown}{\mathbf{T}}_o + \overset{\triangledown}{\overline{\hat{\mathbf{T}}\Upsilon}}\Big) = 2\Big(\mu_o + \mu_\varphi\hat{\varphi}\Upsilon\Big)\Big(\mathbf{D}_o + \hat{\mathbf{D}}\Upsilon\Big). \tag{21}$$

Notice that the upper convected derivative is

$$\overset{\triangledown}{(.)} = \frac{\partial(.)}{\partial t} + \Big[\big(\mathbf{v}_o + \hat{\mathbf{v}}\Upsilon\big)\cdot\nabla\Big](.) - \big(\mathbf{L}_o + \hat{\mathbf{L}}\Upsilon\big)(.) - (.)\big(\mathbf{L}_o + \hat{\mathbf{L}}\Upsilon\big)^T, \tag{22}$$

so that linearization yields

$$\overset{\triangledown}{\mathbf{T}}_o = \big(\hat{\mathbf{v}}\cdot\nabla\big)\mathbf{T}_o\Upsilon - \big(\mathbf{L}_o\mathbf{T}_o + \mathbf{T}_o\mathbf{L}_o^T\big) - \big(\hat{\mathbf{L}}\mathbf{T}_o + \mathbf{T}_o\hat{\mathbf{L}}^T\big)\Upsilon \tag{23}$$

and

$$\overset{\triangledown}{\overline{\hat{\mathbf{T}}\Upsilon}} = \sigma\hat{\mathbf{T}}\Upsilon + i\alpha u_o\hat{\mathbf{T}}\Upsilon - \big(\mathbf{L}_o\hat{\mathbf{T}} + \hat{\mathbf{T}}\mathbf{L}_o^T\big)\Upsilon. \tag{24}$$

Furthermore, it is easy to check that $\big(\hat{\mathbf{v}}\cdot\nabla\big)\mathbf{T}_o = \hat{v}\mathbf{T}_o'$ so that inserting (23)-(24) into (21) and rearranging we find

$$\hat{\mathbf{T}}g - \mathcal{W}\lambda_o\Big(\mathbf{L}_o\hat{\mathbf{T}} + \hat{\mathbf{T}}\mathbf{L}_o^T\Big) = \mathcal{W}\lambda_\varphi\hat{\varphi}\Big(\mathbf{L}_o\mathbf{T}_o + \mathbf{T}_o\mathbf{L}_o^T\Big),$$

$$-\mathcal{W}\,\lambda_o\Big[\hat{v}\mathbf{T}_o' - \big(\hat{\mathbf{L}}\mathbf{T}_o + \mathbf{T}_o\hat{\mathbf{L}}^T\big)\Big] - 2\mu_\varphi\hat{\varphi}\mathbf{D}_o + 2\mu_o\hat{\mathbf{D}}, \tag{25}$$

where we have set

$$\sigma = -i\alpha c, \qquad c \in \mathbb{C}, \qquad g = \Big[1 + \mathcal{W}\,\lambda_o i\alpha\big(u_o - c\big)\Big]. \tag{26}$$

Equation (25) provides the link between the perturbed stress tensor $\hat{\mathbf{T}}$ and the kinematics of the perturbation expressed in terms of $\hat{\mathbf{v}}$, $\hat{\mathbf{L}}$, $\hat{\mathbf{D}}$, $\hat{\varphi}$. Notice that $\hat{\varphi}$ can be expressed in terms of $\hat{v}$ exploiting (17). Equation (25) is a linear system for the unknowns $\hat{T}_{ij}(y)$ that can be easily solved, for instance, by simple Gaussian elimination. The tensor $\hat{\mathbf{T}}$ is linear in $\hat{\mathbf{v}}$ and in the first derivatives of $\hat{\mathbf{v}}$, with coefficients depending on $y$ through the functions $u_o$, $\mu_o$, $\lambda_o$ and their derivatives and through the basic flow. In the Appendix we have reported the exact expressions of the tensor components obtained from (25).

We now introduce the perturbed flow (16) into the momentum equation and linearize the system. We get

$$\begin{aligned}
&i\alpha(u_o - c)\hat{u} + u_o'\hat{v} = -i\alpha\hat{p} + \frac{1}{\mathcal{R}}\Big[i\alpha\hat{T}_{11} + \hat{T}_{12}' + i\beta\hat{T}_{13}\Big],\\
&i\alpha(u_o - c)\hat{v} = -\hat{p}' + \frac{1}{\mathcal{R}}\Big[i\alpha\hat{T}_{12} + \hat{T}_{22}' + i\beta\hat{T}_{23}\Big],\\
&i\alpha(u_o - c)\hat{w} = -i\beta\hat{p} + \frac{1}{\mathcal{R}}\Big[i\alpha\hat{T}_{13} + \hat{T}_{23}' + i\beta\hat{T}_{33}\Big].
\end{aligned} \tag{27}$$

