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Interpolating estimates with applications to some quantitative symmetry results†

Rolando Magnanini[1] **and Giorgio Poggesi**[2,*]

[1] Dipartimento di Matematica ed Informatica "U. Dini", Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy

[2] Department of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley, Perth, WA 6009, Australia

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* **Correspondence:** Email: giorgio.poggesi@uwa.edu.au.

Abstract: We prove interpolating estimates providing a bound for the oscillation of a function in terms of two L^p norms of its gradient. They are based on a pointwise bound of a function on cones in terms of the Riesz potential of its gradient. The estimates hold for a general class of domains, including, e.g., Lipschitz domains. All the constants involved can be explicitly computed. As an application, we show how to use these estimates to obtain stability for Alexandrov's Soap Bubble Theorem and Serrin's overdetermined boundary value problem. The new approach results in several novelties and benefits for these problems.

Keywords: interpolating estimates; Serrin's overdetermined problem; Alexandrov's Soap Bubble Theorem; constant mean curvature; stability; quantitative estimates

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For $1 \le p \le \infty$ the number $\|f\|_{p,\Omega}$, will denote the L^p-norm of a measurable function $f : \Omega \to \mathbb{R}$ with respect to the normalized Lebesgue measure $d\mu_x = dx/|\Omega|$.

In Theorems 2.4 and 2.7, for $N < q \le \infty$, we prove the following interpolating inequalities, which hold true for any $f \in W^{1,q}(\Omega)$:

$$\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f \le c \begin{cases} \|\nabla f\|_{p,\Omega} & \text{for } p > N \\ \|\nabla f\|_{N,\Omega} \log(e\,\|\nabla f\|_{q,\Omega}/\|\nabla f\|_{N,\Omega}), & \text{for } p = N, \\ \|\nabla f\|_{p,\Omega}^{\alpha_{p,q}} \|\nabla f\|_{q,\Omega}^{1-\alpha_{p,q}}, & \text{for } 1 \le p < N. \end{cases} \tag{1.1}$$

Here,

$$\alpha_{p,q} = \frac{p\,(q-N)}{N\,(q-p)} \text{ for } N < q < \infty, \quad \alpha_{p,\infty} = \frac{p}{N}.$$

Notice that simply combining the Morrey-Sobolev embedding $W^{1,r} \hookrightarrow C^{0,1-N/r}$ for $r > N$ and the classical interpolation of L^p spaces (i.e., Hölder's inequality) for $p < r < q$ is not sufficient to obtain (1.1). In fact, we would find that

$$\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f \le c\,\|\nabla f\|_{r,\Omega} \le c\,\|\nabla f\|_{p,\Omega}^{\alpha_{p,q,r}} \|\nabla f\|_{q,\Omega}^{1-\alpha_{p,q,r}}, \tag{1.2}$$

where

$$\alpha_{p,q,r} = \frac{q-r}{r(q-p)}.$$

Now, as $r \to N^+$, we see that $\alpha_{p,q,r}$ tends to the exponent $\alpha_{p,q}$ appearing in (1.1). However, in this limit, the first inequality in (1.2) fails to be true, as one can see by taking $f(x) = \log\log(1 + 1/|x|)$ in the unit ball B in $\mathbb{R}^N$ for $N \ge 2$. In fact, f belongs to $W^{1,N}(B)$, but has infinite oscillation. Nevertheless, (1.1) still holds in the relevant case.

In order to prove (1.1), a different approach is needed. The one we present here has also the advantage to give a unified treatment for all the cases of $p \in [1, \infty]$. These not only include the inequalities for $1 \le p \le N$ and the noteworthy logarithmic profile in the threshold case $p = N$, but also the classical case $p > N$, which may be directly deduced from the classical Morrey-Sobolev embedding. In addition, our proof clearly unveils the dependence of the constant c in (1.1) on the relevant geometric parameters in hand.

Our proof holds when Ω is a bounded domain satisfying a uniform interior cone condition (see Section 2.2 for the definition). Notice that some regularity of Ω (or, alternatively, some information on the boundary traces) is needed for the validity of (1.1), as a simple counterexample shows. In fact, if in the planar domain

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1,\ |x_2| < x_1^r\}, \quad \text{for } r > 1,$$

we consider the function $f(x) = x_1^{-1/q}$, we see that $f \in W^{1,q}(\Omega)$ if $q < r$. Nevertheless, the oscillation (and the L^∞-norm) of f is infinite. Thus, for any $q < \infty$, we can find r such that (1.1) fails on Ω.

The proof of (1.1) is given in Section 2 and is based on a pointwise bound on cones for f in terms of the Riesz potential of its gradient (see Lemma 2.1). When $1 \le p \le N$, (1.1) is obtained by combining that bound with an interpolation procedure performed on cones (see Lemma 2.5).

As an application of our inequalities, we shall use them to give an alternative way to obtain, and even improve, certain estimates proved by the authors in [10, Theorems 2.10 and 2.8]. These have been a crucial ingredient to obtain the stability of the spherical configuration for Alexandrov's Soap Bubble Theorem (SBT), Serrin's and other related overdetermined problems (see, e.g., [3, 4, 8–10, 13–15]).

More precisely, (1.1) can be used as a substitute of [15, Lemma 3.14] when aiming to obtain those stability results in the spirit of [10, Theorems 2.10 and 2.8]. We shall detail in Section 3 how this agenda can be carried out. We emphasize that, while [15, Lemma 3.14] can only be proved for sub-harmonic functions (see also [11] for a version for sub-solutions of elliptic equations in divergence form), our new bounds do not need this requirement. Thanks to this feature, they can also be useful in different and more general contexts. More on this will be clarified in forthcoming research. See also the recent paper [16].

In the remainder of this introduction, for the case of the SBT, we briefly describe the main steps of the argument that motivates the application of our interpolating inequalities. Alexandrov's SBT states that a closed surface Γ, embedded in $\mathbb{R}^N$, and that has constant *mean curvature* H must be a sphere. Roughly speaking, by stability of the spherical configuration in this problem, we mean an inequality of the type:

$$\text{measure of closeness to a sphere} \le \Psi(\|H - H_0\|).$$

Here, Ψ is a non-negative continuous function vanishing at 0 and $\|H - H_0\|$ is the deviation of H from a reference constant H_0, in a suitable norm. In the literature, there are many different ways to quantify the deviations of H from H_0 and of a surface from a sphere (we refer the reader to the works [7, 10, 15] for a quite exhaustive list of references). It is clear that the weaker the norm $\|H-H_0\|$ is and the stronger the distance of Γ from a sphere is, the better the estimate is. On the other hand, in such a weak-strong setting, it may be difficult to obtain for Ψ the most desirable linear profile: $\Psi(\sigma) = c\,\sigma$. Here, c is some constant depending on some geometric parametes of the surface, easy to compute if possible. When this occurs, the optimality can be proved by considering sequences of ellipsoids.

In the works [8–10], we assume Γ to be the boundary of a bounded domain Ω, we set H_0 to be the ratio $|\Gamma|/N|\Omega|$, and we adopt an $L^2(\Gamma)$ (or even $L^1(\Gamma)$) deviation of H from H_0. Also, we measure the distance of Γ from a sphere, by the quantity $\rho_e - \rho_i$, where ρ_i and ρ_e, $\rho_i \le \rho_e$, are the radii of the best spherical annulus containing Γ. This will be given by $B_{\rho_e}(z) \setminus \overline{B_{\rho_i}(z)}$ for some $z \in \Omega$. In other words, we obtained a bound of this type:

$$\rho_e - \rho_i \le c\,\Psi\left(\|H - H_0\|_{L^2(\Gamma)}\right).$$

In this setting, in [10] we obtained a linear profile for Ψ in low dimension ($N = 2, 3$) and a Hölder profile with exponent $2/(N-2)$, for $N \ge 5$. For the threshold case $N = 4$, we got, in a sense, a profile "arbitrarily close to a linear one" (see Remark 3.10 or [10], for details).

In Section 3 of this paper, for surfaces of class C^2, we show that the interpolating bounds obtained in Section 2 help to improve the profile for $N \ge 4$. In fact, in Theorem 3.9, for $N = 4$ we improve the older estimate (that was $\Psi(\sigma) = c_\varepsilon\,\sigma^{1-\varepsilon}$, for any fixed $\varepsilon > 0$) to a sharper and more plausible one: $\Psi(\sigma) = c\,\sigma \log(1/\sigma)$. Moreover, when $N \ge 5$, we are able to upgrade the profile $\Psi(\sigma) = c\,\sigma^{2/(N-2)}$ to $\Psi(\sigma) = c_\varepsilon\,\sigma^{4/N-\varepsilon}$, for any fixed $\varepsilon > 0$. This profile can be further improved to $\Psi(\sigma) = c\,\sigma^{4/N}$, if we consider surfaces of class $C^{2,\gamma}$, $1 < \gamma \le 1$. For $N = 2, 3$, we just show that the new bounds in (1.1) provide an alternative way to recover the optimal profile previously obtained in [9, 10].

