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GROUPS WHOSE PRIME GRAPH ON CLASS SIZES HAS A CUT VERTEX

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Abstract. Let G be a finite group, and let $\Delta(G)$ be the *prime graph* built on the set of conjugacy class sizes of G: this is the simple undirected graph whose vertices are the prime numbers dividing some conjugacy class size of G, two vertices p and q being adjacent if and only if pq divides some conjugacy class size of G. In the present paper, we classify the finite groups G for which $\Delta(G)$ has a *cut vertex*.

1. Introduction

Given a finite group G, the *prime graph* $\Delta(G)$ on the set of conjugacy class sizes of G is the simple undirected graph defined as follows: the vertex set $\mathrm{V}(G)$ of $\Delta(G)$ consists of the prime numbers dividing the size of some conjugacy class of G, and two vertices p and q are adjacent in $\Delta(G)$ if and only if there exists a conjugacy class of G having size divisible by the product pq.

A well-established research field in the theory of finite groups investigates the interplay between graph-theoretical properties of $\Delta(G)$ and the structure of G itself. As a general remark, several results in the literature show that the graph $\Delta(G)$ "tends to have many edges", in the sense that non-adjacency of two vertices highly constrains the group structure of G (we recall for instance, as proved in Theorem 9 of [3], that $\Delta(G)$ is a complete graph if the Fitting subgroup of G is trivial).

An extreme situation is the case when $\Delta(G)$ is a disconnected graph (see Theorem 4 in [6]): this happens if and only if G is a semidirect product AB, where $A \trianglelefteq G$ and B are abelian subgroups of coprime order, and the factor group of G over its centre Z is a Frobenius group whose Frobenius kernel is AZ/Z, whereas BZ/Z is a Frobenius complement (in this case we say that G is a $\mathcal{D}$*-group*). Here, the vertex sets of the (two) connected components of $\Delta(G)$ turn out to be respectively the set of prime divisors of $|AZ/Z|$ and that of $|BZ/Z|$, and both the connected components are cliques (i.e., complete subgraphs) of $\Delta(G)$. Note that, in particular, $\mathrm{V}(G)$ is partitioned in two subsets of pairwise adjacent vertices.

Interestingly, the feature of $\Delta(G)$ described in the last sentence of the paragraph above, regarding the disconnected case, turns out to hold in full generality: as Corollary B of [8] shows, *for every finite group G* the vertex set of $\Delta(G)$ can be

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partitioned in two subsets of pairwise adjacent vertices. This result will be crucial for the analysis carried out in the present paper.

Non-adjacency between vertices of $\Delta(G)$ has been studied and exploited in various forms; for instance, in [4] the authors consider the situation when there exists at most one vertex of $\Delta(G)$ that is adjacent to all the other vertices (a so-called *complete vertex*), proving that such a group G is solvable with Fitting height at most 3. In the same paper, as a generalization of the disconnected case, it is also proved that a finite group G for which $\Delta(G)$ has no complete vertices is a semidirect product of two abelian groups having coprime orders, and the case when $\Delta(G)$ is a (non-complete) regular graph is characterized. Even more generally, the main result of [5] shows that, denoting by π_0 the set of non-complete vertices of $\Delta(G)$, there exist Hall π_0-subgroups for the group G and they are metabelian.

The present paper is a contribution in this framework. Let Δ be a graph with n connected components; denoting by V the vertex set of Δ, an element $r \in V$ is called a *cut vertex* of Δ if the number of connected components of the subgraph induced by $V \setminus \{r\}$ in Δ (i.e., the graph obtained by removing the vertex r and all edges incident to r from Δ) is larger than n. If Δ is connected and it has a cut vertex, then Δ is said to be 1-*connected*. Our aim here is to *describe the finite groups G such that the graph $\Delta(G)$ has a cut vertex*, clearly a situation in which there are many non-adjacencies between vertices of $\Delta(G)$. Note that, under this assumption, $\Delta(G)$ is in fact 1-connected. This follows from the aforementioned Theorem 4 of [6]: if $\Delta(G)$ is disconnected then (it has two connected components and) the connected components are complete subgraphs, so $\Delta(G)$ cannot have any cut vertex in this case.

We will show that $\Delta(G)$ has at most two cut vertices, and we will provide a complete characterization of the structure of the group G, as well as of the graph $\Delta(G)$, in both the cases when $\Delta(G)$ has either one or two cut vertices. In the following statements, given a graph Δ with vertex set V, for $v \in V$ we denote by $\Delta - v$ the subgraph induced by $V \setminus \{v\}$ in Δ. Moreover, we recall that a solvable group is called an $\mathcal{A}$-*group* if all its Sylow subgroups are abelian.

Theorem A. *Let G be a finite group such that $\Delta(G)$ has a cut vertex r. Then the following conclusions hold.*

(a) *G is a solvable group whose Fitting height is at most 3, and whose Sylow p-subgroups are abelian for every prime $p \neq r$.*
(b) *$\Delta(G) - r$ is a graph with two connected components, that are both complete graphs.*
(c) *If r is a complete vertex of $\Delta(G)$, then it is the unique complete vertex and the unique cut vertex of $\Delta(G)$. If r is non-complete, then $\Delta(G)$ is a graph of diameter 3, and it can have at most two cut vertices; moreover, G is a metabelian $\mathcal{A}$-group.*

The main result of this paper provides a complete characterization of the finite groups G such that $\Delta(G)$ has a cut vertex. The full statement, which is somewhat technical, will be given in Section 3 (Theorem 3.3), whereas here we will state a partial version of that result as Theorem B. In what follows, for a group G as in Theorem A, we will denote by α and β the vertex sets of the two complete connected components of $\Delta(G) - r$; moreover, R will denote a Sylow r-subgroup of G, and A, B will be Hall subgroups of G for the sets of primes α and β respectively, such that AB is a subgroup of G.

disegno1.png

disegno2.png

Figure 1. Examples of $\Delta(G)$ with a cut vertex and with two cut vertices.

Theorem B. *Let G be a finite group such that the graph $\Delta(G)$ has a cut vertex r. Then AB is a $\mathcal{D}$-group (so, in particular, both A and B are abelian) and either R or AB is normal in G.*

Furthermore, either $G = Z \times R \times AB$, with $Z \leq \mathbf{Z}(G)$, or there exists a normal subgroup N of G such that, setting $\overline{G} = G/N$, we have $\mathbf{F}(\overline{G}) = \overline{\mathbf{F}(G)}$, $\mathbf{Z}(\overline{G}) = 1$ and, up to interchanging α and β, one of the following holds.

- **(i)** *$\mathbf{F}(\overline{G}) = \overline{R}$, $\overline{A} = \mathbf{F}(\overline{AB})$ is cyclic and acts irreducibly on $\overline{R}$, and $\overline{B}$ has prime order q (so, β consists of the single prime q).*
- **(ii)** *Both $\overline{A}$ and $\overline{R}$ are normal in $\overline{G}$, and $\mathbf{C}_{\overline{G}}(\overline{x}) \leq \mathbf{C}_{\overline{G}}(\overline{A})$ for every non-trivial $\overline{x} \in \overline{R}$.*
- **(iii)** *$[\overline{A}, \overline{B}]\overline{BR}$ is a Frobenius group with kernel $[\overline{A}, \overline{B}]$ and $\overline{R}\mathbf{Z}(\overline{AB})$ is a non-abelian group.*

We observe that, in Theorem B, case **(i)** occurs if G has Fitting height 3, and cases **(ii)** or **(iii)** if G has Fitting height 2. In Section 4 we will discuss the various types of groups that appear in the statement of the above theorem (and of Theorem 3.3), and we will describe the structure of the corresponding graphs.

Moving now to the analysis of the finite groups G whose graph $\Delta(G)$ has two cut vertices, we recall that if, for a prime p, the Sylow p-subgroups of G are abelian, then p does not divide $|\mathbf{Z}(G) \cap G'|$ ([10, Theorem 5.3]); furthermore, if Z is a normal subgroup of G such that $Z \cap G' = 1$, then the set of conjugacy class sizes of G/Z is the same as that of G (Proposition 3.1). As a consequence, if G is an $\mathcal{A}$-group, then the set of conjugacy class sizes of G and of the factor group $\overline{G} = G/\mathbf{Z}(G)$ coincide (thus $\Delta(\overline{G}) = \Delta(G)$); moreover, $\overline{G}$ has trivial centre.

It will also be useful to introduce the notation described in the following remark.

Remark 1.1. In view of the aforementioned Corollary B of [8] (which ensures that for every finite group G the graph $\Delta(G)$ is covered by two complete subgraphs), one sees that if $\Delta(G)$ has two cut vertices r and t, then $\Delta(G)$ consists of two disjoint complete subgraphs, whose vertex sets we denote by $\gamma \cup \{r\}$ and $\delta \cup \{t\}$, which are connected in $\Delta(G)$ only by the edge $\{r, t\}$. In this setting, C and D will denote a Hall γ- and a Hall δ-subgroup of G (respectively), whereas R and T will be a Sylow r- and a Sylow t-subgroup of G, such that CR and DT are subgroups of G; such subgroups do exist, as G is solvable. We also assume, as we may up to interchanging the two sets of primes, that $|\mathbf{F}(G)/\mathbf{Z}(G)|$ is divisible by some prime in $\gamma \cup \{r\}$.

Theorem C. *Let G be a finite group such that the graph $\Delta(G)$ has two cut vertices r and t, and set $\overline{G} = G/\mathbf{Z}(G)$. Then G is an $\mathcal{A}$-group and, using the notation in Remark 1.1, we have that $\mathbf{F}(\overline{G}) = \overline{C} \times \overline{R}$, the cyclic group $\overline{DT}$ acts fixed-point freely on $[\overline{R}, \overline{D}]$, $[\overline{C}, \overline{D}] = 1$ and, for every non-trivial element $\overline{x} \in \mathbf{F}(\overline{G})$, we have $\mathbf{C}_{\overline{T}}(\overline{x}) \leq \mathbf{C}_{\overline{T}}(\overline{C})$.*

Conversely, if G is an $\mathcal{A}$-group such that $\overline{G} = G/\mathbf{Z}(G)$ satisfies the conditions above, with $\overline{C}, \overline{D}, \overline{R}, \overline{T} \neq 1$, then r and t are cut vertices of $\Delta(G)$.

Remark 1.2. As an application of the above results, in Corollary 3.4 we obtain a classification of the finite groups G such that the graph $\Delta(G)$ is acyclic (i.e., it doesn't have any cycle as an induced subgraph).

As regards the graphs having a cut vertex that can occur as $\Delta(G)$ for a finite group G, we prove what follows.

