



Backus problem in geophysics: a resolution near the dipole in fractional Sobolev spaces

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Abstract. We consider Backus's problem in geophysics. This consists in reconstructing a harmonic potential outside the Earth when the intensity of the related field is measured on the Earth's surface. Thus, the boundary condition is (severely) nonlinear. The gravitational case is quite understood. It consists in the local resolution near a monopole, i.e. the potential generated by a point mass. In this paper, we consider the geomagnetic case. This consists in linearizing the field's intensity near the so-called dipole, a harmonic function which models the solenoidal potential of a magnet. The problem is quite difficult, because the resolving operator related to the linearized problem is generally unbounded. Indeed, existence results for Backus's problem in this framework are not present in the literature. In this work, we locally solve the geomagnetic version of Backus's problem in the axially symmetric case. In mathematical terms, we show the existence of harmonic functions in the exterior of a sphere, with given (boundary) field's intensity sufficiently close to that of a dipole and which have the same axial symmetry of a dipole. We also show that unique solutions can be selected by prescribing the average of the potential on the equatorial circle of the sphere. We obtain those solutions as series of spherical harmonics. The functional framework entails the use of fractional Sobolev Hilbert spaces on the sphere, endowed with a spectral norm. A crucial ingredient is the algebra structure of suitable subspaces.

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1. Introduction

One of the most interesting problems in geophysics is the determination of the gravitational or magnetic field of the Earth from surface measurements of its intensity. It is in fact much more convenient to measure field intensities rather

than field directions. In mathematical terms, the problem can be formulated as follows.

We shall represent the Earth's surface by the unit sphere $\mathcal{S} \subset \mathbb{R}^3$ (centered at the origin). We will denote by Ω the exterior of \mathcal{S} , i.e. the unbounded component of $\mathbb{R}^3 \setminus \mathcal{S}$. Also, we suppose a positive continuous function g is given on \mathcal{S} . If u represents the Earth's external scalar potential associated to the gravitational or magnetic field, then $|\nabla u|$ represents the field's intensity. Backus's problem then consists in finding solutions $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of the following nonlinear boundary value problem:

$$\Delta u = 0 \text{ in } \Omega, \quad |\nabla u| = g \text{ on } \mathcal{S}, \quad u \rightarrow 0 \text{ as } |x| \rightarrow \infty. \quad (1.1)$$

It is also important to specify sufficient conditions that ensure the uniqueness of a solution.

A large part of the results known about this problem is contained in the pioneering work of G. E. Backus [4]. There, the problem of finding solutions of the first two equations in (1.1) is first examined for the two-dimensional case in a bounded domain and, by conformal mappings, also in exterior domains. A complete analysis is carried out: existence of solutions is proved, a severe lack of uniqueness is pointed out, and conditions which restore uniqueness are stated. Related to this case, questions about the regularity of solutions are investigated in [12], together with an analysis of the case in which the data g may vanish at isolated points.

In [4], a condition for uniqueness of a solution for problem (1.1) is given in the gravitational case. Non-uniqueness for the geomagnetic case is noted in [5]; [10] contains a general uniqueness condition for (1.1), which includes both the gravitational and geomagnetic case. The solution of the linearized problem near the dipole is also constructed in [10] by means of an expansion in spherical harmonics. By a similar technique, the linearization near quadripoles is considered in [1] and solved.

For what concerns the existence of a solution of (1.1) in physical dimension, a first conclusive result is contained in [9]. There, it is proved local existence and uniqueness of a solution in the gravitational case. It is obtained by linearizing (1.1) at the so-called *monopole*,

$$\Phi(x) = \frac{1}{|x|}, \quad x \in \bar{\Omega},$$

which is nothing else than a normalized version of the fundamental solution of Laplace's equation. More precisely, one can prove existence and uniqueness of a solution of the form

$$u(x) = \Phi(x) + w(x), \quad x \in \bar{\Omega}, \quad (1.2)$$

provided g is sufficiently close to $|\nabla \Phi| \equiv 1$ in the norm of $C^{0,\alpha}(\mathcal{S})$, $0 < \alpha < 1$, the space of α -Hölder continuous functions on \mathcal{S} . In fact, plugging the *ansatz* (1.2) into (1.1) gives that w must solve the problem:

$$\Delta w = 0 \text{ in } \Omega, \quad w_\nu + \frac{1}{2} |\nabla w|^2 = \frac{g^2 - 1}{2} \text{ on } \mathcal{S}, \quad w \rightarrow 0 \text{ as } |x| \rightarrow \infty. \quad (1.3)$$

Here, ν denotes the unit normal to \mathcal{S} , exterior to Ω .

In [9] (see also [13]), the nonlinear problem (1.3) is solved by a Neumann series, essentially based on the following fixed point argument. In fact, one can introduce a (nonlinear) operator \mathcal{T}_Φ by formally setting

$$\mathcal{T}_\Phi[f] = |\nabla v|^2,$$

where v is the solution of the Neumann problem:

$$\Delta v = 0 \text{ in } \Omega, \quad v_\nu = \frac{1}{2} f \text{ on } \mathcal{S}, \quad v \rightarrow 0 \text{ as } |x| \rightarrow \infty. \quad (1.4)$$

Since we know that (1.4) always has a unique solution $v \in C^{1,\alpha}(\bar{\Omega})$ for any $f \in C^{0,\alpha}(\mathcal{S})$, then \mathcal{T}_Φ turns out to be well-defined as an operator on $C^{0,\alpha}(\mathcal{S})$ into itself. The problem (1.3) is thus converted into the fixed-point equation:

$$f + \mathcal{T}_\Phi[f] = g^2 - 1 \text{ on } \mathcal{S}.$$

This can be uniquely solved by a function $f^* \in C^{0,\alpha}(\mathcal{S})$, provided $g^2 - 1$ is sufficiently small in the norm of $C^{0,\alpha}(\mathcal{S})$. A solution of (1.1) is therefore obtained by means of (1.2), where w is the solution of (1.4) corresponding to f^* .

We conclude our review of known results with a couple of papers, [7,8], which provide a genuinely nonlinear approach to problem (1.1). In [7], (1.1) is converted into a boundary value problem in the unit ball B :

$$\Delta U = 0 \text{ in } B, \quad (U + U_\nu)^2 + |\nabla_{\mathcal{S}} U|^2 = g^2 \text{ on } \mathcal{S}, \quad (1.5)$$

where $\nabla_{\mathcal{S}} U$ denotes the tangential gradient of U on \mathcal{S} . Here, U is the Kelvin's transformation of u , which is such that

$$u(x) = |x|^{-1} U(x/|x|^2) \text{ for } x \in \Omega.$$

Then, a solution of (1.5) is obtained by solving the following boundary value problem:

$$\Delta U = 0 \text{ in } B, \quad U + U_\nu = \sqrt{(g^2 - |\nabla_{\mathcal{S}} U|^2)_+} \text{ on } \mathcal{S}. \quad (1.6)$$

Since $U + U_\nu = -u_\nu$ on \mathcal{S} , the corresponding solution u of (1.1) is such that $u_\nu \leq 0$ on \mathcal{S} . It turns out that harmonic functions in Ω , which vanish at infinity and are subject to the constraint $u_\nu \leq 0$ on \mathcal{S} , satisfy some sort of comparison principle. This property is then instrumental to the definition of maximal and minimal solutions of (1.1) such that $u_\nu \leq 0$ on \mathcal{S} . This fact allows the construction of a suitably defined viscosity solution of (1.6). In [8], a numerical scheme to construct a maximal solution is proposed.

The aim of this paper is to study the local resolution of the geomagnetic case, i.e. the (local) existence and uniqueness of solutions of (1.1) near the *dipole* defined by

$$d(x) = \frac{x_3}{|x|^3} \text{ for } x \in \bar{\Omega}.$$

In spherical coordinates $x = r(\cos \theta \cos \varphi, \cos \theta \sin \varphi, \sin \theta)$, d can be written as

$$d = \frac{\sin \theta}{r^2}.$$

Here $r = |x|$, $-\pi/2 \leq \theta \leq \pi/2$ is the *latitude*, and $-\pi \leq \varphi < \pi$ is the *longitude* on the Earth \mathbb{S} . To the best of our knowledge, existence results for this problem are not present in the literature.

Thus, similarly to the gravitational case, we linearize problem (1.1) by setting $u = d + w$ and obtain the following problem for w :

$$\begin{aligned} \Delta w &= 0 \quad \text{in } \Omega, \\ \nabla d \cdot \nabla w + \frac{1}{2} |\nabla w|^2 &= \frac{g^2 - |\nabla d|^2}{2} \quad \text{on } \mathbb{S}, \\ w &\rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned} \tag{1.7}$$

Note that, being as $\nu(x) = -x$, we have that

$$\nabla d(x) = \tau(x) + 2x_3 \nu(x) \quad \text{for } x \in \mathbb{S}. \tag{1.8}$$

The vector field

$$\tau(x) = e_3 + x_3 \nu(x) \quad \text{for } x \in \mathbb{S},$$

is tangential to S and is obtained by projecting $e_3 = (0, 0, 1)$ on the tangent plane of \mathbb{S} at $x \in \mathbb{S}$. Notice that $\nabla d(x)$ has intensity $|\nabla d(x)| = \sqrt{1 + 3x_3^2}$ for $x \in \mathbb{S}$, and points outward to the Earth’s surface on the south hemisphere, becomes tangential on the equator $\mathcal{E} = \{x \in \mathbb{S} : x_3 = 0\}$, and points inward on the north hemisphere. This behavior of ∇d tells us that neither d nor any solution of (1.1) sufficiently close to d falls within the class of solutions studied in [7, 8].

Proceeding as in the monopole case gives the (*irregular*) *oblique derivative* problem:

$$\Delta v = 0 \quad \text{in } \Omega, \quad \nabla d \cdot \nabla v = \frac{1}{2} f \quad \text{on } \mathbb{S}, \quad v \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \tag{1.9}$$

Differently from *regular* oblique derivative problems, in which the relevant directional derivative is controlled by a vector field that points either inward or outward *on the whole boundary*, the irregular ones may present at least two setbacks. These are caused precisely by a change of direction, as described for ∇d .

The former is a non-trivial lack of uniqueness. In fact, uniqueness can be obtained only by prescribing Dirichlet boundary values on the subset of the boundary in which the relevant vector field becomes tangential (the equator \mathcal{E} in the problem at stake). In other words, the homogeneous problem related to (1.9) has infinitely many solutions.

The latter is the so-called *loss of derivatives*. In fact, it may happen that suitably determined unique solutions of oblique boundary value problems with, say, $C^{0,\alpha}$ -regular boundary data, do not gain $C^{1,\alpha}$ -regularity up to the boundary, as it does happen for the Neumann problem or the regular oblique derivative problem (see for instance [2, 15, 16]). In other words, the linear operator on $C^{0,\alpha}(\mathbb{S})$ associating the oblique derivative data to the (trace on the boundary of the) solution of the problem may be unbounded. A similar behavior also occurs if we try to solve the oblique derivative problem in the scale of fractional Sobolev spaces $H^s(\mathbb{S})$ (this fact can be detected by an inspection of the solution obtained in [10]).

