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Existence and regularity results for semi-linearized compressible 2D fluids with generalized diffusion

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We consider semi-linearized compressible Navier–Stokes equations, in a two-dimensional periodic domain, introducing in this scheme generalized diffusive terms. After rewriting the equations in terms of a potential flow for the velocity, we prove existence and regularity for a suitable class of weak solutions.

KEYWORDS

Navier–Stokes equations, PDEs in connection with fluid mechanics, compressible fluids, existence and regularity

MSC CLASSIFICATION

35Q30; 35Q35; 76N10

1 | INTRODUCTION

In this paper we consider a two-dimensional modified version of the Navier–Stokes equations for barotropic flows on $Q_T = (0, T) \times D$, with $T > 0$ and $D = (0, 1)^2$. The starting model reads as follows

$$
\begin{aligned}
&\bar{\rho}\,(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p = \mu\Delta\tilde{F}(\mathbf{u}) + (\mu+\lambda)\nabla(\mathrm{div}\mathbf{u}) + \mu_1\nabla(\mathrm{div}\tilde{F}(\mathbf{u})), && \text{in } Q_T,\\
&\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, && \text{in } Q_T,
\end{aligned}
\tag{1.1}
$$

where $\mathbf{u} = (u_1, u_2)$, ρ and $p = p(\rho)$ are, respectively, velocity, density, and pressure. Also, $\bar{\rho}$, λ, μ, and μ_1 are constants that satisfy the constraints $\bar{\rho} > 0$, $\mu > 0$, $\mu_1 > 0$, $\mu + \lambda \geq 0$. The presence of the terms $\mu\Delta\tilde{F}(\mathbf{u})$ and $\mu_1\nabla(\mathrm{div}\tilde{F}(\mathbf{u}))$, with $\tilde{F}(\mathbf{u}) = \left(\tilde{F}_1(u_1, u_2), \tilde{F}_2(u_1, u_2)\right)$, is the deviation with respect to standard description (see[1]), by means of Navier–Stokes equations, of barotropic flows (see, e.g., previous works[2,3]). More details on the structure of $\tilde{F}(\mathbf{u})$ will be provided in the sequel.

The basic choice behind the introduction of $\tilde{F}(\mathbf{u})$ in the above model, is related to the idea of Prouse[1] who suggested the possibility that the standard viscous stress $\mathbb{S}$ for the incompressible Navier–Stokes equations is suitable in the case of relatively low flow velocities, while for rather high velocities a modification of the type introduced above is reasonable. Here, we make similar assumptions, although for moderately high values of the flow velocity (also assuming moderate mean flow velocity values), and not necessarily in the presence of turbulence: Due to these properties of the fluid flow we consider an intermediate scheme between the Navier–Stokes equations and system (1.1), that is,

$$
\begin{aligned}
&\bar{\rho}\,(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p = \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla(\mathrm{div}\mathbf{u}) + \mu_1\nabla(\mathrm{div}\tilde{F}(\mathbf{u})), && \text{in } Q_T,\\
&\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, && \text{in } Q_T.
\end{aligned}
\tag{1.2}
$$

In this specific circumstance, instead of taking the viscous stress $\mathbb{S}$ of the usual type $\mathbb{S} = 2\mu\mathbb{D}(\mathbf{u}) + \lambda\mathrm{div}\mathbf{u}\,\mathbb{I}$, for compressible flows, with $\mathbb{D}(\mathbf{u}) = sym\nabla\mathbf{u} = \left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right)/2$ and $\mathbb{I} := I_{2\times 2} = I$ the second-rank unit tensor, we assume for $\mathbb{S}$ an augmented structure given by

$$\mathbb{S} = 2\mu\mathbb{D}(\mathbf{u}) + \lambda \mathrm{div}\mathbf{u}\,\mathbb{I} + \mu_1 \mathrm{div}\tilde{F}(\mathbf{u})\,\mathbb{I},$$

that, in components, can be rewritten as follows

$$\mathbb{S}_{ij} = \mu\left(\partial_i u_j + \partial_j u_i\right) + \lambda\delta_{ij}\partial_k u_k + \mu_1\delta_{ij}\partial_k \tilde{F}_k(\mathbf{u}),$$

with $i, j, k = 1, 2$ and where δ_{ij} denotes the Kronecker delta function. We emphasize that with respect to $(1.1)_1$, here $\mu\left(\partial_i \tilde{F}_j(\mathbf{u}) + \partial_j \tilde{F}_i(\mathbf{u})\right)$, $i, j = 1, 2$, has been replaced with $\mu\left(\partial_i u_j + \partial_j u_i\right)$, $i, j = 1, 2$.

Here and in what follows, all the fields involved in the considered problem are assumed to be periodic in $x = (x_1, x_2)$ with period 1.

We study a semi-linearized version of system (1.2) taking into account a simplified setting, under ad hoc assumptions on the convective term: First, by using Helmholtz's filter we perform the following decoupling

$$\mathbf{u} = \bar{\mathbf{u}} + \hat{\mathbf{u}},$$

where $\bar{\mathbf{u}} = (I - \Delta)^{-1}\mathbf{u}$ and $\hat{\mathbf{u}} = \mathbf{u} - \bar{\mathbf{u}}$. This separation yields

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (\bar{\mathbf{u}}\cdot\nabla)\mathbf{u} + (\hat{\mathbf{u}}\cdot\nabla)\mathbf{u},$$

and, neglecting $(\hat{\mathbf{u}}\cdot\nabla)\mathbf{u}$, the system (1.2) reduces to

$$\begin{aligned}
&\bar{\rho}\left(\mathbf{u}_t + (\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}\right) + \nabla p = \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla(\mathrm{div}\mathbf{u}) + \mu_1\nabla(\mathrm{div}\tilde{F}(\mathbf{u})), && \text{in } Q_T,\\
&\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, && \text{in } Q_T,\\
&\bar{\mathbf{u}} = (I-\Delta)^{-1}\mathbf{u}, && \text{in } Q_T.
\end{aligned}$$

As a further mathematical simplification, we substitute $\bar{\mathbf{u}}$ with a suitable constant velocity vector field $\bar{\mathbf{v}}$ related to the flow characteristics, with respect to a proper limited range of length and time scales (which can be derived, for instance, by using suitable numerical methods and simulations). Then, we focus on the following semi-linearized problem, that is,

$$\begin{aligned}
&\bar{\rho}\left(\mathbf{u}_t + (\bar{\mathbf{v}}\cdot\nabla)\mathbf{u}\right) + \nabla p = \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla(\mathrm{div}\mathbf{u}) + \mu_1\nabla(\mathrm{div}\tilde{F}(\mathbf{u})), && \text{in } Q_T,\\
&\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, && \text{in } Q_T.
\end{aligned} \tag{1.3}$$

We also assume as a constitutive relation

$$p(\rho) = c\rho, \tag{1.4}$$

with c a positive constant. For the usual simplicity, we set the constants c, μ_1, $\bar{\rho}$ equal to 1 and the further constants λ and μ will be properly selected later.

Let us now consider a suitable potential y for the velocity $\mathbf{u}$ in (1.3), that is,

$$\mathbf{u} = \nabla y, \tag{1.5}$$

with y a scalar function periodic in $x = (x_1, x_2)$ with period 1. By using relation (1.5), our system can be rewritten as follows (see also previous works[2,3])

$$\begin{aligned}
&\nabla y_t = -\nabla\left((\bar{\mathbf{v}}\cdot\nabla)y\right) + (2\mu+\lambda)\nabla\Delta y + \nabla\mathrm{div}(\tilde{F}(\nabla y)) - \nabla\rho, && \text{in } Q_T,\\
&\rho_t + \mathrm{div}(\rho\nabla y) = 0, && \text{in } Q_T,
\end{aligned} \tag{1.6}$$

where

$$\rho|_{t=0} = \rho_0(x) \geq 0, \qquad \int_D \rho_0(x)\mathrm{d}x = 1,$$

and

$$y|_{t=0} = y_0(x), \qquad \int_D y_0(x)dx = 0.$$

For this system, after integrating in space the momentum equation $(1.6)_1$, we provide existence an regularity results for two specific choices of the vector field $\tilde{F}$ (see Section 3, see also previous works,[1,4,5] for similar situations about the structure of $\tilde{F}$).

In Section 4, we consider a preliminary case (see Theorem 3.2 below), with $\tilde{F} = (F(y) + \varsigma\nabla y)$, $\varsigma > 0$, which is closer to the situation treated in Bessaih,[2] and we require $|F_i(s)| \le c_1|s| + c_2|s|^{1+m}$ for $i = 1, 2$ and $m > 1/2$. Then, in Section 5, we take into account an enriched scheme, with $\tilde{F} = \left(\hat{F}(\nabla y) + \varsigma\nabla y\right)$, $\varsigma > 0$, and $\hat{F}(\nabla y) = (\eta + |\nabla y|)^m \nabla y$, $\eta > 0$ and $-1 < m \le 0$, for which we prove the main result of the paper, that is, Theorem 3.1.

2 | PRELIMINARIES

Given $q \ge 1$, by $L^q(D)$, we denote the customary Lebesgue spaces with norm $\|\cdot\|_q$. In particular, we have that $\|\cdot\| := \|\cdot\|_{L^2(D)}$ and $(\cdot,\cdot)$ denotes the L^2-inner product. Moreover, by $W^{k,q}(D)$, k a non-negative integer and q as before, we denote the usual Sobolev spaces with norm $\|\cdot\|_{k,q}$. For $k > 0$, and $q = 2$, we use the notation $H^k = W^{k,2}(D)$. The dual of $W^{1,q}(D)$ is denoted by $W^{-1,q'}$ with norm $\|\cdot\|_{-1,q'}$; the dual of H^q is denoted by H^{-q}, and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H^{-1},H^1}$ indicates the duality pairing. Similarly we define Lebesgue and Sobolev spaces on $Q_T = (0,T) \times D$, $T > 0$, that is, $L^q(Q_T)$, $q \ge 1$, with norm $\|\cdot\|_{L^q(Q_T)}$. Let us recall that all the considered fields are periodic of period 1 in the variables x_1 and x_2, with $x = (x_1, x_2) \in D$.

In what follows, in order to keep the notation compact, we use the same kind of symbols for scalar and vector fields (the same convention is used also for the related spaces, without specifying the space dimension or the number of the components), distinguishing the different cases only when it is strictly required by the context.

Let X be a real Banach space with norm $\|\cdot\|_X$. We consider the usual Bochner spaces $L^q(0,T;X)$, with norm denoted by $\|\cdot\|_{L^q(0,T;X)}$. Moreover, $C([0,T];X)$ denotes the space of the continuous functions taking values in X. We also use the Orlicz space $L_\phi(D)$ associated to the convex function $\phi = \phi(r)$, for $r \ge 0$, given by $\phi(r) = (1+r)\ln(1+r) - r$ (which is equal to $\rho\ln\rho - \rho + 1$, when $\rho = 1 + r$), see, for example, Vaigant and Kazhikhov.[3] Again, in this case, we use the notation $\|\cdot\|_\phi := \|\cdot\|_{L_\phi(D)}$.

In most of the cases we omit the explicit dependence on D in the considered Lebesgue, Sobolev, Bochner, and Orlicz spaces.

Here and in the sequel, we denote by C or c positive constants that may assume different values, even in the same equation; when the constants depend on quantities of interest, these are explicitly placed in parentheses or as subscripts.

2.1 | Basic estimates

Let us recall classical Ladyzhenskaya's inequality (see, e.g., Ladyzhenskaya[6]). For $v \in H^1$ we have

$$\|v\|_4 \le C\|v\|^{\frac{1}{2}}\|\nabla v\|^{\frac{1}{2}},$$

with C depending only on the domain D.

We will also use the following Gagliardo–Nirenberg inequalities on D (see, e.g., Nirenberg[7]), that is,

$$\|v\|_{4m} \le C\|v\|^{\frac{1}{2m}}\|\nabla v\|^{\frac{2m-1}{2m}}, \tag{2.1}$$

$$\|v\|_{8m} \le C\|v\|^{\frac{1}{4m}}\|\nabla v\|^{\frac{4m-1}{4m}}, \tag{2.2}$$

for $m \ge 1/2$, and

$$\|v\|_8 \le C\|v\|_4^{\frac{3}{4}}\|\nabla v\|_4^{\frac{1}{4}}, \tag{2.3}$$

$$\|v\|_\infty \le C\|v\|^{\frac{1}{3}}\|\nabla v\|_4^{\frac{2}{3}}, \tag{2.4}$$

where C depends on Lebesgue space indexes, the exponents to the interpolating norms, the order of the involved derivatives, as well as on the domain D. Furthermore, we will also exploit Hölder's and Young's inequalities.

3 | MAIN RESULTS

Introducing suitable assumptions on $\tilde{F}$, we state the main theorems of this paper.

3.1 | System (1.6) in reduced form

Integrating the momentum equation $(1.6)_1$ one obtains the following

$$y_t - (\mu + \lambda)\Delta y = -(\bar{\mathbf{v}} \cdot \nabla)y + \mathrm{div}\tilde{F}(\nabla y) + 1 - \rho, \text{ in } Q_T,$$

coupled with $\rho_t + \mathrm{div}\,(\rho \nabla y) = 0$ in Q_T. As mentioned in Section 1, the structure of $\tilde{F}$, in the principal part of this study, is as follows

$$\tilde{F} = \hat{F}(\nabla y) + \varsigma \nabla y, \tag{3.1}$$

with $\varsigma > 0$. In the sequel we always assume $\lambda = \mu = 1/4$, and $\varsigma = 1/2$, to get

$$\begin{aligned} y_t - \Delta y &= -(\bar{\mathbf{v}} \cdot \nabla)y + \mathrm{div}\hat{F}(\nabla y) + 1 - \rho, && \text{in } Q_T, \\ \rho_t + \mathrm{div}\,(\rho \nabla y) &= 0, && \text{in } Q_T. \end{aligned} \tag{3.2}$$

For this system, which is supplied with periodic boundary conditions, we also require that

$$\begin{aligned} \rho|_{t=0} &= \rho_0(x) \geq 0, \quad && \int_D \rho_0(x)\mathrm{d}x = 1, \\ y|_{t=0} &= \varphi_0(x), && \int_D \varphi_0(x)dx = 0. \end{aligned} \tag{3.3}$$

In addition, we impose the following requirements on the solution, that is,

$$\rho(t, x) \geq 0 \text{ in } Q_T, \tag{3.4}$$

and also that

$$\int_D y(t, x)\mathrm{d}x = 0 \text{ in } t \in [0, T]. \tag{3.5}$$

The term $\hat{F}(\nabla y)$ mimics—at least in its form—the extra stress-tensor in the case of some types of generalized Newtonian fluids (see, e.g., previous works[8–11]). We set

$$\hat{F}(\nabla y) = S(\nabla y) := (\eta + |\nabla y|)^m \nabla y, \tag{3.6}$$

where $\eta > 0$ and $m = p - 2$, with $1 < p \leq 2$.

