

A WEAK VERSION OF THE STRONG EXPONENTIAL CLOSURE*

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ABSTRACT

Assuming Schanuel's Conjecture we prove that for any irreducible variety $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ over \mathbb{Q}^{alg} , of dimension n , and with dominant projections on both the first n coordinates and the last n coordinates, there exists a generic point $(\bar{a}, e^{\bar{a}}) \in V$. We obtain in this way many instances of the Strong Exponential Closure axiom introduced by Zilber in [20].

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1. Introduction

In [20] Zilber conjectured that the complex exponential field is quasi-minimal, i.e., every subset of \mathbb{C} definable in the language of rings expanded by the exponential function is either countable or co-countable. If the conjecture is true the complex exponential field should have good geometric properties.

He introduced and studied a class of new exponential fields now known as Zilber fields via axioms of algebraic and geometrical nature. There are many novelties in his analysis, including a reinterpretation of Schanuel's Conjecture in terms of Hrushovski's very general theory of predimension and strong extensions, [12]. Zilber proved that his axioms are uncountably categorical, and all models are quasiminimal (see [13, 20]).

Zilber conjectured that the one in cardinal 2^{\aleph_0} is \mathbb{C} as exponential field. Comparing the complex exponential field and Zilber fields has been object of study in [16], [6], [7], [17], [8], [11], [15].

In this paper we will analyze one of the axioms introduced by Zilber, the Strong Exponential Closure (SEC), in the complex exponential field. Modulo Schanuel's Conjecture, (SEC) is the only axiom still unknown for (\mathbb{C}, \exp) . Some instances of (SEC) for (\mathbb{C}, \exp) have been proved in [16], [15], [5]. Here we obtain a more general result which includes those in [5].

Let $G_n(\mathbb{C}) = \mathbb{C}^n \times (\mathbb{C}^*)^n$ be the algebraic group. Let $1 \leq k \leq n$ and $M = (m_{ij})$ be a $k \times n$ matrix of integers and

$$[M] : G_n(\mathbb{C}) \rightarrow G_k(\mathbb{C})$$

be the homomorphism given by

$$(x_1, \dots, x_n, y_1, \dots, y_n) \rightarrow (x'_1, \dots, x'_k, y'_1, \dots, y'_k)$$

where

$$x'_i = m_{i1}x_1 + \dots + m_{in}x_n \quad \text{and} \quad y'_i = y_1^{m_{i1}} \dots y_n^{m_{in}},$$

for $i = 1, \dots, k$. We recall that a variety V is **rotund** if for every nonzero matrix $M \in \mathcal{M}_{k \times n}(\mathbb{Z})$, $\dim([M](V)) \geq \text{rank}(M)$, i.e., all the images of V under suitable homomorphisms are of large dimension.

A variety V is **free** if V does not lie inside any subvariety of the form either $\{(\bar{x}, \bar{y}) : r_1x_1 + \dots + r_nx_n = b\}$, where $r_i \in \mathbb{Z}$, r_i not all 0, $b \in \mathbb{C}$, or $\{(\bar{x}, \bar{y}) : y_1^{r_1} \dots y_n^{r_n} = b\}$, where $r_i \in \mathbb{Z}$, r_i not all 0, $b \in \mathbb{C}^*$.

STRONG EXPONENTIAL CLOSURE. If $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ is a rotund and free algebraic variety of dimension n , and \bar{a} is a finite tuple of elements of \mathbb{C} then there is $\bar{z} \in \mathbb{C}^n$ such that $(\bar{z}, e^{\bar{z}}) \in V$, and is generic in V over \bar{a} , i.e.,

$$\text{t.d.}_{\mathbb{Q}(\bar{a})}(\bar{z}, e^{\bar{z}}) = \dim(V).$$

The hypotheses of rotundity and freeness on the variety V guarantee that the only relations among the coordinates of points in V are those coming from V itself and the laws of exponentiation.