Another novelty of this paper is that we show that our new improvements also hold if we enforce the quantity $\rho_e - \rho_i$ by replacing it with the stronger deviation:

$$\rho_e - \rho_i + R\,\left\|\nu - \frac{\nabla Q^z}{R}\right\|_{2,\Gamma}$$

Here, $R = 1/H_0$, ν is the exterior unit normal vector to Γ, and Q^z is defined by

$$Q^z(x) = \frac{|x-z|^2}{2} \text{ for } x, z \in \mathbb{R}^N. \tag{1.3}$$

(Also in this case, the relevant norm is defined in the the corresponding normalized measure $dS_x/|\Gamma|$.)

Thus, the smallness of this new measure of closeness to a sphere tells us not only that Γ is uniformly close to a sphere, but also that the Gauss map of Γ is quantitatively close *in the average* to that of the same sphere. Therefore, all in all, in Theorem 3.9, we enhance the last up-to-date bounds of [10] for the stability of the SBT as follows:

$$\rho_e - \rho_i + R \left\| \nu - \frac{\nabla Q^z}{R} \right\|_{2,\Gamma} \le c\, \Psi\left(\|H - H_0\|_{L^2(\Gamma)} \right), \tag{1.4}$$

where

$$\Psi(\sigma) = \begin{cases} \sigma & \text{if } N = 2, 3, \\ \sigma \max\left[\log(1/\sigma), 1\right] & \text{if } N = 4, \\ \sigma^\tau & \text{if } N \ge 5, \end{cases} \tag{1.5}$$

where $\tau = 4/N$ if Γ is of class $C^{2,\gamma}$, $1 < \gamma \le 1$. If Γ is of class C^2, instead, when $N \ge 5$, we obtain that, for any sufficiently small $\varepsilon > 0$, there exists a constant $c = c_\varepsilon$ such that (1.4) and (1.5) hold with $\tau = 4/N - \varepsilon$. The constant c only depends on N, the diameter d_Ω of Ω, and parameters associated with the assumed regularity of Γ. If Γ is of class C^2, these are the radii r_i and r_e of the uniform interior and exterior ball condition (see Section 3). If Γ is of class $C^{2,\gamma}$, c depends on a suitable modulus of $C^{2,\gamma}$-continuity for Γ. For details, see Theorem 3.9 and Remark 3.8. We stress that, for the second summand on the left-hand side of (1.4), we can actually obtain an optimal linear profile of stability, in every dimension (see (3.10)).

We spend a few final words to explain how the bounds derived in Section 2 come into play to obtain (1.4). To this aim, we let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be the solution of the problem:

$$\Delta u = N \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma.$$

Also, we define the harmonic function $h = u - Q^z$. Notice that, if $z \in \Omega$, then we have that

$$\frac{1}{2}\left(\frac{|\Omega|}{|B|}\right)^{1/N} (\rho_e - \rho_i) \le \frac{1}{2}\,(\rho_e^2 - \rho_i^2) = \max_\Gamma h - \min_\Gamma h.$$

Here, B is a unit ball in $\mathbb{R}^N$. Thus, a bound for the term $\rho_e - \rho_i$ in (1.4) descends from the following identity

$$\frac{1}{N-1} \int_\Omega |\nabla^2 h|^2 dx + \frac{1}{R} \int_\Gamma (u_\nu - R)^2 dS_x = \int_\Gamma (H_0 - H)\,(u_\nu)^2 dS_x, \tag{1.6}$$

which was proved in [8]. In fact, since the right-hand side can be easily bounded in terms of the $L^2(\Gamma)$ norm of $H - H_0$, then the desired bound for $\rho_e - \rho_i$ can be obtained if we can control the oscillation of h on Γ in terms of the first summand in (1.6). This goal is achieved by combining the bounds (1.1) (applied to h and its gradient) with some Poincaré-type inequality.

The second summand on the left-hand side of (1.4) can instead be estimated by observing that

$$|R\,\nu - \nabla Q^z| \le |R\,\nu - u_\nu\,\nu| + |\nabla u - \nabla Q^z| = |R - u_\nu| + |\nabla h| \ \text{ on } \ \Gamma.$$

The two quantities on the rightest-hand side can be estimated in $L^2(\Gamma)$-norm by means of (1.6) and, again, by some inequalities derived in [10]. These involve a trace-type formula,

$$\int_\Gamma |\nabla h|^2 dS_x \le c \int_\Omega (-u)\,|\nabla^2 h|^2\, dx,$$

and another identity (stated in [8] and proved in [9]):

$$\int_\Omega (-u)\,|\nabla^2 h|^2\, dx = \frac{1}{2}\int_\Gamma (u_\nu^2 - R^2)\, h_\nu\, dS_x. \tag{1.7}$$

This last identity, immediately gives radial symmetry for Ω in Serrin's overdetermined problem (that prescribes that u_ν is constant on Γ, see [17,18]). Together with the arguments used to obtain (1.4), (1.7) will also help us to upgrade an analogous stability bound for radial symmetry in Serrin's problem. This task will be accomplished in Theorem 4.4.

2. Interpolating estimates for Sobolev functions

Let $\mathbb{S}^{N-1}$ be the unit sphere in the Euclidean space $\mathbb{R}^N$, $N \ge 2$. For $\theta \in [0, \pi/2]$ and $e \in \mathbb{S}^{N-1}$, we set

$$\mathcal{S}_\theta = \{\omega \in \mathbb{S}^{N-1} : \cos\theta < \langle \omega, e\rangle\}.$$

This is a spherical cap with axis e and opening width θ. We also denote by

$$C_{x,a} = \{x + a\omega : \omega \in \mathcal{S}_\theta,\ 0 < s < a\},$$

the finite right spherical cone with vertex at x, axis in some direction e, and height $a > 0$. In what follows, $|C_{x,a}|$ and $|\mathcal{S}_\theta|$ will denote indifferently the N-dimensional Lebesgue measure of $C_{x,a}$ and the $(N-1)$-dimensional surface measure of $\mathcal{S}_\theta$.

2.1. Pointwise estimates on cones

We start by proving some useful pointwise estimates in cones (see also [1]). In what follows, we set $C_x = C_{x,a}$ and use the normalized Lebesgue measure $d\mu_y = dy/|E|$ for any measurable set $E \subset \mathbb{R}^N$ of finite measure.

Lemma 2.1. *For any $f \in C^1(\overline{C}_x)$, it holds that*

$$\left|f(x) - f_{C_x}\right| \le \int_{C_x} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\,\frac{a^N - |y-x|^N}{N}\, d\mu_y. \tag{2.1}$$

In particular, we have that

$$\left|f(x) - f_{C_x}\right| \le \frac{a^N}{N}\int_{C_x} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, d\mu_y. \tag{2.2}$$

Proof. By the change of variables $y = x + s\,\omega$ for $s \in (0, a)$ and $\omega \in \mathcal{S}_\theta$, we write:

$$f(x) - f_{C_x} = f(x) - \int_{C_x} f\,d\mu_y = \frac{1}{|C_x|}\int_{C_x}[f(x) - f(y)]\,dy =$$
$$\frac{1}{|C_x|}\int_0^a s^{N-1}\left\{\int_{\mathcal{S}_\theta}[f(x) - f(x + s\,\omega)]\,dS_\omega\right\}ds,$$

where dS_ω denotes the surface element on $\mathbb{S}^{N-1}$. Next, the fundamental theorem of calculus gives:

$$f(x) - f(x + s\,\omega) = -\int_0^s \omega \cdot \nabla f(x + t\,\omega)\,dt.$$

Thus, we can infer that

$$\left|f(x) - f_{C_x}\right| \le \frac{1}{|C_x|}\int_0^a \int_{\mathcal{S}_\theta} s^{N-1}\left[\int_0^s |\nabla f(x + t\,\omega)|\,dt\right] dS_\omega\,ds =$$
$$\frac{1}{|C_x|}\int_0^a s^{N-1}\int_{\mathcal{S}_\theta}\left[\int_0^s \frac{|\nabla f(x + t\,\omega)|}{|x + t\,\omega - x|^{N-1}}\,t^{N-1}\,dt\right] dS_\omega\,ds =$$
$$\frac{1}{|C_x|}\int_0^a s^{N-1}\left[\int_{C_{x,s}} \frac{|\nabla f(y)|}{|y - x|^{N-1}}\,dy\right] ds.$$

Now, by an application of Fubini's theorem we obtain that

$$\int_0^a\left(\int_{C_{x,s}} \frac{|\nabla f(y)|}{|y - x|^{N-1}}\,dy\right) ds = \int_{C_x} \frac{|\nabla f(y)|}{|y - x|^{N-1}}\,\frac{a^N - |y - x|^N}{N}\,dy.$$

Thus, (2.1) and (2.2) easily follow. □

As a corollary, we have the following *Morrey-Sobolev-type* inequality. The relevant Lebesgue norms are defined with respect to the normalized measure $d\mu_y$.

Corollary 2.2. *If $N < p \le \infty$ and $f \in C^1(\overline{C}_x)$, we have that*

$$\left|f(x) - f_{C_x}\right| \le \frac{a}{N}\,\beta\left(1 - \frac{p'}{N'}, p' + 1\right)^{1/p'} \|\nabla f\|_{p,C_x}, \tag{2.3}$$

where $\beta(\xi, \eta)$ denotes Euler's beta function. *When $p = \infty$,* (2.3) *reads as*

$$\left|f(x) - f_{C_x}\right| \le \frac{a}{N}\,\beta\left(\frac{1}{N}, 2\right)\|\nabla f\|_{\infty,C_x},$$

which can be obtained by taking the limit as $p \to \infty$ in (2.3).