Let us now introduce the notion of "*regular weak solution*" for the considered system.

Definition 3.1. A regular weak solution to the problem (3.2)–to–(3.6) taken with (3.4)–(3.5), is a pair of functions (y, ρ) such that

(i) $y \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$, and $\rho \in L^\infty(0, T; L_\phi(D)) \cap L^2(Q_T)$;
(ii) $y_t \in L^\infty(0, T; L^2(D))$, and $\rho_t \in L^2(0, T; H^{-1})$;
(iii) (y, ρ) satisfies $(3.2)_1$ in $L^2(Q_T)$ and $(3.2)_2$ in the distributions sense;
(iv) (y, ρ) are weakly continuous with respect to t in the above cited spaces and satisfy (3.3), (3.4) and (3.5).

The expression regular weak solutions has been introduced in Berselli and Lewandowski[12] and, although the context is different, the basic idea is the same: The term "weak" refers to the fact that the equations are satisfied in distributional sense, while the solutions are called "regular" because of the properties of the spaces to which they belong. However, in order to be concise, in the following we will always refer to weak solutions rather than regular weak solutions.

Remark 3.1. Let $T > 0$. As a consequence of $(3.2)_2$, and of the regularity claimed in Definition 3.1, it follows that (see Vaigant and Kazhikhov[3])

$$\|\rho(t)\|_1 = \|\rho_0\|_1, \text{ a.e. on } [0, T],$$
$$\rho(x, t) \geq 0, \text{ a.e. on } Q_T.$$

For the problem (3.2)–to–(3.6), we state the following existence theorem which will be proved in Section 5.

Theorem 3.1. *Let $\hat{F}(\nabla y) = S(\nabla y) = (\eta + |\nabla y|)^m \nabla y$, and let $T > 0$ any given time. Assume $y_0 \in H^2$ and $\rho_0 \in L^3(D)$. Then, there exists at least one regular weak solution (y, ρ) for the problem (3.2)–(3.3), such that*

$$\begin{aligned} &y \in L^\infty(0,T;W^{1,4}(D)) \cap L^4(0,T;W^{2,4}(D)),\\ &y_t \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;H^1),\\ &\rho \in L^\infty(0,T;L_\phi(D)) \cap L^4(Q_T). \end{aligned} \tag{3.7}$$

3.2 | Preliminary problem

In this case, we actually substitute $\hat{F} = \hat{F}(\nabla y)$, in relation (3.1) (and consequently also in the system (3.2)), with $F := F(y)$, and so

$$\tilde{F} = F(y) + \varsigma \nabla y, \ \varsigma > 0, \tag{3.8}$$

where $F \in C^2(\mathbb{R})$ and, following the same scheme as in Temam,[5] we require that

$$|F_i(s)| \le c_1|s| + c_2|s|^{1+m} \quad \text{and} \quad |F_i'(s)| \le c_1 + c_2|s|^m, \tag{3.9}$$

for $i = 1, 2$ and $m \ge 1/2$. Here, c_1 and c_2 are positive constants. This model will be considered in Section 4.

The notion of regular weak solution can be directly derived by adapting that one given in Definition 3.1, taking into account relation (3.9) in place of (3.6).

We have the following existence criterion.

Theorem 3.2. *Let $F = F(y)$ and assume that the properties in (3.9) are satisfied. Then, there exists a time $T_0 > 0$ and at least one regular weak solution (y, ρ) for the problem (3.2)–(3.3) defined in any interval $[0, T]$, $T < T_0$ and such that*

$$\begin{aligned} &y \in L^\infty(0,T;W^{1,q}(D)) \cap L^4(0,T;W^{2,4}(D)), \ \textit{with} \ \ q \ge 2,\\ &y_t \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;H^1),\\ &\rho \in L^\infty(0,T;L_\phi(D)) \cap L^4(Q_T). \end{aligned} \tag{3.10}$$

Moreover, if it holds that $\inf_{x\in D}\rho_0(x) > 0$ and that $\sup_{x\in D}\rho_0(x) < \infty$, then it follows that $\inf_{Q_T}\rho(t,x) > 0$, and that $\sup_{Q_T}\rho(t,x) < \infty$.

The calculations we perform to prove this result will then be used, in Section 5, also to prove Theorem 3.1.

Remark 3.2. In the case of Theorem 3.1, assuming that $\inf_{x\in D}\rho_0(x) > 0$ and that $\sup_{x\in D}\rho_0(x) < \infty$, in order to get the bounds $\inf_{Q_T}\rho(t,x) > 0$ and $\sup_{Q_T}\rho(t,x) < \infty$, more regularity on the considered solutions seems to be needed (see Appendix A for more details).

4 | SYSTEM (3.2) UNDER THE HYPOTHESES (3.8)–(3.9)

Let us consider the system

$$\begin{aligned} &y_t - \Delta y = -(\bar{\mathbf{v}} \cdot \nabla)y + \mathrm{div}F(y) + 1 - \rho, && \text{in } Q_T,\\ &\rho_t + \mathrm{div}\,(\rho \nabla y) = 0, && \text{in } Q_T. \end{aligned} \tag{4.1}$$

For this problem, we prove existence and regularity results for the weak solutions by using suitable energy estimates.

4.1 | A priori estimates

Let us start with the following result based on formal estimates that, however, can be made more rigorous by introducing a suitable approximating Galerkin scheme (see, e.g., Vaigant and Kazhikhov[3]).

Lemma 4.1. *If the initial data* (ρ_0, y_0) *are such that* $\rho_0 \in L_\phi(D)$ *and* $y_0 \in H^1$ *then there exists a time* $T_0 = T_0(\|\rho_0\|_\phi, \|y_0\|_{H^1})$ *such that, for any* $0 < t < T_0$*, the following inequality holds true, that is,*

$$\|\nabla y(t)\|^2 + 2\int_D (\rho \ln \rho - \rho + 1)\,\mathrm{d}x \le \frac{\left(\|\nabla y_0\|^2 + 2\|\rho_0\|_\phi\right) e^{ct}}{\left[1 - \left(\|\nabla y_0\|^2 + 2\|\rho_0\|_\phi\right)^{2m}(e^{2mct} - 1)\right]^{\frac{1}{2m}}}, \tag{4.2}$$

where C and c are two positive constants. Moreover, we also have that

$$\int_0^t \|\Delta y(s)\|^2 \mathrm{d}s \le CT, \tag{4.3}$$

for any $0 < T < T_0$ *and* $0 \le t \le T$.

Proof. Multiplying equation $(4.1)_1$ by Δy in L^2, and integrating by parts over D, we find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla y\|^2 + \|\Delta y\|^2 + \int_D \nabla\rho \cdot \nabla y\,\mathrm{d}x &\le \|(\bar{\mathbf{v}} \cdot \nabla) y\|\|\Delta y\| + \|\mathrm{div}F(y)\|\|\Delta y\| \\ &\le \varepsilon\|\Delta y\|^2 + c_\varepsilon(\|\nabla y\|^2 + \|\mathrm{div}F(y)\|^2), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \|\mathrm{div}F(y)\|^2 &\le \int_D |F_i'(y)\partial_i y|^2 \mathrm{d}x \\ &\le \int_D |F'(y)|^2 |\nabla y|^2 \mathrm{d}x \\ &\le c\|\nabla y\|_4^2 \left(\int_D \left(c_1 + c_2|y|^m\right)^4 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\le c\|\Delta y\|\|\nabla y\| \left(1 + \|y\|_{4m}^{2m}\right) \\ &\le \varepsilon_1\|\Delta y\|^2 + c_{\varepsilon_1}\|\nabla y\|^2 \left(1 + \|y\|_{4m}^{4m}\right) \\ &\le \varepsilon_1\|\Delta y\|^2 + c_{\varepsilon_1}\|\nabla y\|^2 \left(1 + \left(\|y\|^{\frac{1}{2m}}\|\nabla y\|^{\frac{2m-1}{2m}}\right)^{4m}\right) \qquad \text{(using (2.1))} \\ &\le \varepsilon_1\|\Delta y\|^2 + c_{\varepsilon_1}\|\nabla y\|^2 \left(1 + \|y\|^2\|\nabla y\|^{2(2m-1)}\right), \end{aligned}$$

where we used Hölder's, Young's and Gagliardo–Nirenberg's inequalities. Hence, by using Poincaré's inequality we get

$$\begin{aligned} \|\mathrm{div}F(y)\|^2 &\le \varepsilon_1\|\Delta y\|^2 + c_{\varepsilon_1}\|\nabla y\|^2 \left(1 + c\|\nabla y\|^2\|\nabla y\|^{2(2m-1)}\right) \\ &= \varepsilon_1\|\Delta y\|^2 + c_{\varepsilon_1}\|\nabla y\|^2 \left(1 + c\|\nabla y\|^{4m}\right) \\ &\le \varepsilon_1\|\Delta y\|^2 + c_{\varepsilon_1}\left(\|\nabla y\|^2 + \|\nabla y\|^{4m+2}\right), \end{aligned}$$

and plugging this estimate into (4.4), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla y\|^2 + (1 - \varepsilon - \varepsilon_1)\int_D |\Delta y|^2 \mathrm{d}x + \int_D \nabla\rho \cdot \nabla y\,\mathrm{d}x \le c_{\varepsilon,\varepsilon_1}\left(\|\nabla y\|^2 + \|\nabla y\|^{4m+2}\right). \tag{4.5}$$

Now, multiplying equation $(4.1)_2$ by $\ln\rho$, in L^2, and performing an integration by part, we get

$$\frac{d}{dt}\int_D (\rho\ln\rho - \rho + 1)\mathrm{d}x - \int_D \nabla\rho \cdot \nabla y\,\mathrm{d}x = 0. \tag{4.6}$$

Adding (4.5) and (4.6), we infer

$$\begin{aligned}
&\frac{d}{dt}\int_D \left(\frac{1}{2}|\nabla y|^2 + \rho\ln\rho - \rho + 1\right) \mathrm{d}x + \bar{c}\|\Delta y\|^2 \\
&\le c_{\varepsilon,\varepsilon_1}\left(\|\nabla y\|^2 + \|\nabla y\|^{4m+2}\right) \\
&\le c\int_D \left(\frac{1}{2}|\nabla y|^2 + \rho\ln\rho - \rho + 1\right) \mathrm{d}x + c_{\varepsilon,\varepsilon_1}\left(\|\nabla y\|^2\right)^{2m+1} \\
&\le c\left[\int_D \left(\frac{1}{2}|\nabla y|^2 + \rho\ln\rho - \rho + 1\right) \mathrm{d}x + \left(\|\nabla y\|^2\right)^{2m+1}\right],
\end{aligned} \tag{4.7}$$

where $\bar{c} = (1 - \varepsilon - \varepsilon_1)$, and $c = 2(1 + c_{\varepsilon,\varepsilon_1} + \hat{c})$, with $\tilde{T} > 0$ large enough and properly chosen later. Then, setting

$$x(t) = \int_D \left(|\nabla y|^2 + 2(\rho\ln\rho - \rho + 1)\right) \mathrm{d}x,$$

from (4.7) we infer

$$x' \le c\left(x + x^{2m+1}\right),$$

and so

$$x^{2m}(t) \le \frac{x_0^{2m} e^{2mct}}{1 - x_0^{2m}(e^{2mct} - 1)},$$

which is defined for

$$t < T_0 := \ln\left(\frac{1 + x_0^{2m}}{x_0^{2m}}\right)^{\frac{1}{2mc}}.$$

Here, $x_0 = \|\nabla y_0\|^2 + 2\|\rho_0\|_\phi$ and we redefine $\tilde{T} := \max\{\tilde{T}, T_0\}$. □

Lemma 4.2. *If $\rho_0 \in L^{p-1}(D)$, $2 < p < \infty$, then there exists a constant C depending on p and D such that the inequality*

$$\begin{aligned}
\|\rho(t)\|_{p-1}^{p-1} + c\int_0^t \|\rho(s)\|_p^p \mathrm{d}s \le C\Bigg(\|\rho_0\|_{p-1}^{p-1} + \int_0^t \|y_t(s)\|_p^p \mathrm{d}s + \int_0^t \|y(s)\|_{2mp}^{2mp} \mathrm{d}s \\
+ \int_0^t \|\nabla y(s)\|_{2p}^{2p} \mathrm{d}s\Bigg) + CT,
\end{aligned} \tag{4.8}$$

holds for any $t \in [0, T]$, $T < T_0$.