We recall

SCHANUEL CONJECTURE (SC): *Let $z_1, \dots, z_n \in \mathbb{C}$. Then*

$$\text{t.d.}_{\mathbb{Q}}(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}) \geq \text{l.d.}(z_1, \dots, z_n).$$

(SC) has many consequences in exponential algebra, see [14, 18].

In this paper, assuming Schanuel’s Conjecture we prove the Strong Exponential Closure for (\mathbb{C}, \exp) for certain varieties defined over \mathbb{Q}^{alg} . We denote the projections on the first n coordinates and on the last n coordinates by $\pi_1 : V \rightarrow \mathbb{C}^n$ and $\pi_2 : V \rightarrow (\mathbb{C}^*)^n$, respectively.

MAIN RESULT (SC): *Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ be an irreducible variety defined over the algebraic closure of \mathbb{Q} , such that $\dim V = n$, and both projections π_1 and π_2 are dominant. Then there is a Zariski dense set of generic points $(\bar{z}, e^{\bar{z}})$ in V .*

We recall that π_1 and π_2 being dominant means that $\pi_1(V)$ and $\pi_2(V)$ are Zariski dense in \mathbb{C}^n . As observed in [1], π_1 being dominant implies that V is rotund and both projections being dominant imply that V is free.

There are examples of free and rotund varieties with projections not dominant, e.g., $\{(x_1, x_2, y_1, y_2) : x_2 = x_1^2 \text{ and } y_2 = y_1 + 1\}$.

In Lemma 2.10 in [5] (see also [4]) the existence of a Zariski dense set of solutions of V is proved under the hypothesis that π_1 is dominant. No appeal to Schanuel’s conjecture is necessary, and moreover there is no restriction on the set of parameters. For the new result on the existence of generic solutions, Schanuel’s Conjecture is crucial and there are restrictions on the set of parameters defining the variety V .

Recently, Bays and Kirby in [1] proved the quasi-minimality of (\mathbb{C}, \exp) assuming a weaker condition than the strong exponential closure, requiring only the existence of a point $(\bar{z}, e^{\bar{z}})$ in V under the same hypothesis on the variety. No appeal to Schanuel’s Conjecture is made.

Some instances of quasi-minimality are known, e.g., if X is a subset of \mathbb{C} defined by either quantifier-free formulas or by $\forall \bar{y}(P(x, \bar{y}) = 0)$ where P is a term in the language $\{+, \cdot, 0, 1, \exp\}$, then X is either countable or co-countable. Boxall in [3] extends this result to sets defined by an existential formula,

$$\exists \bar{y}(P(x, \bar{y}) = 0),$$

where P is a term in the language $\{+, \cdot, 0, 1, \exp\}$ together with parameters from \mathbb{C} .

2. Preliminaries

We recall that the definable subsets of \mathbb{C}^n (in the language of rings) in the sense of model theory coincide with the constructible sets in algebraic geometry. We briefly review some basic facts about the notion of dimension associated to a definable set in \mathbb{C}^n which will be used in the proof of the main theorem; for details see [9] and [10].

We will always allow a finite or a countable set of parameters P . If not necessary we will not specify the set of parameters P .

Every definable (with parameters in P) set X has a dimension

$$\dim(X) = \max\{d : \exists \bar{x} \in X \text{ t. d.}_P(\bar{x}) = d\}.$$

Let $\overline{X}^{\text{Zar}}$ denote the Zariski closure of X . Then

$$\dim(X) = \dim(\overline{X}^{\text{Zar}}).$$

Moreover, for algebraically closed fields the model-theoretic algebraic closure (acl) coincides with the usual field-theoretic algebraic closure.

Fact 1: $\dim(X)$ is well-defined, i.e., it does not depend on the choice of the set P of parameters in the definition of X .

Fact 2: Let X be a definable set in \mathbb{C}^n . The dimension of X is 0 iff X is finite and nonempty. We use the convention that the empty set has dimension -1 .