Theorem D. *Let Δ be a graph having a cut vertex. Then there exists a finite group G such that $\Delta = \Delta(G)$ if and only if Δ is connected (hence, 1-connected) and the vertex set of Δ can be partitioned in two subsets of pairwise adjacent vertices.*

To close with, we mention that the study of cut vertices for the *character degree graph* of finite groups (i.e., the graph obtained by considering the degrees of irreducible characters, instead of the sizes of the conjugacy classes) has been carried out by M.L. Lewis and Q. Meng in [11].

All the groups considered in the following discussion are tacitly assumed to be finite groups.

2. Preliminary results

For a positive integer n, we define $\pi(n)$ to be the set of prime divisors of n; if G is a group, $\pi(G)$ will stand for $\pi(|G|)$.

Next, we gather some well-known facts concerning conjugacy class sizes of a group. Given an element x of the group G, denote by x^G the conjugacy class of

x in G, and by $\pi_G(x)$ the set of prime divisors of $|x^G|$: if N is a normal subgroup of G then, for any $x \in G$, we have $\pi_{G/N}(xN) \subseteq \pi_G(x)$ and, for $y \in N$, we have $\pi_N(y) \subseteq \pi_G(y)$. Another elementary remark is that a prime number p does not belong to $\mathrm{V}(G)$ if and only if G has a central Sylow p-subgroup.

In the following proposition, we recall the description of the $\mathcal{D}$-groups, i.e. the groups G such that $\Delta(G)$ is disconnected. The notation $\mathrm{cs}(G)$ is used for the set of conjugacy class sizes of the group G.

Proposition 2.1 ([6, Theorem 4])**.** *Let G be a group, and set $Z = \mathbf{Z}(G)$. Then the graph $\Delta(G)$ is disconnected if and only if $G = AB$, where $A \trianglelefteq G$ and B are abelian Hall subgroups of G of coprime order, and G/Z is a Frobenius group with Frobenius kernel AZ/Z. In this case $\mathrm{cs}(G) = \{1, |AZ/Z|, |BZ/Z|\}$ and $\Delta(G)$ has two connected components, that are both complete graphs.*

The next lemma is well known and easy to prove. After that, we recall some statements that will come into play, dealing with non-complete vertices of $\Delta(G)$.

Lemma 2.2. *Let G be a group and let $x, y \in G$ be such that one of the following holds.*

(a) *x and y have coprime orders and they commute.*
(b) *$x \in X$ and $y \in Y$, where X and Y are normal subgroups of G such that $X \cap Y = 1$.*

Then $\pi_G(x) \cup \pi_G(y) \subseteq \pi_G(xy)$.

Given a prime p, as customary, we say that a group is p-nilpotent if it has a normal Hall p'-subgroup.

Proposition 2.3. *Let G be a group; then the following holds.*

(a) *Let p, q be non-adjacent vertices of $\Delta(G)$. Then G is either p-nilpotent or q-nilpotent, with both abelian Sylow p-subgroups and Sylow q-subgroups.*
(b) *If π is a set of vertices which are all non-adjacent to a vertex p in $\Delta(G)$, then G is π-solvable with abelian Hall π-subgroups, and the vertices in π are pairwise adjacent.*

Proof. Part (a) comes from [3, Lemma 2 and Theorem B] and part (b) from [5, Theorem C]. □

We remark that the last conclusion in part (b) of Proposition 2.3 follows from a much more general fact, that will be crucial in our discussion, and that was already mentioned in the Introduction. This is Corollary B in [8]:

Theorem 2.4. *Let G be a group. Then the vertex set of $\Delta(G)$ can be partitioned into two subsets, each inducing a complete subgraph of $\Delta(G)$.*

Lemma 2.5. *Let p, r, q be three distinct primes and let $G = PRQ$, where $P \in \mathrm{Syl}_p(G)$, $R \in \mathrm{Syl}_r(G)$, $Q \in \mathrm{Syl}_q(G)$, $RQ \leq G$, and both P and PR are normal subgroups of G. If $\{p, q\}$ is not an edge of $\Delta(G)$, then R centralizes either P or Q.*

Proof. Note that, as $PR \trianglelefteq G$, we have $R = PR \cap RQ \trianglelefteq RQ$. Also, we can assume that both p and q are vertices of $\Delta(G)$, as otherwise either P or Q are central in G. Now, Theorem 24 of [1] yields that either $R \trianglelefteq G$, and hence $[R, P] = 1$, or $PQ \trianglelefteq G$. In the latter case, as above, we have $Q = PQ \cap RQ \trianglelefteq RQ$; therefore both R and Q are normal subgroups of RQ, and $[R, Q] = 1$. □

The following lemma introduces an important characteristic subgroup of G, that we denote by $K_p(G)$, associated to a non-complete vertex p of $\Delta(G)$. Before stating it, we introduce some more notation.

Definition 2.6. For a group G, we denote by $\nu(G)$ the set of the primes $t \in \pi(G)$ such that G has a *normal* Sylow t-subgroup.

Lemma 2.7. [8, Lemma 2.3]. *Let G be a group, let p be a non-complete vertex of $\Delta(G)$ and P a Sylow p-subgroup of G. Then G is p-solvable, P is abelian, and $[G, P]$ has a normal p-complement $K_p(G)$. Furthermore, $[K_p(G), P] = K_p(G)$ and, if $p \notin \nu(G)$, then there are elements x in $K_p(G)$ such that $p \in \pi_G(x)$.*

We note that, using the bar convention in a factor group $\overline{G} = G/N$ (for $N \trianglelefteq G$), we have $\overline{[G, P]} = [\overline{G}, \overline{P}]$, so the image of $K_p(G)$ along the canonical projection is the normal p-complement of $[\overline{G}, \overline{P}]$. In particular, if p is a non-complete vertex also for $\Delta(\overline{G})$, then $\overline{K_p(G)} = K_p(\overline{G})$ holds. We also observe that $p \in \nu(G)$ if and only if $K_p(G) = 1$.

Further, we need a basic result related to the existence of regular orbits in coprime actions of abelian groups.

Lemma 2.8. [8, Lemma 2.4]. *Let G be a group such that $G/\mathbf{F}(G)$ is abelian. Then there exists an element $g \in G$ such that the set of all prime divisors of $|G/\mathbf{F}(G)|$ is contained in $\pi_G(g)$.*

Finally, we are ready to state a key preliminary result. We refer to the notation introduced in Lemma 2.7.

Proposition 2.9. *Let G be a group. Assume that p and q are non-adjacent vertices of $\Delta(G)$, and denote by P and Q a Sylow p-subgroup and a Sylow q-subgroup of G, respectively. Assume further that $M = K_p(G)$ is a minimal normal subgroup of G, and that Q is not normal in G. Then M is abelian, it has a complement in G, and the following conclusions hold.*

(a) $\mathbf{O}_q(G) = Q \cap \mathbf{C}_G(M)$.
(b) *$\overline{G} = G/\mathbf{C}_G(M)$ is a q-nilpotent group, $\mathbf{F}(\overline{G})$ is a cyclic group acting fixed-point freely and irreducibly on M, and $\overline{G}/\mathbf{F}(\overline{G})$ is cyclic as well. Also, $1 \neq \overline{P} \leq \mathbf{F}(\overline{G})$ and $\overline{Q} \cap \mathbf{F}(\overline{G}) = 1$.*
(c) *Setting $|M| = r^m$, we have that $|\overline{Q}|$ divides m; also, q does not divide $r^m - 1$, and $(r^m - 1)/(r^{m/|\overline{Q}|} - 1)$ divides $|\mathbf{F}(\overline{G})|$.*
(d) *If N is a normal subgroup of G such that $N \cap M = 1$, then $Q \leq \mathbf{C}_G(N)$.*
(e) $\mathbf{O}_p(G) = P \cap \mathbf{Z}(G)$.

Proof. This is a reformulation of Proposition 3.1 in [5] and Proposition 2.5 in [8]; the proof of [8, Proposition 2.5] includes an explanation of the fact that the hypotheses of [5, Proposition 3.1] are fulfilled under our assumptions. □

We conclude this preliminary section with an application of the tools introduced so far.

Proposition 2.10. *Let G be a group, and let α, β be non-empty and disjoint vertex subsets of $\Delta(G)$ such that there are no edges of $\Delta(G)$ having one extreme in α and the other in β. Assume also that $\nu(G) \cap \alpha = \emptyset = \nu(G) \cap \beta$ (recall Definition 2.6). Then, up to interchanging α and β, there exists a normal subgroup K of G such that $K = K_p(G)$ for all $p \in \alpha$ and $K < K_q(G)$ for all $q \in \beta$.*

Proof. For $p \in \alpha$ and $q \in \beta$, consider the subgroups $K_p = K_p(G)$ and $K_q = K_q(G)$: we will first show that, say, $K_p < K_q$. Set $N = K_p \cap K_q$ and assume, working by contradiction, that N is a proper subgroup of both K_p and K_q. In particular, p and q are both (non-complete) vertices of $\Delta(G/N)$ as well, therefore, as remarked in the paragraph following Lemma 2.7, we have $K_p/N = K_p(G/N)$ and $K_q/N = K_q(G/N)$. Now, an application of Lemma 2.7 to the factor group G/N yields that there exist two elements $x \in K_p$, $y \in K_q$ such that $p \in \pi_{G/N}(xN)$ and $q \in \pi_{G/N}(yN)$; by Lemma 2.2(b), we see that pq divides $|(xyN)^{G/N}|$, thus it divides $|(xy)^G|$ contradicting the fact that p and q are non-adjacent in $\Delta(G)$. We conclude that (say) $K_p = N$, whence $K_p \leq K_q$. Also, if L is a normal subgroup of G such that K_p/L is a chief factor of G (so, as above, $K_p/L = K_p(G/L)$), then we can apply Proposition 2.9(b) to the group G/L, obtaining that $\overline{G} = G/\mathbf{C}_G(K_p/L)$ has a normal Sylow p-subgroup, and a (non-trivial) Sylow q-subgroup intersecting $\mathbf{F}(\overline{G})$ trivially. In particular, the roles of p and q are not symmetric, and therefore the inclusion of K_p in K_q must be proper. Up to interchanging p and q, we thus have $K_p < K_q$.

Next, we claim that $K_{p_0} < K_q$ for every choice of $p_0 \in \alpha$. In fact, assuming this does not hold, the paragraph above yields $K_q < K_{p_0}$; working in the factor group $\overline{G} = G/\mathbf{C}_G(K_p/L)$ as above, by Lemma 2.8 we have that p_0 does not divide $|\overline{G}/\mathbf{F}(\overline{G})|$, so $\overline{K_{p_0}} = 1$, a contradiction as $\overline{K_{p_0}} \geq \overline{K_q} > 1$. Note that, by essentially the same argument, we can see that $K_p < K_{q_0}$ holds as well for every choice of $q_0 \in \beta$.