Proof. The desired result follows from (2.1) by applying Hölder's inequality to the right-hand side and the calculation:

$$\int_{C_x}\left(\frac{a^N - |y - x|^N}{|y - x|^{N-1}}\right)^{p'} d\mu_y = \frac{|\mathcal{S}_\theta|}{|C_x|}\int_0^r \left(\frac{a^N - s^N}{s^{N-1}}\right)^{p'} s^{N-1}ds =$$
$$a^{p'}\int_0^1 (1 - t)^{p'} t^{-\frac{N-1}{N}p'}dt = a^{p'}\beta\left(1 - \frac{N-1}{N}\,p', p' + 1\right).$$

Since $\beta(\xi, \eta)$ is well-defined only if $\xi, \eta > 0$, we get the restriction $p > N$. As already mentioned, the case $p = \infty$ can be derived by taking the limit as $p \to \infty$. □

2.2. Global estimates for the oscillation of functions

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain (i.e., a connected bounded open set) with boundary Γ. Given $a > 0$ and $\theta \in [0, \pi/2]$, we say that Ω satisfies the (θ, a)-uniform interior cone condition, if for every $x \in \overline{\Omega}$ there exists a cone C_x with opening width θ and height a, such that $C_x \subset \Omega$ and $\overline{C}_x \cap \Gamma = \{x\}$, whenever $x \in \Gamma$. The following result easily follows from Corollary 2.2.

Corollary 2.3. *Let $N < p \leq \infty$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain that satsfies the uniform interior (θ, a)-cone property. For every $x \in \overline{\Omega}$ and $f \in W^{1,p}(\Omega)$, we have that*

$$|f(x) - f_\Omega| \leq k(N, p, \theta)\, a^{1-N/p}\, |\Omega|^{1/p} \|\nabla f\|_{p,\Omega},$$

for some constant $k(N, p, \theta)$ only depending on N, p, and θ.

Proof. For any $x \in \overline{\Omega}$, there is a cone C_x contained in Ω. Hence, we apply (2.3) to the function $f - f_\Omega + f_{C_x}$ and infer that

$$\begin{aligned} |f(x) - f_\Omega| \leq \frac{a}{N} \beta\left(1 - \frac{p'}{N'}, p' + 1\right)^{1/p'} \|\nabla f\|_{p,C_x} \leq \\ \frac{a}{N} \beta\left(1 - \frac{p'}{N'}, p' + 1\right)^{1/p'} \left(\frac{|\Omega|}{|C_x|}\right)^{1/p} \|\nabla f\|_{p,\Omega} \leq \\ \frac{\beta\left(1 - \frac{p'}{N'}, p' + 1\right)^{1/p'}}{N^{1/p'}\, |\mathcal{S}_\theta|^{1/p}}\, a^{1-N/p}\, |\Omega|^{1/p} \|\nabla f\|_{p,\Omega}. \end{aligned}$$

In the second inequality, we use the monotonicity of Lebesgue's integral with respect to set inclusion. □

In this section, we aim to derive inequalities that bound from above the oscillation on $\overline{\Omega}$ of a function f with the L^p-norm of its gradient on Ω.

Theorem 2.4 (The case $p > N$)**.** *Set $N < p \leq \infty$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain satisfying the (θ, a)-uniform interior cone condition.*

There exists a constant $k(N, p, \theta)$ only depending on N, p, and θ such that, for any $f \in W^{1,p}(\Omega)$, it holds that

$$\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f \leq k(N, p, \theta)\, a^{1-N/p}\, |\Omega|^{1/p}\, \|\nabla f\|_{p,\Omega}. \tag{2.4}$$

Proof. Notice that the oscillation of f at the left-hand side of (2.4) is well defined, since f is continuous on $\overline{\Omega}$.

Let $x_m, x_M \in \overline{\Omega}$ be points at which f attains its minimum and maximum. Then, we have that

$$\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f \leq f(x_M) - f_\Omega + f_\Omega - f(x_m)$$

and we conclude by applying twice Corollary 2.3. □

It is clear that the proof of Corollary 2.2 fails when $1 \leq p \leq N$, because of the singularity at x. However, in this case, we can still obtain a slightly different estimate by means of an interpolation procedure, if information on higher integrability of the gradient of f is available.

Lemma 2.5. *Let $f \in C^1(\overline{C}_x)$. Let $1 \le p \le N$, $N < q \le \infty$, and set*

$$\alpha_{p,q} = \frac{p\,(q-N)}{N\,(q-p)}. \tag{2.5}$$

(i) If $1 \le p < N$, we have that

$$a^{N-1} \int_{C_x} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, d\mu_y \le k_{N,p,q}\, \|\nabla f\|_{q,C_x}^{1-\alpha_{p,q}}\, \|\nabla f\|_{p,C_x}^{\alpha_{p,q}}, \tag{2.6}$$

for some positive constant $k_{N,p,q}$ only depending on N, p, and q.

(ii) If $p = N$. we have that

$$a^{N-1} \int_{C_x} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, d\mu_y \le \frac{2q}{q-N}\, \|\nabla f\|_{N,C_x} \log\left(\frac{e\, \|\nabla f\|_{q,C_x}}{\|\nabla f\|_{N,C_x}}\right). \tag{2.7}$$

Proof. For any $\sigma \in (0,a)$, we compute that

$$\begin{aligned}\int_{C_x} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, dy = \int_{C_{x,\sigma}} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, dy + \int_{C_x \setminus C_{x,\sigma}} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, dy \le \\ \left[\int_{C_{x,\sigma}} \frac{dy}{|y-x|^{q'(N-1)}}\right]^{1/q'} \left(\int_{C_{x,\sigma}} |\nabla f(y)|^q dy\right)^{1/q} + \\ \left[\int_{C_x \setminus C_{x,\sigma}} \frac{dy}{|y-x|^{p'(N-1)}}\right]^{1/p'} \left(\int_{C_x \setminus C_{x,\sigma}} |\nabla f|^p dy\right)^{1/p},\end{aligned} \tag{2.8}$$

by Hölder's inequality. Now, a direct computation shows that

$$\begin{aligned}&\left[\int_{C_{x,\sigma}} \frac{dy}{|y-x|^{q'(N-1)}}\right]^{1/q'} = \left[\frac{q-1}{q-N}\, |\mathcal{S}_\theta|\right]^{1/q'} \sigma^{\frac{q-N}{q}}, \\ &\left[\int_{C_x \setminus C_{x,\sigma}} \frac{dy}{|y-x|^{p'(N-1)}}\right]^{1/p'} = \begin{cases} \left[\frac{p-1}{N-p} |\mathcal{S}_\theta| \left(\sigma^{-\frac{N-p}{p-1}} - a^{-\frac{N-p}{p-1}}\right)\right]^{1/p'} & \text{if } 1 \le p < N, \\ \left[|\mathcal{S}_\theta|\, \log \frac{a}{\sigma}\right]^{1/N'} & \text{if } p = N. \end{cases}\end{aligned} \tag{2.9}$$

For $p = 1$, this formula must be intended in the limit as $p \to 1$.

(i) Let $1 \le p < N$. By (2.8), (2.9), and some algebraic manipulations, we can infer that

$$\begin{aligned}a^{N-1} \int_{C_x} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, d\mu_y \le \left[\frac{N\,(q-1)}{q-N}\right]^{1-1/q} \|\nabla f\|_{q,C_x} \left(\frac{\sigma}{a}\right)^{1-N/q} + \\ \left[\frac{N\,(p-1)}{N-p}\right]^{1-1/p} \|\nabla f\|_{p,C_x} \left(\frac{\sigma}{a}\right)^{1-N/p}\end{aligned} \tag{2.10}$$

for any $\sigma \in (0,a]$. The minimum of the right-hand side is attained either at

$$\overline{\sigma} = a \left[\frac{N(p-1)}{N-p}\right]^{\frac{q(p-1)}{N(q-p)}} \left[\frac{q-N}{N(q-1)}\right]^{\frac{p(q-1)}{N(q-p)}} \left(\frac{1-\alpha_{p,q}}{\alpha_{p,q}}\, \frac{\|\nabla f\|_{p,C_x}}{\|\nabla f\|_{q,C_x}}\right)^{\frac{pq}{N(q-p)}},$$

or at $\sigma = a$. In the former case, we plug $\overline{\sigma}$ into (2.10) and obtain (2.6) with some computable constant k'. In the latter case, we have that

$$\left[\frac{N(p-1)}{N-p}\right]^{\frac{q(p-1)}{N(q-p)}}\left[\frac{q-N}{N(q-1)}\right]^{\frac{p(q-1)}{N(q-p)}}\left(\frac{1-\alpha_{p,q}}{\alpha_{p,q}}\,\frac{\|\nabla f\|_{p,C_x}}{\|\nabla f\|_{q,C_x}}\right)^{\frac{pq}{N(q-p)}} > 1,$$

since $\overline{\sigma} > a$. Hence, by means of this inequality and the fact that we have that

$$\int_{C_x}\frac{|\nabla f(y)|}{|y-x|^{N-1}}d\mu_y \le \frac{1}{a^{N-1}}\left[\frac{N(q-1)}{q-N}\right]^{1-1/q}\|\nabla f\|_{q,C_x}\left(\frac{\sigma}{a}\right)^{1-N/q},$$

thanks to (2.8), we again obtain (2.6) for some possibly different computable constant k''. Thus, we conclude that (2.6) holds true with $k_{N,p,q} = \max(k', k'')$.