Besides causing a loss of surjectivity of the relevant operator, more importantly, the loss of derivatives disrupts the iterative scheme on which a classical contraction argument is based. Thus, the strategy of defining an operator $\mathcal{J}_{d,h}$ by first setting

$$\mathcal{J}_{d,h}[f] = |\nabla v|^2,$$

where, for some fixed $h : \mathcal{E} \rightarrow \mathbb{R}$, v is the solution of (1.9), subject to the Dirichlet-type condition

$$v = h \quad \text{on } \mathcal{E}, \tag{1.10}$$

and then solving the equation

$$f + \mathcal{J}_{d,h}[f] = g^2 - |\nabla d|^2,$$

may miserably fail.

To by-pass these difficulties, a standard idea would be to use the *Nash-Moser implicit function theorem*. This often works when a loss of derivatives occurs. The second author and M. C. Jorge have tried this pathway, but with no success. The main difficulty is the lack of sufficiently precise estimates for the relevant oblique derivative problems involved. In fact, in such approach, one would need to precisely control estimates not only for the solution of (1.9), but also for a class of oblique derivative problems obtained by perturbing ∇d .

In this paper, we turn back to a fixed-point approach and obtain local existence near the dipole for the nonlinear problem (1.1) in the case the boundary data g is *axially symmetric* around the Earth's axis. This result is obtained in the framework of fractional Sobolev spaces and is made possible from the discovery that the relevant oblique derivative problem (1.9) no longer loses derivatives in presence of axially symmetric data. Hence, a fixed-point scheme still works for problem (1.7).

From a technical viewpoint, we construct solutions of (1.9)-(1.10) by means of series expansions of spherical harmonics as done in [10]. This time, we trace more carefully the dependence on the data f and h of the coefficients of the relevant expansion. The aim is to obtain precise estimates for the operator $\mathcal{J}_{d,h}$ in the scale of Sobolev spaces $H^s(\mathcal{S})$. It turns out that $\mathcal{J}_{d,h}$ is well-defined as an operator on the subspace $H_{\text{ax}}^s(\mathcal{S})$ of axially symmetric functions on \mathcal{S} . (For a precise definition of $H^s(\mathcal{S})$ and $H_{\text{ax}}^s(\mathcal{S})$, see Sect. 2.)

Our main result is then the following existence and uniqueness theorem.

Theorem 1.1. *Suppose that $g \in H_{\text{ax}}^s(\mathcal{S})$ for some $s > 1$ and that $h \equiv h_0$ on \mathcal{S} with $h_0 \in \mathbb{R}$. If $\|g - |\nabla d|\|_{H^s(\mathcal{S})}$ and $|h_0|$ are sufficiently small, then problem (1.1) has a unique solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying (1.10). Moreover, we have that $u|_{\mathcal{S}} \in H_{\text{ax}}^{s+1}(\mathcal{S})$ and*

$$\|u - d\|_{H^{s+1}(\mathcal{S})} \leq C (\|g - |\nabla d|\|_{H^s(\mathcal{S})} + |h_0|) \tag{1.11}$$

for some constant $C > 0$.

From the continuous embedding of $H^s(\mathcal{S})$ into $C(\mathcal{S})$ (see Proposition 2.1 below) and the maximum principle for harmonic functions, we see that (1.11) holds if the left-hand side is replaced with $\|u - d\|_{C(\bar{\Omega})}$.

We conclude this introduction with one more important technical remark about the proof of Theorem 1.1.

In fact, it should be noticed that the Hilbert-space structure we adopt for $H^s(\mathcal{S})$ and $H_{\text{ax}}^s(\mathcal{S})$ is based on an inner product of a spectral-type. In other words, the relevant inner product is defined in terms of the coefficients in the spherical-harmonics series expansions of the functions at stake. In order to deal with problem (1.7), which shows a *quadratic nonlinearity* in the gradient, a *Banach-algebra structure* for $H^s(\mathcal{S})$ is desirable. As a matter of fact, Banach-algebra structures for Sobolev spaces on Riemannian manifolds are systematically studied in [6] (see [6, Theorem 24] for the most relevant result), in which the norm is defined by

$$\|u\|_{H^{s,p}(\mathcal{S})} = \|u\|_{L^p(\mathcal{S})} + \|(-\Delta)^{s/2}u\|_{L^p(\mathcal{S})},$$

where the fractional Laplace-Beltrami operator $(-\Delta)^{s/2}$ is defined by the Dunford integral. It is shown that $(-\Delta)^{s/2}u$ is comparable to an integral representation of the fractional derivative of u , and hence an algebra structure is derived by Leibniz' rule. However, to the best of our knowledge, there is no explicit comparison between our spectral norm $\|u\|_{H^s(\mathcal{S})}$ and $\|u\|_{H^{s,2}(\mathcal{S})}$ in the literature. Thus, in Theorem 2.4, we provide our own proof for the case of axially symmetric functions. This is based on the series expansions in spherical harmonics of products of spherical harmonics, and the so-called *Wigner 3j-symbols*. (In Proposition 2.5, we show that such a structure also holds in other instances.)

The paper is organized as follows. We begin with Sect. 2, in which we collect all the functional analytical results on the spaces $H^s(\mathcal{S})$ and $H_{\text{ax}}^s(\mathcal{S})$ useful for our purposes. Then, in Sect. 3, we derive the appropriate estimates for the relevant oblique derivative problem. The proof of Theorem 1.1 is given in Sect. 4.

2. The fractional spectral Sobolev space $H^s(\mathcal{S})$ and its subspace $H_{\text{ax}}^s(\mathcal{S})$

In this section, we collect old and new results on the spectral Hilbert space $H^s(\mathcal{S})$.

2.1. Spherical harmonics and the space $H^s(\mathcal{S})$

As mentioned in the introduction, we adopt a system of spherical coordinates by setting

$$x = r (\cos \theta \cos \varphi, \cos \theta \sin \varphi, \sin \theta), \quad r > 0, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad -\pi \leq \varphi < \pi;$$

we call θ the *latitude* and φ the *longitude* on the earth surface \mathcal{S} . With this parametrization, the surface element on \mathcal{S} is given by $dS_x = \cos \theta \, d\theta \, d\varphi$.

We denote by $\{Y_l^m(\theta, \varphi)\}_{|m| \leq l, l=0,1,\dots}$ the spherical harmonics system of functions. We have that

$$Y_l^m(\theta, \varphi) = \alpha_l^m P_l^{|m|}(\sin \theta) e^{im\varphi} \quad (|m| \leq l, \quad l = 0, 1, \dots),$$

where $P_l^m(z)$ is the associated Legendre polynomial of degree l and order m , and α_l^m is defined by

$$\alpha_l^m = (-1)^{\frac{m+|m|}{2}} \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}.$$

It is well-known that $\{Y_l^m(\theta, \varphi)\}_{|m|\leq l, l=0,1,\dots}$ forms an orthonormal basis of $L^2(\mathbb{S})$, that is, the equality

$$\begin{aligned} \psi(\theta, \varphi) &= \sum_{l=0}^{\infty} \sum_{m=-l}^l \widehat{\psi}_l^m Y_l^m(\theta, \varphi) \quad \text{with} \\ \widehat{\psi}_l^m &= \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi(\theta, \varphi) \overline{Y_l^m(\theta, \varphi)} \cos \theta \, d\theta d\varphi \end{aligned}$$

holds in $L^2(\mathbb{S})$ for any function $\psi \in L^2(\mathbb{S})$.

For any non-negative real number s , we define the *fractional Sobolev space*:

$$H^s(\mathbb{S}) = \left\{ \psi \in L^2(\mathbb{S}) : \sum_{l=0}^{\infty} \sum_{m=-l}^l (l+1)^{2s} |\widehat{\psi}_l^m|^2 < +\infty \right\}.$$

Thus, we have that $H^0(\mathbb{S}) = L^2(\mathbb{S})$ and we know that $H^s(\mathbb{S})$ is a Hilbert space endowed with the inner product

$$\langle \psi, \phi \rangle_{H^s(\mathbb{S})} = \sum_{l=0}^{\infty} \sum_{m=-l}^l (l+1)^{2s} \widehat{\psi}_l^m \overline{\widehat{\phi}_l^m}.$$

Similarly, we set

$$\begin{aligned} H^s(\mathcal{E}) &= \left\{ \Psi \in L^2(\mathcal{E}) : \sum_{m=-\infty}^{\infty} (|m|+1)^{2s} |\widehat{\Psi}_m|^2 < +\infty \right\}, \\ \text{where } \widehat{\Psi}_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(\varphi) e^{-im\varphi} \, d\varphi, \end{aligned}$$

and write the norm on $H^s(\mathcal{E})$ as

$$\|\Psi\|_{H^s(\mathcal{E})} = \sqrt{\sum_{m=-\infty}^{\infty} (|m|+1)^{2s} |\widehat{\Psi}_m|^2}.$$

Then, any function $\Psi \in H^s(\mathcal{E})$ can be expanded in $H^s(\mathcal{E})$ as

$$\Psi(\varphi) = \sum_{m=-\infty}^{\infty} \widehat{\Psi}_m e^{im\varphi}.$$

The following properties of $H^s(\mathbb{S})$ will be useful in the sequel.

Proposition 2.1. *Set $s > 1$. Then, the Sobolev space $H^s(\mathbb{S})$ is continuously embedded into $C(\mathbb{S})$.*

Proof. It is known that the identity

$$\begin{aligned} & \sum_{m=-l}^l Y_l^m(\theta_1, \varphi_1) \overline{Y_l^m(\theta_2, \varphi_2)} \\ &= \frac{2l+1}{4\pi} P_l(\sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos(\varphi_2 - \varphi_1)) \end{aligned} \tag{2.1}$$

holds for any $l = 0, 1, \dots$, $-\pi/2 \leq \theta_1, \theta_2 \leq \pi/2$, and $-\pi \leq \varphi_1, \varphi_2 < \pi$ (see [3, (16.57), (16.59)]). In particular, this identity and the fact that $P_l(1) = 1$ give

$$\sum_{m=-l}^l |Y_l^m(\theta, \varphi)|^2 = \frac{2l+1}{4\pi}. \tag{2.2}$$

Hence, by the Cauchy-Schwarz inequality, we have that

$$\sum_{m=-l}^l |\widehat{u}_l^m Y_l^m(\theta, \varphi)| \leq \sqrt{\frac{2l+1}{4\pi}} \sqrt{\sum_{m=-l}^l |\widehat{u}_l^m|^2}.$$

Taking the sum in l and using the Cauchy-Schwarz inequality again, we obtain

$$\begin{aligned} \sum_{l=0}^{\infty} \sum_{m=-l}^l |\widehat{u}_l^m Y_l^m(\theta, \varphi)| &\leq \sqrt{\sum_{l=0}^{\infty} \frac{(l+1)^{-2s}(2l+1)}{4\pi}} \sqrt{\sum_{l=0}^{\infty} \sum_{m=-l}^l (l+1)^{2s} |\widehat{u}_l^m|^2} \\ &\leq \sqrt{\sum_{l=0}^{\infty} \frac{(l+1)^{1-2s}}{2\pi}} \|u\|_{H^s(\mathcal{S})}. \end{aligned}$$

Thus, the series (of continuous functions) on the left-hand side defines a continuous function, since it converges uniformly and absolutely for $s > 1$. \square

Proposition 2.2. *Let k be any natural number. Then, $C^k(\mathcal{S})$ is continuously embedded into $H^k(\mathcal{S})$.*

Proof. Assume $u \in C^k(\mathcal{S})$. Let $\nabla_{\mathcal{S}}$ and $\Delta_{\mathcal{S}}$ denote the gradient and the Laplace-Beltrami operator on \mathcal{S} , respectively. In order to prove the proposition, we verify that the equalities

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l [l(l+1)]^{2j} |\widehat{u}_l^m|^2 = \|(-\Delta_{\mathcal{S}})^j u\|_{L^2(\mathcal{S})}^2, \tag{2.3}$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l [l(l+1)]^{2j+1} |\widehat{u}_l^m|^2 = \|\nabla_{\mathcal{S}} [(-\Delta_{\mathcal{S}})^j u]\|_{L^2(\mathcal{S})}^2 \tag{2.4}$$

hold for any nonnegative integer j with $2j \leq k$, $2j + 1 \leq k$, respectively.