Substituting the expressions for $\hat{T}_{ij}$ that are reported in the appendix into (27) we find

$$\begin{aligned}
\mathcal{R}\Big[i\alpha(u_o - c)\hat{u} + u_o'\hat{v}\Big] = -i\alpha\mathcal{R}\,\hat{p} + \left[\mathsf{L}_{\alpha,\mathcal{W}} - \frac{\mu_o(\alpha^2+\beta^2)}{g}\right]\hat{u} + \\
+\left[\mathsf{M}_{\alpha,\mathcal{W}} + \left(\frac{i\alpha\mu_o'}{g} + \frac{\alpha^2\mu_o\lambda_o'\mathcal{W}\,(u_o - c)}{g^2} - (\alpha^2+\beta^2)\frac{\mathcal{W}\,\lambda_o\mu_o u_o'}{g^2}\right)\right]\hat{v},
\end{aligned} \tag{28}$$

$$\mathcal{R}\,i\alpha(u_o - c)\hat{v} = \left[\mathsf{N}_{\alpha,\mathcal{W}} - \frac{\mu_o(\alpha^2+\beta^2)}{g}\right]\hat{v}, \tag{29}$$

$$\mathcal{R}\, i\alpha(u_o - c)\hat{w} = -i\beta\mathcal{R}\,\hat{p} + \left[\mathbf{L}_{\alpha,\mathcal{W}} - \frac{\mu_o(\alpha^2+\beta^2)}{g}\right]\hat{w} +$$

$$+\left(\frac{i\beta\mu_o'}{g} + \frac{\alpha\beta\mu_o\lambda_o'\mathcal{W}\,(u_o - c)}{g^2}\right)\hat{v}, \tag{30}$$

where $\mathbf{L}_{\alpha,\mathcal{W}}$, $\mathbf{M}_{\alpha,\mathcal{W}}$, $\mathbf{N}_{\alpha,\mathcal{W}}$ are the linear differential operators

$$\mathbf{L}_{\alpha,\mathcal{W}} = \left(\frac{\mu_o}{g}\right)\mathsf{D}^2 + \left[\left(\frac{\mu_o}{g}\right)' + \frac{i\alpha\mathcal{W}\,\lambda_o\mu_o u_o'}{g} + \frac{i\alpha\mathcal{W}\,\lambda_o\mu_o(1+g)u_o'}{g^2}\right]\mathsf{D} +$$

$$+\left[\left(\frac{i\alpha\mathcal{W}\,\lambda_o\mu_o u_o'}{g}\right)' - \frac{\alpha^2\mathcal{W}^2\lambda_o^2\mu_o(2g+1)u_o'^2}{g^2}\right], \tag{31}$$

$$\mathbf{M}_{\alpha,\mathcal{W}} = \left(\frac{\lambda_o\mu_o\mathcal{W}\,u_o'}{g^2}\right)\mathsf{D}^2 + \left[\frac{2i\alpha\mathcal{W}^2\lambda_o^2\mu_o(2+g)u_o'^2}{g^3} + \left(\frac{\mathcal{W}\,\lambda_o\mu_o(2+g)u_o'}{g^2}\right)' +\right.$$

$$\left. -\frac{\mathcal{W}\,\lambda_o\mu_o g u_o'' - i\alpha\mathcal{W}^2\lambda_o^2\mu_o u_o'^2 + \mathcal{W}\,\lambda_o\mu_o' g u_o'}{g^2} - \frac{\mathcal{W}\,\lambda_o u_o'\varphi_o'\mu_\varphi}{g(g-1)}\right]\mathsf{D} +$$

$$\left[-\frac{2\mathcal{W}\,\lambda_o u_o'}{g^3}\Big(i\alpha\mathcal{W}\,\lambda_o\mu_o g(2g+1)u_o'' + 2\alpha^2\mathcal{W}^2\lambda_o^2\mu_o(g+1)u_o'^2 +\right.$$

$$+ i\alpha\mathcal{W}\,g u_o'\big(\lambda_o(g+1)\mu_o' + \lambda_o' g\mu_o\big)\Big) - \frac{2i\alpha\mathcal{W}^2\lambda_o\varphi_o'(\lambda\mu_\varphi + \lambda_\varphi g\mu_o)u_o'^2}{g^2(g-1)} +$$

$$\left. -\left(\frac{\mathcal{W}\,\lambda_o\mu_o g u_o'' - 2i\alpha\mathcal{W}^2\lambda_o^2\mu_o(g+1)u_o'^2 + \mathcal{W}\,\lambda_o\mu_o' g u_o'}{g^2} + \frac{\mathcal{W}\,\lambda_o\mu_\varphi\varphi_o' u_o'}{g(g-1)}\right)'\right], \tag{32}$$