We work now to show that, for every choice of $p, p_0 \in \alpha$, we have $K_p = K_{p_0}$. First, let us see that one of these two subgroups is contained in the other. For a proof by contradiction, assume that $N = K_p \cap K_{p_0}$ is properly contained in both K_p and K_{p_0}. So, we can take normal subgroups L and L_0 of G, containing N, such that K_p/L and K_{p_0}/L_0 are chief factors of G. Let Q be a Sylow q-subgroup of G, where q lies in β; by Proposition 2.9(d) applied to the factor group G/L, the normal subgroup $K_{p_0}L/L$ (which intersects K_p/L trivially) is centralized by QL/L, therefore $[K_{p_0}, Q] \leq L$. But clearly $[K_{p_0}, Q]$ also lies in K_{p_0}, hence it lies in N. In particular, QL_0/L_0 centralizes K_{p_0}/L_0, and thus Proposition 2.9(a) (applied to G/L_0) yields $QL_0/L_0 \trianglelefteq G/L_0$, so $K_q \leq L_0 \leq K_p$, a contradiction by the previous paragraph. Our conclusion so far is that (say) $K_p \leq K_{p_0}$, and it remains to show that equality holds. To this end, setting $\overline{G} = G/\mathbf{C}_G(K_p/L)$, observe first that $\overline{K_{p_0}} = 1$. Otherwise, setting P_0 to be a Sylow p_0-subgroup of G, $\overline{K_{p_0}}$ would be a non-trivial (normal) p_0'-subgroup of $[\overline{G}, \overline{P_0}]$, thus $\overline{G}$ would not have a normal Sylow p_0-subgroup, yielding $p_0 \mid |\overline{G}/\mathbf{F}(\overline{G})|$; but Proposition 2.9(b) ensures that also q divides $|\overline{G}/\mathbf{F}(\overline{G})|$, so that (by Lemma 2.8) $p_0 q$ divides the size of some conjugacy class of $\overline{G}$, a contradiction. Finally, we know by Proposition 2.9 that K_p/L has a complement H/L in G/L, so, in particular, $K_{p_0} = K_p(K_{p_0} \cap H)$; as $(K_{p_0} \cap H)/L$ is normal in H/L and it centralizes K_p/L, we get that $(K_{p_0} \cap H)/L$ is a normal subgroup of G/L intersecting K_p/L trivially. An application of Proposition 2.9(d) to the factor group G/L gives $[K_{p_0} \cap H, Q] \leq L$, whence $[K_{p_0}, Q] = [K_p(K_{p_0} \cap H), Q] \leq K_p$. But now, if K_{p_0} is strictly larger than K_p, we can take a subgroup L_0 of G, containing K_p, such that K_{p_0}/L_0 is a chief factor of G. Proposition 2.9(a) applied to G/L_0 yields $[K_{p_0}, Q] \not\leq L_0$, a contradiction. We conclude that, in fact, $K_{p_0} = K_p$ holds.

Therefore, we have proved that $K = K_p < K_q$ for all $p \in \alpha$ and $q \in \beta$. $\square$

3. Proof of the main results

Our proof of Theorem A relies essentially on Theorem 2.4, and on some easy graph-theoretical considerations.

Proof of Theorem A. By Theorem 2.4, the vertex set $\mathrm{V}(G)$ of $\Delta(G)$ can be partitioned in two subsets, each inducing a complete subgraph of $\Delta(G)$: we write the part containing r as $\{r\} \cup \alpha$, and we denote by β the other one (note that both α and β are non-empty in this situation).

Since the graph $\Delta(G) - r$ is not connected, there are no edges of $\Delta(G)$ having one extreme in α and the other extreme in β. We conclude that $\Delta(G) - r$ is a graph whose connected components are the two cliques α and β, so (b) is proved.

On the other hand, since the existence of a cut vertex r for $\Delta(G)$ implies that $\Delta(G)$ is connected by Proposition 2.1, r must be adjacent to some vertex of β, and we have the following dichotomy that proves (c).

- *The cut vertex r is a complete vertex.* Then, r is obviously the unique complete vertex and the unique cut vertex of $\Delta(G)$.
- *The graph $\Delta(G)$ has no complete vertices at all.* In this situation, it follows at once that a minimal path connecting a vertex in α to a vertex (in β) not adjacent to r has length 3. Recalling that, whenever $\Delta(G)$ is connected, its diameter is at most 3 (and a characterization of groups for which the bound is attained can be found in [2]), the claim of (c) concerning the diameter is proved. Also, if t is another cut vertex of $\Delta(G)$, then it is easily seen that t lies in β, and $\{r, t\}$ is the unique edge of $\Delta(G)$ involving a vertex in $\{r\} \cup \alpha$ and a vertex in β. As a consequence, $\Delta(G)$ has at most two cut vertices.

Finally note that, in both the situations described above, the graph $\Delta(G)$ has at most one complete vertex, namely r. Therefore we can apply Theorem A, Theorem C and Theorem 1.5 of [4], which yield conclusion (a) and (c) and complete the proof. □

In order to introduce the setting for Theorem B (and its full version, Theorem 3.3), the following terminology will be useful: we say that a finite group G is *reduced* if it does not have any non-trivial normal (equivalently, central) subgroup Z with $G' \cap Z = 1$. Note that, for a reduced group G, we have $\mathrm{V}(G) = \pi(G)$.

It is not difficult to see that if $Z \leq \mathbf{Z}(G)$ is maximal with respect to the property $Z \cap G' = 1$, then G/Z is reduced. In view of the following proposition, it will be not restrictive to focus on reduced groups for the purposes of this paper.

Proposition 3.1. *Let Z be a normal subgroup of G such that $G' \cap Z = 1$. Then $Z \leq \mathbf{Z}(G)$ and the set of conjugacy class sizes of G/Z is the same as the set of conjugacy class sizes of G.*

Proof. As $[G, Z] \leq G' \cap Z$, it is clear that Z is contained in the center of G. It will be enough to show that, for every $x \in G$, we have $\mathbf{C}_{G/Z}(xZ) = \mathbf{C}_G(x)/Z$. In fact, if yZ lies in $\mathbf{C}_{G/Z}(xZ)$, we get $[x, y] \leq G' \cap Z = 1$, and therefore y lies in $\mathbf{C}_G(x)$; this proves that $\mathbf{C}_{G/Z}(xZ) \subseteq \mathbf{C}_G(x)/Z$, and equality clearly holds. □

We will now tackle the substantial part of our analysis, and we will start by treating separately, in Theorem 3.2, one of the cases that occur in Theorem 3.3. Recall that, for a group G, we defined $\nu(G)$ as the set of the primes $t \in \pi(G)$

such that G has a normal Sylow t-subgroup; also, in the following statement, $\mathbf{\Phi}(G)$ denotes the Frattini subgroup of the group G.

Theorem 3.2. *Let G be a reduced group such that $\Delta(G)$ has a cut vertex r, and let R be a Sylow r-subgroup of G. Denoting by α and β the vertex sets of the two complete connected components of $\Delta(G) - r$, assume that $\nu(G) \cap \alpha = \emptyset = \nu(G) \cap \beta$. Then, up to interchanging α and β, the following conclusions hold.*

(a) $\mathbf{F}(G) = R$.
(b) *Set $\Phi = \mathbf{\Phi}(R)$ and $K = K_p(G)$, for some $p \in \alpha$. Then we have $R/\Phi = K\Phi/\Phi \times \mathbf{Z}(G/\Phi)$, and $K\Phi/\Phi$ is a chief factor of G whose centralizer in G is R. Furthermore, setting $\overline{G} = G/R$, we have that $\mathbf{F}(\overline{G})$ is cyclic, it is the α-Hall subgroup of $\overline{G}$, and it acts fixed-point freely and irreducibly on $K\Phi/\Phi$. Finally, β consists of a single prime q, G is q-nilpotent and $|\overline{G}/\mathbf{F}(\overline{G})| = q$.*

Proof. An application of Proposition 2.10 to the sets α and β yields (up to interchanging α and β) that $K = K_p(G) \trianglelefteq G$ for all $p \in \alpha$, and $K < K_q(G)$ for all $q \in \beta$. As $\pi(G) = \{r\} \cup \alpha \cup \beta$, and $K_t(G)$ is a t'-subgroup for all $t \in \alpha \cup \beta$, we see that K is an r-group, so $K \leq R$.

Let now $L \trianglelefteq G$ be such that K/L is a chief factor of G. An application of Proposition 2.9(b) to the factor group G/L (together with Theorem 2.1 of [13]) yields that $\overline{G} = G/\mathbf{C}_G(K/L)$ is a subgroup of the group of semilinear maps $\Gamma(K/L)$ on K/L, with the cyclic group $\mathbf{F}(\overline{G})$ lying in the subgroup $\Gamma_0(K/L)$ of multiplication maps, and acting (fixed-point freely and) irreducibly on K/L. Also, we get that $\mathbf{F}(\overline{G})$ is the α-Hall subgroup of $\overline{G}$ and, taking into account Lemma 2.8, $\beta \subseteq \pi(\overline{G}/\mathbf{F}(\overline{G})) \subseteq \beta \cup \{r\}$. As we will see, it turns out that $\mathbf{C}_G(K/L)$ is in fact R. We proceed through a number of steps.

Step 1. *The order of $\overline{G}/\mathbf{F}(\overline{G})$ is a power of a prime q in β* (hence β consists of a single prime).

