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Bounding the fitting height in terms of the exponent

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Abstract

Every finite solvable group G has a normal series with nilpotent factors. The smallest possible number of factors in such a series is called the Fitting height $h(G)$. In the present paper, we derive an upper bound for $h(G)$ in terms of the exponent of G . Our bound constitutes a considerable improvement of an earlier bound obtained in Shalev (Proc Am Math Soc 126(12):3495–3499, 1998).

Keywords Solvable group · Fitting height · Exponent

Mathematics Subject Classification 20D10 · 20F14 · 20F16

1 Introduction

Let $F(G)$ denote the Fitting subgroup of the group G . The *Fitting series* in G is defined via

$$F_0(G) = 1 \text{ and } F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G)) \text{ for every } i \geq 1.$$

When G is finite, this series reaches G if and only if G is solvable. The number of non-trivial members of the Fitting series is called the *Fitting height* $h(G)$ of G .

The Fitting height seems to have a strong influence on the structure of a finite solvable group. Quite a few investigations have been devoted to finding relations involving the Fitting height and other invariants. The interested reader may get an idea of such kind of problems by consulting the survey [5] and its bibliography.

In memory of our friend Carlo Casolo.

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In this short note we will focus on the interplay between the Fitting height and the exponent. Shalev proved in [3, Lemma 2.5] that

$$h(G) < \prod_{i=1}^k (2e_i + 1).$$

whenever G is a finite solvable group with exponent $\exp(G) = p_1^{e_1} \dots p_k^{e_k}$, where the p_i are pairwise distinct primes. Let $\Omega(m)$ denote the number of prime divisors of the natural number m , counted with multiplicities. Shalev's result implies the exponential bound $h(G) < 3^{\Omega(\exp(G))}$.

Let $\Omega_1(m)$ denote the number of odd prime divisors of m , which are Fermat primes (counted with multiplicities). We shall improve Shalev's inequality to a linear bound as follows.

Theorem 1 *Consider a finite solvable group G . Then*

$$h(G) \leq \Omega(\exp(G)) + \Omega_1(\exp(G)) \leq 2\Omega(\exp(G)).$$

A slightly nicer bound holds for groups of odd order.

Theorem 2 *Consider a finite group G of odd order. Then*

$$h(G) \leq \Omega(\exp(G)).$$

The bound given by Theorem 1 is tight, whenever the order of G is not divisible by any Fermat prime. To see this, let P_i be an elementary abelian p_i -group, for distinct non-Fermat primes p_1, \dots, p_n . The iterated wreath product

$$((\dots(P_1 \wr P_2) \wr \dots) \wr P_{n-1}) \wr P_n$$

of the groups P_i in their regular representations has Fitting height n and exponent $m = p_1 \dots p_n$; therefore $n = \Omega(m)$. This kind of example also shows, that the bound given in Theorem 2 is tight. It remains uncertain, whether equality can hold in Theorem 1 in the presence of Fermat primes.

It is worth remarking, that other common invariants of finite solvable groups cannot be bounded by a function of the exponent. For example, as pointed out in [6, p. 267], the derived length of the largest finite m -generated group of prime power exponent $p^n \geq 4$ is at least $\lfloor \log_2 m \rfloor$. Therefore, the derived length of a p -group cannot be bounded by a function of its exponent.

2 The result

For any group G , let

$$\rho_0(G) = G, \quad \rho_1(G) = \bigcap_{i \in \mathbb{N}} \gamma_i(G) \quad \text{and} \quad \rho_n(G) = \rho_1(\rho_{n-1}(G)) \quad \text{for all } n \geq 2.$$

It is clear that G has Fitting height n if and only if n is minimal with respect to $\rho_n(G) = 1$.

Let G be a finite group. A *tower* of height n in G is a family $\{P_n, \dots, P_1\}$ of non-trivial subgroups of G satisfying

- (1) every P_i is a p_i -group for some prime p_i , where $p_{i+1} \neq p_i$ for $i = n - 1, \dots, 1$,
- (2) P_i normalizes P_j whenever $i < j$, and
- (3) $[P_i, P_{i-1}] = \bar{P}_i$ for $i = n, \dots, 2$, where we use the notation $\bar{P}_i = P_i/C_i$ with $C_n = 1$ and $C_i = C_{P_i}(\bar{P}_{i+1})$ for $i = n - 1, \dots, 1$.

Note, that property (3) implies the non-triviality of the groups \bar{P}_i whenever $P_n \neq 1$. The above definition is a slightly weaker form of the concept of a tower as introduced in [4].