By repeated integration by parts and the fact that

$$-\Delta_{\mathcal{S}} Y_l^m = l(l+1) Y_l^m,$$

the Fourier-Laplace coefficient of $(-\Delta_S)^j u$ is computed as

$$\begin{aligned} \left[\widehat{(-\Delta_S)^j u} \right]_l^m &= \int_S [(-\Delta_S)^j u] \overline{Y_l^m} dS \\ &= \int_S u [(-\Delta_S)^j \overline{Y_l^m}] dS = [l(l+1)]^j \widehat{u}_l^m. \end{aligned} \tag{2.5}$$

Hence (2.3) follows. To derive (2.4), we set $w = (-\Delta_S)^j u$. For the moment, we suppose that $w \in C^2(\mathcal{S})$. Then, we have that

$$\|\nabla_S w\|_{L^2(\mathcal{S})}^2 = - \int_S w \Delta_S w dS = - \sum_{l=0}^{\infty} \sum_{m=-l}^l \widehat{w}_l^m \int_S \overline{Y_l^m} \Delta_S w dS,$$

where we used integration by parts and the expansion

$$w = \sum_{l=0}^{\infty} \sum_{m=-l}^l \widehat{w}_l^m Y_l^m = \sum_{l=0}^{\infty} \sum_{m=-l}^l \widehat{\widehat{w}_l^m} \overline{Y_l^m}.$$

From the definition of w and (2.5), we see that

$$-\widehat{w}_l^m \int_S \overline{Y_l^m} \Delta_S w dS = \left[\widehat{(-\Delta_S)^j u} \right]_l^m \left[\widehat{(-\Delta_S)^{j+1} u} \right]_l^m = [l(l+1)]^{2j+1} |\widehat{u}_l^m|^2,$$

and therefore

$$\|\nabla_S w\|_{L^2(\mathcal{S})}^2 = \sum_{l=0}^{\infty} \sum_{m=-l}^l [l(l+1)]^{2j+1} |\widehat{u}_l^m|^2.$$

By approximation, this equality also holds if $w \in C^1(\mathcal{S})$. We thus obtain (2.4).

Using (2.3) and (2.4), we deduce that

$$\begin{aligned} \|u\|_{H^k(\mathcal{S})}^2 &\leq C \sum_{l=0}^{\infty} \sum_{m=-l}^l \left([l(l+1)]^k + 1 \right) |\widehat{u}_l^m|^2 \\ &= \begin{cases} C \left(\left\| (-\Delta_S)^{\frac{k}{2}} u \right\|_{L^2(\mathcal{S})}^2 + \|u\|_{L^2(\mathcal{S})}^2 \right) & \text{if } k \text{ is even,} \\ C \left(\left\| \nabla_S \left[(-\Delta_S)^{\frac{k-1}{2}} u \right] \right\|_{L^2(\mathcal{S})}^2 + \|u\|_{L^2(\mathcal{S})}^2 \right) & \text{if } k \text{ is odd} \end{cases} \\ &\leq C \|u\|_{C^k(\mathcal{S})}^2, \end{aligned}$$

where $C > 0$ is some constant. This proves the proposition. □

2.2. The subspace $H_{\text{ax}}^s(\mathcal{S})$ and its Banach-algebra structure

In our analysis, the subspace of *axially symmetric* functions defined by

$$H_{\text{ax}}^s(\mathcal{S}) = \{ \psi \in H^s(\mathcal{S}) : \psi \text{ does not depend on } \varphi \}$$

will play a crucial role. It is clear that

$$H_{\text{ax}}^s(\mathcal{S}) = \{ \psi \in H^s(\mathcal{S}) : \widehat{\psi}_l^m = 0, 1 \leq |m| \leq l, l = 1, 2, \dots \}.$$

This section is the technical core of this paper. In fact, observe that, for the solvability of the nonlinear problem (1.7), we need to deal with the quadratic term $|\nabla w|^2$. In other words, in the functional framework adopted,

we must be sure that the product of two functions in the relevant space still belongs to the same space. We shall show that the subspace $H_{\text{ax}}^s(\mathcal{S})$ enjoys this property, i.e. it is a Banach algebra with respect to the pointwise product.

To this aim, we recollect some notations and results about products of spherical harmonics. We recall (see [14, Appendix III]) that the product of two spherical harmonics $Y_{l_1}^{m_1}, Y_{l_2}^{m_2}$ is represented by the formula:

$$Y_{l_1}^{m_1}(\theta, \varphi)Y_{l_2}^{m_2}(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l (-1)^m \sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -m \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} Y_l^m(\theta, \varphi).$$

In this formula, the so-called *Wigner 3-j symbol* is defined by

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \sqrt{\frac{(-l_1+l_2+l_3)!(l_1-l_2+l_3)!(l_1+l_2-l_3)!(l_3-m_3)!(l_3+m_3)!}{(l_1+l_2+l_3+1)!(l_1-m_1)!(l_1+m_1)!(l_2-m_2)!(l_2+m_2)!}} \times \sum_k \frac{(-1)^{k+l_1+m_2-m_3} (l_2+l_3+m_1-k)!(l_1-m_1+k)!}{k!(-l_1+l_2+l_3-k)!(l_3-m_3-k)!(l_1-l_2+m_3+k)!},$$

if $m_1 + m_2 + m_3 = 0$, $|l_1 - l_2| \leq l_3 \leq l_1 + l_2$, $|m_1| \leq l_1$, $|m_2| \leq l_2$ and $|m_3| \leq l_3$; the symbol is set to be zero otherwise. The summation in the formula is taken over all integers k for which all the factorials in the sum have nonnegative arguments.

By using the product formula, we find that the product uv of $u, v \in H^s(\mathcal{S})$ has the following Fourier-Laplace coefficients:

$$\widehat{uv}_l^m = \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sum_{l_2=0}^{\infty} \sum_{m_2=-l_2}^{l_2} \sqrt{(2l_1+1)(2l_2+1)(2l+1)} W_{m_1, m_2, m}^{l_1, l_2, l} \widehat{u}_{l_1}^{m_1} \widehat{v}_{l_2}^{m_2},$$

where

$$W_{m_1, m_2, m}^{l_1, l_2, l} := \frac{(-1)^m}{\sqrt{4\pi}} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -m \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

Our proof of the Banach-algebra property of $H_{\text{ax}}^s(\mathcal{S})$ is based on an l -sum relation (see [14, (7.61)]) satisfied by the Wigner 3- j symbols, that is

$$\sum_{l_j=0}^{\infty} (2l_j + 1) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 = 1 \quad (j = 1, 2, 3),$$

if $m_1 + m_2 + m_3 = 0$ and $|m_i| \leq l_i$ for $i \neq j$. This yields that

$$\begin{aligned} \sup_{l_2, m_2, l, m} \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} (2l_1 + 1) |W_{m_1, m_2, m}^{l_1, l_2, l}| &\leq \frac{1}{\sqrt{4\pi}}, \\ \sup_{l_1, m_1, l, m} \sum_{l_2=0}^{\infty} \sum_{m_2=-l_2}^{l_2} (2l_2 + 1) |W_{m_1, m_2, m}^{l_1, l_2, l}| &\leq \frac{1}{\sqrt{4\pi}}, \\ \sup_{l_1, m_1, l_2, m_2} \sum_{l=0}^{\infty} \sum_{m=-l}^l (2l + 1) |W_{m_1, m_2, m}^{l_1, l_2, l}| &\leq \frac{1}{\sqrt{4\pi}}, \end{aligned} \tag{2.7}$$

by the Cauchy-Schwarz inequality.

We conclude our preliminaries with the following simple result.

Lemma 2.3. *For any $\sigma < s - 1/2$, there is a constant $C > 0$ such that*

$$\sum_{l=0}^{\infty} (2l + 1)^\sigma |\widehat{u}_l^0| \leq C \|u\|_{H^s(\mathbb{S})}$$

for any $u \in H^s_{\text{ax}}(\mathbb{S})$.

Proof. The Cauchy-Schwarz inequality implies that

$$\sum_{l=0}^{\infty} (2l + 1)^\sigma |\widehat{u}_l^0| \leq \sqrt{\sum_{l=0}^{\infty} (2l + 1)^{2(\sigma-s)}} \sqrt{\sum_{l=0}^{\infty} (2l + 1)^{2s} |\widehat{u}_l^0|^2},$$

where the first series on the right hand side converges if $2(\sigma - s) < -1$. \square

We are now ready to state and prove the main result of this section.

Theorem 2.4. ($H^s_{\text{ax}}(\mathbb{S})$ is a Banach algebra). *Let $s > 1$. If $u, v \in H^s_{\text{ax}}(\mathbb{S})$, then $uv \in H^s_{\text{ax}}(\mathbb{S})$ and*

$$\|uv\|_{H^s(\mathbb{S})} \leq C \|u\|_{H^s(\mathbb{S})} \|v\|_{H^s(\mathbb{S})}$$

for some constant $C > 0$ independent of u, v .

Proof. Let us simply write $\widehat{u}_l, \widehat{v}_l, \widehat{uv}_l$ for $\widehat{u}_l^0, \widehat{v}_l^0, \widehat{uv}_l^0$, respectively. Since $W_{0,0,0}^{l_1,l_2,l}$ is nonzero only when $l \leq l_1 + l_2$, and in this situation $l_1 \leq l_2$ implies $l \leq 2l_2$, while $l_1 \geq l_2$ implies $l \leq 2l_1$, we have

$$\begin{aligned} (l + 1)^s |\widehat{uv}_l| &\leq \sqrt{2} \sum_{l_1=0}^{\infty} \sum_{l_2=l_1}^{\infty} \sqrt{2l_1 + 1} (2l_2 + 1)^s (l + 1) |W_{0,0,0}^{l_1,l_2,l}| |\widehat{u}_{l_1}| |\widehat{v}_{l_2}| \\ &\quad + \sqrt{2} \sum_{l_2=0}^{\infty} \sum_{l_1=l_2}^{\infty} (2l_1 + 1)^s \sqrt{2l_2 + 1} (l + 1) |W_{0,0,0}^{l_1,l_2,l}| |\widehat{u}_{l_1}| |\widehat{v}_{l_2}|. \end{aligned}$$

In what follows, we denote the two summands in this formula by I_l and J_l , respectively.