For a proof by contradiction, assume that $|\overline{G}/\mathbf{F}(\overline{G})|$ is divisible by two distinct primes q and t (where $q \in \beta$ and possibly $t = r$), let $\overline{Q}$ be a Sylow q-subgroup and $\overline{T}$ a Sylow t-subgroup of $\overline{G}$; setting $|K/L| = r^m$, we observe that $\overline{T}$ is cyclic, hence the order of $\mathbf{C}_{\Gamma_0(K/L)}(\overline{T})$ is $r^{m/|\overline{T}|} - 1$ (see [7, Lemma 3(i)]). Observe also that there exists a primitive prime divisor s of $r^{m/|\overline{T}|} - 1$: in fact, this is not the case only if $m/|\overline{T}| = 2$ or $r^{m/|\overline{T}|} = 2^6$. But in the former situation, by Proposition 2.9(c), we have $q = 2$ against the fact that q does not divide $r^m - 1$; on the other hand, if $r^{m/|\overline{T}|} = 2^6$, then $q = 3$ divides $2^6 - 1$, again a contradiction. Now, s is certainly a divisor of $r^m - 1$, but in fact it also divides $(r^m - 1)/(r^{m/|\overline{Q}|} - 1)$; otherwise, s is a common divisor of $r^{m/|\overline{T}|} - 1$ and $r^{m/|\overline{Q}|} - 1$, thus it divides $r^d - 1$ where $d = \text{g.c.d.}(m/|\overline{Q}|, m/|\overline{T}|)$ and (since s is a primitive prime divisor of $r^{m/|\overline{T}|} - 1$) we get that $m/|\overline{T}|$ divides $m/|\overline{Q}|$, a clear contradiction. Again by Proposition 2.9(c), it follows that s divides $|\mathbf{F}(\overline{G})|$, i.e., there exists an element $\overline{x}$ of $\mathbf{F}(\overline{G})$ whose order is s; recalling that $\Gamma_0(K/L)$ is cyclic and it has a unique subgroup of order s, we deduce that $\overline{x}$ is centralized by $\overline{T}$. Since, as already observed, s does not divide $r^{m/|\overline{Q}|} - 1 = |\mathbf{C}_{\Gamma_0(K/L)}(\overline{Q})|$, we deduce that $\overline{x}$ is not centralized by any Sylow q-subgroup of $\overline{G}$, whence q lies in $\pi_{\overline{G}}(\overline{x})$. Also, if $\overline{y}$ is a generator of $\overline{T}$, certainly $\overline{y}$ does not centralize $\mathbf{F}(\overline{G})$; as a consequence, $\pi_{\overline{G}}(\overline{y})$ contains a prime p in α. We conclude that $|(\overline{xy})^{\overline{G}}|$ is divisible by pq, which is not the case. This contradiction shows that $|\overline{G}/\mathbf{F}(\overline{G})|$ is a power of $q \in \beta$, as claimed.

Step 2. *G is q-nilpotent.*

In fact, if we assume the contrary, then G, and hence $\overline{G}$, is p-nilpotent for every p in α (see Proposition 2.3), but this implies that $\mathbf{F}(\overline{G})$ is central in $\overline{G}$, which is definitely not the case.

Step 3. *The order of $\overline{G}/\mathbf{F}(\overline{G})$ is q.*

For a proof by contradiction, assume $|\overline{G}/\mathbf{F}(\overline{G})| = q^a$ with $a > 1$. Let $\overline{Q}$ be a Sylow q-subgroup of $\overline{G}$ and consider a subgroup $\overline{Q_0}$ of $\overline{Q}$ such that $|\overline{Q_0}| = q^{a-1}$. Writing $m = q^a b$, we have $|\mathbf{C}_{\Gamma_0(K/L)}(\overline{Q_0})| = r^{bq} - 1$, whereas $|\mathbf{C}_{\Gamma_0(K/L)}(\overline{Q})| = r^b - 1$. If $|\mathbf{C}_{\mathbf{F}(\overline{G})}(\overline{Q})|$ is strictly smaller than $|\mathbf{C}_{\mathbf{F}(\overline{G})}(\overline{Q_0})|$, then we can choose $\overline{x} \in \mathbf{C}_{\mathbf{F}(\overline{G})}(\overline{Q_0})$ whose conjugacy class size in $\overline{G}$ is divisible by q; on the other hand, a generator $\overline{y}$ of $\overline{Q_0}$ does not centralize $\mathbf{F}(\overline{G})$, hence its conjugacy class in $\overline{G}$ has a size divisible by a prime $p \in \alpha$. But now we get the contradiction that pq divides $|(\overline{xy})|^{\overline{G}}$. In view of this, it will be enough to show that $|\mathbf{C}_{\mathbf{F}(\overline{G})}(\overline{Q})| < |\mathbf{C}_{\mathbf{F}(\overline{G})}(\overline{Q_0})|$ holds.

Recalling that $\Gamma_0(K/L)$ is a cyclic group, what we need to prove is

$$\text{g.c.d.}(r^b - 1,\ |\mathbf{F}(\overline{G})|) \neq \text{g.c.d.}\left((r^b-1)\left(\frac{r^{bq}-1}{r^b-1}\right),\ |\mathbf{F}(\overline{G})|\right).$$

Assuming the contrary, and considering that $(r^{bq}-1)/(r^b-1)$ is a divisor of $|\mathbf{F}(\overline{G})|$ by Proposition 2.9(c), we would get that $(r^{bq}-1)/(r^b-1)$ divides $r^b - 1$, hence $r^{bq}-1$ divides $(r^b-1)^2$. Since it is not difficult to see, as we did above, that $r^{bq}-1$ has a primitive prime divisor, we reached a contradiction, and our claim is proved.

Step 4. *R is a normal subgroup of G.*

Recalling that every prime of α is not adjacent to q in $\Delta(G)$, Proposition 2.3(b) yields that there exists an α-Hall subgroup A of G, and A is abelian. Observe also that AK is a normal subgroup of G, as $KP = [G, P]P \trianglelefteq G$ for every $P \in \mathrm{Syl}_p(G)$ and $p \in \alpha$. Choosing (again) $L \trianglelefteq G$ such that K/L is a chief factor of G, for our purposes (and for this step only) we can clearly assume that the r-subgroup L is trivial. By Proposition 2.9, we know that K has a complement H in G, and this H can be chosen to contain A, so that $A = AK \cap H$ is a normal subgroup of H. Setting $A_0 = A \cap \mathbf{C}_H(K)$ we observe that, for every $p \in \alpha$, we have $\mathbf{O}_p(A_0) \trianglelefteq H$ because $A_0 \trianglelefteq H$; but $\mathbf{O}_p(A_0)$ is clearly normalized by K as well, so we have $\mathbf{O}_p(A_0) \leq \mathbf{O}_p(G)$. Now, Proposition 2.9(e) yields that $\mathbf{O}_p(A_0)$ lies in $\mathbf{Z}(G)$ and, as this holds for every choice of $p \in \alpha$, we deduce that $A_0 \leq \mathbf{Z}(G)$; in particular, A_0 centralizes a Sylow r-subgroup R_0 of $\mathbf{C}_H(K)$. Recalling that $H/\mathbf{C}_H(K)$ is an r'-group (because $\overline{G}/\mathbf{F}(\overline{G})$ is a q-group by step 1), we have that R_0K is a Sylow r-subgroup of G, and it is enough to show that R_0 is normal in $\mathbf{C}_H(K)$ (thus in H) in order to get $R_0K \trianglelefteq G$. But the normality of R_0 in $\mathbf{C}_H(K)$ follows at once from the fact that $\mathbf{C}_H(K)$ is q-nilpotent, with normal q-complement $R_0 \times A_0$.

Step 5. *We have $\mathbf{F}(G) = R$.*

Let U be a complement for R in G: we have to show that $\mathbf{F}(G) \cap U = 1$. Setting $S = \mathbf{F}(G) \cap U$, our first remark is that S lies in $\mathbf{Z}(G)$. In fact, S is certainly normal in G; thus, writing $S = S_q \times S_\alpha$ as a direct product of its Sylow q-subgroup and its Hall α-subgroup, we have that both S_q and S_α are normal in G. But G is q-nilpotent with abelian Sylow q-subgroups, therefore S_q is central in G. On the other hand, considering the usual normal subgroup L of G such that K/L is a chief factor of G, we have that $S_\alpha L/L$ is a normal subgroup of G/L intersecting K/L trivially, so Proposition 2.9(d) yields $[S_\alpha, Q] \leq L$ where Q is a Sylow q-subgroup of G. Now, $[S_\alpha, Q] \leq L \cap S_\alpha = 1$, thus S_α is centralized by a Sylow q-subgroup of G. Since G has abelian Hall α-subgroups and S_α centralizes R, we conclude that S_α lies in $\mathbf{Z}(G)$ as well, so $S \leq \mathbf{Z}(G)$. This step can be concluded by observing that no prime divisor of $|S|$ can divide $|G' \cap \mathbf{Z}(G)|$, because G has abelian Sylow subgroups for each of these primes (see [10, Theorem 5.3]); as a consequence, $G' \cap S = 1$, and our assumption that G is a reduced group forces $S = 1$. Thus, we proved claim (a) of our statement.

Step 6. The last step is devoted to the proof of claim (b). We start by observing that, for every prime p in α and $P \in \mathrm{Syl}_p(G)$, we have $K = [R, P]$. In fact, we know that $K = [K, P] \leq [R, P]$; on the other hand, $[R, P]$ is a p'-subgroup of $[G, P]$, and it is therefore contained in the normal p-complement K of $[G, P]$. Taking into account that, as remarked in step 4, an α-Hall subgroup A of G is abelian, we thus get $K = [R, A]$. Also, an application of [5, Proposition 3.1] to the factor group G/Φ (recall that here Φ is defined as $\mathbf{\Phi}(R)$) yields that $K\Phi/\Phi$ is a minimal normal subgroup of G/Φ, so L can be chosen to be $\Phi \cap K$, and $K\Phi/\Phi$ is isomorphic to K/L as a G-module. Now, by Fitting's decomposition we have $R/\Phi = K\Phi/\Phi \times Z/\Phi$, where Z/Φ is set to be $\mathbf{C}_{R/\Phi}(A)$ (note that Z is a normal subgroup of G, as $AR \trianglelefteq G$); but since Z/L is a normal subgroup of G/L intersecting K/L trivially, Proposition 2.9(d) yields that $[Z, Q] \leq L$ (where Q is a Sylow q-subgroup of G) and, in particular, Q centralizes Z/Φ. We conclude that Z/Φ lies in $\mathbf{Z}(G/\Phi)$ (in fact, equality clearly holds), and $\mathbf{C}_G(K/L) = \mathbf{C}_G(K\Phi/\Phi) = \mathbf{C}_G(R/\Phi) = R$. Now all the remaining claims in (b) follow by the description of $\overline{G} = G/R$ that we made in the previous parts of this proof. $\square$

We are now ready to state a complete characterization of the groups G such that $\Delta(G)$ has a cut vertex. As previously remarked, it is sensible and not restrictive to focus on reduced groups.