$$\mathbf{N}_{\alpha,\mathcal{W}} = \left(\frac{\mu_o}{g}\right)\mathsf{D}^2 + \left[\left(\frac{2\mu_o}{g}\right)' + \frac{2i\alpha\mathcal{W}\,\lambda_o\mu_o(1+g)u_o'}{g^2}\right]\mathsf{D} + \left[\left(\frac{2i\alpha\mathcal{W}\,\lambda_o\mu_o u_o'}{g}\right)' +\right.$$

$$\left. -\frac{i\alpha\mathcal{W}\,\lambda_o\mu_o g u_o'' + 2\alpha^2\mathcal{W}^2\lambda_o^2\mu_o(g+1)u_o'^2 + i\alpha\mathcal{W}\,\lambda_o\mu_o' g u_o'}{g^2} - \frac{i\alpha\mathcal{W}\,\mu_\varphi u_o'\varphi_o'\lambda_o}{g(g-1)}\right],$$

where $\mathsf{D} = d/dy$. We now define the following Squire's transformation

$$\overline{\alpha}^2 = \alpha^2 + \beta^2, \quad \overline{\alpha}\,\overline{u} = \alpha\hat{u} + \beta\hat{w}, \quad \overline{c} = c, \quad \overline{v} = v, \tag{33}$$

$$\overline{\alpha}\overline{\mathcal{W}} = \alpha\mathcal{W}\,, \qquad \overline{\alpha}\overline{\mathcal{R}} = \alpha\mathcal{R}\,, \qquad \overline{\mathcal{R}}\,\overline{p} = \mathcal{R}\,\hat{p} \tag{34}$$

and we notice that

$$\mathbf{L}_{\alpha,\mathcal{W}} = \mathbf{L}_{\overline{\alpha},\overline{\mathcal{W}}}\,, \qquad \alpha\mathbf{M}_{\alpha,\mathcal{W}} = \overline{\alpha}\mathbf{M}_{\overline{\alpha},\overline{\mathcal{W}}}\,, \qquad \mathbf{N}_{\alpha,\mathcal{W}} = \mathbf{N}_{\overline{\alpha},\overline{\mathcal{W}}}\,.$$

Let us now multiply equation (28) by $i\alpha$, equation (30) by $i\beta$ and then add the two equations. The system (28)-(30) reduces to

$$\overline{\mathcal{R}}\left[i\overline{\alpha}(u_o - \overline{c})\overline{u} + u_o'\overline{v}\right] = -i\overline{\alpha}\overline{\mathcal{R}}\,\overline{p} + \left[\mathbf{L}_{\overline{\alpha},\overline{\mathcal{W}}} - \frac{\overline{\alpha}^2\mu_o}{g}\right]\overline{u} +$$

$$+\left[\mathbf{M}_{\overline{\alpha},\overline{\mathcal{W}}} + \left(\frac{i\overline{\alpha}\mu_o'}{g} + \frac{\overline{\alpha}^2\mu_o\lambda_o'\overline{\mathcal{W}}\,(u_o - \overline{c})}{g^2} - \overline{\alpha}^2\,\frac{\overline{\mathcal{W}}\,\lambda_o\mu_o u_o'}{g^2}\right)\right]\overline{v}, \tag{35}$$

$$\overline{\mathcal{R}}\,i\overline{\alpha}(u_o - \overline{c})\overline{v} = \left[\mathbf{N}_{\overline{\alpha},\overline{\mathcal{W}}} - \frac{\mu_o\overline{\alpha}^2}{g}\right]\overline{v}. \tag{36}$$

The system (35)-(36) has the same structure of the system (28)-(30) with $w = \beta = 0$. The classical no-slip boundary conditions are transformed accordingly to the Squire's transformation (33)-(34). Thus, we have proved the following:

**Theorem 1.** *(***Squire***) In the case of non constant relaxation time and molecular viscosity, if a three-dimensional mode is unstable, a two-dimensional mode is unstable at a lower Reynolds number.*

We recall that the validity of Squire's theorem for constant $\lambda$ and $\mu$ was proved in [13]. As a consequence, it follows that for the study of linear modal stability it is sufficient to consider only a two dimensional perturbation, which shall lead to a generalization of the Orr-Sommerfeld equation.