Lemma 1 *Every non-trivial finite solvable group with Fitting height n contains a tower of height n .*

Proof We shall call a family $\{P_n, \dots, P_1\}$ of subgroups in G a *weak tower* of height n , if it satisfies properties (1) and (2) of a tower, and in addition

$$(3') \quad \bar{P}_i \neq 1 \text{ for all } i.$$

By [4, Lemma 1.4], it is sufficient to show the existence of a weak tower of height n in G .

To this end, consider the Fitting series $1 = F_0 < F_1 < \dots < F_n = G$ in G . Here we have $F_n = G$, because G has Fitting height n . We shall proceed by induction over n in order to produce a weak tower $\{P_n, \dots, P_1\}$ in G such that

$$(\star) \quad P_{n-i}F_i/F_i \text{ is a Sylow subgroup in } F_{i+1}/F_i \text{ for } 0 \leq i \leq n - 1.$$

When $n = 1$, choose any non-trivial Sylow subgroup P_1 in G . Suppose then, that $n > 1$. By induction, there exists a weak tower T in G/F_1 satisfying (\star) . From [4, Lemma 1.6], this leads to a weak tower $\{P_{n-1}, \dots, P_1\}$ in G satisfying (\star) , such that T consists of the groups P_iF_1/F_1 ($n - 1 \geq i \geq 1$). For some prime $p_n \neq p_{n-1}$ the group F_1 contains a Sylow p_n -subgroup P_n satisfying $[P_n, P_{n-1}] \neq 1$, because otherwise the subgroup $P_{n-1}F_1$ would be a nilpotent normal subgroup in F_2 and thus be contained in F_1 , a contradiction. Now $\{P_n, \dots, P_1\}$ is a weak tower in G . \square

Lemma 2 *Let G be a non-trivial finite solvable group with Fitting height n . Suppose, that $G = P_n \cdots P_1$ for some tower $\{P_n, \dots, P_1\}$ in G of height n . If $N \trianglelefteq G$ and $P_nN/N \neq 1$, then $\{P_nN/N, \dots, P_1N/N\}$ is a tower in G/N .*

Proof Let $Q_i = P_iN/N$ and $\bar{Q}_i = Q_i/D_i$ for all i , where $D_n = 1$ and $D_i = C_{Q_i}(\bar{Q}_{i+1})$ for $i = n - 1, \dots, 1$. Obviously, the family $\{Q_n, \dots, Q_1\}$ of subgroups of G/N inherits properties (1) and (2) of a tower from the given tower in G . Since $Q_n \neq 1$ by hypothesis, the non-triviality of the groups Q_i will be a consequence of property (3). Therefore it just remains to establish (3) for the family $\{Q_n, \dots, Q_1\}$.

To this end, we will show first, that $C_{P_i}(\bar{P}_{i+1})N/N \leq D_i$ for $i = n - 1, \dots, 1$. This inclusion obviously holds for $i = n - 1$. We proceed by recursion and assume, that there exists $k \in \{n - 1, \dots, 2\}$ such that the inclusion has already been shown for $i = n - 1, \dots, k$. Then $\bar{Q}_k = Q_k/D_k$ is a homomorphic image of $P_kN/C_{P_k}(\bar{P}_{k+1})N$, hence of $\bar{P}_k = P_k/C_{P_k}(\bar{P}_{k+1})$. Since the involved homomorphisms are projections, and since all the involved groups are normalized by P_{k-1} , we have that \bar{Q}_k is P_{k-1} -isomorphic to a quotient of \bar{P}_k . Therefore,

$C_{P_{k-1}}(\bar{P}_k)$ acts trivially on \bar{Q}_k , and we obtain $C_{P_{k-1}}(\bar{P}_k)N/N \leq D_{k-1}$. This completes the recursion.

Now, for each $i \in \{n-1, \dots, 1\}$, the quotient \bar{Q}_i is P_{i-1} -isomorphic to an image of \bar{P}_i . Therefore property (3) of the given tower in G implies that $[\bar{Q}_i, P_{i-1}] = \bar{Q}_i$, and it follows that $[\bar{Q}_i, Q_{i-1}] = \bar{Q}_i$ for each $i \in \{n-1, \dots, 1\}$. \square

Lemma 3 *Let $\{P_n, \dots, P_1\}$ be a tower in a finite solvable group. If $G = P_n \cdots P_1$, then $\rho_{n-1}(G) = P_n$.*

Proof The claim is true when $n < 2$. So we suppose now, that $n \geq 2$ and argue by induction. Again, we consider the centralizer $C = C_{n-1}$. Let $X = P_{n-1} \cdots P_1$. By Lemma 2, the images in G/C of the groups P_n, \dots, P_1 form a tower of height n in G/C . Since $C_{P_{n-1}/C}(P_n C/C) = C_{n-1}/C = 1$, the images in G/C of the groups P_{n-1}, \dots, P_1 form a tower of height $n-1$, and X/C is the product of these images. Therefore our inductive hypothesis yields $P_{n-1} = \rho_{n-2}(X)C \leq \rho_{n-2}(G)C$.