Proof. For $r > 1$, multiply equation $(4.1)_2$ against ρ^{r-1} to get

$$\partial_t \rho^r + r\rho^{r-1}\operatorname{div}(\rho\nabla y) = 0$$

that can be rewritten as

$$\begin{aligned}
\partial_t \rho^r + \operatorname{div}\left(\rho^r \nabla y\right) &= (1 - r)\rho^r \Delta y \qquad \text{(using (4.1))} \\
&= (r - 1)\rho^r \left(1 - \rho - y_t - (\bar{\mathbf{v}}\cdot\nabla)y + \operatorname{div}F(y)\right).
\end{aligned}$$

Estimating $\rho^r|y_t|$ and $\rho^r|\operatorname{div}F(y)|$ by the Young inequality, that is,

$$\begin{aligned}
&\rho^r|y_t| \le \varepsilon_1\rho^{r+1} + C_{\varepsilon_1}|y_t|^{r+1}, \\
&\rho^r|(\bar{\mathbf{v}}\cdot\nabla)y| \le \varepsilon_{\varepsilon_2}\rho^{r+1} + C_2|\nabla y|^{r+1},
\end{aligned}$$

and

$$\begin{aligned}\rho^r|\mathrm{div}F(y)| &\le \varepsilon_3\rho^{r+1} + C_{\varepsilon_3}|\mathrm{div}F(y)|^{r+1}\\ &\le \varepsilon_3\rho^{r+1} + C_{\varepsilon_3}|F'(y)|^{r+1}|\nabla y|^{r+1}\\ &\le \varepsilon_3\rho^{r+1} + C\left(1+|y|^{m(r+1)}\right)|\nabla y|^{r+1}\\ &\le \varepsilon_3\rho^{r+1} + C\left(1+|y|^{2m(r+1)}\right) + C|\nabla y|^{2(r+1)}.\end{aligned}$$

Then, integrating over D, and setting $\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, we get

$$\begin{aligned}\frac{d}{dt}\|\rho\|_r^r + (r-1-\varepsilon)\|\rho\|_{r+1}^{r+1} &\le (r-1)\|\rho\|_r^r + C\left(1+\|y\|_{2m(r+1)}^{2m(r+1)}\right)\\ &\quad + C\left(\|y_t\|_{r+1}^{r+1} + \|\nabla y\|_{r+1}^{r+1} + \|\nabla y\|_{2(r+1)}^{2(r+1)}\right)\\ &\le C_{\varepsilon_4} + \varepsilon_4\|\rho\|_{r+1}^{r+1} + C\left(\|y\|_{2m(r+1)}^{2m(r+1)}\right.\\ &\quad \left. + \|y_t\|_{r+1}^{r+1} + \|\nabla y\|_{2(r+1)}^{2(r+1)}\right),\end{aligned}$$

from which (4.8) follows directly, by integrating in time over $[0,t]$, $t \le T$, and setting $p = r+1$. □

Lemma 4.3. *If $\rho_0 \in L^{p-1}(D)$, $2 < p < \infty$, then there exists a constant C depending on p such that the inequality*

$$\begin{aligned}\|\Delta y\|_{L^p(Q_T)} \le C\Big(&\|\rho_0\|_{L^{p-1}(D)} + \|y_t\|_{L^p(Q_T)} + \|y\|_{L^{2mp}(Q_T)}^{2m}\\ &+ \|\nabla y\|_{L^p(Q_T)} + \|\nabla y\|_{L^{2p}(Q_T)}^2\Big) + CT,\end{aligned} \tag{4.9}$$

holds for any $T < T_0$.

Proof. Estimate (4.9) is a consequence of (4.8), (3.9) and equation $(4.1)_1$: The latter is multiplied by $-|\Delta y|^{p-2}\Delta y$ and subsequently integrated over Q_T. Indeed, in such a case we have that

$$\int_0^T \|\Delta y(s)\|_p^p \mathrm{d}s \le \int_0^T \int_D \left[|y_t(s)| + (1+\rho(s)) + (C + |\nabla y(s)|)(1+|y(s)|^m)\right]|\Delta y(s)|^{p-1}\mathrm{d}x\mathrm{d}s$$

where $C = C(|\bar{\mathbf{v}}|)$. The conclusion follows by using calculations similar to those present in the proof of Lemma 4.1 and Lemma 4.2. □

In order, to get improved a priori estimates, we differentiate $(4.1)_1$ with respect to x_1, x_2, and with respect to t to get, respectively, the estimates

$$\nabla y_t - \nabla\Delta y = \nabla(\mathrm{div}F(y)) - \nabla((\bar{\mathbf{v}}\cdot\nabla)y) - \nabla\rho, \tag{4.10}$$

and

$$y_{tt} - \Delta y_t - \partial_t(\mathrm{div}F(y)) + (\bar{\mathbf{v}}\cdot\nabla)\partial_t y = -\rho_t = \mathrm{div}(\rho\nabla y). \tag{4.11}$$

In the following we omit the sums on the indices to keep the notation as concise as possible. For arbitrary $q \ge 2$, we multiply (4.10) by $q|\nabla y|^{q-2}\nabla y$ and integrate on D to get

$$\begin{aligned}&\frac{d}{dt}\|\nabla y\|_q^q + q\int_D(\partial_j\partial_k y)^2|\nabla y|^{q-2}\mathrm{d}x + q(q-2)\int_D(\partial_j\partial_k y)(\partial_k y)(\partial_j\partial_l y)(\partial_l y)|\nabla y|^{q-4}\mathrm{d}x\\ &= -q\int_D \mathrm{div}F(y)\mathrm{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x + q\int_D(\bar{\mathbf{v}}\cdot\nabla)y\,\mathrm{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x\\ &\quad + q\int_D \rho\,\mathrm{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x\\ &= q\int_D(\rho - \mathrm{div}F(y) + (\bar{\mathbf{v}}\cdot\nabla)y)\,\mathrm{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x.\end{aligned}$$

Then, substituting $\rho - \mathrm{div}F(y) + (\bar{\mathbf{v}}\cdot\nabla)y = 1 + \Delta y - y_t$ in such a relation, we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla y\|_q^q &+ q\int_D (\partial_j\partial_k y)^2|\nabla y|^{q-2}\mathrm{d}x\\ &+ q(q-2)\int_D (\partial_j\partial_k y)(\partial_k y)(\partial_j\partial_l y)(\partial_l y)|\nabla y|^{q-4}\mathrm{d}x\\ &= q\int_D (1+\Delta y - y_t)\mathrm{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x.\end{aligned} \tag{4.12}$$

Let $s \geq 2$. Multiplying (4.11) by $s|y_t|^{s-2}y_t$ in L^2, and substituting $\rho = 1 + \Delta y + \mathrm{div}F(y) - (\bar{\mathbf{v}}\cdot\nabla)y - y_t$ in the so obtained relation, we obtain

$$\begin{aligned}\frac{d}{dt}\|y_t\|^s &+ s(s-1)\int_D |\nabla y_t|^2|y_t|^{s-2}\mathrm{d}x\\ &= -s(s-1)\int_D \rho\partial_j y\partial_j y_t|y_t|^{s-2}\mathrm{d}x - s(s-1)\int_D F_j'(y)\partial_j y_t|y_t|^{s-2}y_t\mathrm{d}x\\ &\quad - s\int_D ((\bar{\mathbf{v}}\cdot\nabla)y_t)\,|y_t|^{s-2}y_t\mathrm{d}x\\ &= -s(s-1)\int_D (1+\Delta y + \mathrm{div}F(y) - (\bar{\mathbf{v}}\cdot\nabla)y - y_t)\,\partial_j y\partial_j y_t|y_t|^{s-2}\mathrm{d}x\\ &\quad - s(s-1)\int_D F_j'(y)\partial_j y_t\, y_t|y_t|^{s-2}\mathrm{d}x - s\int_D ((\bar{\mathbf{v}}\cdot\nabla)y_t)\,|y_t|^{s-2}y_t\mathrm{d}x.\end{aligned} \tag{4.13}$$

Thus, the sum of the above two relations is as follows

$$\begin{aligned}\frac{d}{dt}\left(\|\nabla y\|_q^q + \|y_t\|^s\right) &+ q\int_D (\partial_j\partial_k y)^2|\nabla y|^{q-2}\mathrm{d}x + s(s-1)\int_D |\nabla y_t|^2|y_t|^{s-2}\mathrm{d}x\\ &= -q(q-2)\int_D (\partial_j\partial_k y)(\partial_j\partial_l y)(\partial_k y)(\partial_l y)|\nabla y|^{q-4}\mathrm{d}x\\ &\quad - s(s-1)\int_D (1+\Delta y + \mathrm{div}F(y) - (\bar{\mathbf{v}}\cdot\nabla)y - y_t)\,\partial_j y\partial_j y_t|y_t|^{s-2}\mathrm{d}x\\ &\quad + q\int_D (1+\Delta y - y_t)\mathrm{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x - s(s-1)\int_D F_j'(y)\partial_j y_t\, y_t|y_t|^{s-2}\mathrm{d}x\\ &\quad - s\int_D ((\bar{\mathbf{v}}\cdot\nabla)y_t)\,|y_t|^{s-2}y_t\mathrm{d}x.\end{aligned} \tag{4.14}$$

To bound the terms on the right-hand side of the above relation, we observe that

$$\||\nabla y|^2\|_4 \leq \||\nabla y|^2\|^{\frac{1}{2}}\|\nabla|\nabla y|^2\|^{\frac{1}{2}}, \tag{4.15}$$

where we used Ladyzhenskaya's inequality. Moreover, we have

$$\||\nabla y|^2\| = \left(\int_D ||\nabla y|^2|^2\mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_D |\nabla y|^4\mathrm{d}x\right)^{\frac{1}{2}} = \|\nabla y\|_4^2 \tag{4.16}$$

$$\|\nabla|\nabla y|^2\| = \left(\int_D |\nabla|\nabla y|^2|^2\mathrm{d}x\right)^{\frac{1}{2}} \leq 2\left(\int_D |\nabla y|^2(\partial_j\partial_k y)^2\mathrm{d}x\right)^{\frac{1}{2}}, \tag{4.17}$$

and, for $q{\geq}4$, we also have

$$\||\nabla y|^{\frac{q}{2}}\| = \left(\int_D ||\nabla y|^{\frac{q}{2}}|^2\mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_D |\nabla y|^q\mathrm{d}x\right)^{\frac{1}{2}} = \|\nabla y\|_q^{\frac{q}{2}}, \tag{4.18}$$

$$\|\nabla|\nabla y|^{\frac{q}{2}}\| = \left(\int_D |\nabla|\nabla y|^{\frac{q}{2}}|^2\mathrm{d}x\right)^{\frac{1}{2}} \leq \frac{q}{2}\left(\int_D |\nabla y|^{q-2}(\partial_j\partial_k y)^2\mathrm{d}x\right)^{\frac{1}{2}}. \tag{4.19}$$

Theorem 4.1. *Let $T > 0$ with $T < T_0$. If $\rho_0 \in L^3(D)$, $y_0 \in H^2$, then there exists a constant C depending on T_0 such that the inequalities*

$$\sup_{0<s<t} \left(\|\nabla y\|_4^4 + \|y_t\|_2^2 \right) + \int_0^t \|\nabla y_t(s)\|^2 \mathrm{d}s \leq C(T), \tag{4.20}$$

and

$$\int_0^t \|\rho(s)\|_4^4 \mathrm{d}s + \int_0^t \|\Delta y(s)\|_4^4 \mathrm{d}s \leq C(T), \tag{4.21}$$

hold true for any $t \in [0, T]$.

Proof. By taking $q = 4$ and $s = 2$ in (4.14), making derivatives explicit and rearranging the terms in such a relation (especially those coming from the first addendum on the right-hand side), we have

$$\begin{aligned}
&\frac{d}{dt} \left(\|\nabla y\|_4^4 + \|y_t\|^2 \right) + 4 \int_D (\partial_j \partial_k y)^2 |\nabla y|^2 \mathrm{d}x + \|\nabla y_t\|^2 \\
&= 4 \int_D \Delta y |\nabla y|^2 + 4 \int_D |\Delta y|^2 |\nabla y|^2 - 4 \int_D (\Delta y) y_t |\nabla y|^2 \mathrm{d}x \\
&+ 8 \int_D (\partial_j y)(\partial_k y)(\partial_j \partial_k y) \mathrm{d}x + 8 \int_D \Delta y (\partial_j y)(\partial_k y)(\partial_j \partial_k y) \mathrm{d}x \\
&- 8 \int_D y_t (\partial_j y)(\partial_k y)(\partial_j \partial_k y) \mathrm{d}x - 2 \int_D \nabla y \cdot \nabla y_t - 2 \int_D \Delta y \nabla y \cdot \nabla y_t \mathrm{d}x \\
&+ 2 \int_D y_t \nabla y \cdot \nabla y_t - 8 \int_D (\partial_j \partial_k y)(\partial_j \partial_l y)(\partial_k y)(\partial_l y) \mathrm{d}x \\
&- 2 \int_D \mathrm{div} F(y) \nabla y \cdot \nabla y_t \mathrm{d}x + 2 \int_D y_t F'(y) \cdot \nabla y_t \mathrm{d}x \\
&- 2 \int_D (\bar{\mathbf{v}} \cdot \nabla) y_t \cdot y_t \mathrm{d}x + 2 \int_D ((\bar{\mathbf{v}} \cdot \nabla) y) \nabla y \cdot \nabla y_t \mathrm{d}x =: \sum_{i=1}^{14} I_i,
\end{aligned} \tag{4.22}$$

where, to keep the notation concise, we omit the summations made on the indices for the derivatives and the map components.