**Remark 2.** *We believe that the substitution of the rate-type model with one based on a different objective time derivative (e.g. Jaumman or Oldroyd) does not invalidate Squire's theorem. The type of calculations leading to (35)-(36) remains essentially the same, so we expect that Squire's theorem remains valid also for other rate-type models. Of course, this is true as long as the rheological* *parameters do not depend on the strain rate. Indeed, when this is the case it is sufficient to consider the constant power law model (which can be obtained setting the relaxation time to zero and assuming that the viscosity is a function of the second invariant of the strain-rate only) for which Squire theorem does not hold, [5].*

## 5. Generalized Orr-Sommerfeld equation

We consider the perturbation (16) with $\beta = w = 0$. In this case $\hat{T}_{13} = \hat{T}_{23} = \hat{T}_{33} = 0$ and the system reduces to $(27)_{1,2}$, in which we set $\beta = w = 0$. Eliminating the pressure and recalling that $\hat{v}' = -i\alpha\hat{u}$ we find

$$-\mathcal{R}\left[(u_o - c)(\mathsf{D}^2 - \alpha^2) - u_o''\right]\hat{v} = \left[i\alpha\mathsf{D}(\hat{T}_{11} - \hat{T}_{22}) + (\mathsf{D}^2 + \alpha^2)\hat{T}_{12}\right]. \qquad (37)$$

Substituting the expressions for the stress components (with $\beta = w = 0$ and with $\hat{v}' = -i\alpha\hat{u}$) given in the appendix we get a polynomial eigenvalue problem of the type

$$\mathcal{L}\hat{v} = 0,$$

where $\mathcal{L}_j$ are the differential operators

$$\mathcal{L} = \sum_{m=0}^{5} \mathcal{L}_m c^m, \qquad \mathcal{L}_j = \sum_{k=0}^{4} L_{jk}(y)\mathsf{D}^k \qquad (38)$$

and where $L_{jk}(y)$ are coefficients depending on the basic solution, the Weissenberg and Reynolds numbers and on the wave number $\alpha$.

### *5.1. Constant coefficients*

As a benchmark problem we consider the one in which $\varphi_o = 0$ (no dispersed particles), which implies $\lambda_o = \mu_o = 1$, $\varphi_o' = 0$. This is exactly the case investigated in [13]. Exploiting the expressions of the stress components with $\beta = w = 0$ given in the appendix and substituting into (37) we find:

$$i\alpha g\mathcal{R}\left[(u_o - c)(\mathsf{D}^2 - \alpha^2) - u_o''\right]\hat{v} = \mathsf{D}^4\hat{v} + b_3(y)\mathsf{D}^3\hat{v} + b_2(y)\mathsf{D}^2\hat{v} + b_1(y)\mathsf{D}\hat{v} + b_o(y), \qquad (39)$$

which is equal to equation (15) of [13] and where the basic velocity profile (15) is now $u_o(y) = 1 - y^2$. The no-slip condition implies $\hat{v}(\pm 1) = \mathsf{D}\hat{v}(\pm 1) = 0$ while the coefficients $b_j(y)$ are given by

$$b_3(y) = 2g'\left(1 - \frac{1}{g}\right), \qquad b_2(y) = -2\alpha^2 + 3g''\left(1 - \frac{1}{g}\right) + 2g'^2\left(1 - \frac{1}{g}\right)^2,$$

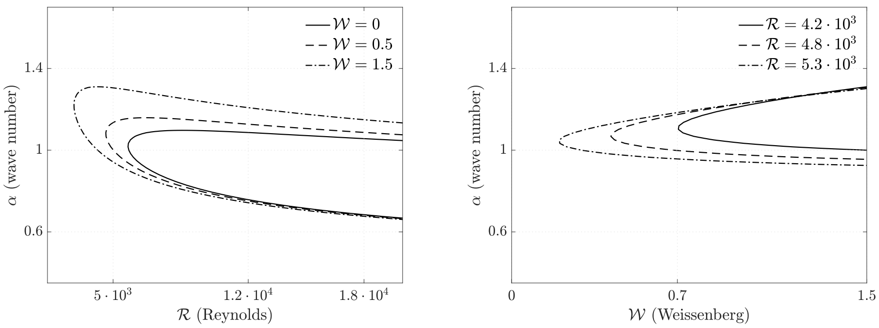
$$b_1(y) = -2\alpha^2 g'\left(1 - \frac{1}{g}\right) + 2g'''\left(1 - \frac{1}{g}\right) + 4g'g''\left(1 - \frac{1}{g}\right)^2 - \frac{4g'^3}{g}\left(1 - \frac{1}{g}\right),$$

$$b_o(y) = \alpha^4 + g^{(iv)} - \alpha^2 g''\left(1 - \frac{1}{g}\right) - 2\alpha^2 g'^2\left(1 + \frac{1}{g^2}\right) - 4\frac{g'g'''}{g} - \frac{3g''^2}{g}$$