(ii) Let $p = N$. We proceed as in the case (i) by putting together (2.8) and (2.9). After some calculation, we obtain:

$$a^{N-1}\int_{C_x}\frac{|\nabla f(y)|}{|y-x|^{N-1}}\,d\mu_y \le \left[\frac{N(q-1)}{q-N}\right]^{1-1/q}\|\nabla f\|_{q,C_x}\left(\frac{\sigma}{a}\right)^{1-N/q} + \left(N\,\log\frac{a}{\sigma}\right)^{1-1/N}\|\nabla f\|_{N,C_x}.$$

If we assume that $0 < \sigma < a/e$, being as $N(q-1) \ge q-N$, we can simplify this inequality to get that

$$\frac{a^{N-1}}{N}\int_{C_x}\frac{|\nabla f(y)|}{|y-x|^{N-1}}\,d\mu_y \le \frac{q-1}{q-N}\|\nabla f\|_{q,C_x}\left(\frac{\sigma}{a}\right)^{1-N/q} + \|\nabla f\|_{N,C_x}\log\frac{a}{\sigma}. \tag{2.11}$$

Thus, the minimum of the right-hand side is attained either at

$$\sigma = \overline{\sigma} = a\left[\frac{q'\,\|\nabla f\|_{N,C_x}}{\|\nabla f\|_{q,C_x}}\right]^{\frac{q}{q-N}} \quad \text{or at } \sigma = a/e.$$

In the former case, we get that

$$\frac{a^{N-1}}{N}\int_{C_x}\frac{|\nabla f(y)|}{|y-x|^{N-1}}\,d\mu_y \le \frac{2q}{q-N}\|\nabla f\|_{N,C_x}\log\left(\frac{e\,\|\nabla f\|_{q,C_x}}{q'\,\|\nabla f\|_{N,C_x}}\right),$$

and hence (2.7) holds true, being as $q' \ge 1$. In the latter case, we have that

$$e^{-1} \le \left[\frac{q'\,\|\nabla f\|_{N,C_x}}{\|\nabla f\|_{q,C_x}}\right]^{\frac{q}{q-N}},$$

since $\overline{\sigma} \ge a/e$. Thus, we get that

$$\frac{a^{N-1}}{N}\int_{C_x}\frac{|\nabla f(y)|}{|y-x|^{N-1}}\,d\mu_y \le \frac{q-1}{q-N}\|\nabla f\|_{q,C_x}e^{N/q-1} + \|\nabla f\|_{N,C_x} \le$$
$$\frac{2q-N}{q-N}\|\nabla f\|_{N,C_x} \le \frac{2q-N}{q-N}\|\nabla f\|_{N,C_x}\,\log\left(\frac{e\,\|\nabla f\|_{q,C_x}}{\|\nabla f\|_{N,C_x}}\right),$$

being as $\|\nabla f\|_{N,C_x} \le \|\nabla f\|_{q,C_x}$. Since $2q-N \le 2q$, (2.7) still holds true. □

We obtain the following consequence.

Corollary 2.6. *For any cone $C_x \subset \Omega$ of height a and opening width θ, it holds that*

$$\int_{C_x} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, d\mu_y \le k(N,p,q,\theta)\, \frac{|\Omega|}{a^{2N-1}}\, \|\nabla f\|_{q,\Omega}^{1-\alpha_{p,q}}\, \|\nabla f\|_{p,\Omega}^{\alpha_{p,q}},$$

for $1 \le p < N$ and, if $p = N$,

$$\int_{C_x} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, d\mu_y \le k(N,p,q,\theta)\, \frac{|\Omega|^{1/N}}{a^{N}}\, \|\nabla f\|_{N,\Omega} \log\left(\frac{e\, \|\nabla f\|_{q,\Omega}}{\|\nabla f\|_{N,\Omega}}\right),$$

for some constant $k(N,p,q,\theta)$ only depending on $N, p, q,$ and θ.

Proof. The monotonicity of Lebesgue's measure with respect to set inclusion and (2.11) easily give:

$$\frac{a^{N-1}}{N} \int_{C_x} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, d\mu_y \le \frac{q-1}{q-N} \left(\frac{|\Omega|}{|C_x|}\right)^{1/q} \|\nabla f\|_{q,\Omega} \left(\frac{\sigma}{a}\right)^{1-N/q} + \left(\frac{|\Omega|}{|C_x|}\right)^{1/N} \|\nabla f\|_{N,\Omega} \log\frac{a}{\sigma}.$$

By using that $(|\Omega|/|C_x|)^{1/q-1/N} \le 1$ (being as $C_x \subset \Omega$ and $q > N$), the above inequality becomes:

$$\frac{a^{N-1}}{N} \int_{C_x} \frac{|\nabla f(y)|}{|y-x|^{N-1}}\, d\mu_y \le \left(\frac{|\Omega|}{|C_x|}\right)^{1/N} \left\{\frac{q-1}{q-N} \|\nabla f\|_{q,\Omega} \left(\frac{\sigma}{a}\right)^{1-N/q} + \|\nabla f\|_{N,\Omega} \log\frac{a}{\sigma}\right\}.$$

Thus, we can proceed as in the last part of the proof of Lemma 2.5, with similar algebraic manipulations. □

In light of Corollary 2.6, we can somewhat extend the bound (2.4) to the case $1 \le p \le N$, provided $f \in W^{1,q}(\Omega)$ for $q > N$. The proof is straightforward and runs as that of Theorem 2.4.

Theorem 2.7. *Let $1 \le p \le N$, $N < q \le \infty$, and set $\alpha_{p,q}$ as in* (2.5). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain satisfying the (θ, a)-uniform interior cone condition.*

For any $f \in W^{1,q}(\Omega)$, it holds that

$$\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f \le k(N,p,q,\theta)\, \frac{|\Omega|^{1/p}}{a^{N/p-1}}\, \|\nabla f\|_{p,\Omega}^{\alpha_{p,q}} \|\nabla f\|_{q,\Omega}^{1-\alpha_{p,q}},$$

if $1 \le p < N$ and, if $p = N$,

$$\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f \le k(N,p,q,\theta)\, \frac{|\Omega|^{1/N}}{a^{N}}\, \|\nabla f\|_{N,\Omega} \log\left(\frac{e\, \|\nabla f\|_{q,\Omega}}{\|\nabla f\|_{N,\Omega}}\right).$$

Here, $k(N,p,q,\theta)$ is some constant only depending on N, p, q, θ.

3. Application to quantitative symmetry for the Soap Bubble Theorem

As already mentioned, Theorems 2.4 and 2.7 give an alternative way to obtain, and even upgrade, the bounds in [10, Theorems 2.10 and 2.8]. As a by-product, we also obtain new upgraded versions of stability estimates for the Soap Bubble Theorem and Serrin's symmetry result. In this and the next section, we shall give some details on how to obtain the new versions of those stability results. Of course, a similar reasoning can be applied to other stability results contained in [4, 8, 10, 15].

3.1. Preliminary notations and useful bounds

For a point $z \in \Omega$, ρ_i and ρ_e shall denote the radius of the largest ball contained in Ω and that of the smallest ball that contains Ω, both centered at z; in formulas,

$$\rho_i = \min_{x \in \Gamma} |x - z| \text{ and } \rho_e = \max_{x \in \Gamma} |x - z|. \tag{3.1}$$

We say that Ω satisfies a *uniform interior sphere condition* (with radius r) if for every $p \in \Gamma$ there exists a ball $B_r \subset \Omega$ such that $\partial B_r \cap \Gamma = \{p\}$; Ω satisfies a *uniform exterior sphere condition* if $\mathbb{R}^N \setminus \overline{\Omega}$ satisfies a uniform interior sphere condition. From now on, we will consider a bounded domain Ω with boundary Γ of class C^2, so that Ω satisfies both a uniform interior and exterior sphere condition. We shall denote by r_i and r_e the relevant respective radii. It is trivial to check that when Ω satisfies the interior condition with radius r_i, then it satisfies the uniform interior (θ, a)-cone condition with

$$\theta = \frac{\sqrt{2}}{2}, \quad a = r_i. \tag{3.2}$$

Next, we consider the solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ of

$$\Delta u = N \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma. \tag{3.3}$$

It is well-known that $u \in C^{m,\gamma}(\overline{\Omega})$ if Γ is of class $C^{m,\gamma}$, $0 < \gamma \leq 1$, for $m = 1, 2, \cdots$.

By M we denote a uniform upper bound for the gradient of u on $\overline{\Omega}$, in formulas,

$$M \geq \max_{\overline{\Omega}} |\nabla u| = \max_{\Gamma} u_\nu.$$

As shown in [8, Theorem 3.10], we can choose an explicit value for M:

$$M = (N+1)\, \frac{d_\Omega (d_\Omega + r_e)}{2 r_e}. \tag{3.4}$$

By following [10, 15], we consider the harmonic function

$$h = Q^z - u,$$

where Q^z is defined in (1.3). Notice that, if $z \in \Omega$, it holds that

$$\max_{\Gamma} h - \min_{\Gamma} h = \frac{1}{2}\, (\rho_e^2 - \rho_i^2) \geq \left(\frac{|\Omega|}{|B|} \right)^{1/N} \frac{\rho_e - \rho_i}{2} \geq \frac{r_i}{2}\, (\rho_e - \rho_i). \tag{3.5}$$

The left-hand side of this inequality can be estimated by Theorems 2.4 and 2.7.

