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Existence and multiplicity results for nonlinear elliptic problems

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Abstract

In this work of thesis, we investigate existence and multiplicity results for a class of nonlinear elliptic problems. First, we deal with problems involving the p-Laplacian operator on bounded smooth domains, where a diffusion term appears into the nonlinearity. For this reason, variational methods cannot be used. Secondly, we treat existence and multiplicity of weak solutions for (p, q)-Laplacian equations, as well as for singular p-Laplacian Schrödinger equations, in the entire $\mathbb{R}^N$ whose nonlinearity combines a power-type term at critical level with a subcritical term, involving also nontrivial weights and a positive parameter λ. This latter case, considered also in a symmetric setting, allows us to use variational methods, but in the delicate situation of lack of compactness, so that classical results cannot be directly used, they need to be adapted.

Contents

Introduction

In this work of thesis we investigate existence and multiplicity results to nonlinear elliptic problems involving p-Laplacian operator, $1 < p < N$, as well as (p,q)-Laplacian operator, namely

$$-\Delta_q u - \Delta_p u = f(x,u,Du) \quad \text{in } \Omega \subseteq \mathbb{R}^N, \tag{1}$$

with $N > 1$ and $1 < q \leq p < N$, where $\Delta_m u = \operatorname{div}\big(|Du|^{m-2}Du\big)$, $m > 1$ and f is a nonlinearity whose properties will be specified later. We are also interested in the study of singular p-Laplacian Schrödinger equations in $\mathbb{R}^N$, whose model is strictly related to problem (1) when $p = q$, but a diffusion term appears.

Typically, elliptic problems of the form (1) come from the study of stationary solutions of reaction-diffusion systems of the form

$$u_t = div[A(|Du|)Du] + f(x,u,Du),$$

which naturally arise in a variety of contexts in General Topology, Geometric Analysis, Functional and Convex Analysis, Game Theory, Mathematical Economics, more recently, in life sciences and in other branches of pure and applied science. None of these fields can be investigated without taking into account nonlinear phenomena. In such applications, the function u describes a concentration, the first term corresponds to the diffusion with a (generally non-constant) diffusion coefficient $A(|Du|)$, i.e. $A = |Du|^{p-2}$, $p > 1$, in the p-Laplacian case or $A = |Du|^{p-2} + |Du|^{q-2}$, $p, q > 1$, in the (p,q)-Laplacian case, whereas the term f is the reaction and relates to sources and loss processes. For instance, in chemical and biological applications, the reaction term f has a polynomial form with respect to the concentration u and eventually to the gradient. More specifically, the p-Laplacian and the (p,q)-Laplacian appear in many fields such as turbulent filtrations in porous media, blood flow ($p > 2$) and in general in non-Newtonian fluid flow ($p \neq 2$), fluid mechanics, rheology, material science, nonlinear elasticity, glaciology and so on.

Our study is directed at two specific prototypes which lead us to completely different approaches. Precisely, we first consider nonlinearities f depending on the gradient, then we treat the case when the reaction f has a polynomial form in u which involves a critical term, that is

(a) $f(x,u,\eta) = |u|^{\mathfrak{m}} + C(|u|^s + |\eta|^\theta)$, with $0 < p-1 < s < \mathfrak{m} < p^*-1$ and $p-1 < \theta < \mathfrak{m}p/(\mathfrak{m}+1)$,

(b) $f(x,u) = \lambda V(x)|u|^{k-2}u + K(x)|u|^{p^*-2}u$, with $1 < k < p^*$ and $\lambda > 0$,

where $p^* = Np/(N-p)$ is the critical Sobolev exponent, while the weights K and V in (b) satisfy natural conditions, specified later.

Consequently, the goal of this thesis is twofold. In case (a) we prove an existence result for a Dirichlet problem involving the p-Laplacian operator (i.e. $q = p$) on a bounded domain, while when we deal with case (b) we are interested in existence and multiplicity results for (p,q)-Laplacian problems, as well as singular p-Laplacian Schrödinger equations, in the entire $\mathbb{R}^N$.