$$+\frac{4g'^4}{g^2} - \frac{6g'^2 g''}{g}\left(1 - \frac{1}{g}\right).$$

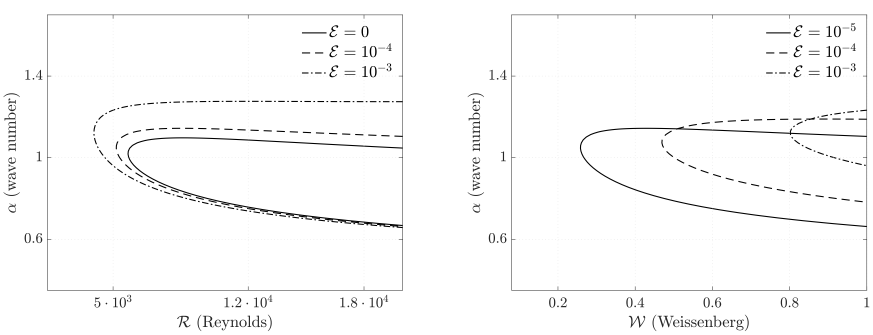
We recall that the quantity $g$ is given by (26). The coefficients above present some differences if compared to those of [13]. Indeed, the $b_j(y)$ determined in [13] are not correct and this can be easily proved deriving the perturbation equation and exploiting equations (10), (11) of [13]. The correctness of the coefficients presented here has also been checked using the symbolic software wxMaxima. Problem (39) is solved using a spectral collocation method based on Chebyshev modes [11].

In Fig. 1a, 1b we show the marginal stability curves obtained solving the eigenvalue problem (39) numerically and considering $\mathcal{W}$ and $\mathcal{R}$ as independent ($\mathcal{E}$ is thus not a fixed constant). Because of the discrepancy in the coefficients we obtain curves that are slightly different from the one of [13]. Looking at Fig. 1a we observe that the increase of the Weissenberg number has a destabilizing effect, since the critical Reynolds number is reduced when $\mathcal{W}$ is increased. From Fig. 1b we see that the increase of the Reynolds number results in a decrease of the critical Weissenberg number. In Fig. 2a, 2b we show the marginal stability curves in the plane $(\mathcal{R}, \alpha)$ and in the plane $(\mathcal{W}, \alpha)$ assuming that $\mathcal{E}$ is a fixed constant. In this case $\mathcal{W}$ and $\mathcal{R}$ are not independent. As one can see the critical Reynolds number is a decreasing function of $\mathcal{E}$, while the critical Weissenberg




(a) $\mathcal{W} = 0,\ 0.5,\ 1.5$. (b) $\mathcal{R} = 4200,\ 4800,\ 5300$.

Figure 1: Marginal stability curves for $\lambda_o = \mu_o = 1$, $\varphi_o' = 0$, ($\mathcal{W}$ independent of kinematics).




(a) $\mathcal{E} = 0,\ 10^{-4},\ 10^{-3}$. (b) $\mathcal{E} = 10^{-5},\ 10^{-4},\ 10^{-3}$.

Figure 2: Marginal stability curves for $\lambda_o = \mu_o = 1$, $\varphi_o' = 0$, ($\mathcal{W}$ dependent on kinematics).

number is an increasing function of $\mathcal{E}$. We may conclude that the increase of the relaxation time $\lambda_p^*$ (i.e. the time needed to gradually release the stress in the fluid) or reduction of the kinematical diffusion time $\lambda_k^*$ has a destabilizing effect on the flow.

### 5.2. Low Weissenberg number

In this section we investigate the general 2D problem with non constant coefficients. Due to the remarkable complexity of this problem we limit ourselves to consider the case in which the Weissenberg number is "small" and in which the quantities $\mu_o' u_o'$, $\lambda_o' u_o'$ are negligible. In practice we linearize the components of the stress tensor given in the appendix around $\mathcal{W} = 0$ so that

$$\hat{T}_{11} = -\left[\frac{4\lambda_o \mu_o u_o' \mathcal{W}}{i\alpha}\right] \mathsf{D}^2 \hat{v} - \left[\frac{2i\alpha\mu_o + 2\lambda_o \alpha^2 \mu_o (u_o - c)\mathcal{W}}{i\alpha}\right] \mathsf{D}\hat{v}$$

$$+ \left[\frac{2i\big(\lambda_o \mu_o' u_o' + \lambda_o' u_o' \mu_o\big) u_o'}{\alpha(u_o - c)} + 2i\alpha\lambda_o \mu_o u_o'\right] \mathcal{W}\, \hat{v},$$

$$\hat{T}_{12} = -\left[\frac{\mu_o - i\alpha\lambda_o \mu_o (u_o - c)\mathcal{W}}{i\alpha}\right] \mathsf{D}^2 \hat{v} + \left[3\lambda_o \mu_o u_o' \mathcal{W} - \frac{i\alpha\lambda_o \mu_o u_o' \mathcal{W}}{i\alpha}\right] \mathsf{D}\hat{v}$$

$$\left[\left(\frac{i\mu_o' u_o'}{\alpha(u_o - c)} + i\alpha\mu_o\right) + \Big(-\lambda_o \mu_o u_o'' + \lambda_o \alpha^2 \mu_o (u_o - c)\Big)\mathcal{W}\right] \hat{v},$$

$$\hat{T}_{22} = \Big[2\mu_o - 2i\alpha\lambda_o \mu_o (u_o - c)\mathcal{W}\Big] \mathsf{D}\hat{v} + \Big[2i\alpha\lambda_o \mu_o u_o' \mathcal{W}\Big] \hat{v}.$$