Theorem 3.3. *Let G be a finite reduced group and assume that the graph $\Delta(G)$ has a cut vertex r. Then, denoting by α and β the vertex sets of the two complete connected components of $\Delta(G) - r$, we have $G = ABR$ where $AB \in \mathrm{Hall}_{\alpha\cup\beta}(G)$, $A \in \mathrm{Hall}_\alpha(G)$, $B \in \mathrm{Hall}_\beta(G)$, $R \in \mathrm{Syl}_r(G)$ are all non-trivial, A and B are abelian, and (up to interchanging α and β) one of the following holds.*

(I) *The Fitting subgroup $\mathbf{F}(G)$ is R, and the set β consists of a single prime q. Also, $\mathbf{F}(AB) = A$ is cyclic, and $|B| = q$, so G is nilpotent by metacyclic, of Fitting height* 3. *Furthermore, for all $x \in R$, either*
- **(i):** *$A^y \leq \mathbf{C}_G(x)$ for some $y \in R$ or*
- **(ii):** *$B^g \leq \mathbf{C}_G(x)$ for some $g \in G$ and $\mathbf{C}_A(x) \leq \mathbf{Z}(AB)$.*

(II) *$\mathbf{F}(G) = A \times R$ (so G is nilpotent by abelian, in fact metabelian if R is abelian), $Z = \mathbf{Z}(AB) < B$, AB is a $\mathcal{D}$-group, and either*
- **(IIa):** *R is abelian, $\mathbf{C}_B(R) = 1$, $Z \neq 1$ and $\mathbf{C}_B(x) \leq Z$ for every non-trivial $x \in R$; or*
- **(IIb):** *R is non-abelian and either*
  - **(IIb(i)):** *$G = R \times AB$; or*

(IIb(ii)): *$\mathbf{C}_B(x) \leq Z$ for all (non-trivial) $x \in R$ such that $|x^G|$ is not a power of r.*

(III) *Up to replacing R by a G-conjugate of it, we have that BR is a nilpotent subgroup of G; furthermore, $\mathbf{F}(G) = A \times R_0$ with $R_0 < R$, $\mathbf{C}_A(R) = 1$, and $[A, B]BR/R_0$ is a Frobenius group with kernel $[A, B]R_0/R_0$. In particular, G is metanilpotent, in fact metabelian if R is abelian; in this case, we also have $R_0 = 1$ and $\mathbf{C}_A(B) \neq 1$.*

Conversely, assume $G = ABR$, where A and B are abelian Hall subgroups of G, R is a Sylow r-subgroup of G, and $|A|$, $|B|$, $|R|$ are pairwise coprime. If the structure of G is as in **(I)**, **(II)** *or* **(III)** *above, then the prime r is a cut vertex of $\Delta(G)$.*

Proof. We start by assuming that G is a reduced group whose graph $\Delta(G)$ has a cut vertex r and, as usual, we denote by α and β the vertex sets of the two complete connected components of $\Delta(G) - r$. By Theorem A, we know that G is solvable. Let $A \in \mathrm{Hall}_\alpha(G)$, $B \in \mathrm{Hall}_\beta(G)$ and $R \in \mathrm{Syl}_r(G)$ be such that AB and AR are subgroups of G. Since no vertex of α is adjacent in $\Delta(G)$ to any vertex of β, Proposition 2.3 yields that both A and B are abelian.

Recalling that $\nu(G)$ is the set of the prime divisors t of $|G|$ such that G has a normal Sylow t-subgroup, let us first assume that $\nu(G) \cap (\alpha \cup \beta) = \emptyset$. Our aim is to show that conclusion (**I**) holds in this case. By Theorem 3.2, we have that $R = \mathbf{F}(G)$; moreover, $H = AB$ has a Sylow q-subgroup B of order q, where $\{q\} = \beta$, and a cyclic normal q-complement $A = \mathbf{F}(H)$.

Now, assume that $x \in R$ does not centralize any conjugate A^y with $y \in R$ (hence, any G-conjugate of A at all). As a consequence, there exists a prime $p \in \pi(A)$ which divides the size of x^G. Since, as remarked above, p is not adjacent to q, certainly x is centralized by a Sylow q-subgroup of G; moreover, if there exists an element w in $\mathbf{C}_A(x) \setminus \mathbf{Z}(H)$, then some prime in $\pi(H)$ has to divide $|w^H|$, and this prime is certainly q because A is abelian. But now q divides $|w^G|$ as well (because H is isomorphic to G/R), so pq divides $|(xw)^G|$, a contradiction. We deduce that $\mathbf{C}_A(x)$ lies in $\mathbf{Z}(H)$, and we get case **(I)**.

Assume now, by the symmetry of α and β, that there exists a prime $t \in \nu(G) \cap \alpha$. Note that this implies, by Proposition 2.3(a), that G is q-nilpotent for all $q \in \beta$. Hence, the β-complement AR is a normal subgroup of G, and $G' \leq AR$.

Observe first that $\nu(G) \cap \beta = \emptyset$. In fact, if $q \in \nu(G) \cap \beta$, then $G = \mathbf{C}_G(T) \cup \mathbf{C}_G(Q)$, where $T \in \mathrm{Syl}_t(G)$ and $Q \in \mathrm{Syl}_q(G)$, which is not possible. Next, we claim that $\alpha \subseteq \nu(G)$. In fact assume, working by contradiction, that $\pi = \alpha \setminus \nu(G)$ is non-empty; then, as shown in step 5 of the proof of [8, Theorem A], we have $K_q(G) < K_p(G)$ for all $q \in \beta$ and $p \in \pi$. Also, Proposition 2.10 yields that there exists $K \trianglelefteq G$ such that $K_q(G) = K$ for all $q \in \beta$. In particular, this implies that $\pi(K) \subseteq \nu(G) \cup \{r\}$. Let now $L \leq K$ be a normal subgroup of G such that K/L is a chief factor of G and let $\overline{G} = G/\mathbf{C}_G(K/L)$. Observe that, as the Fitting subgroup of G centralizes every chief factor of G, the group $\overline{G}$ is a $\nu(G)'$-group. By Proposition 2.9(b), for all $p \in \pi$ the Sylow p-subgroup $\overline{P}$ of $\overline{G}$ intersects $\mathbf{F}(\overline{G})$ trivially, and $\overline{B}$ acts fixed-point freely on K/L. As $\overline{B}$ is central in $\overline{G}$ (because G is q-nilpotent for every q in β) and $\overline{A}$ is abelian, it follows that $K_p(\overline{G}) = [\overline{G}, \overline{P}]$ is an r-group. Hence, r does not divide $|K/L|$, so K/L is a t-group for some $t \in \nu(G)$. For any non-trivial $xL \in K/L$, we have $\pi(\overline{B}) \subseteq \pi_G(x)$, so x is centralized by a Sylow p-subgroup P_0 of G. Since P_0

is not contained in $\mathbf{C}_G(K/L)$, there exists $y \in P_0$ such that $t \in \pi_G(y)$, and hence $\pi_G(xy)$ contains both $\pi(\overline{B})$ and t, a contradiction.

Hence, $\alpha \subseteq \nu(G)$ and A is a normal subgroup of G. We will show, next, that either R or AB is a normal subgroup of G. We first observe that, for every $q \in \beta$, there exists $Q \in \mathrm{Syl}_q(G)$ such that RQ is a subgroup of G. As $G_0 = PRQ$ is isomorphic to a normal section of G, the graph $\Delta(G_0)$ is a subgraph of $\Delta(G)$ so, in particular, $\{p, q\}$ is not an edge of $\Delta(G_0)$. As both P and PR are normal subgroups of G_0, by Lemma 2.5, R commutes with either P or Q; in the first case R is normal in G_0 and in the second case PQ is normal in G_0. Thus the subgroup AB is non-abelian; otherwise, either P or Q would be central in G, a contradiction. So, by a suitable choice of $p \in \alpha$ and $q \in \beta$, we can assume that $[P, Q] \neq 1$. Since PQ is either a normal subgroup of G_0 or isomorphic to a quotient of G_0, there are elements $x \in P$ and $y \in Q$ such that $q \in \pi_{G_0}(x) \subseteq \pi_G(x)$ and $p \in \pi_{G_0}(y) \subseteq \pi_G(y)$. Assume first that $[R, P] = 1$ (so $R \trianglelefteq G_0$) and let $t \in \alpha$ and $T \in \mathrm{Syl}_t(G)$. If $[R, T] \neq 1$, we consider $w \in R$, $w \notin \mathbf{C}_R(T)$ and get $\{t, q\} \subseteq \pi_G(xw)$, a contradiction. So, in this case, R commutes with A and hence R is a normal subgroup of G.

Assume, on the other hand, $[R, Q] = 1$. Let $\overline{G} = G/A$. If $[\overline{R}, \overline{B}] \neq 1$, then there is $w \in R$ and $t \in \beta$ such that $t \in \pi_{\overline{G}}(\overline{w})$. Thus, $\{t, p\} \subseteq \pi_G(yw)$, a contradiction. Therefore, in this case, $G/A \simeq R \times B$ and AB is the normal r-complement of G.

We now suppose that R is normal in G. Hence, $AR = A \times R = \mathbf{F}(G)$, because $B \cap \mathbf{F}(G) \leq \mathbf{Z}(G)$ has trivial intersection with G' and G is reduced. Let $Z = \mathbf{Z}(AB)$ and note that $Z \cap A = \mathbf{C}_A(B)$ is (by Fitting's decomposition) a central direct factor of G, so $Z = \mathbf{O}_\beta(AB) \leq B$ as G is reduced. Note that $Z < B$, as otherwise A would be central in G.

Let $b \in B \setminus Z$ and $a \in \mathbf{C}_A(b)$. If $a \neq 1$, then there exists $q \in \beta$ such that $q \in \pi_G(a)$. Also, there is $p \in \alpha$ such that $p \in \pi_G(b)$, so we get the contradiction $\{p, q\} \subseteq \pi_G(ab)$. Hence AB/Z is a Frobenius group, with kernel AZ/Z.

If $[R, B] = 1$, then $G = R \times AB$ and R is non-abelian; so we are in case **(IIb(i))**.

If R is abelian, then $\mathbf{C}_R(B)$ is a central direct factor of G and hence $\mathbf{C}_R(B) = 1$ as G is reduced. So, for every non-trivial $x \in R$ we have $\pi_G(x) \cap \beta \neq \emptyset$ and hence $\mathbf{C}_B(x) \leq \mathbf{C}_B(A) = Z$ by Lemma 2.2. Note also that in this case $Z \neq 1$, as otherwise the graph $\Delta(G)$ would be disconnected by Proposition 2.1. Finally, as $\mathbf{C}_B(R) \leq Z$ we see that $\mathbf{C}_B(R) \leq \mathbf{Z}(G)$. Since $B \cap G' = 1$ and G is reduced, we see that $\mathbf{C}_B(R) = 1$. Thus, we have case **(IIa)**.

Assume now that R is non-abelian and that $[R, B] \neq 1$. Consider an element $x \in R$ such that $|x^G|$ is not a power of r, i.e. such that $\mathbf{C}_G(x)$ does not contain any conjugate of B in G. Then there exists a prime $q \in \beta$ such that $q \in \pi_G(x)$ and again Lemma 2.2 implies that $\mathbf{C}_B(x) \leq Z$. So, we have case **(IIb(ii))**.