It is clear that we need to approach the two problems with different techniques because of the intrinsic nature of each equation considered. Indeed, in case (a) variational methods cannot be used since the presence of a convection term f typically destroys the variational structure, even if the variational structure is recovered by the associated limit problem. While case (b) allows us to base our proofs on variational methods but in the very delicate situation of loss of compactness. Thus, the main mathematical interest lies in the fact that some classical theorems of Functional Analysis cannot be directly used for the above problems, but they need to be adapted. Precisely, the main results of this work of thesis for nonlinearities of type (a) are contained in [12], while the ones concerning nonlinearities of type (b) have been proved in [13, 14, 15, 16].

Nonlinearities of the form (a), treated in Chapter 1, were introduced in [39] by Chipot, Weissler in order to investigate the possible effect of a damping gradient term on global existence or nonexistence and can also be used to describe the evolution of the population density of a biological species, under the effect of certain natural mechanisms. In particular, the gradient term, in the dissipative form, represents the action of a predator which destroys the individuals during their displacements (it is assumed that the preys are not vulnerable at rest). For the Laplacian case, we mention the pioneering paper with gradient terms by Brezis and Turner [28], and then that by Ghergu and Rădulescu [70] in which the convection term has the form $|\nabla u|^a$, $0 < a \le 2$, see also [51] where a competition between an anisotropic potential, a convection term $|\nabla u|$ and a singular nonlinearity is taken into account. For further details, we refer to the survey [127] and also to [18] where a destruction term $u^s|Du|^\theta$ is introduced leading to a variant model widely described. Among problems with gradient terms, we mention also [69] (see also the references therein) for results dealing with coercive p-Laplacian problems and [71] for singular elliptic equations.

To obtain the main existence result in a bounded domain Ω, in Chapter 1 we adapt the technique developed by Barrios et al. in [17] for nonlocal elliptic problems, which is a modification of the classical scaling blow up method for semilinear problems due to Gidas and Spruck in [73, 74]. The blow up method is based on a priori estimate deduced by a scaling procedure and on the use of a Liouville theorem either on the entire $\mathbb{R}^N$ or in the halfspace $\mathbb{R}^N_+$ depending on whether the limit point of the blow up sequence belongs to the interior or to the boundary of the domain Ω. When dealing with quasilinear operators, say the p-Laplacian, the main difficulty in applying the blow up technique relies on the use of Liouville type theorems in the halfspace, since under rather general hypotheses they are not common in literature, we mention just some recent contributions [56, 57, 45, 52, 58, 146]. For this reason, after the work of Gidas and Spruck, a series of papers have been developed with hypotheses, mainly of geometrical type on the domain cfr. [10, 41], to avoid the case of the halfspace. In this direction falls the paper of Ruiz [119] in which rather than assuming geometrical conditions on the domain, he produces a slight modification of the blow up method by applying the rescaling argument around a fixed point in Ω, instead of around a sequence of suitable points of Ω. This new technique avoids the possibility, for the limit point of the blow up sequence built, to belong to the boundary of Ω, consequently no Liouville theorem in the halfspace is required. Unfortunately, this tool produces a restriction of the range of $\mathfrak{m}$, due to the use of an Harnack type inequality by Trudinger [134], precisely $\mathfrak{m} < p_* - 1$, where $p_* = p(N-1)/(N-p)$ is the Serrin exponent with $p_* < p^*$. A recent extension of this result to a more general nonlinearity f can be found in [59], but again Harnack type inequality is used so that $\mathfrak{m} < p_* - 1$.

In order to improve the range of $\mathfrak{m}$ up to $p^* - 1$, we use the scaling method in [17], which requires, on one side, "good" estimates for solutions, both in the interior and at the boundary, and, on the other side, the use of a Liouville theorem in the halfspace for the p-Laplacian limit problem with no convection terms, proved by Zou in [146] some years later than the paper of Ruiz [119].

The second type of nonlinearity, the one whose prototype is (b), is a model for a reaction which combines a power-type nonlinearity at critical level with a subcritical term, involving also nonnegative nontrivial weights and a positive parameter λ.

The proofs of the existence and multiplicity results for problem (1)-(b) make use of variational methods but in the very delicate situation when "double" lack of compactness appears due to the entire space $\mathbb{R}^N$ and to the presence of the critical exponent p^*. For this reasons, critical problems in $\mathbb{R}^N$ represent one of the most dramatic cases of loss of compactness and have been studied intensively in the last 25 years.