As in [10], it will be convenient to choose $z \in \Omega$ as a global minimum point of u. We know from [12] that, in this case, the distance $\delta_\Gamma(z)$ of z to Γ can be estimated from below in terms of the *inradius* r_Ω (the radius of a maximal ball contained in Ω). In fact, in light of [12, Theorem 1.1], it holds that

$$\delta_\Gamma(z) \geq \frac{r_\Omega}{\sqrt{N}}, \tag{3.6}$$

if Γ is mean convex (i.e., $H \geq 0$). If Γ is a general surface of class C^2, [12, Corollary 2.7] gives instead the slightly poorer bound:

$$\delta_\Gamma(z) \geq \frac{r_\Omega}{\sqrt{N}} \left[1 + \frac{N^2 - 1}{2N}\, \frac{d_\Omega}{r_e} \left(1 + \frac{d_\Omega}{r_e} \right) \right]^{-1/2}. \tag{3.7}$$

Remark 3.1 (On the normalized norms)**.** For the sake of consistency with the previous sections, we will continue to denote by $\|\cdot\|_{p,\Omega}$ and $\|\cdot\|_{p,\Gamma}$ the L^p-norms in the relevant normalized measure. Since it holds that

$$|B|\, r_\Omega^N \le |\Omega| \le |B|\, d_\Omega^N \quad \text{and} \quad N\, |B|\, r_\Omega^{N-1} \le |\Gamma| \le N\, \frac{|\Omega|}{r_i},$$

such norms are equivalent to the standard ones. The first three inequalities follow from the inclusions $B_{r_\Omega} \subset \Omega \subset B_{d_\Omega}$. The last inequality is obtained by putting together the identity

$$N|\Omega| = \int_\Gamma u_\nu \, dS_x$$

with the inequality $u_\nu \ge r_i$, which holds true at any point in Γ, by an adaptation of Hopf's lemma (see [8, Theorem 3.10]).

Notice that, since $r_\Omega \ge r_i$, r_Ω can be replaced by r_i in all the relevant formulas.

In what follows, we use the letter c to denote a constant whose value may change line by line. The dependence of c on the relevant parameters will be indicated whenever it is important. All the constants c can be explicitly computed (by following the steps in the relevant proofs) and estimated in terms of the indicated parameters only.

3.2. *Bounds for $\rho_e - \rho_i$ in terms of h*

By applying Theorems 2.4 and 2.7 to h, we easily obtain the starting point of our analysis.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with boundary Γ of class C^2. Let z be a point in Ω, and consider the function $h = Q^z - u$, with Q^z defined in* (1.3).

There exists a constant $c = c(N, p, r_i)$ such that

$$\rho_e - \rho_i \le c \begin{cases} \|\nabla h\|_{p,\Omega} & \text{if } p > N; \\ \|\nabla h\|_{N,\Omega} \log\left(\dfrac{e\, \|\nabla h\|_{\infty,\Omega}}{\|\nabla h\|_{N,\Omega}} \right) & \text{if } p = N; \\ \|\nabla h\|_{\infty,\Omega}^{(N-p)/N} \|\nabla h\|_{p,\Omega}^{p/N} & \text{if } 1 \le p < N. \end{cases}$$

Proof. We apply Theorems 2.4 and 2.7, with $f = h$ and $q = \infty$. By taking into account (3.5) and (3.2), the desired estimates easily follow. (Notice that (3.2) informs us that in Theorems 2.4 and 2.7 we can take $a = r_i$.) □

Remark 3.3 (Weighted Poincaré inequality)**.** Here, we recall a bound for the gradient of h, which we will need in the sequel. Since z is a critical point of h (being as $\nabla h(z) = \nabla Q^z(z) - \nabla u(z) = 0$), we know from [10, Corollary 2.3] that h satisfies the weighted Poincaré inequality

$$\|\nabla h\|_{r,\Omega} \le c\, \|\delta_\Gamma^\alpha \nabla^2 h\|_{p,\Omega}.$$

Here, r, p, α are three numbers such that

$$1 \le p \le r \le \frac{N\, p}{N - p\,(1-\alpha)}, \quad p\,(1-\alpha) < N, \quad 0 \le \alpha \le 1.$$

The constant c can be explicitly estimated by putting together item (iii) of [10, Remark 2.4], (3.7), and the normalizations discussed in Remark 3.1. In detail, we can compute that

$$c \le k_{N,r,p,\alpha} \, |\Omega|^{\frac{1-\alpha}{N}} (d_\Omega/r_i)^N \left[N + (N^2-1) \frac{d_\Omega}{2r_e} \left(1 + \frac{d_\Omega}{r_e} \right) \right]^{N/2},$$

for some constant $k_{N,r,p,\alpha}$ only depending on N, r, p, α. When Γ is mean convex, by using (3.6) in place of (3.7), we obtain the finer bound:

$$c \le k_{N,r,p,\alpha} \, |\Omega|^{\frac{1-\alpha}{N}} (d_\Omega/r_i)^N.$$

As described in the introduction, in order to obtain stability estimates for the Soap Bubble Theorem, we must associate the difference $\rho_e - \rho_i$ with the L^2-norm of the hessian matrix $\nabla^2 h$. The following result gives this association.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with boundary Γ of class C^2. Let $z \in \Omega$ be a global minimum point of u in $\overline{\Omega}$ and set $h = Q^z - u$. Then, there exists a constant $c = c(N, r_i, r_e, d_\Omega)$ such that*

$$\rho_e - \rho_i \le c \begin{cases} \|\nabla^2 h\|_{2,\Omega} & \text{for } N = 2, 3; \\ \|\nabla^2 h\|_{2,\Omega} \max \left[\log \left(\dfrac{e \, \|\nabla h\|_{\infty,\Omega}}{\|\nabla^2 h\|_{2,\Omega}} \right), 1 \right], & \text{for } N = 4; \\ \|\nabla h\|_{\infty,\Omega}^{\frac{N-4}{N-2}} \, \|\nabla^2 h\|_{2,\Omega}^{\frac{2}{N-2}}, & \text{for } N \ge 5. \end{cases}$$

Proof. (i) Lemma 3.2 with $p = 6$ gives that

$$\rho_e - \rho_i \le c \, \|\nabla h\|_{6,\Omega} \le c \, \|\nabla^2 h\|_{2,\Omega}.$$

The last inequality follows from Remark 3.3 with $r = 6$, $p = 3/2$, and $\alpha = 0$, and Hölder's inequality, for $N = 2$, and directly from Remark 3.3 with $r = 6$, $p = 2$, and $\alpha = 0$, for $N = 3$.

(ii) Let $N = 4$. We use Lemma 3.2 with $p = N = 4$ and get:

$$\rho_e - \rho_i \le c \, \max \left\{ \|\nabla^2 h\|_{4,\Omega} \log \left(\frac{e \, \|\nabla h\|_{\infty,\Omega}}{\|\nabla^2 h\|_{4,\Omega}} \right), \|\nabla^2 h\|_{4,\Omega} \right\}.$$

Next, Remark 3.3 with $r = 4$, $p = 2$, $\alpha = 0$, gives:

$$\|\nabla h\|_{4,\Omega} \le c \, \|\nabla^2 h\|_{2,\Omega}.$$

Thus, the desired conclusion ensues by invoking the monotonicity of the function $t \mapsto t \max\{\log(A/t), 1\}$ for every $A > 0$.

(iii) When $N \ge 5$, we can use Lemma 3.2 with $p = 2N/(N-2)$ and put it together with Remark 3.3 with $r = 2N/(N-2)$, $p = 2$, and $\alpha = 0$. □

Remark 3.5. For $N \ge 4$ the estimates of this theorem depend on $\|\nabla h\|_{\infty,\Omega}$. Thus, as done in [10], since we know that

$$\|\nabla h\|_{\infty,\Omega} \le M + d_\Omega,$$

we can easily bound $\rho_e - \rho_i$ in terms of some constant (which possibly depends on r_i, r_e, and d_Ω, thanks to (3.4)) and the number $\|\nabla^2 h\|_{2,\Omega}$. Thanks to identity (1.6), this number is connected to the deviation $H - H_0$. This will lead to the asymptotic profile in the quantitative symmetry estimate for the Soap Bubble Theorem obtained in [10], with an improvement for the case $N = 4$.

However, notice that, when Ω is near a ball in some good topology, the function h tends to be a constant, and hence $\|\nabla h\|_{\infty,\Omega}$ tends to be zero. Thus, we expect to improve the relevant bounds in Theorem 3.4, once we can control $\|\nabla h\|_{\infty,\Omega}$ in terms of $\|\nabla^2 h\|_{2,\Omega}$. This control will in turn benefit the quantitative symmetry estimate we are aiming to. It turns out that an adaptation of our Theorem 2.7 gives such desired bound for $\|\nabla h\|_{\infty,\Omega}$, if an a priori bound for $\|\nabla^2 h\|_{q,\Omega}$ for large q is available, as the following corollary states.