Now, Lemma 2.3 shows that

$$\sum_{l_1=0}^{\infty} \sqrt{2l_1 + 1} |\widehat{u}_{l_1}| \leq C \|u\|_{H^s(\mathbb{S})}, \quad \sum_{l_2=0}^{\infty} \sqrt{2l_2 + 1} |\widehat{v}_{l_2}| \leq C \|v\|_{H^s(\mathbb{S})}. \quad (2.8)$$

Thus, we use Lemma A.3 with the settings

$$\begin{aligned} p_k &= \sqrt{2k + 1} |\widehat{u}_k|, \quad q_k = (2k + 1)^s |\widehat{v}_k|, \quad \text{and} \\ r_{i,j,k} &= \begin{cases} (k + 1) |W_{0,0,0}^{i,j,k}| & \text{for } i \leq j, \\ 0 & \text{for } i > j, \end{cases} \end{aligned}$$

and, by recalling the l -sum relation (2.7), we see that

$$\sqrt{\sum_{l=0}^{\infty} |I_l|^2} \leq \frac{C}{\sqrt{2\pi}} \|u\|_{H^s(\mathbb{S})} \|v\|_{H^s(\mathbb{S})}.$$

A similar formula can be obtained for the terms J_l . Therefore, the proof is completed. \square

For future reference, we conclude this section by showing that the algebra structure still holds for the entire space $H^s(\mathcal{S})$ if $s > 3/2$.

Proposition 2.5. *If $u, v \in H^s(\mathcal{S})$ with $s > 3/2$, then $uv \in H^s(\mathcal{S})$ and*

$$\|uv\|_{H^s(\mathcal{S})} \leq C\|u\|_{H^s(\mathcal{S})}\|v\|_{H^s(\mathcal{S})}.$$

Proof. The proof runs similarly to that of Theorem 2.4. All what is needed is an extension of (2.8) to the case of two independent variables l and m . Indeed, we simply have that the inequality

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l \sqrt{2l+1} |\widehat{u}_l^m| \leq C\|u\|_{H^s(\mathcal{S})}$$

holds for $u \in H^s(\mathcal{S})$ if $s > 3/2$, as in the proof of Lemma 2.3. \square

2.3. The square of the gradient of a function in $H_{\text{ax}}^s(\mathcal{S})$

In the case of axially symmetric functions, we have that

$$|\nabla w|^2 = w_r^2 + w_\theta^2.$$

The following two lemmas will be decisive for the proof of existence for problem (1.1) of Sect. 4.

Lemma 2.6. *Let u, v be harmonic functions in Ω , continuous up to the boundary \mathcal{S} , and such that $u, v \rightarrow 0$ as $|x| \rightarrow \infty$. If $u, v \in H_{\text{ax}}^{s+1}(\mathcal{S})$ for $s > 1$, then $u_r v_r \in H_{\text{ax}}^s(\mathcal{S})$ and*

$$\|u_r v_r\|_{H^s(\mathcal{S})} \leq C\|u\|_{H^{s+1}(\mathcal{S})}\|v\|_{H^{s+1}(\mathcal{S})}$$

for some $C > 0$ independent of u, v .

Proof. If $u = \sum_{l=0}^{\infty} r^{-l-1} \widehat{u}_l Y_l^0$, then $u_r = -\sum_{l=0}^{\infty} r^{-l-2} (l+1) \widehat{u}_l Y_l^0$, and similarly for v . Hence, Theorem 2.4 gives

$$\begin{aligned} \|u_r v_r\|_{H^s(\mathcal{S})} &\leq C\|u_r\|_{H^s(\mathcal{S})}\|v_r\|_{H^s(\mathcal{S})} \\ &= C\sqrt{\sum_{l=0}^{\infty} (l+1)^{2s+2} |\widehat{u}_l|^2} \sqrt{\sum_{l=0}^{\infty} (l+1)^{2s+2} |\widehat{v}_l|^2} \\ &= C\|u\|_{H^{s+1}(\mathcal{S})}\|v\|_{H^{s+1}(\mathcal{S})}, \end{aligned}$$

as desired. \square

We now turn to the estimate for v_θ^2 in the $H^s(\mathcal{S})$ -norm. Unlike the case of v_r , we do not have a simple expression of the spherical harmonics expansion of v_θ . Nevertheless, we will show that there is one for v_θ^2 , by using the fact that the Legendre polynomial $P_l^0(z)$ of degree l , which we will denote here by $P_l(z)$, solves the differential equation:

$$[(1-z^2) P_l']' + l(l+1)P_l = 0. \tag{2.9}$$

Lemma 2.7. *Let u, v be harmonic functions in Ω , continuous up to the boundary \mathcal{S} , and such that $u, v \rightarrow 0$ as $|x| \rightarrow \infty$. If $u, v \in H_{\text{ax}}^{s+1}(\mathcal{S})$ for $s > 1$, then $u_\theta v_\theta \in H_{\text{ax}}^s(\mathcal{S})$ and*

$$\|u_\theta v_\theta\|_{H^s(\mathcal{S})} \leq C \|u\|_{H^{s+1}(\mathcal{S})} \|v\|_{H^{s+1}(\mathcal{S})}$$

for some $C > 0$ independent of u, v .

Proof. Recall that

$$Y_l^0(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\sin \theta).$$

Hence, if $u(\theta) = \sum_{i=0}^\infty \widehat{u}_i Y_i^0(\theta, \varphi)$ and $v(\theta) = \sum_{j=0}^\infty \widehat{v}_j Y_j^0(\theta, \varphi)$, we infer that

$$\begin{aligned} u_\theta(\theta) v_\theta(\theta) &= \sum_{i=0}^\infty \sum_{j=0}^\infty \widehat{u}_i \widehat{v}_j \sqrt{\frac{(2i+1)(2j+1)}{(4\pi)^2}} \cos^2 \theta P'_i(\sin \theta) P'_j(\sin \theta) \\ &= \sum_{l=0}^\infty c_l Y_l^0(\theta, \varphi), \end{aligned}$$

where

$$\begin{aligned} c_l &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u_\theta(\theta) v_\theta(\theta) Y_l^0(\theta, \varphi) \cos \theta \, d\theta \\ &= 2\pi \sum_{i=0}^\infty \sum_{j=0}^\infty \widehat{u}_i \widehat{v}_j \sqrt{\frac{(2i+1)(2j+1)(2l+1)}{(4\pi)^3}} \int_{-1}^1 (1-z^2) P'_i(z) P'_j(z) P_l(z) \, dz. \end{aligned}$$

The last integral can be represented by the symbols $W_{0,0,0}^{i,j,l}$ defined in (2.6). Indeed, (2.9) gives that

$$\{(1-z^2) [P'_i(z) P_l(z) - P_i(z) P'_l(z)]\}' P_j = [l(l+1) - i(i+1)] P_i(z) P_j(z) P_l(z).$$

An integration by parts then gives:

$$\begin{aligned} [i(i+1) - l(l+1)] \int_{-1}^1 P_i P_j P_l \, dz &= \int_{-1}^1 (1-z^2) (P'_i P_l - P_i P'_l) P'_j \, dz \\ &= 2 \int_{-1}^1 (1-z^2) P'_i P'_j P_l \, dz \\ &\quad - j(j+1) \int_{-1}^1 P_i P_j P_l \, dz. \end{aligned}$$

Hence, we compute:

$$\begin{aligned} c_l &= \sum_{i,j=0}^\infty \widehat{u}_i \widehat{v}_j [i(i+1) + j(j+1) - l(l+1)] \sqrt{\frac{(2i+1)(2j+1)(2l+1)}{64\pi}} \int_{-1}^1 P_i P_j P_l \, dz \\ &= \sum_{i,j=0}^\infty \frac{i(i+1) + j(j+1) - l(l+1)}{2} \sqrt{(2i+1)(2j+1)(2l+1)} W_{0,0,0}^{i,j,l} \widehat{u}_i \widehat{v}_j. \end{aligned}$$

Note that $W_{0,0,0}^{i,j,l}$ is nonzero only when $|i - j| \leq l \leq i + j$, so that we have:

$$-ij \leq \frac{i(i + 1) + j(j + 1) - l(l + 1)}{2} \leq ij.$$

Therefore, as in the proof of Theorem 2.4, we can split up the sum into two summands and obtain the inequality:

$$\begin{aligned} (l + 1)^s |c_l| &\leq \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} (2i + 1)^{3/2} (2j + 1)^{s+1} (l + 1) |W_{0,0,0}^{i,j,l}| |\widehat{u}_i| |\widehat{v}_j| \\ &\quad + \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} (2i + 1)^{s+1} (2j + 1)^{3/2} (l + 1) |W_{0,0,0}^{i,j,l}| |\widehat{u}_i| |\widehat{v}_j|. \end{aligned}$$

The desired estimate then follows thanks to the same arguments as those in the proof of Theorem 2.4. □

3. Series solution for the linearized problem

In this section, we collect the results on the linearized problem (1.9)–(1.10), which will be instrumental for the proof of our main theorem in Sect. 4.

In the spherical system of coordinates, we can compute that

$$d = \frac{\sin \theta}{r^2} \quad \text{and} \quad \nabla u \cdot \nabla v = u_r v_r + \frac{u_\theta v_\theta}{r^2} + \frac{u_\varphi v_\varphi}{r^2 \cos^2 \theta}.$$

Thus, problem (1.9)–(1.10) reads as

$$\begin{aligned} \frac{1}{r^2} (r^2 v_r)_r + \frac{1}{r^2 \cos \theta} (\cos \theta v_\theta)_\theta + \frac{1}{r^2 \cos^2 \theta} v_\varphi \varphi = 0 \\ \text{for } r \geq 1, \quad -\pi/2 \leq \theta \leq \pi/2, \quad -\pi \leq \varphi < \pi, \end{aligned} \tag{3.1a}$$

$$-2 \sin \theta v_r + \cos \theta v_\theta = \frac{1}{2} f \quad \text{for } r = 1, \quad -\pi/2 \leq \theta \leq \pi/2, \quad -\pi \leq \varphi < \pi, \tag{3.1b}$$

$$v \rightarrow 0 \quad \text{uniformly in } -\pi/2 \leq \theta \leq \pi/2, \quad -\pi \leq \varphi < \pi \quad \text{as } r \rightarrow \infty, \tag{3.1c}$$

$$v = h \quad \text{for } r = 1, \quad \theta = 0, \quad -\pi \leq \varphi < \pi. \tag{3.1d}$$

The main result of this section is the following.

Theorem 3.1. (Unique existence and fractional Sobolev estimates). *Suppose that $f \in H^s(\mathcal{S})$, with $f_\varphi \in H^{s-1/2}(\mathcal{S})$, and $h \in H^{s+3/4}(\mathcal{E})$ for some $s > 1$. Then (3.1) has a unique classical solution $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Furthermore, the solution satisfies $v|_{\mathcal{S}} \in H^{s+1}(\mathcal{S})$ and*

$$\|v\|_{H^{s+1}(\mathcal{S})} \leq C (\|f\|_{H^s(\mathcal{S})} + \|f_\varphi\|_{H^{s-1/2}(\mathcal{S})} + \|h\|_{H^{s+3/4}(\mathcal{E})}), \tag{3.2}$$

where $C > 0$ is a constant independent of f and h .

In particular, if $f \in H_{\text{ax}}^s(\mathcal{S})$ and h is a constant, then $v|_{\mathcal{S}} \in H_{\text{ax}}^{s+1}(\mathcal{S})$ and

$$\|v\|_{H^{s+1}(\mathcal{S})} \leq C (\|f\|_{H^s(\mathcal{S})} + |h|).$$

The proof of Theorem 3.1 is given in Sect. 3.2.