Notation: Let $Y \subseteq \mathbb{C}^{n+m}$. For every $\bar{x} \in \mathbb{C}^n$,

$$Y_{\bar{x}} = \{\bar{y} \in \mathbb{C}^m : (\bar{x}, \bar{y}) \in Y\} \quad \text{and} \quad Y^{\bar{y}} = \{\bar{x} \in \mathbb{C}^n : (\bar{x}, \bar{y}) \in Y\}.$$

If $Y \subseteq \mathbb{C}^{n+m}$ is definable then $(Y_{\bar{x}})_{\bar{x} \in \mathbb{C}^n}$ is a definable family.

Fact 3: Let $(Y_{\bar{x}})_{\bar{x} \in \mathbb{C}^n}$ be a definable family of subsets of \mathbb{C}^m . For every $d \in \mathbb{N}$, the set $\{\bar{x} \in \mathbb{C}^n : \dim(Y_{\bar{x}}) = d\}$ is definable, with the same parameters as $(Y_{\bar{x}})_{\bar{x} \in \mathbb{C}^n}$. For $d = 0$ this gives that $\{\bar{x} \in \mathbb{C}^n : Y_{\bar{x}} \text{ is finite}\}$ is also definable.

Fact 4: Let $(Y_{\bar{x}})_{\bar{x} \in \mathbb{C}^n}$ be a definable family of subsets of \mathbb{C}^m . Then the family $(\overline{Y_{\bar{x}}}^{\text{Zar}})_{\bar{x} \in \mathbb{C}^n}$ of the Zariski closures is still a definable family.

Let $\pi_1 : Y \rightarrow \mathbb{C}^n$ and $\pi_2 : Y \rightarrow \mathbb{C}^m$ be the projections on the first n and the last m coordinates, respectively.

LEMMA 2.1: *Let $Y \subseteq \mathbb{C}^{n+m}$ be definable over P , and $X := \pi_1(Y)$. Assume that, for every $\bar{x} \in X$, $\dim(Y_{\bar{x}}) = d$. Then, $\dim(Y) = \dim(X) + d$. In particular, if $Y_{\bar{x}}$ is infinite for every $\bar{x} \in X$, then $\dim(Y) > \dim(X)$.*

Notice that an equivalent result holds in the case of $X := \pi_2(Y)$. For the proof of the above lemma see [9].

Simple calculations give the following result.

LEMMA 2.2: *Let $Y \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be a definable set over P , such that $\dim(Y) \leq n$. Let $\bar{c} \in \mathbb{C}^n$ be generic over P , i.e., $\text{t.d.}_P(\bar{c}) = n$. Then, the fiber*

$$Y_{\bar{c}} := \{\bar{z} \in \mathbb{C}^m : (\bar{c}, \bar{z}) \in Y\}$$

is finite.

Brownawell and Masser in [4] develop a very powerful criterion for solvability of systems of exponential equations. Proposition 2 in [4] implies the following result (see also [5]).

THEOREM 2.3: *Let $W \subseteq G_n(\mathbb{C})$ be an irreducible algebraic variety such that $\pi_1(W)$ is Zariski dense in \mathbb{C}^n . Then, the set $\{\bar{a} \in \mathbb{C}^n : (\bar{a}, e^{\bar{a}}) \in W\}$ is Zariski dense in \mathbb{C}^n .*

The hypothesis that $\pi_1(W)$ is Zariski dense is a non-trivial condition, and it implies that the variety is rotund. Theorem 2.3 states an even stronger property than the Exponential-Algebraic Closedness for (\mathbb{C}, \exp) for irreducible variety W with π_1 dominant. Indeed, there is a Zariski-dense sets of points $(\bar{a}, e^{\bar{a}})$ in W . A major problem is to replace the hypothesis that π_1 is dominant with much weaker ones like rotundity and freeness while still retaining the conclusion of the theorem.

Notice that no restriction is made on the coefficients of the polynomials defining W , and the result is independent from Schanuel’s Conjecture.