Corollary 3.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class C^2. Let $1 \le p < N$, $N < q \le \infty$, and set $\alpha_{p,q}$ as in* (2.5). *Then, if $h \in W^{2,q}(\Omega)$, it holds that*

$$\|\nabla h\|_{\infty,\Omega} \le c\, \frac{|\Omega|^{1/p}}{r_i^{N/p-1}}\, \|\nabla^2 h\|_{p,\Omega}^{\alpha_{p,q}} \|\nabla^2 h\|_{q,\Omega}^{1-\alpha_{p,q}}.$$

Here, c is a constant only depending on N, p, q.

Proof. Since Γ is of class C^2 , Ω has the uniform interior cone property with $\theta = \sqrt{2}/2$ and $a = r_i$. Let $x \in \overline{\Omega}$ and let ℓ be any unit vector. Applying Theorem 2.7 and using that, with our choice of z, $|h_\ell(x)| = |h_\ell(x) - h_\ell(z)|$, we have that

$$|h_\ell(x)| \le k(N,p,q)\, \frac{|\Omega|^{1/p}}{r_i^{N/p-1}}\, \|\nabla h_\ell\|_{p,\Omega}^{\alpha_{p,q}} \|\nabla h_\ell\|_{q,\Omega}^{1-\alpha_{p,q}} \le k(N,p,q)\, \frac{|\Omega|^{1/p}}{r_i^{N/p-1}}\, \|\nabla^2 h\|_{p,\Omega}^{\alpha_{p,q}} \|\nabla^2 h\|_{q,\Omega}^{1-\alpha_{p,q}},$$

where we used the pointwise inequality $|\nabla h_\ell| \le |\nabla^2 h|$. Hence, taking the supremum over all directions ℓ yields the desired conclusion.

An inspection of the proof tells us that the corollary could be stated for a domain satisfying an interior cone condition. □

This corollary allows us to upgrade Theorem 3.4 for $N \ge 5$. Notice that, for $N = 4$, we would not get any subtantial improvement, due to the presence of the logarithm in the relevant claim of that theorem.

Corollary 3.7. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 5$, be a bounded domain with boundary Γ of class C^2. Let $z \in \Omega$ be a global minimum point of u in $\overline{\Omega}$, set $h = Q^z - u$, and suppose that $h \in W^{2,q}(\Omega)$. Then, for every $q \in (N, \infty]$, there exists a constant $c = c(N, q, r_i, r_e, d_\Omega)$ such that*

$$\rho_e - \rho_i \le c\, \|\nabla^2 h\|_{q,\Omega}^{\frac{q(N-4)}{(q-2)N}} \|\nabla^2 h\|_{2,\Omega}^{\frac{4}{N} - \frac{2(N-4)}{N(q-2)}}.$$

Proof. Our claim simply follows by combining Theorem 3.4 and Corollary 3.6 with the choice $p = 2$. □

3.3. Quantitative symmetry results

We are now in position to obtain our new quantitative estimates of radial symmetry per the Soap Bubble Theorem. As already mentioned, all we have to do is to relate the norm $\|\nabla^2 h\|_{2,\Omega}$ to the deviation of H from H_0 in some norm.

The quantities $\|\nabla h\|_{\infty,\Omega}$ and $\|\nabla^2 h\|_{q,\Omega}$ in Theorem 3.4 and Corollary 3.7 will contribute to the computation of the constant in the desired stability profile, as explained in the next remark.

Remark 3.8. We shall consider two regularity assumptions on Γ.

(i) When Γ is of class C^2, we have that $u \in W^{2,q}(\Omega)$ for any $q \in [1,\infty)$ and an a priori bound for $\|\nabla^2 h\|_{q,\Omega}$ can be obtained, by the standard L^q estimates for elliptic equations, being as $\nabla^2 h = I - \nabla^2 u$. In fact, by putting together [6, Theorems 914 and 9.15], even under the weaker assumption of $\Gamma \in C^{1,1}$, we can obtain for u the bound

$$\|\nabla^2 u\|_{q,\Omega} \le C \ \text{ for } \ N < q < \infty,$$

where C only depends on $N, q, |\Omega|$, and the regularity Ω (and may blow up as $q \to \infty$). It is well known that Γ is of class $C^{1,1}$ if and only if it satisfies both the interior and exterior ball condition. Thus, we can claim that C only depends on N, q, d_Ω, r_i, and r_e.

(ii) When Γ is of class $C^{2,\gamma}$ with $0 < \gamma \le 1$, we can obtain an a priori bound also for $\|\nabla^2 h\|_{\infty,\Omega}$, by standard Schauder's estimates for $\nabla^2 u$ (see [6]), in terms of the $C^{2,\gamma}$-modulus of continuity $\omega_{2,\gamma}$ of Γ. (For a definition of $\omega_{2,\gamma}$, see e.g., [2, Remark 1].)

The following theorem clearly gives (1.4).

Theorem 3.9 (Soap Bubble Theorem: enhanced stability)**.** *Let $N \ge 2$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class C^2. Denote by H the mean curvature of Γ and set $R = N|\Omega|/|\Gamma|$ and $H_0 = 1/R$.*

Let $z \in \Omega$ be a global minimum point of the solution u of (3.3) *and let ρ_i and ρ_e be defined by* (3.1)*. Then, the following inequalities hold true.*

(i) If $2 \le N \le 4$, there exists a constant $c = c(N, d_\Omega, r_i, r_e)$ such that

$$\rho_e - \rho_i \le c \begin{cases} \|H_0 - H\|_{2,\Gamma}, & \text{if } N = 2,3, \\ \|H_0 - H\|_{2,\Gamma} \max\left[\log\left(\frac{1}{\|H_0 - H\|_{2,\Gamma}}\right), 1\right], & \text{if } N = 4. \end{cases} \tag{3.8}$$

(ii) If $N \ge 5$, for any $q \in (N,\infty)$, there exists a constant $c = c(N, q, d_\Omega, r_i, r_e)$ such that

$$\rho_e - \rho_i \le c\, \|H_0 - H\|_{2,\Gamma}^{\frac{4}{N} - \frac{2(N-4)}{N(q-2)}}. \tag{3.9}$$

Moreover, (for any $N \ge 2$) we have that

$$R \left\| \nu - \frac{\nabla Q^z}{R} \right\|_{2,\Gamma} \le c\, \|H_0 - H\|_{2,\Gamma}. \tag{3.10}$$

If Γ is of class $C^{2,\gamma}$, $0 < \gamma \le 1$, the exponent in (3.9) *can be replaced by its limit as $q \to \infty$, i.e., $4/N$. In this case, the relevant constant c only depends on N, d_Ω, and the $C^{2,\gamma}$-modulus of continuity of Γ.*

Proof. Inequalities (3.8) and (3.9) will simply follow from the inequality:

$$\|\nabla^2 h\|_{2,\Omega} \le c\, \|H - H_0\|_{2,\Gamma}. \tag{3.11}$$

This was proved in [10].

For the reader's convenience, we summarize the main steps in the proof of [10, Theorem 3.5], which lead to (3.11), with the necessary modifications. As usual, the constant c may change from line to line and only depends on quantities (e.g., R, $\|u_\nu\|_{\infty,\Gamma}$, $\|Q^z_\nu\|_{\infty,\Gamma}$) that, in turn, can be bounded in terms of the parameters indicated in the statement.

The starting point is a modification of the fundamental identity (1.6):

$$\frac{1}{N-1}\int_\Omega |\nabla^2 h|^2 dx + \frac{1}{R}\int_\Gamma (u_\nu - R)^2 dS_x = -\int_\Gamma (H_0 - H)\, h_\nu\, u_\nu\, dS_x + \int_\Gamma (H_0 - H)\,(u_\nu - R)\, Q^z_\nu\, dS_x.$$

Next, if we discard the first summand in this identity, by Cauchy-Schwarz inequality we obtain that

$$\|u_\nu - R\|^2_{2,\Gamma} \le c\, \|H - H_0\|_{2,\Gamma}(\|h_\nu\|_{2,\Gamma} + \|u_\nu - R\|_{2,\Gamma}). \tag{3.12}$$

Instead, if we discard the second summand, we can infer that

$$\int_\Omega |\nabla^2 h|^2 dx \le c\, \|H - H_0\|_{2,\Gamma}(\|h_\nu\|_{2,\Gamma} + \|u_\nu - R\|_{2,\Gamma}). \tag{3.13}$$

Now, we use the fact that we can control ∇h (and hence h_ν) on Γ in terms of the deviation $u_\nu - R$. This is obtained by combining a trace-type inequality for h derived in [10, Lemma 2.5] and identity (1.7), as follows:

$$\int_\Gamma |\nabla h|^2 dS_x \le c \int_\Omega (-u)\, |\nabla^2 h|^2\, dx = \frac{1}{2}\, c \int_\Gamma (u_\nu^2 - R^2)\, h_\nu\, dS_x \le \\ c\, \|u_\nu - R\|_{2,\Gamma}\|h_\nu\|_{2,\Gamma} \le c\, \|u_\nu - R\|_{2,\Gamma}\, \|\nabla h\|_{2,\Gamma}.$$

This then gives:

$$\|h_\nu\|_{2,\Gamma} \le \|\nabla h\|_{2,\Gamma} \le c\, \|u_\nu - R\|_{2,\Gamma}. \tag{3.14}$$

Thus, inserting this inequality into (3.12) gives that

$$\|u_\nu - R\|_{2,\Gamma} \le c\, \|H - H_0\|_{2,\Gamma}. \tag{3.15}$$

Also, by plugging it into (3.13), we infer that

$$\int_\Omega |\nabla^2 h|^2 dx \le c\, \|H - H_0\|_{2,\Gamma}\|u_\nu - R\|_{2,\Gamma} \le c\, \|H - H_0\|^2_{2,\Gamma}.$$

Therefore, (3.11) follows at once.