3.1. Formal derivation of a series solution

We start by formally deriving a representation formula of a solution v of problem (3.1) (computations here will be verified in Proposition 3.7 below). The formula is given by

$$v(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l (b_l^m + c_l^m) r^{-l-1} Y_l^m(\theta, \varphi), \tag{3.3}$$

where b_l^m and c_l^m are defined as follows. We set

$$\begin{aligned} \beta_l^m &= \sqrt{\frac{(l - |m|)(l + |m|)}{(2l - 1)(2l + 1)}}, \\ \gamma_l^m &= -\frac{(l + 1) \beta_l^m}{3(l + 2) \beta_{l+1}^m} = -\frac{l + 1}{3(l + 2)} \sqrt{\frac{(2l + 3)(l - |m|)(l + |m|)}{(2l - 1)(l + 1 - |m|)(l + 1 + |m|)}} \\ &\text{for } m = 0, \pm 1, \dots, \quad l = |m|, |m| + 1, \dots, \end{aligned}$$

and put

$$\Gamma_0^m = 1, \quad \Gamma_k^m = \prod_{j=1}^k \gamma_{|m|+2j-1}^m \quad \text{for } m = 0, \pm 1, \dots, \quad k = 1, 2, \dots$$

For $f \in L^2(\mathbb{S})$ and $h \in L^2(\mathcal{E})$, we write

$$a_l^m = \frac{1}{6(l + 2) \beta_{l+1}^m} \widehat{f}_l^m, \quad \tilde{a}^m = \frac{\widehat{h}_m - \sum_{k=0}^{\infty} b_{|m|+2k}^m \alpha_{|m|+2k}^m P_{|m|+2k}^{|m|}(0)}{\sum_{k=0}^{\infty} \Gamma_k^m \alpha_{|m|+2k}^m P_{|m|+2k}^{|m|}(0)}$$

(note that the denominator in the definition of \tilde{a}^m is nonzero due to the inequality (3.8) of Lemma 3.3 below). b_l^m is then defined by the recurrence relation

$$\begin{aligned} b_{|m|-1}^m &= b_{|m|}^m = 0 && \text{for } m = 0, \pm 1, \dots, \\ b_{l+1}^m &= \gamma_l^m b_l^m + a_l^m && \text{for } m = 0, \pm 1, \dots, \quad l = |m|, |m| + 1, \dots, \end{aligned}$$

and c_l^m is given by

$$c_{|m|+2k-1}^m = 0, \quad c_{|m|+2k}^m = \Gamma_k^m \tilde{a}^m \quad \text{for } m = 0, \pm 1, \dots, \quad k = 0, 1, \dots$$

Let us derive (3.3). Let $\widehat{v}_l^m(r)$ denote the Fourier-Laplace coefficients of $v(r, \cdot, \cdot)$, that is,

$$\widehat{v}_l^m(r) = \langle v(r, \cdot, \cdot), Y_l^m \rangle_{L^2(\mathbb{S})} = \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v(r, \theta, \varphi) \overline{Y_l^m(\theta, \varphi)} \cos \theta \, d\theta d\varphi.$$

For abbreviation, we write \widehat{v}_l^m instead of $\widehat{v}_l^m(1)$ and, for convenience, we set $\widehat{v}_l^m = 0$ if $l < |m|$. First, we observe that v has the form

$$v(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \widehat{v}_l^m r^{-l-1} Y_l^m(\theta, \varphi). \tag{3.4}$$

This follows from (3.1a) and (3.1c). Indeed, multiplying (3.1a) by $\overline{Y_l^m}$, integrating over \mathbb{S} and using the fact that $-\Delta_{\mathbb{S}} Y_l^m = l(l+1)Y_l^m$, we see that $\widehat{v}_l^m(r)$ satisfies

$$\frac{d^2 \widehat{v}_l^m}{dr^2}(r) + \frac{2}{r} \frac{d \widehat{v}_l^m}{dr}(r) - \frac{l(l+1)}{r^2} \widehat{v}_l^m(r) = 0 \quad \text{for } r > 1.$$

This together with the condition (3.1c) gives $\widehat{v}_l^m(r) = \widehat{v}_l^m r^{-l-1}$, and hence we obtain (3.4).

Next, we derive a recurrence relation for \widehat{v}_l^m from (3.1b). It is known that the following recurrence relations hold (see [3, (15.88)] for the first equality and [3, (15.88), (15.89), (15.92)] for the second equality):

$$\begin{aligned} z P_l^{|m|}(z) &= \frac{l - |m| + 1}{2l + 1} P_{l+1}^{|m|}(z) + \frac{l + |m|}{2l + 1} P_{l-1}^{|m|}(z), \\ (1 - z^2) \frac{d P_l^{|m|}}{dz}(z) &= -\frac{l(l - |m| + 1)}{2l + 1} P_{l+1}^{|m|}(z) + \frac{(l + 1)(l + |m|)}{2l + 1} P_{l-1}^{|m|}(z). \end{aligned}$$

Hence we have that

$$\begin{aligned} \sin \theta Y_l^m &= \beta_{l+1}^m Y_{l+1}^m + \beta_l^m Y_{l-1}^m, \\ \cos \theta \frac{\partial Y_l^m}{\partial \theta} &= -l \beta_{l+1}^m Y_{l+1}^m + (l + 1) \beta_l^m Y_{l-1}^m. \end{aligned}$$

Here, $Y_l^m = 0$ if $l < |m|$. From these identities, we see that

$$\begin{aligned} \langle \sin \theta v_r, Y_l^m \rangle_{L^2(\mathbb{S})} &= \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_r(r, \theta, \varphi) \sin \theta \overline{Y_l^m(\theta, \varphi)} \cos \theta \, d\theta d\varphi \\ &= \frac{d}{dr} (\beta_{l+1}^m \widehat{v}_{l+1}^m(r) + \beta_l^m \widehat{v}_{l-1}^m(r)) \\ &= -(l + 2) \beta_{l+1}^m \widehat{v}_{l+1}^m r^{-l-3} - l \beta_l^m \widehat{v}_{l-1}^m r^{-l-1}, \end{aligned}$$

and

$$\begin{aligned} &\langle \cos \theta v_{\theta}, Y_l^m \rangle_{L^2(\mathbb{S})} \\ &= \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_{\theta}(r, \theta, \varphi) \overline{Y_l^m(\theta, \varphi)} \cos^2 \theta \, d\theta d\varphi \\ &= \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v(r, \theta, \varphi) \left(\cos \theta \overline{\frac{\partial Y_l^m}{\partial \theta}(\theta, \varphi)} - 2 \sin \theta \overline{Y_l^m(\theta, \varphi)} \right) \cos \theta \, d\theta d\varphi \\ &= (l + 2) \beta_{l+1}^m \widehat{v}_{l+1}^m r^{-l-2} - (l - 1) \beta_l^m \widehat{v}_{l-1}^m r^{-l}, \end{aligned}$$

where we used integration by parts. Thus, multiplying (3.1b) by $\overline{Y_l^m}$ and integrating over \mathbb{S} , we find the recurrence relation

$$\widehat{v}_{l+1}^m = \gamma_l^m \widehat{v}_{l-1}^m + a_l^m \quad \text{for } m = 0, \pm 1, \dots, \quad l = |m|, |m| + 1, \dots$$

Finally, we consider the condition (3.1d). Interchanging the sum in (3.4), we have that

$$v(1, \theta, \varphi) = \sum_{m=-\infty}^{\infty} \widehat{v}_m(\theta) e^{im\varphi}, \quad \text{where } \widehat{v}_m(\theta) = \sum_{l=|m|}^{\infty} \widehat{v}_l^m \alpha_l^m P_l^m(\sin \theta).$$

Hence, by (3.1d), we deduce that

$$\widehat{h}_m = \widehat{v}_m(0) = \sum_{l=|m|}^{\infty} \widehat{v}_l^m \alpha_l^m P_l^m(0). \tag{3.5}$$

The recurrence relations for b_l^m and \widehat{v}_l^m show that $d_l^m = \widehat{v}_l^m - b_l^m$ satisfies

$$\begin{aligned} d_{|m|-1}^m &= 0, \quad d_{|m|}^m = \widehat{v}_{|m|}^m && \text{for } m = 0, \pm 1, \dots, \\ d_{l+1}^m &= \gamma_l^m d_{l-1}^m && \text{for } m = 0, \pm 1, \dots, \quad l = |m|, |m| + 1, \dots, \end{aligned}$$

and therefore $d_{|m|+2k-1}^m = 0$ and $d_{|m|+2k}^m = \Gamma_k^m \widehat{v}_{|m|}^m$. Plugging $\widehat{v}_l^m = b_l^m + d_l^m$ into (3.5) and using the fact that $P_l^{|m|}(0) = 0$ if $l - |m|$ is odd (see [3, (15.96)]), we find that

$$\widehat{h}_m = \sum_{k=0}^{\infty} b_{|m|+2k}^m \alpha_{|m|+2k}^m P_{|m|+2k}^{|m|}(0) + \widehat{v}_{|m|}^m \sum_{k=0}^{\infty} \Gamma_k^m \alpha_{|m|+2k}^m P_{|m|+2k}^{|m|}(0).$$

This gives $d_l^m = c_l^m$, and thus we obtain (3.3).

Remark 3.2. Suppose that f and h are independent of φ , that is, $\widehat{f}_l^m = \widehat{h}_m = 0$ if $m \neq 0$. Then, by definition, we see that $b_l^m = c_l^m = 0$ unless $m = 0$. This shows that the function v defined by (3.3) is also independent of φ .

3.2. Proof of Theorem 3.1

We divide the proof of Theorem 3.1 into a sequence of lemmas. First we show some estimates to be mainly used in deriving (3.2).

Lemma 3.3. *The inequalities*

$$\begin{aligned} c \sqrt[4]{\frac{|m| + 2k + 1}{2k + 1}} &\leq (-1)^{\frac{|m|-m}{2}+k} \alpha_{|m|+2k}^m P_{|m|+2k}^{|m|}(0) \leq C \sqrt[4]{\frac{|m| + 2k + 1}{2k + 1}}, \end{aligned} \tag{3.6}$$

$$c \sqrt[4]{\frac{|m| + 1}{(2k + 1)(|m| + 2k + 1)}} \leq (-3)^k \Gamma_k^m \leq C \sqrt[4]{\frac{|m| + 1}{(2k + 1)(|m| + 2k + 1)}}, \tag{3.7}$$

$$c \sqrt[4]{|m| + 1} \leq \left| \sum_{k=0}^{\infty} \Gamma_k^m \alpha_{|m|+2k}^m P_{|m|+2k}^{|m|}(0) \right| \leq C \sqrt[4]{|m| + 1} \tag{3.8}$$

hold with some positive constants c and C .