3. Strong Exponential Closure

We now go back to analyze Zilber's original axiom (SEC), i.e., we want to prove the existence of a point in the variety V of the form $(\bar{a}, e^{\bar{a}})$ which is generic in V . Assuming Schanuel's Conjecture we can prove (SEC) for algebraic varieties satisfying certain conditions.

THEOREM 3.1 (SC): *Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ be an irreducible variety over the algebraic closure of \mathbb{Q} with $\dim V = n$. Assume that both projections π_1 and π_2 are dominant. Then there is $\bar{a} \in \mathbb{C}^n$ such that $(\bar{a}, e^{\bar{a}}) \in V$ and $\text{t. d.}_{\mathbb{Q}}(\bar{a}, e^{\bar{a}}) = n$. In fact, the set*

$$\{\bar{a} \in \mathbb{C}^n : (\bar{a}, e^{\bar{a}}) \in V \text{ and } \text{t. d.}_{\mathbb{Q}}(\bar{a}, e^{\bar{a}}) = n\}$$

is Zariski dense.

In the proof of Theorem 3.1 we will use the following known result.

Let $M \in \mathcal{M}_{m \times n}(\mathbb{Z})$,

$$L_M = \{\bar{x} \in \mathbb{C}^n : M \cdot \bar{x} = \bar{0}\} \quad \text{and} \quad T_M = \{\bar{y} \in (\mathbb{C}^*)^n : \bar{y}^M = \bar{1}\}.$$

By \bar{y}^M we denote the result of the exponential map applied to $M \cdot \bar{x}$, where $y_i = e^{x_i}$ for $i = 1, \dots, n$.

Fact 5: The hyperspace L_M and the algebraic subgroup T_M have the same dimension.

Proof. Let

$$Z_M = \{\bar{x} \in \mathbb{C}^n : e^{M \cdot \bar{x}} = \bar{1}\} = \{\bar{x} \in \mathbb{C}^n : e^{\bar{x}} \in T_M\} = \{\bar{x} \in \mathbb{C}^n : M \cdot \bar{x} \in 2\pi i \mathbb{Z}^m\}.$$

The algebraic subgroup T_M is a closed differential submanifold in $(\mathbb{C}^*)^n$, and since \exp is a local diffeomorphism Z_M is a differential submanifold of \mathbb{C}^n of the same dimension as T_M . Notice that L_M is the tangent space of Z_M at $\bar{0}$, and so $\dim(L_M) = \dim(Z_M) = \dim(T_M)$. ■

Proof of Theorem 3.1. Let $U = \{(\bar{x}, \bar{y}) \in V : |V_{\bar{x}}| \text{ and } |V_{\bar{y}}| \text{ are finite}\}$. Clearly, U is definable and Zariski dense in V . Let $(\bar{a}, e^{\bar{a}}) \in U$, and suppose that $(\bar{a}, e^{\bar{a}})$ is not generic in U , i.e., $\text{t. d.}_{\mathbb{Q}}(\bar{a}, e^{\bar{a}}) = m < n$. The finite cardinality of $V_{\bar{a}}$ implies that all coordinates of the tuple $e^{\bar{a}}$ are algebraic over \bar{a} , since they are in $\text{acl}(\bar{a})$. Exchanging \bar{a} and $e^{\bar{a}}$ we have that each coordinate of the tuple \bar{a} is algebraic over $e^{\bar{a}}$. Hence,

$$(1) \quad m = \text{t. d.}_{\mathbb{Q}}(\bar{a}) = \text{t. d.}_{\mathbb{Q}}(\bar{a}, e^{\bar{a}}) = \text{t. d.}_{\mathbb{Q}}(e^{\bar{a}}).$$

Schanuel’s Conjecture implies $\text{l.d.}(\bar{a}) \leq \text{t.d.}_{\mathbb{Q}}(\bar{a}, e^{\bar{a}}) = m < n$. By equation (1) we can then conclude that $\text{l.d.}(\bar{a}) = m$. Hence, there exists a matrix $M \in \mathcal{M}_{(n-m) \times n}(\mathbb{Z})$ of rank $n - m$ such that $M \cdot \bar{a} = \bar{0}$, which together with its multiplicative version give the following hyperspace and torus:

$$L_M = \{\bar{x} : M \cdot \bar{x} = \bar{0}\} \quad \text{and} \quad T_M = \{\bar{y} : \bar{y}^M = \bar{1}\}.$$