Proceeding as in Bessaih,[2] for the first ten terms from I_1 to I_{10}, we have the following inequalities that we list for the sake of completeness (see also Vaigant and Kazhikhov,[3] for more details on the other terms I_i, $i = 1, \ldots, 10$), that is,

$$\begin{aligned}
I_2 &\leq \varepsilon_1 \|\Delta y\| \|\Delta y\|^2 + \varepsilon_2 \|\nabla |\nabla y|^2\|^2 + C_{\varepsilon_1, \varepsilon_2} \|\Delta y\|^2 \||\nabla y|^2\|^2 \\
&\leq \varepsilon_1 \|\Delta y\| \|\Delta y\|^2 + \varepsilon_2 \int_D |\nabla y|^2 (\partial_j \partial_k y)^2 \mathrm{d}x + C_{\varepsilon_1, \varepsilon_2} \|\nabla y\|^4 \|\Delta y\|^2,
\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}
I_1 &= 4 \int_D |\nabla y|^2 \Delta y \mathrm{d}x \leq \varepsilon I_2 + C_\varepsilon \|\nabla y\|^2, \\
I_3 &\leq \varepsilon_1 \int_D |\nabla y|^2 |\Delta y|^2 \mathrm{d}x + \varepsilon_2 \|\nabla y_t\|^2 + C_{\varepsilon_1, \varepsilon_2} \|y_t\|^2 \|\nabla y\|^2 \|\Delta y\|^2, \\
I_4 &\leq \varepsilon \int_D (\partial_k y)^2 (\partial_j \partial_k y)^2 \mathrm{d}x + C_\varepsilon \|\nabla y\|^2, \\
I_5 &\leq \varepsilon \int_D (\partial_k y)^2 (\partial_j \partial_k y)^2 \mathrm{d}x + C_\varepsilon I_2, \\
I_6 &\leq \varepsilon_1 \int_D (\partial_k y)^2 (\partial_j \partial_k y)^2 \mathrm{d}x + \varepsilon_2 \|\nabla y_t\|^2 + C \|y_t\|^2 \|\nabla y\|^2 \|\partial_j \partial_k y\|^2, \\
I_7 &\leq 2 \left| \int_D \nabla y \cdot \nabla y_t \mathrm{d}x \right| \leq \varepsilon \|\nabla y_t\|^2 + C \int_D |\nabla y|^2 \mathrm{d}x. \\
I_8 &\leq 2 \left| \int_D \Delta y \nabla y \cdot \nabla y_t \mathrm{d}x \right| \leq \varepsilon \|\nabla y_t\|^2 + C I_2 + \|\Delta y\|^2, \\
I_9 &\leq 2 \left| y_t \nabla y \cdot \nabla y_t \mathrm{d}x \right| \leq \varepsilon \|\nabla y_t\|^2 + C_\varepsilon \|y_t\|^2 \|\Delta y\|^2 \|\nabla y\|^2.
\end{aligned}$$

Then, in particular, we have that

$$
\begin{aligned}
I_{10} &\leq \varepsilon_1 \|\partial_j \partial_k y\| \|\partial_j \partial_l y\|_4^2 + \varepsilon_2 \|\nabla |\nabla y|^2\|^2 + C_{\varepsilon_1,\varepsilon_2} \||\nabla y|^2\|^2 \|\partial_j \partial_k y\|^2 \\
&\leq \varepsilon_1 \|\partial_j \partial_k y\| \|\partial_j \partial_l y\|_4^2 + 4\varepsilon_2 \int_D |\nabla y|^2 (\partial_j \partial_k y)^2 \mathrm{d}x + C_{\varepsilon_1,\varepsilon_2} \|\nabla y\|_4^4 \|\Delta y\|^2.
\end{aligned} \tag{4.24}
$$

Let us take into account the terms coming from $(\bar{\mathbf{v}} \cdot \nabla)y$, that is,

$$
I_{13} \leq 2 \left| \int_D (\bar{\mathbf{v}} \cdot \nabla) y_t \cdot y_t \mathrm{d}x \right| \leq \varepsilon \|\nabla y_t\|^2 + C_\varepsilon \|y_t\|^2, \tag{4.25}
$$

and

$$
\begin{aligned}
I_{14} \leq 2 \left| \int_D ((\bar{\mathbf{v}} \cdot \nabla) y) \, \nabla y \cdot \nabla y_t \mathrm{d}x \right| &\leq C \|\nabla y\|_4^2 \|\nabla y_t\| \\
&\leq \varepsilon \|\nabla y_t\|^2 + C_\varepsilon \|\nabla y\|^2 \|\Delta y\|^2.
\end{aligned} \tag{4.26}
$$

Now, consider the new terms coming from $\mathrm{div}F(y)$. Let us start with I_{11} to get

$$
\begin{aligned}
I_{11} &\leq 2 \int_D |\mathrm{div}F(y)| \, |\nabla y| \, |\nabla y_t| \mathrm{d}x \\
&\leq 2 \|\nabla y_t\| \left(\int_D |\nabla y|^2 |F'(y)|^2 |\nabla y|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq 2 \|\nabla y_t\| \|\nabla y\|_4 \left(\int_D |F'(y)|^4 |\nabla y|^4 \mathrm{d}x \right)^{\frac{1}{4}} \\
&\leq \varepsilon_1 \|\nabla y_t\|^2 + C_{\varepsilon_1} \|\nabla y\|_4^2 \left(\int_D |F'(y)|^4 |\nabla y|^4 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq \varepsilon_1 \|\nabla y_t\|^2 + C_{\varepsilon_1} \|\nabla y\|_4^2 \|F'(y)\|_8^2 \|\nabla y\|_8^2 \\
&\leq \varepsilon_1 \|\nabla y_t\|^2 + C_{\varepsilon_1} \|\nabla y\| \|\Delta y\| \left(1 + \|y\|_{8m}^{2m}\right) \|\nabla y\|_8^2 \\
&\leq \varepsilon_1 \|\nabla y_t\|^2 + \varepsilon_2 \left(1 + \|y\|_{8m}^{4m}\right) \|\nabla y\|_8^4 + C_{\varepsilon_1,\varepsilon_2} \|\nabla y\|^2 \|\Delta y\|^2 \\
&\leq \varepsilon_1 \|\nabla y_t\|^2 + \varepsilon_2 \left(1 + \|y\| \|\nabla y\|^{4m-1}\right) \|\nabla y\|_8^4 + C_{\varepsilon_1,\varepsilon_2} \|\nabla y\|^2 \|\Delta y\|^2 \\
&\leq \varepsilon_1 \|\nabla y_t\|^2 + \varepsilon_2 C \|\nabla y\|_8^4 + C_{\varepsilon_1,\varepsilon_2} \|\Delta y\|^2,
\end{aligned} \tag{4.27}
$$

where, to control $\|y\|_{8m}^{4m}$, we used (2.2).

Finally, observe that

$$
\begin{aligned}
I_{12} &\leq 2 \left| \int_D y_t F'(y) \cdot \nabla y_t \mathrm{d}x \right| \\
&\leq 2 \|\nabla y_t\|^2 \left(\int_D |y_t|^2 |F'(y)|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq \varepsilon \|\nabla y_t\|^2 + C_\varepsilon \int_D |y_t|^2 |F'(y)|^2 \mathrm{d}x \\
&\leq \varepsilon \|\nabla y_t\|^2 + C_{\varepsilon_1} \|y_t\|_4^2 \|F'(y)\|_4^2 \\
&\leq \varepsilon \|\nabla y_t\|^2 + C_{\varepsilon_1} \|y_t\| \|\nabla y_t\| \|F'(y)\|_4^2 \\
&\leq \varepsilon_1 \|\nabla y_t\|^2 + \varepsilon_2 \|\nabla y_t\|^2 + C_{\varepsilon_1,\varepsilon_2} \|y_t\|^2 \|F'(y)\|_4^4 \\
&\leq \varepsilon_1 \|\nabla y_t\|^2 + \varepsilon_2 \|\nabla y_t\|^2 + C_{\varepsilon_1,\varepsilon_2} \|y_t\|^2 \left(1 + \|y\|_{4m}^{4m}\right) \\
&\leq (\varepsilon_1 + \varepsilon_2) \|\nabla y_t\|^2 + C_{\varepsilon_1,\varepsilon_2} \|y_t\|^2 \left(1 + \|y\|^2 \|\nabla y\|^{2(2m-1)}\right) \\
&\leq (\varepsilon_1 + \varepsilon_2) \|\nabla y_t\|^2 + C_{\varepsilon_1,\varepsilon_2} \|y_t\|^2,
\end{aligned} \tag{4.28}
$$

where, in particular, we used (2.1) and (4.2).

To close the differential estimate (4.22), we integrate in time (in the time-interval $[0,t]$) such an estimate and we use the previous relations (4.23)–to–(4.28). Let us take into account the worst term coming from (4.24), that is,

$$\begin{aligned}
\varepsilon\int_0^t \|\partial_j\partial_k y\|\|\partial_j\partial_l y\|_4^2 \mathrm{d}s &\le \varepsilon C\int_0^t \|\Delta y\|\|\Delta y\|_4^2 \mathrm{d}s \\
&\le \varepsilon C\left(\int_0^t \|\Delta y\|^2 \mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t \|\Delta y\|_4^4 \mathrm{d}s\right)^{\frac{1}{2}} \quad \text{(using (4.3))} \\
&\le \varepsilon C T^{\frac{1}{2}}\left(\int_0^t \|\Delta y\|_4^4 \mathrm{d}s\right)^{\frac{1}{2}} \\
&\le \varepsilon C\left(\int_0^t \|\Delta y\|_4^4 \mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.29}$$

To control this last term we use, up to integrate in t, relation (4.9): In the present case, that is, $p = 4$ and $s = 2$, relation (4.9) gives

$$\begin{aligned}
\|\Delta y\|_4^4 &\le C\left(\|\rho_0\|_3 + \|y_t\|_4^4 + \|y\|_{8m}^{8m} + \|\nabla y\|_8^8\right) + C \\
&\le C\left(\|y_t\|^2\|\nabla y_t\|^2 + \|y\|_{8m}^{8m} + \|\nabla y\|_8^8\right) + C,
\end{aligned} \tag{4.30}$$

where, in particular, for the term $\|y_t\|_4^4$ we used Gagliardo–Nirenberg's interpolation inequality (2.1). We focus on $\|\nabla y\|_8^8$. Then, after and integration over the interval $[0,t]$, by using (4.15)–to–(4.16) we have that

$$\begin{aligned}
\left(\int_0^t \|\nabla y(s)\|_8^8 \mathrm{d}s\right)^{\frac{1}{2}} &= \left(\int_0^t \||\nabla y(s)|^2\|_4^4 \mathrm{d}s\right)^{\frac{1}{2}} \\
&\le \left(\int_0^t \||\nabla y(s)|^2\|^2\|\nabla|\nabla y(s)|^2\|^2 \mathrm{d}s\right)^{\frac{1}{2}} \\
&= \left(2\int_0^t \|\nabla y(s)\|_4^4\left(\int_D |\nabla y(s)|^2(\partial_j\partial_k y)^2 \mathrm{d}x\right)\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le C\sup_{0<s<t}\|\nabla y(s)\|_4^2\left(\int_0^t\int_D |\nabla y|^2(\partial_j\partial_k y)^2 \mathrm{d}x\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le C_{\varepsilon_1}\sup_{0<s<t}\|\nabla y(s)\|_4^4 + \varepsilon_1\int_0^t\int_D |\nabla y|^2(\partial_j\partial_k y)^2 \mathrm{d}x.
\end{aligned} \tag{4.31}$$

Therefore, using (4.29) along with (4.30) and (4.31), we get

$$\begin{aligned}
\varepsilon\int_0^t \|\partial_j\partial_k y\|\|\partial_j\partial_l y\|_4^2 \mathrm{d}s &\le \varepsilon C\left(\int_0^t \|\Delta y\|_4^4 \mathrm{d}s\right)^{\frac{1}{2}} \\
&\le \varepsilon C\left(\alpha(t)^{\frac{1}{2}}\left(\int_0^t \beta(s)\mathrm{d}s\right)^{\frac{1}{2}} + Ct^{\frac{1}{2}}\right),
\end{aligned}$$

where

$$\alpha(t) := \sup_{0<s<t}\left(\|\nabla y(s)\|_4^4 + \|y_t(s)\|^2\right),$$

and

$$\beta(t) := \int_D (\partial_j\partial_k y)^2|\nabla y|^2 \mathrm{d}x + \|\nabla y_t\|^2.$$

The remaining terms in the right-hand side of (4.22) are easier to deal with: In essence, the same type of estimates used above allows us to bound them (see also Bessaih[2]). Therefore, from (4.22) along with (4.23)–to–(4.31), we infer

$$\alpha(t) + \int_0^t \beta(s)\mathrm{d}s \le \alpha(0) + \varepsilon C\alpha(t)^{\frac{1}{2}} \left(\int_0^t \beta(s)\mathrm{d}s\right)^{\frac{1}{2}} + C\left(1 + \int_0^t \left[(\|\Delta y\|^2 + 1)\alpha(s) + 1\right] \mathrm{d}s\right),$$

where $y_0 \in H^2 \hookrightarrow W^{1,4}(D)$. Using Gronwall's lemma we obtain

$$\sup_{0<s<t} \left(\|\nabla y(s)\|_4^4 + \|y_t(s)\|_2^2\right) + \int_0^t \|\nabla y_t\|^2 \mathrm{d}s \le C(T),\ 0 < t \le T < T_0. \tag{4.32}$$

Also, exploiting this inequality along with (4.8), (4.30) and (4.31), it follows that

$$\int_0^t \|\rho(s)\|_4^4 \mathrm{d}s + \int_0^t \|\Delta y(s)\|_4^4 \mathrm{d}s \le C(T_0),\ 0 < t \le T < T_0. \tag{4.33}$$

In fact, in relation (4.8), for $p = 4$, the worst term on the right-hand side, besides $\int_0^t \|\Delta y(s)\|_4^4 \mathrm{d}s$, still remain $\int_0^t \|\nabla y(s)\|_8^8 \mathrm{d}s$. Thus, the same calculations as above can be used to close this last integral inequality.