Proof. It is known (see [3, (15.96)]) that

$$P_{|m|+2k}^{|m|}(0) = (-1)^{|m|+k} \frac{(2|m| + 2k - 1)!!}{(2k)!!},$$

which gives:

$$\begin{aligned} & (-1)^{\frac{|m|-m}{2}+k} \alpha_{|m|+2k}^m P_{|m|+2k}^{|m|}(0) \\ &= \frac{\sqrt{(2k)!}}{(2k)!} \frac{(2|m|+2k+1)!!}{\sqrt{(2|m|+2k+1)!}} \sqrt{\frac{2|m|+4k+1}{4\pi(2|m|+2k+1)}}. \end{aligned}$$

Also, by definition, we have that

$$\begin{aligned} (-3)^k \Gamma_k^m &= \sqrt{\frac{2|m|+4k+1}{2|m|+1}} \prod_{j=1}^k \frac{|m|+2j}{|m|+2j+1} \sqrt{\frac{2j-1}{2j} \frac{2|m|+2j-1}{2|m|+2j}} \\ &= \frac{(|m|+1)!! (|m|+2k)!!}{|m|!! (|m|+2k+1)!!} \sqrt{\frac{2|m|+4k+1}{2|m|+1}} \\ &\quad \times \sqrt{\frac{(2k-1)!!}{(2k)!} \frac{(2|m|)!!}{(2|m|-1)!!} \frac{(2|m|+2k-1)!!}{(2|m|+2k)!}}. \end{aligned}$$

Therefore (3.6) and (3.7) follow from Lemma A.1 and some simple estimates. The inequality (3.8) is derived easily by (3.6) and (3.7). \square

Next, we derive an estimate of the $H^s(\mathbb{S})$ -norm of the formal solution (3.3). For $a \in \mathbb{R}$, we write $a_+ = \max\{a, 0\}$.

Lemma 3.4. (Fractional Sobolev estimates for v). *Suppose that $f \in H^s(\mathbb{S})$, $f_\varphi \in H^{(s-1/2)+}(\mathbb{S})$ and $h \in H^{s+3/4}(\mathcal{E})$ for some $s \geq 0$. Then the function v defined by (3.3) satisfies $v|_{\mathbb{S}} \in H^{s+1}(\mathbb{S})$ and*

$$\|v\|_{H^{s+1}(\mathbb{S})} \leq C \left(\|f\|_{H^s(\mathbb{S})} + \|f_\varphi\|_{H^{(s-1/2)+}(\mathbb{S})} + \|h\|_{H^{s+3/4}(\mathcal{E})} \right) \quad (3.9)$$

for some constant $C > 0$ independent of f and h .

Proof. Throughout the proof, c and C denote generic positive constants depending only on s , which may change from formula to formula.

Notice that, since $(Y_l^m)_\varphi = imY_l^m$, we have that $(\widehat{f_\varphi})_l^m = im\widehat{f}_l^m$, and hence

$$\|f_\varphi\|_{H^s(\mathbb{S})}^2 = \sum_{l=0}^\infty \sum_{m=-l}^l m^2 (l+1)^{2s} |\widehat{f}_l^m|^2. \quad (3.10)$$

Next, we write $v|_{\mathbb{S}}$ as

$$v|_{\mathbb{S}} = v_1 + v_2, \quad v_1 = \sum_{l=0}^\infty \sum_{m=-l}^l b_l^m Y_l^m, \quad v_2 = \sum_{l=0}^\infty \sum_{m=-l}^l c_l^m Y_l^m,$$

and estimate the norms of v_1 and v_2 separately. We first consider v_1 . It is elementary to show that $|\gamma_l^m| \leq 2/3$, and hence $|b_{l+1}^m| \leq 2|b_{l-1}^m|/3 + |a_l^m|$. Thus, applying Lemma A.2 with $p_k = |b_{|m|+2k}^m|$, $q_k = |a_{|m|+2k+1}^m|$, $\sigma = 2/3$, $\tau_1 = 2(s+1)$, $\tau_2 = 0$, and $\chi = |m|$ gives:

$$\sum_{k=0}^\infty (|m|+2k+1)^{2(s+1)} \left| b_{|m|+2k}^m \right|^2 \leq C \sum_{k=0}^\infty (|m|+2k+1)^{2(s+1)} \left| a_{|m|+2k+1}^m \right|^2.$$

We use Lemma A.2 again with $p_k = |b_{|m|+2k-1}^m|$ and $q_k = |a_{|m|+2k}^m|$, and combine the resulting inequality with the above inequality to obtain that

$$\sum_{l=|m|}^{\infty} (l+1)^{2(s+1)} |b_l^m|^2 \leq C \sum_{l=|m|}^{\infty} (l+1)^{2(s+1)} |a_l^m|^2.$$

Since

$$\begin{aligned} (l+1)^{2(s+1)} |a_l^m|^2 &\leq \frac{(l+1)^{2(s+1)}}{(l+1-|m|)(l+1+|m|)} |\widehat{f}_l^m|^2 \\ &= (l+1)^{2s} |\widehat{f}_l^m|^2 + \frac{m^2(l+1)^{2s}}{(l+1-|m|)(l+1+|m|)} |\widehat{f}_l^m|^2 \\ &\leq (l+1)^{2s} |\widehat{f}_l^m|^2 + m^2(l+1)^{(2s-1)+} |\widehat{f}_l^m|^2, \end{aligned}$$

we find

$$\sum_{l=|m|}^{\infty} (l+1)^{2(s+1)} |b_l^m|^2 \leq C \sum_{l=|m|}^{\infty} (l+1)^{2s} |\widehat{f}_l^m|^2 + C \sum_{l=|m|}^{\infty} m^2(l+1)^{(2s-1)+} |\widehat{f}_l^m|^2.$$

By taking the sum over m and applying (3.10), we conclude that

$$\|v_1\|_{H^{s+1}(S)} \leq C \left(\|f\|_{H^s(S)} + \|f_\varphi\|_{H^{(s-1/2)+}(S)} \right). \quad (3.11)$$

In order to examine v_2 , we estimate \tilde{a}^m . We see from (3.6) and the Cauchy-Schwarz inequality that

$$\begin{aligned} &\left| \sum_{k=0}^{\infty} b_{|m|+2k}^m \alpha_{|m|+2k}^m P_{|m|+2k}^{|m|}(0) \right| \\ &\leq C \sum_{k=0}^{\infty} \sqrt[4]{\frac{|m|+2k+1}{2k+1}} |b_{|m|+2k}^m| \\ &\leq C \sqrt{\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{3/2}}} \sqrt{\sum_{k=0}^{\infty} (2k+1) \sqrt{|m|+2k+1} |b_{|m|+2k}^m|^2}. \end{aligned}$$

Applying Lemma A.2 with $p_k = |b_{|m|+2k}^m|$, $q_k = |a_{|m|+2k+1}^m|$, $\sigma = 2/3$, $\tau_1 = 1/2$, $\tau_2 = 1$, and $\chi = |m|$ yields that

$$\begin{aligned} \sum_{l=0}^{\infty} (2k+1) \sqrt{|m|+2k+1} |b_{|m|+2k}^m|^2 &\leq C \sum_{k=0}^{\infty} (2k+1) \sqrt{|m|+2k+1} |a_{|m|+2k+1}^m|^2 \\ &\leq C \sum_{l=|m|}^{\infty} \sqrt{l+1} (l+1-|m|) |a_l^m|^2 \\ &\leq C \sum_{l=|m|}^{\infty} \frac{1}{\sqrt{l+1}} |\widehat{f}_l^m|^2. \end{aligned}$$

Hence

$$\left| \sum_{k=0}^{\infty} b_{|m|+2k}^m \alpha_{|m|+2k}^m P_{|m|+2k}^{|m|}(0) \right|^2 \leq C \sum_{l=|m|}^{\infty} \frac{1}{(l+1)^{1/2}} |\widehat{f}_l^m|^2.$$

From this and (3.8), we obtain that

$$|\tilde{a}^m|^2 \leq \frac{C}{\sqrt{|m|+1}} \left[|\widehat{h}_m|^2 + \sum_{l=|m|}^{\infty} \frac{1}{\sqrt{l+1}} |\widehat{f}_l^m|^2 \right]. \quad (3.12)$$

Now, we are ready to estimate v_2 . Note that

$$\begin{aligned} \|v_2\|_{H^{s+1}(\mathbb{S})}^2 &= \sum_{l=0}^{\infty} \sum_{m=-l}^l (l+1)^{2(s+1)} |c_l^m|^2 \\ &= \sum_{m=-\infty}^{\infty} \left[\sum_{k=0}^{\infty} (|m|+2k+1)^{2(s+1)} (\Gamma_k^m)^2 \right] |\tilde{a}^m|^2. \end{aligned}$$

By (3.7) and the fact that $|m|+1 \leq |m|+2k+1 \leq (|m|+1)(2k+1)$, we have

$$\begin{aligned} \sum_{k=0}^{\infty} (|m|+2k+1)^{2(s+1)} (\Gamma_k^m)^2 &\leq C \sum_{k=0}^{\infty} \frac{[(|m|+1)(2k+1)]^{2(s+1)}}{3^{2k}} \\ &\leq C (|m|+1)^{2(s+1)}. \end{aligned}$$

This, together with (3.12), gives

$$\begin{aligned} \|v_2\|_{H^{s+1}(\mathbb{S})}^2 &\leq C \sum_{m=-\infty}^{\infty} (|m|+1)^{2s+\frac{3}{2}} |\widehat{h}_m|^2 \\ &\quad + C \sum_{m=-\infty}^{\infty} \sum_{l=|m|}^{\infty} \frac{(|m|+1)^{2s+3/2}}{\sqrt{l+1}} |\widehat{f}_l^m|^2. \end{aligned}$$

Since

$$(|m|+1)^{2s+\frac{3}{2}} \leq 2(m^2+1)(l+1)^{2s-\frac{1}{2}} \leq 2m^2(l+1)^{(2s-1)+\frac{1}{2}} + 2(l+1)^{2s+\frac{1}{2}},$$

we deduce that

$$\begin{aligned} \|v_2\|_{H^{s+1}(\mathbb{S})}^2 &\leq C \sum_{m=-\infty}^{\infty} (|m|+1)^{2s+\frac{3}{2}} |\widehat{h}_m|^2 \\ &\quad + C \sum_{m=-\infty}^{\infty} \sum_{l=|m|}^{\infty} \left[m^2(l+1)^{(2s-1)+} + (l+1)^{2s} \right] |\widehat{f}_l^m|^2 \\ &= C \left(\|h\|_{H^{s+3/4}(\mathcal{E})} + \|f_{\varphi}\|_{H^{(s-1/2)+}(\mathbb{S})} + \|f\|_{H^s(\mathbb{S})} \right). \end{aligned}$$

Combining this and (3.11), we obtain (3.9). Thus the proof is completed. \square

In the next lemma we check that, under the assumptions of Theorem 3.1, the function v given by (3.3) is indeed a classical solution of (3.1).