We see that the presence of the term $\alpha(u_o - c)$ at the denominator introduces a possible singularity in the perturbed stress components. As observed in [4], the singular factor is essentially due to the nature of the transport equation for the particle concentration (transport equation (17) with no diffusion, infinite Péclet number). Although the absence of diffusion allows one to consider any arbitrary even function $\varphi_o(y)$ as the base particle distribution, the transport equation introduces the possible singularity in the perturbation equation. In

[6], [7] we have proved that, for a Newtonian suspension and for a generalized Newtonian suspension, the absence of diffusion is responsible for unconditional instability, i.e. unstable modes exist for all $\mathcal{R}$. Here the situation is analogous, i.e. using the expressions that contain the singular factor $\alpha(u_o - c)$ in the perturbation equation (37) we end up finding unstable modes for each choice of $\mathcal{R}$. On the other hand, if we neglect the singular terms, which is possible only if $\mu_o' u_o'$, $\lambda_o' u_o'$ are “sufficiently small”, we may determine stability regions and neutral stability curves. So let us suppose to neglect all the terms containing $\alpha(u_o - c)$ in the stress components above and insert the latter into (37). We end up with the following eigenvalue problem

$$\Big[c\mathsf{A} + \mathsf{B}\Big]\hat{v} = 0, \tag{40}$$

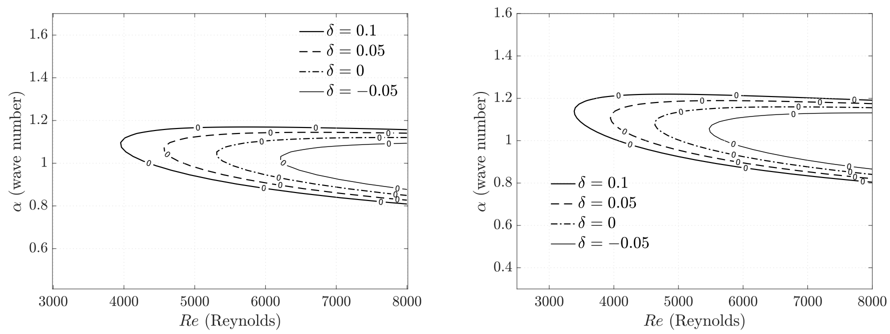
where

$$\begin{gathered}
\mathsf{A} = \Big[\alpha\lambda_o\mu_o\mathcal{W}\Big]\mathsf{D}^4 + \Big[2\alpha(\lambda_o\mu_o)'\mathcal{W}\Big]\mathsf{D}^3 + \Big[\alpha(\lambda_o\mu_o)''\mathcal{W} - 2\alpha^3\lambda_o\mu_o\mathcal{W} + \alpha\mathcal{R}\Big]\mathsf{D}^2 + \\
-\Big[2\alpha^3(\lambda_o\mu_o)'\mathcal{W}\Big]\mathsf{D} + \Big[\alpha^3(\lambda_o\mu_o)''\mathcal{W} + \alpha^5\lambda_o\mu_o\mathcal{W} - \alpha^3\mathcal{R}\Big], \\
\mathsf{B} = \Big[-\alpha\lambda_o\mu_o u_o\mathcal{W} - i\mu_o\Big]\mathsf{D}^4 + \Big[-2\alpha(\lambda_o\mu_o)'u_o\mathcal{W} - 2i\mu_o'\Big]\mathsf{D}^3 + \\
\Big[-2\alpha u_o'(\lambda_o\mu_o)'\mathcal{W} - \alpha(\lambda_o\mu_o)''u_o\mathcal{W} + 2\alpha^3\lambda_o\mu_o u_o\mathcal{W} - \alpha u_o\mathcal{R} - iu_o'' + 2i\alpha^2\mu_o\Big]\mathsf{D}^2 + \\
\Big[-2\alpha(u_o'\lambda_o\mu_o)'\mathcal{W} + 2\alpha^3 u_o(\lambda_o\mu_o)'\mathcal{W} + 2i\alpha^2\mu_o'\Big]\mathsf{D} + \\
\Big[\alpha(\lambda_o\mu_o u_o'')''\mathcal{W} + (u_o'' + \alpha^3 u_o)\mathcal{R} - 2\alpha^3 u_o'(\lambda_o\mu_o)'\mathcal{W} - \alpha^3 u_o(\lambda_o\mu_o)''\mathcal{W} + \\
-\alpha^5\lambda_o\mu_o u_o\mathcal{W} - i\alpha^2\mu_o'' - i\alpha^4\mu_o\Big].
\end{gathered}$$