For the last case, assume that AB is the normal r-complement of G. By the Frattini argument we can choose $R \leq \mathbf{N}_G(B)$; therefore, BR is a subgroup of G and, since $AR \trianglelefteq G$, we have $R = AR \cap RB \trianglelefteq BR$. As a consequence, B and R are direct factors of BR (i.e., BR is nilpotent). Let $R_0 = \mathbf{O}_r(G)$, and observe that we can assume that $R_0 < R$, as otherwise $G = R \times AB$ and we are again in case **(IIb(i))**. As above we observe that, as G is reduced, we have $\mathbf{F}(G) = A \times R_0$. So, $R_0 = \mathbf{C}_{BR}(A)$. Write $A = A_0 \times C$, where $A_0 = [A, B]$ and $C = \mathbf{C}_A(B)$. We show that A_0BR/R_0 is a Frobenius group. In fact, if $x \in A_0 \setminus \{1\}$, then $q \in \pi_G(x)$ for some $q \in \beta$, and hence Lemma 2.2 implies that $\mathbf{C}_{BR}(x) \leq \mathbf{C}_{BR}(A) = R_0$. Moreover, if R is abelian, then $R_0 = 1$ as G is reduced. Hence, if $C = 1$, then

$\Delta(G)$ would be disconnected by Proposition 2.1, against our assumptions, and we reached conclusion **(III)**.

Next, we prove that if G satisfies the conditions described in the statement of the theorem, then $\Delta(G)$ has a cut vertex r.

Recall that, if $\pi_1, \pi_2, \ldots, \pi_n$ are disjoint sets of primes and g is an element of G, one can uniquely write $g = g_{\pi_1} g_{\pi_2} \cdots g_{\pi_n}$, where each g_{π_i} is a π_i-element and a power of g; we call this the *standard decomposition* of g (with respect to $\pi_1, \pi_2, \ldots, \pi_n$). Note that then $\mathbf{C}_G(g) = \bigcap_{i=1}^n \mathbf{C}_G(g_{\pi_i})$.

Let us assume **(I)**: in this case $B = Q$ is a Sylow q-subgroup of G. We first show that q is not adjacent in $\Delta(G)$ to any prime in α.

What we have to prove is that, for a fixed $p \in \pi(A)$ and $g \in G$, the size of g^G is not divisible by pq. We have the standard decomposition $g = g_r g_\alpha g_q$, where we can assume, up to conjugation in G, that $g_r \in R$, $g_\alpha \in A$ and $g_q \in Q_0$, for some $Q_0 \in \mathrm{Syl}_q(G)$. If $g_q \neq 1$, then $\langle g_q \rangle = Q_0$ (recall that $|Q_0| = q$) centralizes g, therefore $q \nmid |g^G|$. To the end of showing that $|g^G|$ is not divisible by pq we will therefore assume $g_q = 1$.

Let us consider the case when g_r is centralized by a conjugate A^v of A, with $v \in R$. Since g_α is a $\pi(A)$-element of $\mathbf{C}_G(g_r)$ and A^v is a Hall $\pi(A)$-subgroup of $\mathbf{C}_G(g_r)$, there exists $c \in \mathbf{C}_G(g_r)$ such that g_α lies in A^{vc}. But A^{vc} is abelian, so g_α is centralized by A^{vc}, as well as g_r. The conclusion is that $g = g_r g_\alpha$ is centralized by the Hall $\pi(A)$-subgroup A^{vc} of G, whence $p \nmid |g^G|$ and we are done in this case.

The last situation that has to be considered is when g_r is not centralized by A^v for any $v \in R$. Set $H = AB$. Then, by our assumptions, a G-conjugate Q^u of Q lies in $\mathbf{C}_G(g_r)$, and $\mathbf{C}_A(g_r) \leq \mathbf{Z}(H)$; in particular, we get $g_\alpha \in \mathbf{Z}(H)$, thus $o(g_\alpha) \mid |\mathbf{Z}(H)|$. Choose now an r-complement H_1 of $\mathbf{C}_G(g_r)$ which contains Q^u, and let A_1 be the (cyclic) $\pi(A)$-Hall subgroup of H_1. Since g_α is a $\pi(A)$-element of $\mathbf{C}_G(g_r)$, there exists $c \in \mathbf{C}_G(g_r)$ such that g_α lies A_1^c. Observe that $o(g_\alpha)$ divides the order of $\mathbf{Z}(H_1^c) \leq A_1^c$ and, A_1^c being cyclic, its unique subgroup of order $o(g_\alpha)$ (i.e., $\langle g_\alpha \rangle$) is forced to lie in $\mathbf{Z}(H_1^c)$. We conclude that g_α lies in $\mathbf{Z}(H_1^c)$, and therefore g_α is centralized by Q^{uc}. But Q^u lies in $\mathbf{C}_G(g_r)$, so the same holds for Q^{uc} (recall that $c \in \mathbf{C}_G(g_r)$) and Q^{uc} centralizes g_r as well. As a consequence, in this situation the size of the conjugacy class of $g = g_r g_\alpha$ in G is not divisible by q.

So we finished the proof that q is not adjacent in $\Delta(G)$ to any prime in α, which also implies (by Proposition 2.3(b)) that the vertices in α are pairwise adjacent in $\Delta(G)$.

Finally, we observe that r is a complete vertex of $\Delta(G)$. In fact, assuming the contrary, our graph would have no complete vertices, and therefore G would be metabelian by Theorem C of [4]. But this is not the case, as G has Fitting height 3. We conclude that r is a cut vertex of $\Delta(G)$ and we are done.

Let us assume now case **(II)**: $\mathbf{F}(G) = A \times R$, $Z = \mathbf{Z}(AB) < B$, and AB/Z is a Frobenius group with kernel AZ/Z (note that $Z = \mathbf{C}_B(A)$ and $AZ/Z \simeq A$).

(IIa) (R is abelian, $\mathbf{C}_B(R) = 1$ and $\mathbf{C}_B(x) \leq Z \neq 1$ for every non-trivial $x \in R$). Note that, as G is reduced, the vertex set of $\Delta(G)$ is $\alpha \cup \beta \cup \{r\}$. We first show that, for $p \in \alpha$ and $q \in \beta$, p and q are non-adjacent in $\Delta(G)$. In fact, let $g \in G$ and consider the standard decomposition $g = g_\alpha g_r g_\beta$, with $g_\alpha \in A$, $g_r \in R$ and, up to conjugation, $g_\beta \in B$. Assuming that pq divides $|g^G|$, we clearly have $p \in \pi_G(g_\beta)$, which implies $g_\beta \notin Z$. Since AB/Z is a Frobenius group with kernel AZ/Z, and

g_β commutes with g_α, we deduce that g_α must be trivial and so $g_r \neq 1$ (otherwise $g = g_\beta$ would not lie in a conjugacy class having size divisible by q). But now we get $g_\beta \in \mathbf{C}_B(g_r) \leq Z$, a contradiction. As in case **(I)**, this also implies that both α and β induce complete subgraphs of $\Delta(G)$. Finally, we observe that $\Delta(G)$ is connected by Proposition 2.1, so r is a cut vertex of G, as wanted.

Note also that, as easily seen, every element in $B \setminus Z$ has a G-conjugacy class size divisible by r and by all the primes in α, therefore $\alpha \cup \{r\}$ induces a complete subgraph of $\Delta(G)$.

(IIb(i)) ($G = R \times AB$). In this case, it is clear that $\Delta(G)$ is the join of a graph with one vertex r and a disconnected graph with connected components of vertex sets α and β.

(IIb(ii)) (R is non-abelian, and $\mathbf{C}_B(x) \leq Z$ for all $x \in R$ such that $|x^G|$ is not a power of r). Let $g \in G$, and write g in its standard decomposition as $g_\alpha g_r g_\beta$, with $g_\alpha \in A$, $g_r \in R$ and, up to conjugation, $g_\beta \in B$. Assume, working by contradiction, that $\{p, q\} \subseteq \pi_G(g)$ for some $p \in \alpha$ and $q \in \beta$; then $p \in \pi_G(g_\beta)$. Thus we have $g_\beta \notin Z$, and hence $g_\alpha = 1$, because g_α commutes with g_β and AB/Z is a Frobenius group with kernel AZ/Z. But also g_r commutes with g_β, therefore, by our assumptions, we have $\mathbf{C}_G(g_r)R = G$; in particular, there exists a Hall β-subgroup B_0 of G lying in $\mathbf{C}_G(g_r)$. Now, g_β is a β-element of G contained in $\mathbf{C}_G(g_r)$, and so there exists $c \in \mathbf{C}_G(g_r)$ such that g_β lies in B_0^c (which is abelian). As a consequence, B_0^c centralizes $g = g_r g_\beta$, and in particular $q \notin \pi_G(g)$, contradicting our assumptions. As in case **(I)**, G being not metabelian, r is a complete vertex of $\Delta(G)$ and it is therefore a cut vertex of $\Delta(G)$, as wanted.

Let us assume the last case **(III)**: BR is a nilpotent subgroup of G; also, $\mathbf{F}(G) = A \times R_0$, with $R_0 < R$, $\mathbf{C}_A(R) = 1$, and $[A, B]BR/R_0$ is a Frobenius group with kernel $[A, B]R_0/R_0$. In the case when R is abelian, in addition we have $R_0 = 1$ and $C = \mathbf{C}_A(B) \neq 1$.

As before, let $g \in G$ and consider the standard decomposition $g = g_\alpha g_{\{r\}\cup\beta}$, with $g_\alpha \in A$, and, up to conjugation, $g_{\{r\}\cup\beta} \in BR$. Assume, working by contradiction, that $\{p, q\} \subseteq \pi_G(g)$, for some $p \in \alpha$ and $q \in \beta$. Then $p \in \pi_G(g_{\{r\}\cup\beta})$. As $A = [A, B] \times C$, write also $g_\alpha = g_0 g_1$ with $g_0 \in [A, B]$ and $g_1 \in C$, and note that $g_{\{r\}\cup\beta}$ centralizes both g_0 and g_1, because $[A, B]$ and C are normal subgroups of G. Since $g_{\{r\}\cup\beta} \notin R_0 = \mathbf{C}_{BR}(A)$, our assumptions imply that $g_0 = 1$, so $g_\alpha \in C$ and hence $q \notin \pi_G(g)$, a contradiction. As usual, what we proved implies also that both α and β induce complete subgraphs of $\Delta(G)$. Finally we observe that, by Proposition 2.1, $\Delta(G)$ is connected both when R is non-abelian (in which case r, as in **(I)**, is a complete vertex of $\Delta(G)$) and when R is abelian; in fact, in the latter case, we get that $\mathbf{Z}(G) = 1$ and G is not a Frobenius group. Thus r is a cut vertex of $\Delta(G)$, and the proof is complete.

We also note that every non-trivial element in $[A, B]$ has a G-conjugacy class size divisible by r and by all the primes in β, therefore $\beta \cup \{r\}$ induces a complete subgraph of $\Delta(G)$. □

As a consequence of Theorem 3.3, we deduce Theorem B.