Now, we proceed to prove (3.8) and (3.9). The cases $N = 2, 3$ easily follow from Theorem 3.4. Thus, we are left to prove it for $N \ge 4$.

For $N = 4$, we simply combine Theorem 3.4 and the first part of Remark 3.5. Indeed, $\|\nabla h\|_{\infty,\Omega}$ is bounded by a constant which only depends on r_i, r_e, and d_Ω.

For $N \geq 5$, instead, we use Corollary 3.7 and Remark 3.8, which give

$$\rho_e - \rho_i \leq c\, \|\nabla^2 h\|_{2,\Omega}^{\frac{4}{N} - \frac{2(N-4)}{N(q-2)}}.$$

Hence, (3.9) ensues from (3.11). The case in which Γ is of class $C^{2,\gamma}$ can be dealt similarly.

To conclude the proof, we are left to show that (3.10) also holds. To this aim, as done in the introduction, we observe that

$$\left| \nu(x) - \frac{x - z}{R} \right| \leq \frac{|R - u_\nu(x)| + |\nabla h(x)|}{R} \quad \text{for} \quad x \in \Gamma.$$

Hence, we infer that

$$R \left(\int_\Gamma \left| \nu(x) - \frac{x - z}{R} \right|^2 \frac{dS_x}{|\Gamma|} \right)^{1/2} \leq \|u_\nu - R\|_{2,\Gamma} + \|\nabla h\|_{2,\Gamma} \leq c\, \|u_\nu - R\|_{2,\Gamma},$$

where we applied the triangle inequality and the second inequality in (3.14). By using (3.15), then (3.10) easily follows from the last inequality above. □

Remark 3.10. In order to compare the results of Theorem 3.9 to previous estimates, we recall what we obtained in [10, Theorem 3.5] — the last up-to-date bound for stability in the Soap Bubble Theorem. In fact, there we obtained the bound

$$\rho_e - \rho_i \leq c\, \Psi\left(\|H - H_0\|_{L^2(\Gamma)} \right),$$

with

$$\Psi(\sigma) = \begin{cases} \sigma & \text{if } N = 2, 3, \\ \sigma^{1-\varepsilon} & \text{if } N = 4, \\ \sigma^{2/(N-2)} & \text{if } N \geq 5, \end{cases}$$

where the case $N = 4$ must be interpreted thus: for any $0 < \varepsilon < 1$, there exists a constant $c = c_\varepsilon$ (which may blow up as $\varepsilon \to 0$), such that case $N = 4$ holds. Theorem 3.9 clearly improves these profiles if Γ is either of class C^2 or $C^{2,\gamma}$. Moreover, it also states that we can control linearly the deviation of the Gauss map from that of a sphere, at least in the L^2-norm.

4. Application to quantitative symmetry in Serrin's overdetermined problem

In order to obtain stability estimates for Serrin's problem, we must use identity (1.7). In fact, this relates the weighted integral at the right-hand side to the deviation $u_\nu - R$. Since the torsion u can be easily bounded below by δ_Γ (see [9, Lemma 3.1]), we understand that this time we must associate the difference $\rho_e - \rho_i$ with the weighted L^2-norm $\|\delta_\Gamma^{1/2} \nabla^2 h\|_{2,\Omega}$. The following result goes in that direction.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary Γ of class C^2 and $z \in \Omega$ be a global minimum point of the solution u of* (3.3). *Consider the function $h = Q^z - u$, with Q^z given by* (1.3). *Then, there exists a constant $c = c(N, d_\Omega, r_i, r_e)$ such that*

$$\rho_e - \rho_i \leq c \begin{cases} \|\delta_\Gamma^{1/2} \nabla^2 h\|_{2,\Omega} & \text{if } N = 2; \\ \|\delta_\Gamma^{1/2} \nabla^2 h\|_{2,\Omega} \max\left[\log\left(\dfrac{e\, \|\nabla h\|_{\infty,\Omega}}{\|\delta_\Gamma^{1/2} \nabla^2 h\|_{2,\Omega}} \right), 1 \right] & \text{if } N = 3; \\ \|\nabla h\|_{\infty,\Omega}^{(N-3)/(N-1)} \|\delta_\Gamma^{1/2} \nabla^2 h\|_{2,\Omega}^{2/(N-1)} & \text{if } N \geq 4. \end{cases}$$

Proof. (i) Let $N = 2$. By using Lemma 3.2 with $p = 4$ we have that

$$\rho_e - \rho_i \le c\, \|\nabla h\|_{4,\Omega}.$$

By applying Remark 3.3 with $r = 4$, $p = 2$, and $\alpha = 1/2$, we obtain that

$$\|\nabla h\|_{4,\Omega} \le c\, \|\delta_\Gamma^{1/2} \nabla^2 h\|_{2,\Omega},$$

and the conclusion follows.

(ii) Let $N = 3$. By using Remark 3.3 with $r = 3$, $p = 2$, $\alpha = 1/2$, we get

$$\|\nabla h\|_{3,\Omega} \le c\, \|\delta_\Gamma^{1/2} \nabla^2 h\|_{2,\Omega}.$$

The conclusion follows by using Lemma 3.2 with $p = N = 3$.

(iii) When $N \ge 4$, we use Lemma 3.2 with $p = 2N/(N-1)$ and put it together with Remark 3.3 with $r = \frac{2N}{N-1}$, $p = 2$, $\alpha = 1/2$. □

By recalling Remark 3.5, to gain better stability for Serrin's problem for $N \ge 3$, we need to obtain a bound similar to that in Corollary 3.6, but with $\|\nabla^2 h\|_{p,\Omega}$ replaced by $\|\delta_\Gamma^{1/2} \nabla^2 h\|_{p,\Omega}$. This time, we proceed differently.

Lemma 4.2. *Set $1 \le p \le \infty$ and $q > N$. Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with boundary Γ of class C^2 and assume that $h \in W^{2,q}(\Omega)$. Then, there exists a constant $c = c(N, p, q)$ such that*

$$\|\nabla h\|_{\infty,\Omega}^{N+p\,(1-N/q)} \le c\, |\Omega|\, \|\nabla h\|_{p,\Omega}^{p\,(1-N/q)} \|\nabla^2 h\|_{q,\Omega}^{N}. \tag{4.1}$$

Proof. For any $x \in \overline{\Omega}$ there is a cone $C_{x,a} \subset \Omega$. Applying (2.3) with $p = q$ to any cone $C_{x,\sigma} \subset C_{x,a}$ gives that

$$|f(x)| \le \int_{C_{x,\sigma}} |f|\, d\mu_y + c\,\sigma\, \|\nabla f\|_{q,C_{x,\sigma}} \le \|f\|_{p,C_{x,\sigma}} + c\,\sigma\, \|\nabla f\|_{q,C_{x,\sigma}},$$

where we used Hölder's inequality at the second inequality. Here, $c = c(N, q)$. Thus, we have that

$$\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f \le 2 \max_{\overline{\Omega}} |f| \le c\, \left(|\Omega|^{1/p} \sigma^{-N/p} \|f\|_{p,\Omega} + c\, |\Omega|^{1/q} \sigma^{1-N/q}\, \|\nabla f\|_{q,\Omega} \right),$$

for every $\sigma \in (0, a)$, where in the second inequality we also used the monotonicity of Lebesgue's integral with respect to set inclusion. Here, $c = c(N, p, q)$ (notice that the dependence on θ can be dropped, since $\theta = \sqrt{2}/2$, being as Γ of class C^2). We now minimize in σ as done before. This time, we omit the details. We end up with the formula:

$$\max_{\overline{\Omega}} f - \min_{\overline{\Omega}} f \le c\, |\Omega|^{\frac{1}{N+p(1-N/q)}} \|f\|_{p,\Omega}^{\frac{p(1-N/q)}{N+p(1-N/q)}} \|\nabla f\|_{q,\Omega}^{\frac{N}{N+p(1-N/q)}}.$$

This holds for any $x \in \overline{\Omega}$ and $1 \le p < q \le \infty$. By choosing f as any directional derivative h_ℓ of h and using that, with our choice of z, $|h_\ell(x)| = |h_\ell(x) - h_\ell(z)|$, we thus get that

$$|h_\ell(x)|^{N+p\,(1-N/q)} \le c\, |\Omega|\, \|h_\ell\|_{p,\Omega}^{p\,(1-N/q)} \|\nabla h_\ell\|_{q,\Omega}^{N} \quad \text{for any} \quad x \in \overline{\Omega}.$$

Hence, (4.1) follows by observing that $|h_\ell| \le |\nabla h|$, $|\nabla h_\ell| \le |\nabla^2 h|$, and by choosing ℓ such that $h_\ell(x) = |\nabla h(x)|$ and $x \in \Gamma$ that maximizes $|\nabla h|$ on $\overline{\Omega}$.