The first remarkable contributions are the milestone papers for the Laplacian by Brezis and Nirenberg in [27], Trudinger in [135] and Aubin [8]. Later, the p-Laplacian case, as well as more general operators, both in bounded sets and in the entire $\mathbb{R}^N$ was investigated by many authors, we refer to the pioneering paper for quasilinear operators by García Azorero, Peral [9] for bounded sets and with a parametrical critical nonlinearity without weights, for which they found two positive values λ_0, λ_1 such that for $\lambda \in (0, \lambda_1)$ existence of infinitely many solution holds if $1 < k < p$, while existence of a nontrivial solution holds if $1 < p < k < p^*$ provided that $\lambda \geq \lambda_0$. Then, Huang [79] introduced weights in the critical nonlinearity, the case of exterior domains was treated in [60] by Filippucci, Pucci, Rădulescu, while Ghoussoub and Robert considered in [72] singular Hardy-Schrödinger differential operators. Finally the case of double critical nonlinearities is developed in [61] by Filippucci, Pucci and Robert, [83] by Kang and [93] by Lin, Li. We refer also to [50], [125], [130], [72] [55], [68] and the references therein.

A further generalization is represented by critical problems for (p, q)-Laplacian. In this case, in order to discuss the competition between the two terms in the nonlinearity of type (b), three situations occur, precisely $1 < k < q \leq p$ or $1 < q \leq p < k < p^*$ or $1 < q < k < p$. The last case, to the best of our knowledge, is not so much investigated in literature.

Among papers on bounded domains, we mention that by Cherfils and Il'yasov [38], which is, as far as we know, one of the very few papers dealing with condition $1 < q < k < p$ but with a subcritical nonlinearity and where, among other results, they prove existence for λ large, by using a suitable nonlinear spectral analysis. While for the case $p < k < p^*$ with singular nonlinearities we refer to [87].

Subcritical problems with $q = 2$, that is $(p, 2)$-Laplacian, in bounded domains, recently was studied by Papageorgiou, Rădulescu, Repovs in [107], [108] and [109], where they prove existence and multiplicity theorems by using a variational approach and Morse theory with $p > 2$. In particular, in [107], they take into account parametric equations when the parameter λ is close to the principal eigenvalue $\lambda_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$, while in [108] and [109], the authors consider equations where the reaction term satisfies particular conditions which imply the resonance of problem at $\pm\infty$ and at $0^{\pm}$. In this direction, we quote also the papers [112, 110] where a more general operator, given by a certain combination of p and q-Laplacian, is investigated. For a detailed theory on the subject we refer to the book [111] by Papageorgiou, Rădulescu, Repovs.

Moving to (p, q)-Laplacian critical problems in the unbounded case, the situation is fairly delicate. In particular, problem (1)-(b) is treated in [80] and in [102] for $1 < k < q$, while the case $p < k < p^*$ is treated in [53] without weights for $p, q \geq 2$ and in [36] with $K \equiv 1$. We refer also to the references therein. Further multiplicity results of a class of superlinear (p, q)-Laplacian type equations in $\mathbb{R}^N$

can be found in [19].

The common strategy adopted to produce existence and multiplicity results for these type of problems consists in constructing solutions as critical points of the energy functional associated to the problem as well as limit functions of Palais Smale sequences for the same functional, after having developed a careful analysis of the Palais Smale property for the corresponding functional. This latter standard and crucial compactness condition is one of the main delicate point, since for critical problems in all of $\mathbb{R}^N$ it is often loss, even on bounded domains.

Roughly speaking, the loss of compactness could produce bounded minimizing sequences that do not converge in a strong enough sense to pass to the limit. The reason for this is the invariance of $\mathbb{R}^{\mathbb{N}}$ with respect to translations, which in turn makes the embedding of $H^1(\mathbb{R}^N)$ into $L^\ell(\mathbb{R}^N)$ not compact for any ℓ. A natural attempt to overcome this problem is to try to work in a space of functions where translations are not allowed. For instance, this is possible if the problem is also invariant under rotations, so that one can try to work in spaces of radial functions or in general to use other special function spaces where the compactness is preserved, such as weighted Sobolev spaces. It appears clear that the presence of weights produces additional serious problems.