Proof of Theorem B. Let G be a group such that $\Delta(G)$ has a cut vertex r. By Theorem A, we know that G is solvable and that $\Delta(G) - r$ is the union of two complete graphs with vertex sets, say, α and β. Let A and B be Hall subgroups of

G for the sets of primes α and β, respectively, such that AB is a Hall $\alpha\cup\beta$-subgroup of G. Let R be a Sylow r-subgroup of G.

Let Z_0 be a subgroup of $\mathbf{Z}(G)$, maximal with respect to the condition $Z_0 \cap G' = 1$. Then $\widetilde{G} = G/Z_0$ is a reduced group and $\Delta(\widetilde{G}) = \Delta(G)$; so we can apply Theorem 3.3 to $\widetilde{G}$.

We first observe that $\widetilde{G}$ has either a normal Sylow r-subgroup or a normal r-complement, so the same holds in G. Hence, $\Delta(AB)$ is a subgraph of $\Delta(G) - r$ and therefore AB is a $\mathcal{D}$-group. We also remark that $G = ABR \times Z$, where Z is the product of the *central* Sylow subgroups of G. (Notice that, as both A and B are abelian, $(\mathbf{Z}(G) \cap AB) \times Z \leq Z_0$.)

If $\widetilde{G}$ is as in case **(IIb(i))** of Theorem 3.3, then $[R, AB] = 1$ and hence $G = Z \times R \times AB$.

If $\widetilde{G}$ satisfies the conditions in **(I)** of Theorem 3.3, then we choose $\widetilde{N} \geq \widetilde{\Phi} = \mathbf{\Phi}(\widetilde{R})$ such that $\widetilde{N}/\widetilde{\Phi} = \mathbf{Z}(\widetilde{G}/\widetilde{\Phi})$. Setting $\overline{G} = G/N$, by Theorem 3.2 we have that $\overline{R} = \mathbf{F}(\overline{G}) = \overline{\mathbf{F}(G)}$, $\overline{G}/\overline{R} \cong \overline{AB}$ and, up to interchanging α and β, $\overline{A} = \mathbf{F}(\overline{AB})$ acts irreducibly and fixed-point freely on $\overline{R}$ and $|\overline{B}| = q$. Note that $\mathbf{Z}(\overline{G}) = 1$ and hence we are in case **(i)** of Theorem B.

If $\widetilde{G}$ is of type **(II)** (but not of type **(IIb(i))** considered above) of Theorem 3.3, then both A and R are normal subgroups of G and G/R' has abelian Sylow subgroups. Setting $N/R' = \mathbf{Z}(G/R')$, then $\overline{G} = G/N$ satisfies the conditions in **(IIa)** of Theorem 3.3 with $1 \neq \overline{Z} = \mathbf{Z}(\overline{AB}) \leq \overline{B}$ and, for every non-trivial element $\overline{x} \in \overline{R}$, $\mathbf{C}_{\overline{G}}(\overline{x}) = \overline{ARZ} = \mathbf{C}_{\overline{G}}(\overline{A})$. Moreover, $\mathbf{Z}(\overline{G}) = 1$ and $\mathbf{F}(\overline{G}) = \overline{AR} = \overline{\mathbf{F}(G)}$. So, we have case **(ii)** of Theorem B.

Finally, if $\widetilde{G}$ is of type **(III)** of Theorem 3.3, then we choose $\widetilde{N} = \mathbf{O}_r(\widetilde{G})$. Observe that $\overline{G} = G/N$ has trivial centre as $[\overline{A}, \overline{B}]\overline{BR}$ is a Frobenius group and $\mathbf{C}_{\overline{A}}(\overline{R}) = 1$, and that $\overline{G}$ satisfies the conditions in **(III)** of Theorem 3.3 with $R_0 = \mathbf{O}_r(\overline{G}) = 1$. So, we have case **(iii)** of Theorem B, as either $\overline{R}$ is non-abelian or $\overline{Z} = \mathbf{Z}(\overline{AB}) = \mathbf{C}_{\overline{A}}(\overline{B}) \neq 1$ and then $[\overline{R}, \overline{Z}] \neq 1$. Clearly, $\mathbf{F}(\overline{G}) = \overline{A} = \overline{\mathbf{F}(G)}$. $\square$

Next, we prove Theorem C.

Proof of Theorem C. We assume that $\Delta(G)$ has two cut vertices r and t. By Theorem A, G is solvable and all Sylow subgroups of G are abelian. Then, as $\mathrm{cs}(G/\mathbf{Z}(G)) = \mathrm{cs}(G)$, we can assume that $\mathbf{Z}(G) = 1$; in particular, G is reduced. We adopt the notation in Remark 1.1 and we observe that, applying Theorem 3.3, G is of type **(IIa)** with respect to the cut vertex r and of type **(III)** with respect to the cut vertex t. In particular, $\mathbf{F}(G) = CR$ and G is the semidirect product of the abelian groups, of coprime orders, CR and DT. Moreover, (by type **(III)** with respect to t) DT acts fixed-point freely on $[CR, D]$ (hence DT is cyclic), $\mathbf{C}_{CR}(T) = 1$ and $\mathbf{C}_{CR}(D) \neq 1$. Take a non-trivial $x \in \mathbf{C}_{CR}(D)$. Then $t \in \pi_G(x)$ and hence (by Lemma 2.2) $\gamma \cap \pi_G(y) = \emptyset$ for every $y \in D$. It follows that D centralizes C; in particular, $[CR, D] = [R, D]$. Now, since G is of type **(IIa)** with respect to the cut vertex r, for every non-trivial $x \in R$ we have $\mathbf{C}_{DT}(x) \leq Z$, where $Z = \mathbf{Z}(CDT) = \mathbf{C}_{DT}(C) = D \times \mathbf{C}_T(C)$. Also, as CDT/Z is a Frobenius group, the same holds for every non-trivial element of C. Hence, we conclude that for every non-trivial $x \in CR$, $\mathbf{C}_T(x) \leq \mathbf{C}_T(C)$.

We now prove the converse. Since G is an $\mathcal{A}$-group, we can assume that $\mathbf{Z}(G) = 1$. Also, we assume the conditions of Theorem C (with respect to the notation in

Remark 1.1, and R, T, C, D non-trivial groups) hold for G. Then by Proposition 2.1, $\Delta(G)$ is connected.

We start by showing that, for every $p \in \gamma$ and $q \in \delta$, p and q are non-adjacent in $\Delta(G)$. Assume, working by contradiction, that there exists an element $g \in G$ such that pq divides $|g^G|$. Up to interchanging g with a suitable conjugate, we can write $g = xy$ where x and y are powers of g, with $x \in CR$ and $y \in DT$. Observe that y does not centralize C, hence we can write $y = y_0 y_1$ with $y_0 \in D$ and $y_1 \in T$, with y_1 a suitable power of y and $y_1 \notin \mathbf{C}_T(C)$. But, as $x \neq 1$, by assumption $\mathbf{C}_T(x) \leq \mathbf{C}_T(C)$, while y_1 centralizes x but does not centralize C, a contradiction.

Assume now that $t \in \pi_G(g)$ for some $g \in G$ and write $g = xy$, as above, with $x \in CR$ and $y \in DT$ powers of g. Then $x \neq 1$, hence $\mathbf{C}_T(x) \leq \mathbf{C}_T(C)$ and so, recalling that also D centralizes C, $y \in \mathbf{C}_{DT}(C)$ and $\gamma \cap \pi_G(g) = \emptyset$. This means that t is not adjacent to any prime in γ in the graph $\Delta(G)$.

Assume finally, working by contradiction, that $\{r, q\} \subseteq \pi_G(xy)$, with $x \in CR$, $y \in DT$ and $xy = yx$, for some $q \in \delta$. Then $y \neq 1$ and x does not centralize D (note that $\mathbf{C}_G(x) \trianglelefteq G$). As $x \in \mathbf{C}_{CR}(D) \times [R, D]$ (because $[CR, D] = [R, D]$), we can write $x = x_0 x_1$, with $1 \neq x_1 \in [R, D]$. Since both $\mathbf{C}_{CR}(D)$ and $[R, D]$ are normal in G, it follows that $y \in \mathbf{C}_{DT}(x_1)$, which is against the Frobenius action of DT on $[R, D]$. Hence r is only adjacent to t in $\Delta(G)$. □

We now consider the situation when $\Delta(G)$ is acyclic (i.e., it does not contain any cycle). This property clearly holds if the graph has at most two vertices; moreover, as we have a complete control of the case when $\Delta(G)$ is disconnected (Proposition 2.1), the relevant question in this context is just to classify the finite reduced groups G such that $\Delta(G)$ is acyclic, connected, with at least three vertices.

Corollary 3.4. *Let G be a finite group such that $\Delta(G)$ is connected and it has at least three vertices. Then $\Delta(G)$ is acyclic if and only if*

- **(a):** *$\Delta(G)$ is a path with three vertices $p - r - q$ and G is one of the groups described in Theorem 3.3, with $\alpha = \{p\}$ and $\beta = \{q\}$;*
- **(b):** *$\Delta(G)$ is a path with four vertices $p - r - t - q$ and G is one of the groups described in Theorem C, with $\gamma = \{p\}$ and $\delta = \{q\}$.*

Proof. Since the vertices of $\Delta(G)$ can be partitioned in two subsets each inducing a complete subgraph (Theorem 2.4), it follows immediatly that if $\Delta(G)$ is acyclic, then it is a path with at most four vertices. So we conclude by recalling Theorem 3.3 and Theorem C. □

We close this section by showing that *every 1-connected graph which is covered by two complete subgraphs does in fact occur as the graph $\Delta(G)$ for a suitable group G.* (Conversely, every graph of the kind $\Delta(G)$ which has a cut vertex is 1-connected, as observed in the Introduction, and it is covered by two complete subgraphs by Theorem 2.4.)

Proof of Theorem D. Let n, m_1 be positive integers and m_0 a non-negative integer. Let $b_0 = q_1 q_2 \cdots q_{m_0}$ and $b_1 = t_1 t_2 \cdots t_{m_1}$ where the q_i and the t_j are distinct primes (meaning also $q_i \neq t_j$, for all i, j). Let r, $p_1, p_2, \ldots, p_n$ be distinct primes such that $r \equiv 1 \pmod{b_0 b_1}$ and $p_i \equiv 1 \pmod{b_1}$ for all $1 \leq i \leq n$; note that they exist by Dirichlet's Theorem on primes in an arithmetic progression.