As a very last point, we observe that (4.32) requires initial data $(\rho_0, y_0) \in L^3(D) \times H^2$, actually ρ_0 is taken in $L^3(D)$ in order to use (4.8), and subsequently get (4.33). Indeed, if (y_k, ρ_k) is the family of Galerkin approximating functions used to make rigorous the previous calculations (see the end of this subsection for more details), then to bound $(\partial_t y_k)(0) = y_{k,t}(0)$, for $\zeta \in L^2(D)$ with $\|\zeta\| \le 1$, consider

$$\begin{aligned} |((\partial_t y_k)(0), \zeta)| &\le | -((\bar{\mathbf{v}} \cdot \nabla) y_0) + \mathrm{div} F(y_0) + \Delta y_0 + 1 - \rho_0, \zeta)| \\ &\le c(1 + \|y_0\|_{4m}^{2m})\|\nabla y_0\|_4^2 + C\|\nabla y_0\| + \|\Delta y_0\| + \|\rho_0\| + c, \end{aligned} \tag{4.34}$$

and this concludes the proof. □

Lemma 4.4. *Let $\rho_0 \in L^3(D)$ and $y_0 \in H^2$. As a consequence of the previous results, follows that $\rho_t \in L^2(0, T; H^{-1})$, $T < T_0$.*

Proof. As a consequence of the previous estimates, we also have that for any $\xi \in H^1$, with $\|\xi\|_{H^1} = 1$, the following relation holds true, that is,

$$\begin{aligned} \langle \rho_t, \xi \rangle &= -\int_D \mathrm{div}(\rho \nabla y)\xi \mathrm{d}x \\ &= \int_D \rho \nabla y \cdot \nabla \xi \mathrm{d}x \\ &\le \|\rho\|_4^2 \|\nabla y\|_4^2 \|\nabla \xi\|, \end{aligned}$$

and so

$$\begin{aligned} \int_0^t \|\rho_t\|_{H^{-1}}^2 \mathrm{d}s &\le C \int_0^t \|\rho\|_4^4 \|\nabla y\|_4^4 \mathrm{d}s \\ &\le C \sup_{0<s<t} \left(\|\nabla y\|_4^4\right) \int_0^t \|\rho\|_4^4 \mathrm{d}s. \end{aligned}$$

As a direct consequence, the conclusion follows. □

4.2 | Higher order estimates

Now, let us consider the Equation (4.12) for $q \geq 4$ and Equation (4.13) for $s = 2$. Adding these relations, without using $\rho = 1 + \Delta y + \operatorname{div}F(y) - (\bar{\mathbf{v}} \cdot \nabla)y - y_t$, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla y\|_q^q + \|y_t\|^2\right) + q\int_D (\partial_j\partial_k y)^2 |\nabla y|^{q-2}\mathrm{d}x + 2\|\nabla y_t\|^2 \\
&= -q(q-2)\int_D (\partial_j\partial_k y)(\partial_j\partial_l y)(\partial_k y)(\partial_l y)|\nabla y|^{q-4}\mathrm{d}x \\
&\quad - 2\int_D \rho\nabla y\cdot\nabla y_t\mathrm{d}x + q\int_D \rho\operatorname{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x \\
&\quad - q\int_D \operatorname{div}F(y)\operatorname{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x + 2\int_D F_j'(y)\partial_j y_t|y_t|^{s-2}y_t\mathrm{d}x \\
&\quad - 2\int_D (\bar{\mathbf{v}}\cdot\nabla)y_t\cdot y_t\mathrm{d}x + q\int_D ((\bar{\mathbf{v}}\cdot\nabla)y)\operatorname{div}(|\nabla y|^{q-2}\nabla y)\mathrm{d}x \\
&\leq -q(q-2)\int_D (\partial_j\partial_k y)(\partial_j\partial_l y)(\partial_k y)(\partial_l y)|\nabla y|^{q-4}\mathrm{d}x \\
&\quad - 2\int_D \rho\nabla y\cdot\nabla y_t\mathrm{d}x + q\int_D \rho|\nabla y|^{q-2}\Delta y\mathrm{d}x \\
&\quad + q(q-2)\int_D \rho|\nabla y|^{q-4}(\partial_k\partial_l y)(\partial_l y)(\partial_k y)\mathrm{d}x \\
&\quad - q(q-2)\int_D (\partial_i F_i(y))|\nabla y|^{q-4}(\partial_k\partial_l y)\partial_l y\partial_k y\mathrm{d}x \\
&\quad - q\int_D (\partial_i F_i(y))\partial_{kk}^2 y|\nabla y|^{q-2}\mathrm{d}x + 2\int_D F_j'(y)\partial_j y_t|y_t|^{s-2}y_t\mathrm{d}x \\
&\quad + C\int_D |\nabla y_t||y_t|\mathrm{d}x + q\int_D ((\bar{\mathbf{v}}\cdot\nabla)y)\operatorname{div}(|\nabla y|^{q-2}\nabla y)\mathrm{d}x := \sum_{i=1}^{9} J_i.
\end{aligned} \tag{4.35}$$

Also, in this case we omit the summations on the indices for the derivatives and the function components.

Lemma 4.5. *Let $T > 0$ fixed with $T < T_0$. If $\rho_0 \in L^3(D)$, $y_0 \in H^2$, then the inequality*

$$\sup_{0<\tau<t} \|\nabla y\|_q^q \leq C, \quad q \geq 2, \tag{4.36}$$

holds true for $t \in [0, T]$.

Proof. Due to the hypotheses on the initial data, we have that $y_0 \in H^2 \hookrightarrow W^{1,q}(D)$, $q \geq 1$. Let us consider the case $q > 4$ for (4.36) (the cases $1 \leq q \leq 4$ are consequence of the previous estimates), and make use of (4.35). When $q = 4$ the calculations are as in the proof of Theorem 4.1. For the first four terms J_i, $i = 1, 2, 3, 4$, on the right-hand side of (4.35), we use the same estimates as in Bessaih;[2] we report in details only the case of J_1, that is,

$$\begin{aligned}
J_1 &= q(q-2)\int_D (\partial_j\partial_k y)(\partial_j\partial_l y)(\partial_k y)(\partial_l y)|\nabla y|^{q-4}\mathrm{d}x \\
&\leq \varepsilon\int_D |\nabla y|^{q-2}(\partial_j\partial_k y)^2\mathrm{d}x + C_\varepsilon\int_D (\partial_j\partial_l y)^2|\nabla y|^{q-2}\mathrm{d}x \\
&\leq \varepsilon\int_D |\nabla y|^{q-2}(\partial_j\partial_k y)^2\mathrm{d}x + C_\varepsilon\|\partial_j\partial_l y\|_4^2\left(\int_D |\nabla y|^{2(q-2)}\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \varepsilon\int_D |\nabla y|^{q-2}(\partial_j\partial_k y)^2\mathrm{d}x + C_\varepsilon\|\partial_j\partial_l y\|_4^2\||\nabla y|^{\frac{q}{2}}\|_{\frac{4(q-2)}{q}}^{\frac{2(q-2)}{q}}.
\end{aligned}$$

Now, using Gagliardo-Nirenberg's inequality (2.1), with $m = (q-2)/q$, and exponent $\sigma = (q-4)/(2(q-2))$, along with (4.18) and (4.19), we obtain

$$\||\nabla y|^{\frac{q}{2}}\|_{\frac{4(q-2)}{q}}^{\frac{2(q-2)}{q}} \leq \left(C\||\nabla y|^{\frac{q}{2}}\|^{\frac{q}{2(q-2)}} \|\nabla|\nabla y|^{\frac{q}{2}}\|^{\frac{q-4}{2(q-2)}} \right)^{\frac{2(q-2)}{q}}$$
$$\leq C\|\nabla y\|_q^{\frac{q}{2}} \left(\frac{q}{2} \int_D |\nabla y|^{q-2} (\partial_j \partial_k y)^2 \mathrm{d}x \right)^{\frac{q-4}{2q}},$$

and, hence, we can conclude that

$$\begin{aligned} J_1 &\leq \varepsilon \int_D |\nabla y|^{q-2} (\partial_j \partial_k y)^2 \mathrm{d}x + C_\varepsilon \|\partial_j \partial_l y\|_4^2 \|\nabla y\|_q^{\frac{q}{2}} \left(\int_D |\nabla y|^{q-2} (\partial_j \partial_k y)^2 \mathrm{d}x \right)^{\frac{q-4}{2q}} \\ &\leq (\varepsilon + \varepsilon_1) \int_D |\nabla y|^{q-2} (\partial_j \partial_k y)^2 \mathrm{d}x + C_{\varepsilon,\varepsilon_1} \|\partial_j \partial_l y\|_4^{\frac{4q}{q+4}} \left(\int_D |\nabla y|^q \right)^{\frac{q}{q+4}}, \end{aligned} \tag{4.37}$$

where we used Young's inequality with exponents $\sigma = 2q/(q-4)$ and $\sigma' = 2q/(q+4)$. For the second term on the right-hand side of the above inequality, we use again (4.9) to bound $\|\Delta y\|_4^4$. As a consequence, we actually have to control $\|\nabla y\|_8^8$. Hence,

$$\begin{aligned} \|\nabla y\|_8^8 &\leq \left(\|\nabla y\|_4^{\frac{3}{4}} \|\Delta y\|_4^{\frac{1}{4}} \right)^8 \\ &\leq \varepsilon_2 \|\Delta y\|_4^4 + C_{\varepsilon_2} \left(\|\nabla y\|_4^4 \right)^3 \\ &\leq \varepsilon_2 \|\Delta y\|_4^4 + C_{\varepsilon_2} \left(\sup_{0<t<T} \|\nabla y\|_4^4 \right)^3, \end{aligned}$$

where we used Gagliardo–Nirenberg's inequality (2.3).

The remaining terms J_2, J_3 and J_4 are simpler and can be treated similarly: We only list here, for the sake of completeness, the related estimates

$$\begin{aligned} J_2 &\leq \int_D |\rho||\nabla y||\nabla y_t| \leq \varepsilon \|\nabla y_t\|^2 + C\|\rho\|_4^2 \|\nabla y\|_4^2, \\ J_3 &\leq \int_D |\rho||\Delta y||\nabla y|^{q-2} \mathrm{d}x, \\ &\leq \varepsilon \int_D (\partial_j \partial_k y)^2 |\nabla y|^{q-2} \mathrm{d}x + C_\varepsilon \|\rho\|_4^{\frac{4q}{q+4}} \left(\int_D |\nabla y|^q \right)^{\frac{q}{q+4}}, \\ J_4 &\leq \varepsilon \int_D |\nabla y|^{q-2} (\partial_j \partial_k y)^2 \mathrm{d}x + C_\varepsilon \|\rho\|_4^{\frac{4q}{q+4}} \left(\int_D |\nabla y|^q \right)^{\frac{q}{q+4}}. \end{aligned}$$

Let us consider the remaining terms. Observe that J_7 can be estimated exactly as in (4.28). We now have that

$$J_8 \leq C\|y_t\| \|\nabla y_t\| \leq \varepsilon \|\nabla y_t\|^2 + C_\varepsilon \|y_t\|^2.$$

and

$$\begin{aligned} J_9 &\leq q \int_D |\nabla y| |\mathrm{div}(|\nabla y|^{q-2} \nabla y)| \mathrm{d}x \\ &\leq q(q-1) \int_D |\nabla y|^{\frac{q}{2}} |\Delta y| |\nabla y|^{\frac{q-2}{2}} \mathrm{d}x \\ &\leq \varepsilon \int_D |\nabla y|^{q-2} |Delta y|^2 \mathrm{d}x + C_\varepsilon \|\nabla y\|_q^q. \end{aligned}$$

Consider J_5 and J_6 to get

$$
\begin{aligned}
J_5+J_6 &= q(q-2)\int_D (\partial_i F_i(y))\,|\nabla y|^{q-4}(\partial_k\partial_l y)\partial_l y\partial_k y\mathrm{d}x \\
&\quad + q\int_D (\partial_i F_i(y))\,\partial^2_{kk} y|\nabla y|^{q-2}\mathrm{d}x \\
&\le q(q-2)\int_D |F'(y)||\nabla y|^{q-1}(\partial_k\partial_l y)\mathrm{d}x + q\int_D |F'(y)||\nabla y|^{q-1}|\Delta y|\mathrm{d}x \\
&\le q(q-2)C\int_D |\nabla y|^{q-1}|\partial_k\partial_l y|\mathrm{d}x + qC\int_D |\nabla y|^{q-1}|\Delta y|\mathrm{d}x \\
&\le q(q-1)C\int_D |\nabla y|^{q-1}|\partial_k\partial_l y|\mathrm{d}x \\
&\le \varepsilon\int_D |\Delta y|^{q-2}|\partial_k\partial_l y|^2\mathrm{d}x + C_\varepsilon\|\nabla y\|_q^q.
\end{aligned}
\tag{4.38}
$$

Consequently, using (4.35) along with (4.37)–to–(4.38), (4.34) and (4.9), we have that

$$
\begin{aligned}
&\|\nabla y\|_q^q + \|y_t\|^2 + \int_0^T \|\nabla y_t(s)\|^2\mathrm{d}s \le C\left(\|\nabla y_0\|_q^q + \|\rho_0\|^2 + \|\Delta y_0\|^2 + c\right) \\
&\quad + (C-\varepsilon_2)\int_0^T \left(\|\rho\|_4^{\frac{4q}{(q+4)}} + \|y_t\|^{\frac{2q}{(q+4)}}\|\nabla y_t\|^{\frac{2q}{(q+4)}} + \|\rho_0\|_3^{\frac{4q}{(q+4)}}\right. \\
&\quad \left. + \|y\|_{8m}^{\frac{8mq}{q+4}} + \|\nabla y\|_4^{\frac{12q}{q+4}} + C\right)\left(\|\nabla y\|_q^q\right)^{\frac{q}{(q+4)}},
\end{aligned}
$$

where we used Ladyzhenskaya's inequality to get $\|y_t\|_4^{4q/q+4} \le \|y_t\|^{2q/(q+4)}\|\nabla y_t\|^{2q/(q+4)}$. The conclusion follows by an application of Gronwall's lemma. □

The obtained estimates are sufficient for proving the existence of solutions. Indeed, we can use the method for constructing solutions given in Vaigant and Kazhikhov.[3] According to this scheme, approximating solutions $\{(y_k,\rho_k)\}$ are found by the Galerkin method (see, e.g., Galdi;[13] see also Bisconti[14]). In particular $\{\nabla y_k\}$ is compact in $L^2(Q_T)$. Thanks to the previous estimates, we can use classical compactness arguments (see, e.g., Temam;[15] see also the argument in previous works[2,3]) to extract a convergent subsequence (still denoted by $\{(y_k,\rho_k)\}$) Thus, the passage to the limit in the nonlinear terms in $(4.1)_2$ is justified. For the nonlinear term in $(4.1)_1$, to pass to the limit, it is enough to observe that since $F(y_k)$ is bounded, uniformly with respect to k, in $L^2(Q_T)$, then $F(y_k)\rightharpoonup A$ in $L^2(Q_T)$. Therefore, using that $F=F(s)$ is continuous along with the fact that $\{y_k\}$ converges a.e. on D to $\tilde{y}$, due to the uniqueness of the limit, it follows that $A=F(\tilde{y})$.

4.3 | Upper and lower bounds for the density

The estimates previously obtained allow us to prove the density ρ is actually bounded provided that the initial density ρ_0 is bounded as well. To this end, we use the same approach provided in Vaigant and Kazhikhov[3] (see also Bessaih[2]). We have the following two lemmas whose proofs, close to those of [2, Lemma 4.6, and Lemma 4.7], are provided in Appendix A.

Lemma 4.6. *If $y_0\in H^2$, and $\rho_0\in L^\infty(D)$ then*

$$
\|\rho(t)\|_\infty \le M,\ \forall t\in[0,T],\ T<T_0,
\tag{4.39}
$$

where $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(D)}$ and $M=M(T)$ is a suitable positive constant.

Lemma 4.7. *If the initial density $\rho_0(x)$ is strictly positive under the hypotheses of Lemma 4.6, then $\rho(t,x)$ remains a strictly positive function in $Q_T,\ T<T_0$, that is,*

$$
\rho(t,x)\ge m>0 \ \ a.e.\ in\ \ Q_T.
\tag{4.40}
$$

As a consequence of the results of Subsections 4.1 and 4.3, then Theorem 3.2 follows directly.