As observed $\dim T_M = \dim L_M = m$. So, \bar{a} is generic in L_M and $e^{\bar{a}}$ is generic in T_M . Then the non-genericity of $(\bar{a}, e^{\bar{a}})$ in U is witnessed by either \bar{a} or $e^{\bar{a}}$.

Without loss of generality we can assume that T_M is irreducible. If not, we consider the irreducible component of T_M containing $\bar{1}$ whose associated matrix we call M' . By results on pages 82-83 in [2] the associate hyperspace $L_{M'}$ coincides with L_M .

For every $N \in \mathcal{M}_{(n-m) \times n}(\mathbb{C})$, define

$$W_N = \{(\bar{x}, \bar{y}) \in U : \bar{x} \in L_N\}.$$

Clearly, W_N is definable, and so $(W_N)_N$ is a definable family.

If $N = M$ then $(\bar{a}, e^{\bar{a}}) \in W_M$, and so $\dim W_M \geq \dim L_M$. Moreover, from the definitions of U and W_N it follows that π_1 restricted to W_M is finite-to-one. Therefore, $\dim W_M = \dim \pi_1(W_M)$, and so $\dim W_M \leq \dim L_M$. Hence, $\dim W_M = \dim L_M$.

Let W'_M be the irreducible component of the Zariski closure of W_M containing the point $(\bar{a}, e^{\bar{a}})$. Since $(\bar{a}, e^{\bar{a}})$ is generic in W'_M , and $e^{\bar{a}} \in \pi_2(W'_M) \cap T_M$ is generic in $\pi_2(W'_M)$ we have that $\pi_2(W'_M) \subseteq T_M$. Hence, the Zariski closure of the projection, $\overline{\pi_2(W'_M)}^{\text{Zar}}$, is contained in T_M . Moreover, $e^{\bar{a}}$ is generic in T_M , and this implies that

$$T_M = \overline{\pi_2(W'_M)}^{\text{Zar}}.$$

Let $(W_i^*)_{i \in I}$ (where I is a definable set) be the definable family of all irreducible components of all $(W_N)_N$ for $N \in \mathcal{M}_{(n-m) \times n}(\mathbb{C})$. In particular, W'_M is one of W_i^* for some $i \in I$. For each $i \in I$, denote S_i the Zariski closure of $\pi_2(W_i^*)$.

Let

$$\mathcal{U}_m = \{S_i : S_i \text{ is a subgroup of } (\mathbb{C}^*)^n\}.$$

Since \mathcal{U}_m is a countable definable family in $(\mathbb{C}^*)^n$, and \mathbb{C} is saturated then \mathcal{U}_m is either finite or uncountable. Then \mathcal{U}_m is necessarily finite, and $T_M \in \mathcal{U}_m$.

Let $\mathcal{U} := \mathcal{U}_1 \cup \dots \cup \mathcal{U}_{n-1}$. Clearly, \mathcal{U} is finite since each \mathcal{U}_j is finite for $j \in \{1, \dots, n-1\}$, so $\mathcal{U} = \{H_1, \dots, H_\ell\}$. Let

$$T = H_1 \cup \dots \cup H_\ell \quad \text{and} \quad C = \{(\bar{x}, \bar{y}) \in U : \bar{y} \in T\}.$$

Then by Theorem 2.3 the set

$$X = \{(\bar{a}, e^{\bar{a}}) : (\bar{a}, e^{\bar{a}}) \in U - C\}$$

is not empty, Zariski dense in U (and hence in V), and $\text{t.d.}_{\mathbb{Q}}(\bar{a}, e^{\bar{a}}) = n$ for every $(\bar{a}, e^{\bar{a}}) \in X$. ■

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