Let B_0 and B_1 be cyclic groups of order b_0 and b_1, and R and A cyclic groups of order r and $p_1 p_2 \cdots p_n$, respectively. Consider the semidirect product $G = (A \times R) \rtimes$

$(B_0 \times B_1)$ with respect to a Frobenius action of $B_0 \times B_1$ on R and of B_1 on A, while B_0 acts trivially on A. Then it is easily seen that the graph $\Delta(G)$ is covered by two complete subgraphs (on the sets $\{r, p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_{m_0}, t_1, \ldots, t_{m_1}\}$), and that r is a cut vertex of $\Delta(G)$ which is adjacent exactly to the primes $\{t_1, \ldots, t_{m_1}\}$ (see Figure 2).

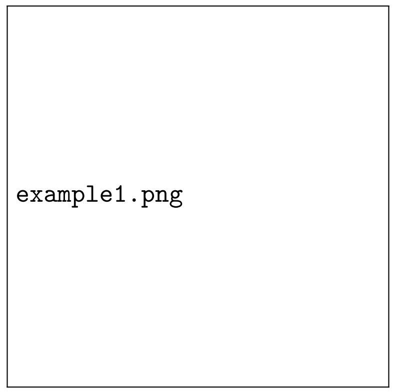

Figure 2

Observe that r is complete if and only if $G = (A \times R) \rtimes B_1$, and that there are two cut vertices if and only if $m_1 = 1$. □

4. Final remarks

Next, we take time for a closer look at the groups that appear in Theorem 3.3, also deriving some more detailed information about the associated graphs.

So, let G be a reduced group such that $\Delta(G)$ has a cut vertex r. As in Theorem 3.3, we denote by α and β the vertex sets of the two connected components of the graph $\Delta(G) - r$ (the description being given up to interchanging α and β).

First of all we stress that, in this setting, the groups as in **(I)** are characterized by the fact that they have a normal Sylow subgroup only for the prime r.

• If G is of type **(I)**, then by Theorem 3.2, writing $Z/\mathbf{\Phi}(R)$ for the centre of $G/\mathbf{\Phi}(R)$, the factor group G/Z is isomorphic to a subgroup of the affine semilinear group $\mathrm{A}\Gamma(R/Z)$ (see [13, Chapter 1, Section 2]). Also, the Fitting subgroup of G/Z is R/Z and, if F/Z is the second Fitting subgroup of G/Z, then F/R is the cyclic Hall α-subgroup of G/R; as for the top section G/F, it is a group of order q. Furthermore, we observe that $\Delta(G/Z)$ is the same as $\Delta(G)$.

It is worth remarking that, as shown by the family of examples in the following paragraph, the nilpotency class of R can be large, as well as the size of $\mathrm{cs}(G)$.

Let r and q be distinct primes and n a positive integer. In [9] and [12] an interesting class of r-groups $R(n, r, q)$ is described, depending on the three paramenters r, q, n satisfying the additional condition $q > \max\{n, r\}$. With these assumption, the r-group $R = R(n, r, q)$ has order r^{nq}, nilpotency class n (each factor of the upper (or lower) central series being elementary abelian of order r^q) and derived length $\lceil \log_2(n+1) \rceil$. On R there is an action of a Frobenius group AB with cyclic kernel A of order $(r^q - 1)/(r - 1)$ and complement B of order q (see [9, Section 4] or [12, Sections 2 and 5]). The action of A on R is fixed-point free, while every element of R is centralized by a conjugate of B in AB ([9, Lemma 4.8]). Moreover, $\mathrm{cs}(R) = \{(r^{q-1})^k \mid k = 0, 1, \ldots, n-1\}$ ([12, Corollary 3.4]) and $|\mathbf{C}_{RA}(x)| = r^n$ for every non-trivial element $x \in B$ ([12, page 210]). Therefore, setting $G = RAB$, it turns out that $\mathrm{cs}(G) = \{qr^{qn}, (r^{(q-1)k})|A|, k = 0, 1, \ldots, n\}$.

As regards the groups in classes **(II)** and **(III)**, they share the property of having normal Sylow subgroups for all the primes in α, whereas the Sylow subgroups for the primes in β are all non-normal. If, in this situation, the group has an abelian normal Sylow r-subgroup, then it lies in **(IIa)**; if it has a non-abelian normal Sylow r-subgroup, then we are in case **(IIb)**. On the other hand, if the group does not have a normal Sylow r-subgroup, then it belongs to class **(III)**.

Some more remarks:

• For a group G as in **(IIa)**, the cut vertex r need not be a complete vertex of $\Delta(G)$. If it is not, as observed in Theorem A, the graph $\Delta(G)$ has diameter 3.

More specifically, r is adjacent to all the primes in α, but it can be non-adjacent to some prime in β: in order to have a better understanding of $\Delta(G)$ in this case, we characterize next the set $\beta^* \subseteq \beta$ of the vertices of our graph that are non-adjacent to r.

Let R be the Sylow r-subgroup of G and, for $q \in \beta$, let Q be in $\mathrm{Syl}_q(B)$. We claim that q lies in β^* if and only if $Q \leq Z = \mathbf{C}_B(A)$ and B acts fixed point-freely on $[R, Q]$. In fact, if $Q \not\leq Z$, then $q \in \pi_G(x)$ for some element $x \in A$. Consider a non-trivial element $y \in Z$ (recall that $Z \neq 1$); then $r \in \pi_G(y)$ and hence $\{r, q\} \subseteq \pi_G(xy)$. If, on the other hand, there exist non-trivial and commuting elements $x \in [R, Q]$ and $y \in B$, then $\pi_G(xy) \supseteq \pi_G(x) \cup \pi_G(y) \supseteq \{r, q\}$ (recall that $\mathbf{C}_B(R) = 1$).

Conversely, let $g = g_\alpha g_r g_\beta$ be the standard decomposition of g, where we can assume, up to conjugation, $g_\alpha \in A$, $g_r \in R$ and $g_\beta \in B$. Assume that $\{r, q\} \subseteq \pi_G(g)$ and that B acts fixed point-freely on $[R, Q]$. As $r \in \pi_G(g)$, then $g_\beta \neq 1$; so, using the Fitting decomposition of the abelian group R with respect to the action of Q, we get $g_r \in \mathbf{C}_R(Q)$. Thus $q \in \pi_G(g)$ implies $Q \not\leq \mathbf{C}_B(g_\alpha)$, and hence $Q \not\leq Z$.

• For the groups in **(IIb)** (as well as for those as in **(I)**), the cut vertex r is a complete vertex of $\Delta(G)$.

A small example of type **(IIb(ii))** is the following: let p, q, r be distinct primes such that q divides both $p - 1$ and $r - 1$, R be an extraspecial group of order r^3 and exponent r, and A and B cyclic groups of order p and q, respectively. Then there is a Frobenius action of B on A, and an action of B on R with $\mathbf{C}_R(B)$ (of order r) non-normal in R. Then the semidirect product $G = (A \times R)B$ is of type **(IIb(ii))**, and $\Delta(G)$ is the path $p - r - q$.

• Finally, let G be as in **(III)**. Then the cut vertex r is always adjacent in $\Delta(G)$ to all the vertices in β, and it is a complete vertex if a Sylow r-subgroup R of G is non-abelian. On the other hand, if R is abelian, r can be non-adjacent to some prime in α (and, if this happens, then $\Delta(G)$ has diameter 3): as we did for class **(IIa)**, we characterize next the set $\alpha^* \subseteq \alpha$ of the vertices of $\Delta(G)$ that are non-adjacent to r in this case.

For $p \in \alpha$ and $P \in \mathrm{Syl}_p(A)$, we show that $p \in \alpha^*$ if and only if $P \leq C = \mathbf{C}_A(B)$ and $\mathbf{C}_R(x) \leq \mathbf{C}_R(P)$ for all non-trivial $x \in A$. In fact, if $P \not\leq C$, then there exists $y \in B$ such that $p \in \pi_G(y)$. Considering a non-trivial $x \in C$, we have $r \in \pi_G(x)$

(as $\mathbf{C}_A(R) = 1$) and hence $\{p, r\} \subseteq \pi_G(xy)$. If, on the other hand, there exist non-trivial elements $y \in R \setminus \mathbf{C}_R(P)$ and $x \in \mathbf{C}_A(y)$, then again $\{p, r\} \subseteq \pi_G(xy)$.

Conversely, let $g = g_\alpha g_r g_\beta$ be the standard decomposition of g, where we can assume, up to conjugation, $g_\alpha \in A$, $g_r \in R$ and $g_\beta \in B$. Assume that $\{p, r\} \subseteq \pi_G(g)$ and that $\mathbf{C}_R(x) \leq \mathbf{C}_R(P)$ for all non-trivial $x \in A$. As R does not centralize g, then $g_\alpha \neq 1$ and hence $g_r \in \mathbf{C}_R(P)$. So $g_\beta \notin \mathbf{C}_G(P)$ and hence $P \not\leq C$.

We conclude by stressing that, unlike the case of the groups G whose graph $\Delta(G)$ is disconnected (where G has only two non-central conjugacy class sizes (see Proposition 2.1)), the condition that $\Delta(G)$ has a cut vertex does not impose an upper bound on $|\mathrm{cs}(G)|$ in any of the cases of Theorem 3.3. This has already been remarked above for groups of type **(I)**.

For groups of type **(IIa)** and $x \in R$, one has $|x^G| = |x^{RZ}||B/Z|$, where $Z = \mathbf{Z}(AB) < B$, and it is not difficult to build examples with arbitrarily many distinct $|x_i^{RZ}|$ for $x_i \in R$. For instance, we can define R as the direct product $R_1 \times \cdots \times R_n$ of cyclic groups of prime order r; also, let $C_0, \ldots, C_n$ be cyclic groups of prime order (for various primes) such that there exists a fixed-point free action of every C_i on a cyclic group of order r. Setting now $Z = C_1 \times \cdots \times C_n$, $B = C_0 \times Z$, and defining an action of B on R so that RC_0, as well as the R_iC_i, are Frobenius groups and $RZ = R_1C_1 \times \cdots \times R_nC_n$, we have an example as discussed above for any suitable choice of A (as prescribed for type **(IIa)**).

As another (obvious) example of type **(IIb(i))**, we can just choose a group R such that $|\mathrm{cs}(R)|$ is large.

Finally, if G is of type **(III)** and $x \in R$, $x \notin R_0$, then $|x^G| = |x^{RZ}||[A, B]|$, where $Z = \mathbf{Z}(AB) = \mathbf{C}_A(B)$. In this case, in order to have many class sizes, one can consider $R_0 = 1$, $R = \langle x \rangle$ a cyclic group of order r^n, (n a positive integer) acting on $Z = C_1 \times \cdots \times C_n$ such that $\mathbf{C}_Z(x^{r^i}) = C_{n-i+1} \times \cdots \times C_n$, for $i = 1, \ldots, n$ and $\mathbf{C}_Z(R) = 1$.

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