5 | CASE OF $\hat{F}(\nabla y) = S(\nabla y)$

Let us consider the following system

$$
\begin{aligned}
&y_t - \Delta y = -(\bar{\mathbf{v}} \cdot \nabla) y + \operatorname{div}\left((\eta + |\nabla y|)^{p-2} \nabla y\right) + 1 - \rho, && \text{in } Q_T,\\
&\rho_t + \operatorname{div}(\rho \nabla y) = 0, && \text{in } Q_T,
\end{aligned}
\tag{5.1}
$$

with $\eta > 0$ and $1 < p \leq 2$. In the sequel we use the notation $S(\nabla y) = (\eta + |\nabla y|)^{p-2}\nabla y$, introduced in (3.6) and we always assume $\eta \geq 1$. Further hypotheses on η will be introduced later. This section devoted to the proof of Theorem 3.1.

5.1 | Energy estimates

Also, in this case we proceed formally by providing a number of a priori estimates that, combined with a compactness criterion à la Aubin–Lions, allow us to prove the existence of a weak solution for the considered system supplied with appropriate initial data. As before, calculations can be made rigorous by using a suitable Galerkin scheme (see, e.g., previous works[3,10,11,13]).

Let us recall some properties characterizing the non-linear term $S(\nabla y)$. It can be proved that $S(v)$, $v \in \mathbb{R}^2$, satisfies the following relation: There exists a positive constant C_1 such that (see, e.g., Diening[16,17])

$$
\frac{\partial S_i(v)}{\partial v_j} w_i w_j \geq C_1 (\eta + |v|)^{p-2} |w|^2, \tag{5.2}
$$

for any vector $w \in \mathbb{R}^2$. Also, for any pair of vectors $v, w \in \mathbb{R}^2$, the following relations hold true, that is,

$$
(S(v) - S(w)) \cdot (v - w) \geq C_2 \frac{|v - w|^2}{(\eta + |v| + |w|)^{2-p}}, \tag{5.3}
$$

and

$$
|S(v) - S(w)| \leq C_3 \frac{|v - w|}{(\eta + |v| + |w|)^{2-p}}, \tag{5.4}
$$

with C_2 and C_3 positive constants. The proofs of these estimates are given[16] in the case of second-rank tensors, but they can easily adapted to our simpler case.

Remark 5.1. It holds that

$$
\begin{aligned}
&\int_D \operatorname{div}\left((\eta + |\nabla y|)^{p-2} \nabla y\right) \Delta y \mathrm{d}x \\
&\quad = \int_D \partial_i \left((\eta + |\nabla y|)^{p-2} \partial_i y\right) \partial^2_{ss} y \mathrm{d}x \\
&\quad = \int_D \partial_s \left((\eta + |\nabla y|)^{p-2} \partial_i y\right) \partial_s \partial_i y \mathrm{d}x \\
&\quad \left(= \int_D \frac{\partial \left((\eta + |\mathrm{D}|)^{p-2} \mathrm{D}_i\right)}{\partial \mathrm{D}_l} \frac{\partial \mathrm{D}_l}{\partial x_s} \frac{\partial \mathrm{D}_i}{\partial x_s} \mathrm{d}x, \text{ with } \mathrm{D} = \nabla y \right) \\
&\quad = \int_D (\eta + |\nabla y|)^{p-2} (\partial_i \partial_s y)^2 \mathrm{d}x \\
&\qquad + (p-2) \int_D (\eta + |\nabla y|)^{p-3} \frac{(\partial_s \partial_l y)(\partial_l y)}{|\nabla y|} (\partial_s \partial_i y)(\partial_i y) \mathrm{d}x \\
&\quad \geq (p-1) \int_D (\eta + |\nabla y|)^{p-2} (\partial_i \partial_s y)^2 \mathrm{d}x,
\end{aligned}
\tag{5.5}
$$

where in the last step we used (5.2), with $C_1 = p - 1$, and the derivatives with respect to D_l are evaluated at the point $\mathrm{D} = \nabla y$. Here $\partial \mathrm{D}_l / \partial x_s = \partial_s \partial_l y$ and $\partial \mathrm{D}_i / \partial x_s = \partial_s \partial_i y$. For further details see, for example, previous works[10,16,17]

Lemma 5.1. *Let $T>0$. If the initial data (ρ_0, y_0) are such that $\rho_0 \in L_\phi(D)$ and $y_0 \in H^1$ then there exists a positive constant C, such that for any $0<t<T$ the following inequality holds true, that is,*

$$\|\nabla y(t)\|^2 + \int_D (\rho \ln \rho - \rho + 1)\,\mathrm{d}x + 2\int_0^t \|\Delta y\|^2 \mathrm{d}x + 2(p-1)\int_0^t \int_D (\eta + |\nabla y|)^{p-2}(\partial_i \partial_s y)^2 \mathrm{d}x \leq \exp(CT). \tag{5.6}$$

Proof. Multiplying equation $(5.1)_1$ by Δy, in L^2, and using relation (5.5), we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla y\|^2 + \|\Delta y\|^2 + (p-1)\int_D (\eta + |\nabla y|)^{p-2}(\partial_i \partial_s y)^2 \mathrm{d}x + \int_D \nabla \rho \cdot \nabla y \,\mathrm{d}x \\ = -\int_D (\bar{\mathbf{v}} \cdot \nabla) y \Delta y \mathrm{d}x \\ \leq \varepsilon \|\Delta y\|^2 + C_\varepsilon \|\nabla y\|^2.\end{aligned}$$

Then, multiplying equation $(5.1)_2$ by $\ln \rho$, in L^2, we get

$$\frac{d}{dt}\int_D (\rho \ln \rho - \rho + 1)\mathrm{d}x - \int_D \nabla \rho \cdot \nabla y \,\mathrm{d}x = 0,$$

and summing them up, and integrating in time, we get

$$\begin{aligned}\|\nabla y(t)\|^2 + 2\int_D (\rho \ln \rho - \rho + 1)\,\mathrm{d}x + 2(1-\varepsilon)\int_0^t \|\Delta y\|^2 \mathrm{d}s \\ + 2\int_0^t \int_D (\eta + |\nabla y|)^{p-2}(\partial_i \partial_s y)^2 \mathrm{d}x\mathrm{d}s \\ \leq C\int_0^t \left(\|\nabla y\|^2 + 2\int_D (\rho \ln \rho - \rho + 1)\mathrm{d}x \right) \mathrm{d}s,\end{aligned}$$

and the conclusion follows by an application of Gronwall's lemma and selecting $\varepsilon = 1/2$. □

Remark 5.2. Let us multiply $(5.1)_1$ by y_t and integrate on D to get

$$\|y_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla y\|^2 - \int_D \left(\mathrm{div}(\eta + |\nabla y|)^{p-2}\nabla y\right) y_t \mathrm{d}x = \int_D (1-\rho) y_t \mathrm{d}x - \int_D (\bar{\mathbf{v}} \cdot \nabla) y y_t \mathrm{d}x. \tag{5.7}$$

Now, we have that

$$\begin{aligned}-\int_D \left(\mathrm{div}(\eta + |\nabla y|)^{p-2}\nabla y\right) y_t \mathrm{d}x = -\left(\mathrm{div}(\eta + |\nabla y|)^{p-2}\nabla y, y_t\right) \\ = \left((\eta + |\nabla y|)^{p-2}\nabla y, \nabla y_t\right) = \frac{d}{dt}\mathcal{L}(y),\end{aligned}$$

where $\mathcal{L}(y) := \int_D L(|\nabla y|)\mathrm{d}x \geq 0$, with

$$L(t) := \int_0^t (\eta + s)^{p-2} s \,ds \geq 0, \text{ for } t \geq 0.$$

Moreover, we have that $L(t) \simeq (\eta + t)^{p-2}t^2$ and also that (see, e.g., Berselli and Bisconti[10])

$$(\eta + t)^{p-2}t^2 \leq t^p, \text{ with } 1 \leq p \leq 2.$$

In particular, this shows that

$$L(y) \leq C_p \|\nabla y\|_p^p, \text{ with } 1 \leq p \leq 2.$$

Then, using (5.7) along with the above relations, we reach

$$\|y_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla y\|^2 + \frac{d}{dt}\mathcal{L}(y) = \int_D (1-\rho) y_t \mathrm{d}x - \int_D (\bar{\mathbf{v}} \cdot \nabla) y y_t \mathrm{d}x.$$

that, however, cannot be closed at this level due to the lack of a direct control on $\|\rho\|$. Therefore, to control $\|y_t\|$ we resort to higher order estimates.

5.2 | Estimates in higher-order norms

Let us start with the following result

Lemma 5.2. *Let $T > 0$ and let $\rho_0 \in L^3(D)$. Then, there exists a positive constant C such that*

$$\|\rho(t)\|_3^3 + \int_0^t \|\rho(s)\|_4^4 \mathrm{d}s \leq C\left(\|\rho_0\|_3^3 + \int_0^t \left(\|y_t(s)\|_4^4 + \|\nabla y(s)\|_4^4\right) \mathrm{d}s + \int_0^t \int_D |divS(y)|^4 \mathrm{d}x\mathrm{d}s\right) + CT. \tag{5.8}$$

Proof. Multiply equation $(5.1)_2$ against $3\rho^2$ to get

$$\partial_t \rho^3 + 3\rho^2 \mathrm{div}\,(\rho \nabla y) = 0,$$

that can be rewritten as

$$\begin{aligned} \partial_t \rho^3 + \mathrm{div}\left(\rho^3 \nabla y\right) &= -2\rho^3 \Delta y \\ &= 2\rho^3\left(1 - \rho - y_t + \mathrm{div}\left((\eta + |\nabla y|)^{p-2}\nabla y\right) - (\bar{\mathbf{v}} \cdot \nabla) y\right), \end{aligned} \tag{5.9}$$

where in the last step we used $(5.1)_1$. By Young's inequality, we have that

$$\begin{aligned} \rho^3|y_t| &\leq \varepsilon\rho^4 + C_\varepsilon |y_t|^4, \\ \rho^3|(\bar{\mathbf{v}} \cdot \nabla) y| &\leq \varepsilon\rho^4 + C_\varepsilon |\nabla y|^4, \end{aligned}$$

and also that

$$\rho^3|\mathrm{div}S(\nabla y)| \leq \varepsilon\rho^4 + C_\varepsilon |\mathrm{div}S(y)|^4.$$

Integrating (5.9) over D, and using the above controls, we obtain

$$\begin{aligned} \frac{d}{dt}\|\rho\|_3^3 + (2 - 3\varepsilon)\|\rho\|_4^4 &\leq 2\|\rho\|_3^3 + C\left(\|y_t\|_4^4 + \|\nabla y\|_4^4 + \int_D |\mathrm{div}S(y)|^4 \mathrm{d}x\right) \\ &\leq \varepsilon\|\rho\|_4^4 + C\left(1 + \|y_t\|_4^4 + \|\nabla y\|_4^4 + \int_D |\mathrm{div}S(y)|^4 \mathrm{d}x\right), \end{aligned}$$

and relation (5.8) follows directly. □

Lemma 5.3. *Let $T > 0$ and let $\rho_0 \in L^3(D)$. Then, for η large enough, there exists a positive constant $C = C(\eta, T)$ such that the following control holds true*

$$\int_0^t \|\Delta y(s)\|_4^4 \mathrm{d}s + \int_0^t \int_D (\eta + |\nabla y|)^{p-2} |\Delta y|^4 \mathrm{d}x\mathrm{d}s \le C\left(\|\rho_0\|_3^3 + \int_0^t \left(\|y_t\|_4^4 + \|\nabla y\|_4^4 \right) \mathrm{d}s + T \right). \tag{5.10}$$

Proof. Multiply $(5.1)_1$ by $-|\Delta y|^2 \Delta y$, and integrate on D, to get

$$\begin{aligned}
&\|\Delta y\|_4^4 + \int_D (\eta + |\nabla y|)^{p-2} |\Delta y|^4 \mathrm{d}x \\
&\le (2-p) \left| \int_D (\eta + |\nabla y|)^{p-3} \frac{(\partial_i \partial_k y)(\partial_k y)}{|\nabla y|} \partial_i y |\Delta y|^2 \Delta y \mathrm{d}x \right| + \int_D [|y_t| + (1+\rho)] \, |\Delta y|^3 \mathrm{d}x \\
&\le \underbrace{(2-p) \int_D (\eta + |\nabla y|)^{p-3} | \frac{|\partial_i \partial_k y| \, |\partial_k y|}{|\nabla y|} |\partial_i y| |\Delta y|^3 \mathrm{d}x}_{=:\Theta_1} + C\left(1 + \|\rho\|_4^4 + \|y_t\|_4^4\right) + \varepsilon \|\Delta y\|_4^4,
\end{aligned}$$

where, in the last step, we exploited Hölder's and Young's inequalities.