We consider the following non-constant functions $\mu_o(y)$ and $\lambda_o(y)$

$$\mu_o(y) = 1 + \delta y^2, \qquad \lambda_o(y) = 1 + \delta y^2, \tag{41}$$

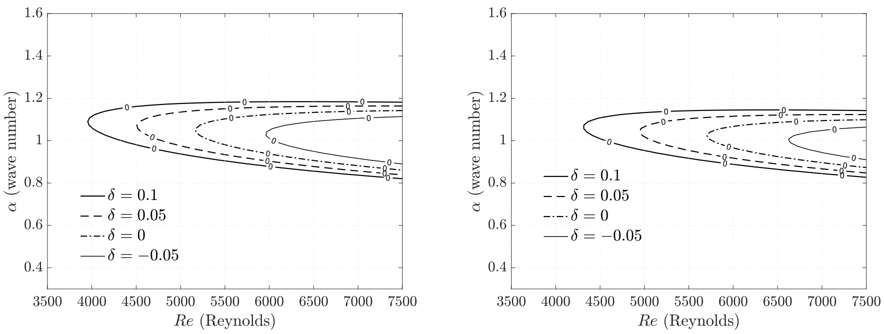
for $\delta = -0.05,\ 0,\ 0.05,\ 0.1$. This choice, that is not dictated by any specific practical application, allows us to investigate how stability is influenced by the basic particle distribution across the channel. In particular we shall see that the most stable flows are the ones in which the concentration of solid particles




(a) $\mathcal{W} = 0.2$. (b) $\mathcal{W} = 0.5$.

Figure 3: Marginal stability curves for $\lambda_o,\ \mu_o$ given by (41). $\mathcal{W}$ independent of $\mathcal{R}$ (varying $\mathcal{E}$).

is reduced towards the channel walls. Once again, we solve problem (40) numerically using a spectral collocation method based on Chebyshev modes [11] and we plot the neutral stability curves for different values of the Weissenberg number. As done in the previous section, we treat the cases $\mathcal{W}$ independent and dependent of $\mathcal{R}$ separately. In Fig. 3 we plot the marginal stability curves for $\mathcal{W}$ independent of $\mathcal{R}$ with the material coefficients given in (41) for different values of $\delta$ and for $\mathcal{W} = 0.2,\ 0.5$. In both cases the decrease of the coefficient $\delta$ results in an increase of the critical Reynolds number, indicating that flows in which the particle distribution is increasing towards the wall are less stable that the ones in the opposite case. This type of behavior is in accordance with the one observed in [6] and [7], where it was shown that flows with larger particle concentration in the center of the channel are more stable than the ones in which the larger concentration is near the wall. Finally, as observed in the constant case, we note that the Weissenberg number has a destabilizing effect on the flow, as the critical Reynolds number is reduced for larger $\mathcal{W}$. In Figs 4 we plot the neutral stability curves assuming that $\mathcal{W}$ depends on $\mathcal{R}$ and we use $\mathcal{E}$ as a variable parameter. Looking at Fig. 4b, 4a we observe that the reduction of $\delta$ produces an increase of the critical Reynolds number, thus enhancing stability. Further, we see that, for a fixed $\delta$, a lower $\mathcal{E}$ results in a smaller value of the




(a) $\mathcal{E} = 10^{-5}$.

(b) $\mathcal{E} = 10^{-4}$.

Figure 4: Marginal stability curves for $\lambda_o$, $\mu_o$ given by (41). $\mathcal{W} = \mathcal{E}\mathcal{R}$.

critical Reynolds number, showing once again that the flow is more stable for larger values of $\mathcal{E}$ (i.e. for larger values of the ratio between the relaxation time and the characteristic time of kinematical diffusion).

## 6. Conclusions

We study the linear stability of an isothermal non-homogeneous dense suspension flowing between parallel plates (Poiseuille flow). The suspension is modeled as a a viscoelastic upper convected Maxwell fluid with rheological parameters depending on the particles concentration. The density of the particles evolves according to a transport equation with negligible diffusion.

We determine the basic unidirectional flow driven by a given pressure gradient and we perturb such a flow with a three dimensional wave-like disturbance, proving the validity of Squire's theorem (in the case of non constant relaxation time and molecular viscosity). The latter allows us to consider only a two-dimensional perturbation in the stream-wise direction. In this case the problem reduces to a generalization of the Orr-Sommerfeld equation which we solve numerically via a spectral collocation method based on Chebyshev polynomials. As a benchmark problem we initially study the case with constant rheological moduli, showing the agreement with the results of [13]. Subsequently we consider the case of low Weissenberg number and non constant parameters.