As for Corollary 3.6, the lemma could be stated for a domain satisfying an interior cone condition. □

Corollary 4.3. *Set* $1 \le p < 2N$ *and* $q > N$. *Under the assumptions of Lemma 4.2, we have that*

$$\|\nabla h\|_{\infty,\Omega}^{2N-p+2p\,(1-N/q)} \le c\, \|\nabla^2 h\|_{q,\Omega}^{2N-p}\, \|\delta_\Gamma^{1/2} \nabla^2 h\|_{p,\Omega}^{2p\,(1-N/q)}. \tag{4.2}$$

Here, the constant c only depends on N, p, q, d_Ω, r_i, *and* r_e.

Proof. We use Remark 3.3 with r, p, and α replaced by $2pN/(2N-p)$, p, and $1/2$, respectively. We thus get that

$$\|\nabla h\|_{\frac{2pN}{2N-p},\Omega} \le c\, \|\delta_\Gamma^{1/2} \nabla^2 h\|_{p,\Omega}.$$

Therefore, (4.2) follows by combining this bound and (4.1) with p replaced by $2pN/(2N-p)$. □

Theorem 4.4 (Serrin's problem: enhanced stability)**.** *Let* $\Omega \subset \mathbb{R}^N$, $N \ge 2$, *be a bounded domain with boundary* Γ *of class* C^2 *and set* $R = N|\Omega|/|\Gamma|$.

Let u be the solution of problem (3.3) *and* $z \in \Omega$ *be a global minimum point of u in* $\overline{\Omega}$. *Then, there exists a constant* $c = c(N, d_\Omega, r_i, r_e)$ *such that*

$$\rho_e - \rho_i \le c \begin{cases} \|u_\nu - R\|_{2,\Gamma} & \text{if } N = 2; \\ \|u_\nu - R\|_{2,\Gamma} \max\left[\log\left(\dfrac{1}{\|u_\nu - R\|_{2,\Gamma}}\right), 1\right] & \text{if } N = 3. \end{cases}$$

When $N \ge 4$, *for any* $q \in (N, \infty)$, *there exists a constant* $c = c(N, q, d_\Omega, r_i, r_e)$ *such that*

$$\rho_e - \rho_i \le c\, \|u_\nu - R\|_{2,\Gamma}^{\frac{4-2N/q}{N+1-2N/q}}. \tag{4.3}$$

Moreover (for any $N \ge 2$*),*

$$R \left\| \nu - \frac{\nabla Q^z}{R} \right\|_{2,\Gamma} \le c\, \|u_\nu - R\|_{2,\Gamma},$$

for some constant $c = c(N, d_\Omega, r_i, r_e)$.

If Γ *is of class* $C^{2,\gamma}$, $0 < \gamma \le 1$, *the stability exponent in* (4.3) *for* $N \ge 4$ *can be replaced its limit as* $q \to \infty$, *i.e.,* $4/(N+1)$. *In this case, c only depends on N,* d_Ω, *and the* $C^{2,\gamma}$*-modulus of continuity of* Γ.

Proof. It is sufficient to notice that, thanks to (1.7) and the pointwise inequality $\delta_\Gamma \le -2u/r_i$, we can infer that

$$\|\delta_\Gamma^{1/2} \nabla^2 h\|_{2,\Omega}^2 \le c \int_\Omega (-u)\, |\nabla^2 h|^2 dx \le c\, \|u_\nu - R\|_{2,\Gamma}\, \|h_\nu\|_{2,\Gamma}.$$

Thus, by (3.14), we obtain that

$$\|\delta_\Gamma^{1/2} \nabla^2 h\|_{2,\Omega} \le c\, \|u_\nu - R\|_{2,\Gamma}.$$

Therefore, with this inequality in hand, we can proceed similarly to the proof of Theorem 3.9 by also taking into account Remark 3.8. For instance, the claim for $N \ge 4$ simply follows from Theorem 4.1 and Corollary 4.3 with $p = 2$.

The remaining claims follow from Theorem 4.1 at once. □

Remark 4.5. In order to compare the results of Theorem 4.4 to previous estimates, it is sufficient to recall what we obtained in [10, Theorem 3.1] — the last up-to-date bound for stability in Serrin's problem. In fact, there we obtained the bound

$$\rho_e - \rho_i \le c\, \Psi\left(\|u_\nu - R\|_{L^2(\Gamma)}\right),$$

with

$$\Psi(\sigma) = \begin{cases} \sigma & \text{if } N = 2, \\ \sigma^{1-\varepsilon} & \text{if } N = 3, \\ \sigma^{2/(N-1)} & \text{if } N \geq 4. \end{cases}$$

The case $N = 3$ must be interpreted thus: for any $0 < \varepsilon < 1$ there exists a constant $c = c_\varepsilon$ (which may blow up as $\varepsilon \to 0$), such that case $N = 3$ holds.

The comparison with Theorem 4.4 is left to the reader.

As already mentioned in the introduction for the Soap Bubble Theorem, if one adopts a stronger norm for the deviation $u_\nu - R$, linear stability can also be obtained (with some restrictions) in general dimension. See for instance [5].

Remark 4.6. A direct inspection of the corresponding proofs tells us that the dependence of the relevant constant c on the parameter r_e can be removed whenever Γ is mean convex. In fact, in this case, the bounds in (3.4), (3.7) and the former inequality for c in Remark 3.3 can be replaced by [10, Formula (2.4)], (3.6) and the latter inequality for c in Remark 3.3.

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Conflict of interest

The authors declare no conflict of interest.

References

1. R. A. Adams, *Sobolev spaces*, New York: Academic Press, 1975.
2. A. Aftalion, J. Busca, W. Reichel, Approximate radial symmetry for overdetermined boundary value problems, *Adv. Differential Equations*, **4** (1999), 907–932.
3. L. Cavallina, G. Poggesi, T. Yachimura, Quantitative stability estimates for a two-phase Serrin-type overdetermined problem, 2021, arXiv:2107.05889.
4. S. Dipierro, G. Poggesi, E. Valdinoci, A Serrin-type problem with partial knowledge of the domain, *Nonlinear Anal.*, **208** (2021), 112330. http://dx.doi.org/10.1016/j.na.2021.112330
5. A. Gilsbach, M. Onodera, Linear stability estimates for Serrin's problem via a modified implicit function theorem, *Calc. Var.*, **60** (2021), 241. http://dx.doi.org/10.1007/s00526-021-02107-1
6. D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Berlin, Heidelberg: Springer, 2001. http://dx.doi.org/10.1007/978-3-642-61798-0

7. R. Magnanini, Alexandrov, Serrin, Weinberger, Reilly: symmetry and stability by integral identities, *Bruno Pini Mathematical Analysis Seminar*, **8** (2017), 121–141. http://dx.doi.org/10.6092/issn.2240-2829/7800
8. R. Magnanini, G. Poggesi, On the stability for Alexandrov's Soap Bubble theorem, *JAMA*, **139** (2019), 179–205. http://dx.doi.org/10.1007/s11854-019-0058-y
9. R. Magnanini, G. Poggesi, Serrin's problem and Alexandrov's Soap Bubble Theorem: enhanced stability via integral identities, *Indiana Univ. Math. J.*, **69** (2020), 1181–1205. http://dx.doi.org/10.1512/iumj.2020.69.7925
10. R. Magnanini, G. Poggesi, Nearly optimal stability for Serrin's problem and the Soap Bubble theorem, *Calc. Var.*, **59** (2020), 35. http://dx.doi.org/10.1007/s00526-019-1689-7
11. R. Magnanini, G. Poggesi, An interpolating inequality for solutions of uniformly elliptic equations, In: *Geometric properties for parabolic and elliptic PDE's*, Cham: Springer. http://dx.doi.org/10.1007/978-3-030-73363-6_11
12. R. Magnanini, G. Poggesi, The location of hot spots and other extremal points, *Math. Ann.*, 2021, in press. http://dx.doi.org/10.1007/s00208-021-02290-8
13. Y. Okamoto, M. Onodera, Stability analysis of an overdetermined fourth order boundary value problem via an integral identity, *J. Differ. Equations*, **301** (2021), 97–111. http://dx.doi.org/10.1016/j.jde.2021.08.017
14. G. Poggesi, Radial symmetry for p-harmonic functions in exterior and punctured domains, *Appl. Anal.*, **98** (2019), 1785–1798. http://dx.doi.org/10.1080/00036811.2018.1460819
15. G. Poggesi, The Soap Bubble Theorem and Serrin's problem: quantitative symmetry, PhD Thesis, Università di Firenze, 2019, arXiv:1902.08584.
16. J. Scheuer, Stability from rigidity via umbilicity, 2021, arXiv:2103.07178.
17. J. Serrin, A symmetry problem in potential theory, *Arch. Rational Mech. Anal.*, **43** (1971), 304–318. http://dx.doi.org/10.1007/BF00250468
18. H. F. Weinberger, Remark on the preceding paper of Serrin, *Arch. Rational Mech. Anal.*, **43** (1971), 319–320. http://dx.doi.org/10.1007/BF00250469