We can use this relation in order to prove, that $P_n \leq \rho_i(G)$ for all $i < n$: Beginning with $P_n = [P_n, P_{n-1}] \leq \gamma_2(G)$, a recursion shows that

$$P_n = [P_n, P_{n-1}] \leq [\gamma_j(G), G] = \gamma_{j+1}(G) \quad \text{for all } j.$$

It follows that $P_n \leq \rho_1(G)$.

Arguing by induction, we suppose next, that $P_n \leq \rho_i(G)$ for some $i \leq n-2$. Then

$$\begin{aligned} P_n &= [P_n, P_{n-1}] \leq [P_n, \rho_{n-2}(G)C] \\ &= [P_n, \rho_{n-2}(G)] \leq [\rho_i(G), \rho_{n-2}(G)] \leq \gamma_2(\rho_i(G)), \end{aligned}$$

and it follows as before, that $P_n \leq \gamma_{j+1}(\rho_i(G))$ for all j . In particular $P_n \leq \rho_{i+1}(G)$. In the end, we obtain $P_n \leq \rho_{n-1}(G)$.

On the other hand, the subgroups $N_i = P_n \cdots P_i$ are normal in G and every P_i is nilpotent. Therefore, $\rho_i(G) \leq P_n \cdots P_i$ for all i . In particular, $\rho_{n-1}(G) \leq P_n$. \square

Corollary 1 *If the finite solvable group G is the product of subgroups, which form a tower of height n , then $h(G) = n$.*

Corollary 2 *Let G be a finite solvable group with a tower $\{P_n, \dots, P_1\}$ such that $G = P_n \cdots P_1$. If a normal subgroup N of G does not contain P_n , then $h(G/N) = h(G) = n$.*

Proof If $h(G/N) < n$, then $\rho_{n-1}(G/N) = 1$ and $P_n = \rho_{n-1}(G) \leq N$. \square

When $\mathcal{T} = \{P_n, \dots, P_1\}$ is a tower in a finite solvable group G , we define

$$m_p(\mathcal{T}) = |\{i \mid P_i \text{ is a } p\text{-group}\}| \quad \text{for each prime } p.$$

Recall, that the p -length $\ell_p(G)$ of a finite solvable group G is the number of p -factors in a shortest normal series in G , whose factors are p -groups or p' -groups.

Proposition 1 *Let the finite solvable group G be the product of the subgroups in a tower \mathcal{T} . Then $\ell_p(G) = m_p(\mathcal{T})$ for all primes p .*

Proof We shall proceed by induction on the Fitting height n of G . Clearly, the claim holds for $n \leq 2$. So we assume now, that $n > 2$ and that the claim is true for all groups of Fitting height $\leq n - 1$. Amongst the groups of Fitting height $\leq n$, we proceed by induction over $|G|$. We may thus assume, that $h(G) = n$ and that the claim holds for all groups H of Fitting height $\leq n$ satisfying $|H| < |G|$. By hypothesis, G is the product of subgroups P_n, \dots, P_1 forming a tower \mathcal{T} .

Suppose, that there exists a non-trivial normal subgroup N in G , which is properly contained in P_n . The factor group G/N has Fitting height n , and Lemma 2 ensures that the subgroups $P_i N/N$ ($n \geq i \geq 1$) form a tower \mathcal{S} in G/N . By minimality of G , we have $\ell_p(G/N) = m_p(\mathcal{S}) = m_p(\mathcal{T})$ for all primes p . Now $O_{p_n}(G/N) = O_{p_n}(G)/N \geq P_n/N \neq 1$ implies $\ell_{p_n}(G) = \ell_{p_n}(G/N) = m_{p_n}(\mathcal{T})$. Moreover, $\ell_p(G) = \ell_p(G/N) = m_p(\mathcal{T})$ holds for all primes $p \neq p_n$ because N is a p_n -group. We have thus shown, that it remains to treat the case, when P_n is a minimal normal subgroup in G .