Lemma 3.5. (Regularity of v). *Suppose that $f \in H^s(\mathcal{S})$, $f_\varphi \in H^{s-1/2}(\mathcal{S})$ and $h \in H^{s+3/4}(\mathcal{E})$ for some $s > 1$, and let v be defined by (3.3). Then v belongs to $C^2(\Omega) \cap C^1(\bar{\Omega})$ and satisfies (3.1) in the classical sense.*

Proof. We know that the Fourier-Laplace coefficients \widehat{v}_l^m of $v|_{\mathcal{S}}$ is given by $\widehat{v}_l^m = b_l^m + c_l^m$. From (2.2) and the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} & \sum_{l=0}^{\infty} \sum_{m=-l}^l |\widehat{v}_l^m r^{-l-1} Y_l^m(\theta, \varphi)| \\ & \leq r^{-1} \sum_{l=0}^{\infty} \sum_{m=-l}^l |\widehat{v}_l^m Y_l^m(\theta, \varphi)| \\ & \leq r^{-1} \|v\|_{H^{s+1}(\mathcal{S})} \sqrt{\sum_{l=0}^{\infty} \sum_{m=-l}^l (l+1)^{-2(s+1)} |Y_l^m(\theta, \varphi)|^2} \\ & \leq r^{-1} \|v\|_{H^{s+1}(\mathcal{S})} \sqrt{\frac{1}{2\pi} \sum_{l=0}^{\infty} (l+1)^{-1-2s}}. \end{aligned} \tag{3.13}$$

Lemma 3.4 and the fact that $s > 0$ show that the right-hand side is bounded by a constant independent of (r, θ, φ) . Hence the series on the left-hand side converges uniformly on $\bar{\Omega}$. Since it is well-known that a uniform limit of a sequence of harmonic functions is smooth and harmonic, we deduce that v is of class C^2 in Ω and satisfies (3.1a). In addition, from (3.13), we see at once that (3.1c) is satisfied.

Next, we show that $v \in C^1(\bar{\Omega})$ and that (3.1b) and (3.1d) hold. It suffices to show that the series

$$\begin{aligned} & \sum_{l=0}^{\infty} \sum_{m=-l}^l (l+1) \widehat{v}_l^m r^{-l-2} Y_l^m(\theta, \varphi) \left(= \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{\partial}{\partial r} (\widehat{v}_l^m r^{-l-2} Y_l^m(\theta, \varphi)) \right), \\ & \sum_{l=0}^{\infty} \sum_{m=-l}^l \widehat{v}_l^m r^{-l-1} \frac{\partial Y_l^m}{\partial \theta}(\theta, \varphi), \quad \sum_{l=0}^{\infty} \sum_{m=-l}^l \widehat{v}_l^m r^{-l-1} \frac{1}{\cos \theta} \frac{\partial Y_l^m}{\partial \varphi}(\theta, \varphi) \end{aligned} \tag{3.14}$$

are uniformly convergent on $\bar{\Omega}$. A computation similar to that in the derivation of (3.13) gives

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l |(l+1) \widehat{v}_l^m r^{-l-2} Y_l^m(\theta, \varphi)| \leq \|v\|_{H^{s+1}(\mathcal{S})} \sqrt{\frac{1}{2\pi} \sum_{l=0}^{\infty} (l+1)^{1-2s}}.$$

Since the right-hand side is finite if $s > 1$, we deduce that the first series of (3.14) converges uniformly. For the second and third series, we use the identities

$$\sum_{m=-l}^l \left| \frac{\partial Y_l^m}{\partial \theta}(\theta, \varphi) \right|^2 = \sum_{m=-l}^l \frac{1}{\cos^2 \theta} \left| \frac{\partial Y_l^m}{\partial \varphi}(\theta, \varphi) \right|^2 = \frac{l(l+1)(2l+1)}{8\pi},$$

which are derived by operating $\partial^2/\partial\theta_1\partial\theta_2$ or $\partial^2/\partial\varphi_1\partial\varphi_2$ to the equality (2.1) and then taking $(\theta_1, \varphi_1) = (\theta_2, \varphi_2) = (\theta, \varphi)$. From these identities and the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} & \sum_{l=0}^{\infty} \sum_{m=-l}^l \left| \widehat{v}_l^m r^{-l-1} \frac{\partial Y_l^m}{\partial\theta}(\theta, \varphi) \right| + \sum_{l=0}^{\infty} \sum_{m=-l}^l \left| \widehat{v}_l^m r^{-l-1} \frac{1}{\cos\theta} \frac{\partial Y_l^m}{\partial\varphi}(\theta, \varphi) \right| \\ & \leq \|v\|_{H^{s+1}(\mathcal{S})} \sqrt{\sum_{l=0}^{\infty} \sum_{m=-l}^l (l+1)^{-2(s+1)} \left| \frac{\partial Y_l^m}{\partial\theta}(\theta, \varphi) \right|^2} \\ & \quad + \|v\|_{H^{s+1}(\mathcal{S})} \sqrt{\sum_{l=0}^{\infty} \sum_{m=-l}^l (l+1)^{-2(s+1)} \frac{1}{\cos^2\theta} \left| \frac{\partial Y_l^m}{\partial\varphi}(\theta, \varphi) \right|^2} \\ & \leq \|v\|_{H^{s+1}(\mathcal{S})} \sqrt{\frac{1}{\pi} \sum_{l=0}^{\infty} (l+1)^{1-2s}}. \end{aligned}$$

This shows that the second and third series of (3.14) are uniformly convergent, and thus the assertion follows. \square

Finally, we prove the uniqueness of a solution of (3.1).

Lemma 3.6. *Let $v_1, v_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy (3.1). Then $v_1 = v_2$ on $\overline{\Omega}$.*

Proof. Although the lemma can be shown in the same way as [10, Theorem], we give a proof for readers' convenience.

We know that the function $w = v_1 - v_2$ satisfies

$$\Delta w = 0 \text{ in } \Omega, \quad \nabla d \cdot \nabla w = 0 \text{ on } \mathcal{S}, \quad w \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad w = 0 \text{ on } \mathcal{E}. \tag{3.15}$$

We show that $w \leq 0$ on $\overline{\Omega}$. On the contrary, suppose that w is positive somewhere. Then, we can take a point $x_0 \in \overline{\Omega}$ such that $w(x_0) = \sup_{\overline{\Omega}} w > 0$, since w decays at infinity. By the last condition of (3.15), we have either $x_0 \in \Omega$ or $x_0 \in \mathcal{S} \setminus \mathcal{E}$. Assume $x_0 \in \Omega$. Then, since w is harmonic in Ω and vanishes on \mathcal{E} , we see from the strong maximum principle that $w(x_0) = 0$, a contradiction. Assume $x_0 \in \mathcal{S} \setminus \mathcal{E}$. In this case, we note that the tangential derivative of w on \mathcal{S} vanishes at x_0 . From (1.8) and the second condition of (3.15), it follows that

$$\frac{\partial w}{\partial \nu} \Big|_{x=x_0} = \frac{1}{2x_3} (\nabla d \cdot \nabla w - \tau \cdot \nabla w) \Big|_{x=x_0} = 0.$$

Hence the Hopf lemma and the last condition of (3.15) give $w(x_0) = 0$, which is impossible. Consequently w is nonpositive everywhere. The fact that $w \geq 0$ can be shown in the same way, and therefore we obtain $w = 0$. \square

We can now prove Theorem 3.1.

Proof of Theorem 3.1. The unique existence of a solution of (3.1) follows from Lemmas 3.5 and 3.6. The inequality (3.2) is a direct consequence of Lemma 3.4.

From Remark 3.2, we see that the solution v satisfies $v|_{\mathcal{S}} \in H_{\text{ax}}^{s+1}(\mathcal{S})$ if $f \in H_{\text{ax}}^s(\mathcal{S})$ and h is a constant. Therefore the proof is completed. \square

3.3. Validity of the formula (3.3)

We have proved in Theorem 3.1 that any classical solution of (3.1) is given by the formula (3.3), provided that f and h are in certain Sobolev spaces. At the end of this section, we prove that this is still true without assuming extra regularity conditions on f and h .

Proposition 3.7. *Any solution $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of (3.1) is of the form (3.3).*

Proof. Most of computations in Sect. 3.1 are valid, since the assumption $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ implies that $\widehat{v}_l^m(r) \in C^2((1, \infty)) \cap C^1([1, \infty))$, $f \in C(\mathcal{S})$ and $h \in C^1(\mathcal{E})$. The only point where we have to verify is (3.5). In order to ensure (3.5), we need to show that the equality (3.1) holds in $L^2(\mathcal{E})$ for $\theta = 0$. For this purpose, we use the inequality (see [11, Corollary 1])

$$|P_l^{|m|}(z)| \leq \sqrt[4]{\frac{64}{\pi^3(2l+1)\sqrt{1-z^2}}} \sqrt{\frac{(l+|m|)!}{(l-|m|)!}},$$

which gives

$$|\alpha_l^m P_l^{|m|}(\sin \theta)| \leq \sqrt[4]{\frac{4(2l+1)}{\pi^5 \cos \theta}}.$$

From this and the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} |\widehat{v}_m(\theta)|^2 &\leq \sqrt{\frac{8}{\pi^5 \cos \theta}} \left(\sum_{l=|m|}^{\infty} \sqrt[4]{l+1} |\widehat{v}_l^m| \right)^2 \\ &\leq \sqrt{\frac{8}{\pi^5 \cos \theta}} \left[\sum_{l=0}^{\infty} \frac{1}{(l+1)^{3/2}} \right] \left[\sum_{l=|m|}^{\infty} (l+1)^2 |\widehat{v}_l^m|^2 \right]. \end{aligned}$$

Since Proposition 2.2 and the assumption $v|_{\mathcal{S}} \in C^1(\mathcal{S})$ show that

$$\sum_{m=-\infty}^{\infty} \sum_{l=|m|}^{\infty} (l+1)^2 |\widehat{v}_l^m|^2 = \|v\|_{H^1(\mathcal{S})} < \infty,$$

we see from the Weierstrass M-test that the series $\sum_{m=-\infty}^{\infty} |\widehat{v}_m(\theta)|^2$ converges locally uniformly in $\theta \in (-\pi/2, \pi/2)$. This means that the right-hand side of (3.1) is convergent in $L^2(\mathcal{E})$ locally uniformly in $\theta \in (-\pi/2, \pi/2)$. Since we know that (3.1) holds in $L^2(\mathcal{S})$, we conclude that (3.1) is valid in $L^2(\mathcal{E})$ for every $\theta \in (-\pi/2, \pi/2)$. Thus (3.5) is verified, and the proof is completed. \square

4. Axially symmetric solutions of Backus problem

In this section, we finally prove the existence of axially symmetric solutions of Backus problem (1.1) near the dipole.

Remark 4.1. Notice that, since $|\nabla d(x)| = \sqrt{1 + 3x_3^2}$ for $x \in \mathcal{S}$, then $|\nabla d| \in H^s(\mathcal{S})$ for any s , thanks to Proposition 2.2, being as $|\nabla d| \in C^\infty(\mathcal{S})$.

Lemma 4.2. *Let u, v be harmonic functions in $\mathbb{R}^N \setminus \overline{B}$, continuous up to the boundary \mathcal{S} , and such that $u, v \rightarrow 0$ as $|x| \rightarrow \infty$. If $u, v \in H_{\text{ax}}^{s+1}(\mathcal{S})$ for $s > 1$. Then, $|\nabla u|^2, |\nabla v|^2 \in H_{\text{ax}}^s(\mathcal{S})$ and*

$$\begin{aligned} \|\nabla u\|^2_{H^s(\mathcal{S})} &\leq C\|u\|^2_{H^{s+1}(\mathcal{S})}, \\ \|\nabla u\|^2 - \|\nabla v\|^2_{H^s(\mathcal{S})} &\leq C(\|u\|_{H^{s+1}(\mathcal{S})} + \|v\|_{H^{s+1}(\mathcal{S})})\|u - v\|_{H^{s+1}(\mathcal{S})}, \end{aligned}$$

for some constant $C > 0$ independent of u, v .