Therefore, integrating in time on $(0, t)$, $t < T$, and using (5.8) to control $\int_0^t \|\rho(s)\|_4^4 \mathrm{d}s$, we infer

$$\begin{aligned}
&(1-\varepsilon) \int_0^t \|\Delta y\|_4^4 \mathrm{d}s + \int_0^t \int_D (\eta + |\nabla y|)^{p-2} |\Delta y|^4 \mathrm{d}x\mathrm{d}s \\
&\le \int_0^t \Theta_1(s) \mathrm{d}s + C\|\rho_0\|_3^3 + C \int_0^t \left(\|y_t\|_4^4 + \|\nabla y\|_4^4 + \int_D |\mathrm{div} S(\nabla y)|^4 \mathrm{d}x \right) \mathrm{d}s + CT.
\end{aligned} \tag{5.11}$$

In particular, for the term Θ, we have that

$$\begin{aligned}
\Theta_1 &= (2-p) \int_D (\eta + |\nabla y|)^{p-2} \frac{|\partial_i \partial_k y| \, |\partial_k y|}{|\nabla y|} \frac{|\partial_i y|}{(\eta + |\nabla y|)} |\Delta y|^3 \mathrm{d}x \\
&\le (2-p) \int_D \left[(\eta + |\nabla y|)^{\frac{3(p-2)}{4}} |\Delta y|^3 \right] \left[(\eta + |\nabla y|)^{\frac{p-2}{4}} |\partial_i \partial_k y| \right] \mathrm{d}x \\
&\le \varepsilon \int_D (\eta + |\nabla y|)^{p-2} |\Delta y|^4 \mathrm{d}x + C_{\varepsilon,p} \int_D (\eta + |\nabla y|)^{p-2} |\partial_i \partial_s y|^4 \mathrm{d}x \qquad (\text{for } \varepsilon = 1/2) \\
&= \frac{1}{2} \int_D (\eta + |\nabla y|)^{p-2} |\Delta y|^4 \mathrm{d}x + \frac{27(2-p)^4}{32\eta^{2-p}} \|\partial_i \partial_k y\|_4^4 \\
&\le \frac{1}{2} \int_D (\eta + |\nabla y|)^{p-2} |\Delta y|^4 \mathrm{d}x + C \frac{27(2-p)^4}{32\eta^{2-p}} \left(\|y\|_4^4 + \|\Delta y\|_4^4 \right) \\
&\le \frac{1}{2} \int_D (\eta + |\nabla y|)^{p-2} |\Delta y|^4 \mathrm{d}x + C \frac{27(2-p)^4}{32\eta^{2-p}} \|\Delta y\|_4^4 + C\|y\|^2 \|\nabla y\|^2,
\end{aligned}$$

where we used Hölder's, Young's and Ladyzhenskaya's inequalities. Hence, we reach

$$\int_0^t \Theta_1(s) \mathrm{d}s \le \frac{1}{2} \int_0^t \int_D (\eta + |\nabla y|)^{p-2} |\Delta y|^4 \mathrm{d}x\mathrm{d}s + C \frac{(2-p)^4}{\eta^{2-p}} \int_0^t \|\Delta y\|_4^4 \mathrm{d}s + \hat{C} T, \tag{5.12}$$

where we exploited Poincaré's inequality along with (5.6), and $\hat{C} = \hat{C}(T)$ is a suitable positive constant. The first two terms in the right-hand side of (5.12) can be easily reabsorbed on the left-hand side of (5.11), provided that η is taken sufficiently large.

Similarly, the worst terms coming from $\int_D |\mathrm{div} S(\nabla y)|^4 \mathrm{d}x$ can be reabsorbed on the left-hand side of (5.11) provided that η is large enough. In fact, we have that

$$\begin{aligned}
\int_0^t \int_D |\mathrm{div} S(\nabla y)|^4 \mathrm{d}x\mathrm{d}s &\leq \int_0^t \int_D \frac{|\Delta y|^4}{(\eta + |\nabla y|)^{4(2-p)}} \mathrm{d}x\mathrm{d}s \\
&\quad + (2-p)^4 \int_0^t \int_D \left((\eta + |\nabla y|)^{p-2} \frac{\partial_i \partial_k y \partial_k y}{|\nabla y|} \frac{\partial_i y}{(\eta + |\nabla y|)} \right)^4 \mathrm{d}x\mathrm{d}s \\
&\leq \frac{1}{\eta^{4(2-p)}} \int_0^t \left(\|\Delta y\|_4^4 + (2-p)^4 \|\partial_i \partial_k y\|_4^4 \right) \mathrm{d}s \\
&\leq \frac{C}{\eta^{4(2-p)}} \int_0^t \left(\|\Delta y\|_4^4 + (2-p)^4 (\|y\|_4^4 + \|\Delta y\|_4^4) \right) \mathrm{d}s \\
&\leq \frac{C}{\eta^{4(2-p)}} \int_0^t \left(\|\Delta y\|_4^4 + (2-p)^4 (\|y\|^2 \|\nabla y\|^2 + \|\Delta y\|_4^4) \right) \mathrm{d}s \\
&\leq \frac{c}{\eta^{4(2-p)}} \int_0^t \|\Delta y\|_4^4 \mathrm{d}s + CT,
\end{aligned}$$

where $c = c(p)$, and in the last two steps we used Ladyzhenskaya's inequality and (5.6). As a consequence of the above control used along with (5.11) and (5.12) (here we set $C := max\{\hat{C}, C\}$), we have that relation (5.10) follows directly. □

Differentiating $(5.1)_1$ with respect to $x = (x_1, x_2)$, and with respect to t, we get, respectively, the following controls

$$\nabla y_t - \nabla \Delta y = \nabla \left(\mathrm{div} \left((\eta + |\nabla y|)^{p-2} \nabla y \right) \right) - \nabla (\bar{\mathbf{v}} \cdot \nabla) y - \nabla \rho, \tag{5.13}$$

and

$$y_{tt} - \Delta y_t - \partial_t \left(\mathrm{div} \left((\eta + |\nabla y|)^{p-2} \nabla y \right) \right) + (\bar{\mathbf{v}} \cdot \nabla) y_t = -\rho_t = \mathrm{div}(\rho \nabla y). \tag{5.14}$$

Now, multiplying (5.14) by y_t in L^2, we reach

$$\frac{1}{2} \frac{d}{dt} \|y_t\|^2 + \|\nabla y_t\|^2 + \int_D \partial_t \left((\eta + |\nabla y|)^{p-2} \nabla y \right) \cdot \nabla y_t \mathrm{d}x + \int_D (\bar{\mathbf{v}} \cdot \nabla) y_t \cdot y_t \mathrm{d}x = -\int_D \rho \nabla y \cdot \nabla y_t \mathrm{d}x,$$

and so

$$\begin{aligned}
&\frac{d}{dt} \|y_t\|^2 + 2\|\nabla y_t\|^2 + 2 \int_D \partial_t \left((\eta + |\nabla y|)^{p-2} \nabla y \right) \cdot \nabla y_t \mathrm{d}x + \int_D (\bar{\mathbf{v}} \cdot \nabla) |y_t|^2 \mathrm{d}x \\
&\quad = -2 \int_D \left[\Delta y + \mathrm{div} \left((\eta + |\nabla y|)^{p-2} \nabla y \right) - (\bar{\mathbf{v}} \cdot \nabla) y - y_t + 1 \right] \nabla y \cdot \nabla y_t \mathrm{d}x.
\end{aligned} \tag{5.15}$$

Observe that

$$\begin{aligned}
\int_D \partial_t \left((\eta + |\nabla y|)^{p-2} \nabla y \right) \cdot \nabla y_t \mathrm{d}x &= \int_D \partial_t \left((\eta + |\nabla y|)^{p-2} \partial_i y \right) \partial_i y_t) \mathrm{d}x \\
&\quad \left(= \int_D \frac{\partial}{\partial \mathrm{D}_k} \left((\eta + |\mathrm{D}|)^{p-2} \mathrm{D}_i \right) \frac{\partial \mathrm{D}_k}{\partial t} \frac{\partial \mathrm{D}_i}{\partial t} \mathrm{d}x, \text{ with } \mathrm{D} = \nabla y \right) \\
&\quad \geq (p-1) \int_D (\eta + |\nabla y|)^{p-2} |\nabla y_t|^2 \mathrm{d}x,
\end{aligned}$$

where we used (5.2), with $C_1 = p - 1$, and the derivatives with respect to $\mathrm{D}_k = \partial_k y$ are evaluated at the point $\mathrm{D} = \nabla y$, and $\partial \mathrm{D}_i / \partial t = \partial_i y_t$ as well as $\partial \mathrm{D}_k / \partial t = \partial_k y_t$.

Therefore, using the above relation along with (5.13), we obtain

$$\begin{aligned}&\frac{d}{dt}\|y_t\|^2+2\|\nabla y_t\|^2+2(p-1)\int_D(\eta+|\nabla y|)^{p-2}|\nabla y_t|^2\mathrm{d}x+\int_D(\bar{\mathbf{v}}\cdot\nabla)|y_t|^2\mathrm{d}x\\&\quad\le -2\int_D\left[\Delta y+\mathrm{div}\left((\eta+|\nabla y|)^{p-2}\nabla y\right)-(\bar{\mathbf{v}}\cdot\nabla)y-y_t+1\right]\nabla y\cdot\nabla y_t\mathrm{d}x.\end{aligned}\tag{5.16}$$

Multiplying (5.13) by $q|\nabla y|^{q-2}\nabla y$, in L^2, integrating by parts, and substituting $\rho-\mathrm{div}\left((\eta+|\nabla y|)^{p-2}\nabla y\right)+(\bar{\mathbf{v}}\cdot\nabla)y=1+\Delta y-y_t$ in such a relation, we infer

$$\begin{aligned}&\frac{d}{dt}\|\nabla y\|_q^q+q\int_D(\partial_j\partial_k y)^2|\nabla y|^{q-2}\mathrm{d}x\\&\quad+q(q-2)\int_D(\partial_j\partial_k y)(\partial_k y)(\partial_j\partial_l y)(\partial_l y)|\nabla y|^{q-4}\mathrm{d}x\\&\quad=q\int_D\left(\rho-\mathrm{div}\left((\eta+|\nabla y|)^{p-2}\nabla y\right)+(\bar{\mathbf{v}}\cdot\nabla)y\right)\mathrm{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x\\&\quad=q\int_D(1+\Delta y-y_t)\,\mathrm{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x.\end{aligned}\tag{5.17}$$

Summing up (5.16) and (5.17), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|y_t\|^2+\|\nabla y\|_q^q\right)+2(p-1)\int_D(\eta+|\nabla y|)^{p-2}|\nabla y_t|^2\mathrm{d}x\\&\quad+q\int_D(\partial_j\partial_k y)^2|\nabla y|^{q-2}\mathrm{d}x+2\|\nabla y_t\|^2\\&\le q\int_D(1+\Delta y-y_t)\mathrm{div}\left(|\nabla y|^{q-2}\nabla y\right)\mathrm{d}x+\int_D\left|(\bar{\mathbf{v}}\cdot\nabla)|y_t|^2\right|\mathrm{d}x\\&\quad+q(q-2)\int_D|(\partial_j\partial_k y)(\partial_k y)(\partial_j\partial_l y)(\partial_l y)|\nabla y|^{q-4}|\mathrm{d}x\\&\quad+2\left|\int_D\left(\Delta y+\mathrm{div}\left((\eta+|\nabla y|)^{p-2}\nabla y\right)-(\bar{\mathbf{v}}\cdot\nabla)y-y_t+1\right)\nabla y\cdot\nabla y_t\mathrm{d}x\right|.\end{aligned}\tag{5.18}$$

We are now ready to proceed with the proof of the following theorem.

Theorem 5.1. *Let $T>0$. If $\rho_0\in L^3(D)$, $y_0\in H^2$, then there exists a constant C depending on T such that the inequalities*

$$\sup_{0<s<t}\left(\|\nabla y(s)\|_4^4+\|y_t(s)\|^2\right)+\int_0^t\|\nabla y_t(s)\|^2\mathrm{d}s\le CT,\tag{5.19}$$

and

$$\int_0^t\|\rho(s)\|_4^4\mathrm{d}t+\int_0^t\|\Delta y(s)\|_4^4\mathrm{d}s\le CT.\tag{5.20}$$

hold true for any $t\in[0,T]$.

Proof. By taking $q = 4$ in (5.18), making derivatives explicit and rearranging the terms in such a relation, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla y\|_4^4 + \|y_t\|^2\right) + (p-1)\int_D (\eta + |\nabla y|)^{p-2}|\nabla y_t|^2 \mathrm{d}x \\
&+ 4\int_D (\partial_j \partial_k y)^2 |\nabla y|^2 \mathrm{d}x + 2\int_D |\nabla y_t|^2 \mathrm{d}x \\
&\leq \sum_{i=1}^{10} I_i + 2\varepsilon \|\nabla y_t\|^2 + C_\varepsilon \left(\|y_t\|^2 + \|\nabla y\|^2 \|\Delta y\|^2\right) \\
&+ \left| \int_D \mathrm{div}\left((\eta + |\nabla y|)^{p-2}\nabla y\right) \nabla y \cdot \nabla y_t \mathrm{d}x \right| .
\end{aligned} \tag{5.21}$$

where the terms I_i, $i = 1, \dots, 10$, are the same we have already considered in Theorem 4.1, and we already used (4.25) and (4.26) for the integral terms I_{13} and I_{14} to get $2\varepsilon\|\nabla y_t\|^2$, $\varepsilon > 0$ (see the right-hand side of (4.22)). Thus, the only new term to be estimated is the last one, and we have that

$$\begin{aligned}
&\int_D \mathrm{div}\left((\eta + |\nabla y|)^{p-2}\nabla y\right) \nabla y \cdot \nabla y_t \mathrm{d}x \\
&\leq \|\nabla y_t\| \left(\int_D \left(\mathrm{div}\left((\eta + |\nabla y|)^{p-2}\nabla y\right)^2 |\nabla y|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \varepsilon \|\nabla y_t\|^2 + C_\varepsilon \int_D (\eta + |\nabla y|)^{2(p-2)} |\Delta y|^2 |\nabla y|^2 \mathrm{d}x \\
&+ C_\varepsilon \int_D (\eta + |\nabla y|)^{2(2-p)} \left(\frac{\partial_i \partial_k y \partial_k y}{|\nabla y|} \frac{\partial_i y}{(\eta + |\nabla y|)} \right)^2 |\nabla y|^2 \mathrm{d}x \\
&\leq \varepsilon \|\nabla y_t\|^2 + C_\varepsilon \int_D |\Delta y|^2 \frac{|\nabla y|^2}{(\eta + |\nabla y|)^{2(2-p)}} \mathrm{d}x + C_\varepsilon \int_D |\nabla^2 y|^2 \frac{|\nabla y|^2}{(\eta + |\nabla y|)^{2(2-p)}} \mathrm{d}x \\
&\leq \varepsilon \|\nabla y_t\|^2 + \frac{C_\varepsilon}{\eta^{2(p-2)}} \left(\int_D |\Delta y|^2 |\nabla y|^2 \mathrm{d}x + \int_D |\nabla^2 y|^2 |\nabla y|^2 \mathrm{d}x \right) \\
&\leq \varepsilon \|\nabla y_t\|^2 + \varepsilon \|\Delta y\|_4^4 + C_{\varepsilon,\eta,p} \left(\|\nabla y\|_4^4 + 1\right),
\end{aligned}$$

Then, estimate (5.21) can be closed exactly as in the proof of Theorem 4.1, by using (5.10) and an application of Gronwall's lemma. □

5.3 | Approximating solutions and passage to the limit

Also, in this case we use a Galerkin scheme $\{(y_k, \rho_k)\}$ for approximating. Thanks to the previously obtained a priori estimates, we can use the Aubin–Lions compactness lemma (see, e.g., Temam[15]) to extract a convergent subsequence $\{(y_k, \rho_k)\}$. The passage to the limit is standard and the only point to pay attention with is the one related to the convergence of the extra nonlinear term. Since $S(\nabla y_k)$ is bounded uniformly, with respect to k, in $L^2(Q_T)$, it follows that $S(\nabla y_k) \rightharpoonup \mathcal{A}$ in $L^2(Q_T)$ for some $\mathcal{A}$. We have now to show that $\mathcal{A} = S(\nabla y)$, where (y, ρ) is the limiting pair, up to a subsequence, for the approximating sequence $\{(y_k, \rho_k)\}$. This is obtained by an application of the monotonicity trick (see, e.g., Lions, 18 Sections 2–5.2]; see also Berselli and Bisconti[10]).