We distinguish between two different situations: i) the Weissenberg number $\mathcal{W}$ and the Reynolds number $\mathcal{R}$ are independent ($\mathcal{W}$ does not depends on the kinematics); ii) the Weissenberg number $\mathcal{W}$ and the Reynolds number $\mathcal{R}$ are dependent. In our opinion hypothesis i) is less significant from a physical stand-point since it implies that a change in the Reynolds number is due to a variation of the material properties of the fluid. We show that flows in which the particle distribution is concentrated in the proximity of the channel walls are less stable that those showing opposite behavior. This latter result is in line with the findings of [6] and [7].

## Appendix

The tensor components obtained solving the linear system (25) are

$$\hat{T}_{11} = \left[\frac{2\lambda_o \mathcal{W}\,(1+g)\mu_o u_o'}{g^2}\right]\mathsf{D}\hat{u} + \left[\frac{2\lambda_o^2 \mathcal{W}^{\,2}(2+g)\mu_o u_o'^2}{g^3}\right]\mathsf{D}\hat{v}$$

$$+ \left[\frac{2i\alpha\mu_o\big(\lambda_o^2\mathcal{W}^{\,2}(2g+1)u_o'^2 + g\big)}{g^2}\right]\hat{u} + \mathcal{H}_{11}\hat{v},$$

$$\hat{T}_{12} = \left[\frac{\mu_o}{g}\right]\mathsf{D}\hat{u} + \left[\frac{\lambda_o\mathcal{W}\,(2+g)\mu_o u_o'}{g^2}\right]\mathsf{D}\hat{v} + \left[\frac{i\lambda_o\mathcal{W}\,\alpha\mu_o u_o'}{g}\right]\hat{u} + \mathcal{H}_{12}\hat{v},$$

$$\hat{T}_{13} = \left[\frac{\lambda_o\mathcal{W}\,(g+1)\mu_o u_o'}{g^2}\right]\mathsf{D}\hat{w} + \left[\frac{i\beta\mu_o}{g}\right]\hat{u} + \left[\frac{i\lambda_o\mathcal{W}\,\beta\mu_o u_o'}{g^2}\right]\hat{v}$$

$$+ \left[\frac{i\alpha\mu_o\big(\lambda_o^2\mathcal{W}^{\,2}(2g+1)u_o'^2 + g\big)}{g^2}\right]\hat{w},$$

$$\hat{T}_{22} = \left[\frac{2\mu_o}{g}\right]\mathsf{D}\hat{v} + \left[\frac{2i\lambda_o\mathcal{W}\,\alpha\mu_o u_o'}{g}\right]\hat{v},$$

$$\hat{T}_{23} = \left[\frac{\mu_o}{g}\right]\mathsf{D}\hat{w} + \left[\frac{i\beta\mu_o}{g}\right]\hat{v} + \left[\frac{i\lambda_o\mathcal{W}\,\alpha\mu_o u_o'}{g}\right]\hat{w},$$

$$\hat{T}_{33} = \left[\frac{2i\beta\mu_o}{g}\right]\hat{w},$$

where

$$\mathcal{H}_{11} = -\Big[\lambda_o\mathcal{W}\,g(2g+1)\mu_o u_o'' - 2i\alpha\lambda_o^2\mathcal{W}^{\,2}(g+1)\mu_o u_o'^2 +$$

$$+\,\mathcal{W}\,g u_o'\Big(\lambda_o(g+1)\mu_o' + \lambda_o' g\mu_o\Big) - i\alpha g\mu_o\Big]\left(\frac{2\lambda_o\mathcal{W}\,u_o'}{g^3}\right) - \frac{2\lambda_o\mathcal{W}^{\,2}\varphi_o'\big(\lambda_o\mu_\varphi + \lambda_\varphi g\mu_o\big)u_o'^2}{g^2(g-1)},$$

$$\mathcal{H}_{12} = \left[\frac{-\lambda_o\mathcal{W}\,g\mu_o u_o'' + 2i\alpha\lambda_o^2\mathcal{W}^{\,2}(g+1)\mu_o u_o'^2 - \lambda_o\mathcal{W}\,g\mu_o' u_o'}{g^2} + \frac{i\alpha\mu_o}{g}\right] - \frac{\lambda_o\mathcal{W}\,\varphi_o'\mu_\varphi u_o'}{g(g-1)}.$$

We recall that the function $g$ is given by

$$g = \Big[1 + \mathcal{W}\,\lambda_o i\alpha\big(u_o - c\big)\Big].$$

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