Let $V = P_n$ and $X = P_{n-1} \cdots P_1$. Since $G = VX$ and since V is abelian, we have $V \cap X \trianglelefteq G$. Hence $V \cap X = 1$ or $V \cap X = V$. However, the latter case cannot occur, because $V \leq X$ and $P_{n-1} \trianglelefteq X$ would imply $V = [V, P_{n-1}] \leq V \cap P_{n-1} = 1$. We have thus shown, that G is the semidirect product of V and X .

Consider the centralizer $C = C_{P_{n-1}}(V)$. Note that $C < P_{n-1}$ because of property (3) of the tower \mathcal{T} . By Lemma 2, the subgroups $P_i C/C$ ($n-1 \geq i \geq 1$) form a tower \mathcal{R} in X/C . By minimality of G , we obtain $m_p(\mathcal{R}) = \ell_p(X/C)$ for all primes p . Since $C < P_{n-1}$, we also have $\ell_p(X/C) = \ell_p(X)$ for all primes p . It follows, that $m_p(\mathcal{T}) = m_p(\mathcal{R}) = \ell_p(X)$ for all primes $p \neq p_n$. It remains to treat the prime $p = p_n$.

Consider the subgroup $O_p(G)$ of G . In the case when $O_p(G) = V$, we have $m_p(\mathcal{T}) = m_p(\mathcal{R}) + 1 = \ell_p(X/C) + 1 = \ell_p(X) + 1 = \ell_p(G)$. Therefore, it remains to treat the case when $V < O_p(G)$. Note that $VO_p(X)$ is a normal p -subgroup in G . Therefore $O_p(G) = VX \cap O_p(G) = V(X \cap O_p(G)) \leq VO_p(X) \leq O_p(G)$, so that equality holds. The non-trivial normal subgroup $Z(O_p(G)) \cap V$ of G must coincide with the minimal normal subgroup V of G . Thus, $Y = O_p(X)$ is centralized by V , hence $Y \trianglelefteq G = VX$.

By Corollary 2, the group $G/Y = (P_n Y/Y) \cdots (P_1 Y/Y)$ has Fitting height n . It follows, that $P_i Y/Y \neq 1$ for $i = n, \dots, 1$. And Lemma 2 ensures, that the subgroups $P_i Y/Y$ form a tower \mathcal{U} in G/Y . By minimal choice of G , we have $m_p(\mathcal{T}) = m_p(\mathcal{U}) = \ell_p(G/Y) = \ell_p(G)$. The proof of Proposition 1 is complete. \square

Proposition 2 *Let G be a finite solvable group. For each prime p , let p^{e_p} be the exponent of the Sylow p -subgroups of G and let ℓ_p denote the p -length of G . Then we have*

- (1) $\ell_p \leq 2e_p$, whenever p is an odd Fermat prime,
1. $[(2)] \ell_p \leq e_p$, whenever $p = 2$ or p is odd and not a Fermat prime.

Proof By [2, Theorem A] we have $e_p \geq [(l_p + 1)/2]$, whenever p is an odd Fermat prime. It follows that $2e_p \geq \ell_p$. In all other cases, $e_p \geq \ell_p$ follows from [2, Theorem A] for odd p and from [1] for $p = 2$. \square

We can now relate the exponent of a finite solvable group to its Fitting height.

Proof of Theorem 1 From Lemma 1 it is enough to prove the claim, when G is the product of subgroups forming a tower \mathcal{T} . The exponent of G is the product of the exponents of

its Sylow subgroups. If p_1, p_2, \dots, p_k are the primes dividing $|G|$, where p_i is Fermat for $i = r + 1, \dots, k$, then we write $p_i^{e_i}$ for the exponent of the Sylow p_i -subgroups of G . Propositions 1 and 2 directly imply

$$\begin{aligned} h(G) &= \sum_{i=1}^k m_{p_i}(G) = \sum_{i=1}^k \ell_{p_i}(G) \\ &\leq \sum_{i=1}^r e_i(G) + 2 \sum_{i=r+1}^k e_i(G) = \Omega(\exp(G)) + \Omega_1(\exp(G)). \end{aligned}$$

□

Proof of Theorem 2 A group of odd order is solvable by Feit-Thompson theorem. Again it is enough to prove the claim when $G = P_n \cdots P_1$ where $\{P_n, \dots, P_1\}$ is a tower. Since 2 does not divide the order of G , it is a consequence of [2, Theorem 2.1.1] and part (ii) of its Corollary, that the inequality $\ell_p \leq e_p$ holds for all prime divisors p of $|G|$. We thus obtain $h(G) \leq \Omega(\exp(G))$ as in the proof of Theorem 1. □

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