Testing $(5.1)_1$—written in terms of (y, ρ)—against y in L^2, we obtain the following energy inequality, for $0 \leq t_0 < t$, that is,

$$\frac{1}{2}\|y(t)\|^2 + \int_{t_0}^t \|\nabla y\|^2 \mathrm{d}s = -\int_{t_0}^t (\mathcal{A}, \nabla y)\,\mathrm{d}s + \frac{1}{2}\|y(t_0)\|^2 - \int_{t_0}^t ((\bar{\mathbf{v}} \cdot \nabla) y, y)\,\mathrm{d}s + \int_{t_0}^t (1 - \rho, y)\mathrm{d}s. \tag{5.22}$$

Let us now return to the Galerkin scheme. Take $t_0 = 0$. Defining for $\xi \in L^2(0,T;H^1)$, which is a test having the same regularity of y_k, the following quantity

$$\Xi_t^k := \int_0^t (S(\nabla y_k) - S(\nabla \xi), \nabla(y_k - \xi))\, \mathrm{d}s + \int_0^t \|\nabla y_k\|^2 \mathrm{d}s + \frac{1}{2}\|y_k(t)\|^2, \tag{5.23}$$

it follows, by using (5.3) and by semi-continuity of the norm, that

$$\liminf_{k\to\infty} \Xi_t^k \geq \frac{1}{2}\|y(t)\|^2 + \int_0^t \|\nabla y\|^2 \mathrm{d}s. \tag{5.24}$$

Now, using (5.23) along with (5.22), and recalling that $\rho_k \rightharpoonup \rho$ and $y_k \to y$ in $L^2(Q_T)$ (actually we also have that $y_k \to y$ in $L^2(0,T;H^1)$), we infer

$$\begin{aligned}\lim_{k\to\infty} \Xi_t^k = \int_0^t (1-\rho, y)\mathrm{d}s - \int_{t_0}^t ((\bar{\mathbf{v}}\cdot\nabla)y, y)\, \mathrm{d}s + \frac{1}{2}\|y_0\|^2 \\ - \int_0^t (\mathcal{A}, \nabla\xi)\, d\tau - \int_0^t (S(\nabla\xi), \nabla(y-\xi))\, \mathrm{d}s.\end{aligned} \tag{5.25}$$

In particular, in passing to the limit in the first term, we used the triangle inequality $|\int_0^t (\rho_k, y_k)\mathrm{d}s - \int_0^t (\rho, y)\mathrm{d}s| \leq |\int_0^t (\rho_k, y - y_k)\mathrm{d}s| + |\int_0^t (\rho_k - \rho, y)\mathrm{d}s|$ along with the strong convergence of y_k and the weak convergence of ρ_k, in $L^2(Q_T)$. Hence, by using (5.22) along with (5.24) and (5.25) (i.e., subtracting (5.22) from (5.25)), we get

$$\int_0^t (\mathcal{A} - S(\nabla\xi), \nabla y - \nabla\xi)\, ds \geq 0, \;\; a.e.\, s \in [0,T].$$

Choosing $\xi = y + \varpi\chi$ for some smooth χ and letting $\varpi \to 0$, we can conclude that $\mathcal{A} = S(\nabla y)$.

Remark 5.3. In this case, due to the form of $S(\nabla y) = (\eta + |\nabla y|)^{p-2}\nabla y$, does not seem possible to reproduce—in an elementary way—the calculations in the proof of Lemma 4.6, which are used to provide upper and lower bounds for ρ. In order to retrive such bounds, we resort to higher order estimates assuming more regularity on the initial data. A sketch of these additional calculations is provided in Appendix A.

As a consequence of the above results, Theorem 3.1 follows directly.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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APPENDIX A

Here we prove the results stated in Subsection 4.3.

Proof of Lemma 4.6. Assuming that $\rho(t,x)>0,\ (t,x)\in Q_T$ and $T<T_0$, and arguing as in Galdi,[3] we can rewrite $(4.1)_2$ in the form

$$\partial_t \ln\rho + \nabla y\cdot\nabla\ln\rho + \Delta y = 0. \tag{A.1}$$

Then, by adding (A.1) to $(4.1)_1$, we get

$$\underbrace{\partial_t(y+\ln\rho) + (\nabla y\cdot\nabla)(y+\ln\rho)}_{=\frac{d}{dt}(y+\ln\rho)} + \rho = 1 + \operatorname{div}F(y) - (\bar{\mathbf{v}}\cdot\nabla)y + \nabla y\cdot\nabla y. \tag{A.2}$$

Now, set

$$\gamma := y + \ln\rho, \text{ and } \gamma_+(t,x) := \max\{0,\gamma(t,x)\}.$$

Considering (A.2) as the transport equation for γ, we can conclude that

$$\begin{aligned}\gamma_+(t,x) &\le \|\gamma_+|_{t=0}\|_\infty + \int_0^t \left(1+\|\nabla y\|_\infty^2 + \|F'(y)\|_\infty\|\nabla y\|_\infty\right)\mathrm{d}s\\ &\le \|\gamma_+|_{t=0}\|_\infty + C\int_0^t \left(1+\|\nabla y\|_\infty^2 + \|y\|_\infty^{2m}\right)\mathrm{d}s,\end{aligned} \tag{A.3}$$

where, as usual, we have that $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(D)}$. As a consequence of (4.20), we have that

$$\|y\|_{L^\infty(Q_T)} \le CT \sup_{0<t<T} \|\nabla y\|_4 \le C(T).$$

We also have that

$$\int_0^T \|\nabla y(s)\|_\infty^2 \mathrm{d}s \le C \int_0^T \left(\|\nabla y\|^{\frac{1}{3}} \|\Delta y\|_4^{\frac{2}{3}} \right)^2 \mathrm{d}s$$
$$\le C \int_0^T \left(\|\nabla y\| + \|\Delta y\|_4^4 \right) \le C(T),$$

where we used (2.4), (4.20) and Young's inequality.

Then, (4.39) follows directly from (A.3) along with the last two inequalities above. In particular the constant M in (4.39) is given by

$$M := \exp\left\{ \|y\|_{L^\infty(Q_T)} + \|\gamma_+|_{t=0}\|_{L^\infty(D)} + C(T) \right\}. \qquad \square$$

Then, we have the following

Proof of Lemma 4.7. Let us change the sign in Equation (4.39) and rewrite it for $-\gamma$. We wish to find an upper bound for the function $\gamma_- = \max\{0, -\gamma\}$. By analogy, we obtain

$$\gamma_-(t,x) \le \|\gamma_-|_{t=0}\|_\infty + C(T) \int_0^T \left(\|\nabla y\|_\infty^2 + \|\rho\|_\infty + 1 \right) \mathrm{d}s.$$

Hence, (4.40) follows with the constant

$$m = \exp\left\{ - \left(\|y\|_{L^\infty(Q_T)} + \|\gamma_-|_{t=0}\|_{L^\infty(D)} + \|\nabla y\|_\infty^2 + 1)\mathrm{d}s + MT \right) \right\}.$$

$\square$

Case of $\hat{F}(\nabla y) = S(\nabla y)$: Upper and lower bounds for ρ by using estimates for higher order derivatives

In order to use an approach similar to the one just exploited to bound the density ρ in (5.1), we require more regularity on the initial data to get improved solutions. Also, the parameter $\eta > 0$ is taken as large as needed.

We ssume that y_t, $\mathrm{D}^3 y$, and $\nabla\rho$, are sufficiently regular (here $\mathrm{D}^3 y = \partial_i \partial_j \partial_k y$, $i, j, k = 1, 2$). Starting from $(5.1)_1$, i.e.

$$\nabla y_t - \nabla \Delta y = \nabla \left(\mathrm{div} \left((\eta + |\nabla y|)^{p-2} \nabla y \right) \right) - \nabla(\bar{\mathbf{v}} \cdot \nabla) y - \nabla \rho,$$

for $1 < q < \infty$, by using the regularity theory for parabolic equations (see previous works[3,19]) we infer the estimate

$$\|\nabla y_t\|_{L^q(Q_T)}^q + \|\mathrm{D}^3 y\|_{L^q(Q_T)}^q \le C \Big(\|\mathrm{D}^3 y_0\|_q^q + \|(\bar{\mathbf{v}} \cdot \nabla) y\|_{L^q(Q_T)}^q + \|\nabla\rho\|_{L^q(Q_T)}^q$$
$$+ \|\nabla \left(\mathrm{div} \left((\eta + |\nabla y|)^{p-2} \nabla y \right) \right)\|_{L^2(Q_T)}^q \Big)$$
$$\le C \Big(\|\mathrm{D}^3 y_0\|_q^q + \|\mathrm{D}^2 y\|_{L^q(Q_T)}^q + \|\nabla\rho\|_{L^q(Q_T)}^q$$
$$+ C_q \int_0^T \|\nabla \left((\eta + |\nabla y|)^{p-2} \mathrm{D}^2 y \right)\|_q^q \mathrm{d}s \Big)$$
$$\le C \Big(1 + \|\mathrm{D}^3 y_0\|_q^q + \varepsilon \|\mathrm{D}^3 y\|_{L^q(Q_T)}^q + \|\nabla\rho\|_{L^q(Q_T)}^q$$
$$+ C_{p,q,\eta} \int_0^T \|(\eta + |\nabla y|)^{p-2} \mathrm{D}^3 y\|_q^q \mathrm{d}s \Big)$$

and so

$$\|\nabla y_t\|_{L^q(Q_T)}^q + \|\mathrm{D}^3 y\|_{L^q(Q_T)}^q \le C \left(1 + \|\mathrm{D}^3 y_0\|_q^q + \varepsilon \|\mathrm{D}^3 y\|_{L^q(Q_T)}^q + \|\nabla\rho\|_{L^q(Q_T)}^q + C_{p,q,\eta} \|\mathrm{D}^3 y\|_{L^q(Q_T)}^q \right), \qquad \text{(A.4)}$$

where $y_0 \in W^{1,q}(D)$, and we used Gagliardo-Nirenberg's inequality to estimate $\|D^2y\|^q_{L^q(Q_T)}$, where $D^2y = \partial_i\partial_j y$, $i = 1, 2$. For the last addendum on the right-hand side of (A.4), which is considered up to lower order terms, the constant $C_{p,q,\eta} = C(p, q, \eta)$ can be assumed as small as needed (i.e., $C_{p,q,\eta} < \epsilon$) by taking η sufficiently large. As a consequence, we have that

$$\|\nabla y_t\|^q_{L^q(Q_T)} + (1 - 2\epsilon)\|D^3 y\|^q_{L^q(Q_T)} \leq C\left(1 + \|D^3 y_0\|^q_q + \|\nabla \rho\|^q_{L^q(Q_T)}\right). \tag{A.5}$$

Let us now differentiate the continuity equation $(5.1)_2$ in order to get

$$\nabla \rho_t + (\nabla y \cdot \nabla)\nabla \rho + (\nabla \rho \cdot \nabla)\nabla y + \Delta y \nabla \rho + \rho \nabla \Delta y = 0, \tag{A.6}$$

and assume $\rho_0 \in W^{1,r}(D),\ r > 2$. Multiplying (A.6) against $r|\nabla \rho|^{r-2}\nabla \rho$, in $L^2(D)$, and proceeding as in [3, Section 5], we infer

$$\frac{d}{dt}\|\nabla \rho(t)\|^r_r \leq c\left(\|\Delta y(s)\|^r_\infty \|\nabla \rho(s)\|^r_r + \|\nabla \Delta y(s)\|^r_r\right). \tag{A.7}$$

Now, assuming that $2 < r = q \leq 4$ in (A.5) and (A.7), and using the embedding $W^{3,q}(D) \hookrightarrow W^{2,\infty}(D)$, we reach

$$\zeta_t \leq c(\zeta + \zeta^2),$$

where $\zeta(t) = 1 + \|\nabla \rho(t)\|^q_q$, and $c = c(\|\nabla \rho_0\|_q)$. Therefore, we get

$$\|\nabla \rho(t)\|^q_q \leq \frac{(1 + \|\nabla \rho_0\|^q_q)e^{ct}}{1 + \|\nabla \rho_0\|^q_q(1 - e^{ct})}, \tag{A.8}$$

for $0 < t < \tilde{T}$, where $\tilde{T}(\|\nabla \rho_0\|_q)$ is a suitable time depending on $\|\nabla \rho_0\|_q$, for which the above relation holds true.

Using (A.8) along with (A.5) we obtain, in particular, the boundedness of $\|D^2y(t)\|_\infty$ in $L^q(0, T)$, $0 < T < \tilde{T}$, which is enough to estimate the integral term

$$\int_0^t \|\mathrm{div}\hat{F}(\nabla y(s))\|_\infty \mathrm{d}s = \int_0^t \|\mathrm{div}\left((\eta + |\nabla y(s)|)^{p-2}\nabla y(s)\right)\|_\infty \mathrm{d}s$$
$$\leq (p-1)\int_0^t \|\left((\eta + |\nabla y(s)|)^{p-2}\mathrm{D}^2 y(s)\|_\infty \mathrm{d}s$$
$$\leq C_{p,\eta}\int_0^t \|\mathrm{D}^2 y(s)\|_\infty \mathrm{d}s,$$

for $0 < t < T$, $T < \tilde{T}$. Hence, we can reproduce the same calculations in (A.3) and extend Lemma 4.6 and Lemma 4.7 to the considered case, for $0 < t < T$, and $T < \tilde{T}$.