Proof. The assertion follows from the decompositions

$$\begin{aligned} |\nabla u|^2 &= u_r^2 + u_\theta^2, \\ |\nabla u|^2 - |\nabla v|^2 &= (u + v)_r(u - v)_r + (u + v)_\theta(u - v)_\theta, \end{aligned}$$

Lemmas 2.6 and 2.7 . □

We are now in position to prove our main result.

Proof of Theorem 1.1. In the proof C denotes a generic positive constant depending only on s . We define operators \mathcal{T} and Ψ by

$$\mathcal{T}[f] = |\nabla v|^2, \quad \Psi[f] = \frac{1}{2}(g^2 - |\nabla d|^2 - \mathcal{T}[f]),$$

where v is a unique solution of (3.1). Due to Theorem 3.1, Remark 4.1 and Lemma 4.2, we see that \mathcal{T} and Ψ are defined as mappings from $H_{\text{ax}}^s(\mathcal{S})$ to $H_{\text{ax}}^s(\mathcal{S})$. Put $\delta = \|g - |\nabla d|\|_{H^s(\mathcal{S})} + |h|$ and define a closed subset X_δ of $H^s(\mathcal{S})$ by

$$X_\delta = \{f \in H_{\text{ax}}^s(\mathcal{S}) : \|f\|_{H^s(\mathcal{S})} \leq M\delta\}.$$

We shall prove that Ψ has a unique fixed point, by showing that it is a contraction mapping on X_δ , for some number $M > 0$.

To this end, we observe that Theorems 2.4 and 3.1, Remark 4.1 and Lemma 4.2 give that

$$\begin{aligned} \|g^2 - |\nabla d|^2\|_{H^s(\mathcal{S})} &\leq C\|g + |\nabla d|\|_{H^s(\mathcal{S})}\|g - |\nabla d|\|_{H^s(\mathcal{S})} \leq C(\delta + 1)\delta, \\ \|\mathcal{T}[f]\|_{H^s(\mathcal{S})} &\leq C\|v\|_{H^{s+1}(\mathcal{S})}^2 \leq C(\|f\|_{H^s(\mathcal{S})}^2 + |h_0|^2) \leq C(M^2 + 1)\delta^2, \end{aligned}$$

for any $f \in X_\delta$. Hence, the inequality

$$\|\Psi[f]\|_{H^s(\mathcal{S})} \leq C_1 [(M^2 + 1)\delta + 1]\delta \tag{4.1}$$

holds for some other positive constant C_1 only depending on s .

Next, let $f_j \in X_\delta$ and let v_j be a unique solution of (3.1) for $f = f_j$ ($j = 1, 2$). We see from Theorem 3.1 and Lemma 4.2 that

$$\begin{aligned} \|\mathcal{T}[f_1] - \mathcal{T}[f_2]\|_{H^s(\mathcal{S})} &\leq C(\|v_1\|_{H^{s+1}(\mathcal{S})} + \|v_2\|_{H^{s+1}(\mathcal{S})})\|v_1 - v_2\|_{H^{s+1}(\mathcal{S})} \\ &\leq C(\|f\|_{H^s(\mathcal{S})} + |h_0|)\|f_1 - f_2\|_{H^s(\mathcal{S})} \\ &\leq C(M + 1)\delta\|f_1 - f_2\|_{H^s(\mathcal{S})}. \end{aligned}$$

Therefore we have

$$\|\Psi[f_1] - \Psi[f_2]\|_{H^s(s)} = \frac{1}{2} \|\mathcal{J}[f_1] - \mathcal{J}[f_2]\|_{H^s(s)} \leq C_2 (M + 1) \delta \|f_1 - f_2\|_{H^s(s)}, \quad (4.2)$$

for a constant $C_2 > 0$, which only depends on s .

Now, in order to show that Ψ is a contraction mapping on X_δ , we must choose the positive parameters M and δ such that

$$C_1 [(M^2 + 1)\delta + 1] < M,$$

so that $\Psi(X_\delta) \subset X_\delta$ thanks to (4.1), and

$$C_2 (M + 1) \delta < 1,$$

from (4.2). The last two inequalities are certainly satisfied if we take $M = 2C_1$ and

$$\delta < \min \left\{ \frac{1}{1 + 4C_1^2}, \frac{1}{C_2(1 + 2C_1)} \right\}.$$

Thus, by the Banach fixed-point theorem, Ψ has a unique fixed point f_* in X_δ . Therefore, we can easily see that the solution v_* of (3.1) with $f = f_*$ is such that $u = d + v_*$ satisfies (1.1) with $u = h$ on \mathcal{E} . Thus, the proof is completed. \square

Remark 4.3. The constant h_0 can be chosen as the average on \mathcal{E} of a function h . Thus, loosely speaking, Theorem 1.1 can be interpreted from a geophysics point of view as: for any field intensity of dipolar character given on the Earth's surface, there exists a unique geomagnetic potential outside the Earth, with that field intensity on its surface, and with given average potential on the equator.

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Appendix A. Technical lemmas

In this appendix, we collect the following simple lemmas for numerical sequences. In what follows, we use the standard notations for the *double factorial*:

$$n!! = \prod_{j=0}^{\lfloor n/2 \rfloor - 1} (n - 2j),$$

where $\lfloor \cdot \rfloor$ is the greatest integer function.

Lemma A.1. *There are constants $c > 0$ and $C > 0$ such that*

$$c \sqrt[4]{n+1} \leq \frac{n!!}{\sqrt{n!}} \leq C \sqrt[4]{n+1},$$

$$c \sqrt{n+1} \leq \frac{(n+1)!!}{n!!} \leq C \sqrt{n+1},$$

for all $n = 0, 1, \dots$.

Proof. By Stirling’s formula, we can check that

$$\lim_{n \rightarrow \infty} \frac{(2n-1)!!}{\sqrt[4]{2n} \sqrt{(2n-1)!}} = \sqrt[4]{\frac{2}{\pi}}, \quad \lim_{n \rightarrow \infty} \frac{(2n)!!}{\sqrt[4]{2n+1} \sqrt{(2n)!}} = \sqrt[4]{\frac{\pi}{2}},$$

$$\lim_{n \rightarrow \infty} \frac{(2n-1)!!}{\sqrt{2n-1} (2n-2)!!} = \sqrt{\frac{2}{\pi}}, \quad \lim_{n \rightarrow \infty} \frac{(2n)!!}{\sqrt{2n} (2n-1)!!} = \sqrt{\frac{\pi}{2}}.$$

The desired inequalities then ensue. □

Lemma A.2. *Let the sequences of non-negative real numbers $\{p_k\}_{k=0,1,\dots}$ and $\{q_k\}_{k=0,1,\dots}$ satisfy the recurrence relations:*

$$p_0 = 0, \quad p_k \leq \sigma p_{k-1} + q_{k-1} \quad \text{for } k = 1, 2, \dots,$$

for some constant $0 \leq \sigma < 1$. Then, for any non-negative numbers τ_1, τ_2 and χ , there exists a positive constant C depending only on σ, τ_1 and τ_2 such that

$$\sum_{k=0}^{\infty} (\chi + 2k + 1)^{\tau_1} (2k + 1)^{\tau_2} p_k^2 \leq C \sum_{k=0}^{\infty} (\chi + 2k + 1)^{\tau_1} (2k + 1)^{\tau_2} q_k^2.$$

Proof. Iterating the recurrence relations gives that

$$p_k \leq \sum_{j=0}^{k-1} \sigma^{k-j-1} q_j \quad \text{for } k = 1, 2, \dots,$$

so that by the Cauchy-Schwarz inequality we infer that

$$\begin{aligned}
 p_k^2 &\leq \left(\sum_{j=0}^{k-1} \sigma^{k-j-1} q_j \right)^2 \leq \left(\sum_{j=0}^{k-1} \sigma^{k-j-1} \right) \left(\sum_{j=0}^{k-1} \sigma^{k-j-1} q_j^2 \right) \\
 &\leq \frac{1}{1-\sigma} \sum_{j=0}^{k-1} \sigma^{k-j-1} q_j^2 \quad \text{for any } k = 1, 2, \dots
 \end{aligned}$$

Next, we compute that

$$\begin{aligned}
 &\sum_{k=0}^{\infty} (\chi + 2k + 1)^{\tau_1} (2k + 1)^{\tau_2} p_k^2 \\
 &\leq \frac{1}{1-\sigma} \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} (\chi + 2k + 1)^{\tau_1} (2k + 1)^{\tau_2} \sigma^{k-j-1} q_j^2 \\
 &= \frac{1}{1-\sigma} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (\chi + 2j + 2i + 3)^{\tau_1} (2j + 2i + 3)^{\tau_2} \sigma^i q_j^2,
 \end{aligned}$$

after switching the two sums in the second line and then setting $k = i + j$. Finally, we apply the simple algebraic inequality $a + b + 3 \leq (a + 1)(b + 3)$ (for $a, b \geq 0$), and hence infer:

$$\begin{aligned}
 &\sum_{k=0}^{\infty} (\chi + 2k + 1)^{\tau_1} (2k + 1)^{\tau_2} p_k^2 \\
 &\leq \frac{1}{1-\sigma} \left[\sum_{i=0}^{\infty} (2i + 3)^{\tau_1 + \tau_2} \sigma^i \right] \left[\sum_{j=0}^{\infty} (\chi + 2j + 1)^{\tau_1} (2j + 1)^{\tau_2} q_j^2 \right].
 \end{aligned}$$

Thus, the lemma follows. □

We conclude this appendix by recalling a well-known result for the standard space l^p ($p \geq 1$) of numerical sequences $\{a_n\}_{n=0,1,\dots}$ such that

$$\sum_{n=0}^{\infty} |a_n|^p < \infty.$$

Lemma A.3. *Let $\mathbf{p} = \{p_i\}_{i=0,1,\dots} \in l^1$, $\mathbf{q} = \{q_j\}_{j=0,1,\dots} \in l^2$ and let the 3-indices sequence $\{r_{i,j,k}\}_{i,j,k=0,1,\dots}$ satisfy*

$$M := \max \left\{ \sup_{i,k} \sum_{j=0}^{\infty} |r_{i,j,k}|, \sup_{i,j} \sum_{k=0}^{\infty} |r_{i,j,k}| \right\} < \infty.$$

Then, the sequence $\mathbf{s} = \{s_k\}_{k=0}^{\infty}$ defined by

$$s_k = \sum_{i,j=0}^{\infty} r_{i,j,k} p_i q_j$$

belongs to l^2 and satisfies

$$\|\mathbf{s}\|_{l^2} \leq M \|\mathbf{p}\|_{l^1} \|\mathbf{q}\|_{l^2}.$$

Proof. The Cauchy-Schwarz inequality gives that

$$\begin{aligned} \|\mathbf{s}\|_{l^2}^2 &= \langle \mathbf{s}, \mathbf{s} \rangle_{l^2} \leq \sum_{k=0}^{\infty} \sum_{i,j=0}^{\infty} |r_{i,j,k} p_i q_j s_k| \\ &\leq \sum_{i=0}^{\infty} |p_i| \sqrt{\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |r_{i,j,k}| |q_j|^2} \sqrt{\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} |r_{i,j,k}| |s_k|^2} \leq M \|\mathbf{p}\|_{l^1} \|\mathbf{q}\|_{l^2} \|\mathbf{s}\|_{l^2}. \end{aligned}$$

The claim then follows at once. \square

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