In particular, when lack of compactness is manifest, in order to understand the consequences of spreading or concentration of mass of Palais Smale sequences, a key tool is the celebrated concentration compactness principles by Lions [94]-[97], see also [22], which involve the notion of tight convergence. Roughly speaking, "tightness" tells that the values of the functions should belong, in a suitable integral sense, to some compact set, see Lemma I.1 in [94]. Actually, tight convergence for a sequence of measures is the weak star convergence of measures in the dual space of bounded functions, which of course is stronger, when the domain is unbounded, than the classical weak star convergence. For this reason, the proof of the tightness property is fairly delicate since often leads to rather cumbersome and tricky calculations, especially when weights are included in the equation. For details in this direction, we refer to Subsection 2.1.3 based on the book by Fonseca and Leoni [63].

As already underlined, a way to recover compactness is to restrict the analysis to a symmetric setting yielding an improvement of the existence and multiplicity theorems respect to the non symmetric setting.

In this context, as far as we know, one of the pioneering paper for critical problems in the entire $\mathbb{R}^N$ under the symmetric setting, in the sense of symmetry respect to a subgroup T of orthogonal transformation, is the one by Bianchi, Chabrowski and Szulkin in [25]. Later, there have been a variety of remarkable results on T-symmetric solutions, cfr. [21], [81], [79], [137], [32] and the reference therein. We adapt the approach in [25] for problem (1)-(b) in the last part of

Chapter 2, whose results are included in the paper [14], extending some theorems in [25], [79] and [80].

The last chapter of the work of thesis, Chapter 3, deals with solutions both with negative and with positive energy of a singular p-Laplacian Schrödinger equation in $\mathbb{R}^N$ with a nonlinearity of type (b). Solutions of this type are related to the existence of standing wave solutions for Schrödinger equations of the form

$$i\partial_t\psi = -\Delta\psi + W(x)\psi - \varphi(|\psi|^2)\psi - \kappa\Delta\varrho(|\psi|^2)\varrho'(|\psi|^2)\psi, \tag{2}$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, W is a given potential, κ is a real constant and φ, ϱ real functions of essentially pure power forms. This equation has been derived as models of several physical phenomena corresponding to various types of ϱ. The semilinear case, corresponding to $\kappa = 0$ was studied intensively in [23, 82], by so called energy methods. Due to a phenomenon called loss of derivatives these standard methods do not apply to quasilinear equations containing nonlinearities including derivatives of the second order.

Quasilinear equations of the form (2) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of ϱ. The case $\varrho(s) = s$ describes the superfluid film equation in plasma physics by Kurihura in [86], while if $\varrho(s) = (1+s)^{1/2}$, equation (2) models the self-channeling of a high-power ultra short laser in matter, see [37] and [118]. For applications in plasma physics and fluid mechanics, or in the theory of Heisenberg ferromagnets and magnons we refer to [99] and the references therein.

Actually, stationary singular p-Laplacian Schrödinger equation in $\mathbb{R}^N$ with a nonlinearity of type (b), can be seen as the p-Laplacian case of (1), but with a diffusion depending on u and a nonlinearity f composed by a gradient term and a polynomial part of the form (b), roughly a sort of combination between nonlinearities of type (b) and a generalization of (a).

The theorems given in Chapter 3, contained in paper [16], are a first attempt in treating these type of problems, indeed, as far as we know, there are only few results in this direction in literature, such as [122, 5, 142]. For a detailed discussion, we refer to the Introduction of Chapter 3.

The proof technique, we developed, relays on the use of a suitable change of variables, which involves a singular function in the case we analyze, giving rise to a reformulation of the original problem in a "more comfortable" variational setting.

This work of thesis is divided into three chapters. Let us analyze and discuss in detail the results of every chapter.

The aim of Chapter 1, motivated by [17], is to establish the existence of positive solutions for the problem

$$\begin{cases} -\Delta_p u = u^{\mathfrak{m}} + h(x, u, Du) & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $1 < p < N$, $\Omega \subseteq \mathbb{R}^N$ is a bounded smooth domain, the exponent $\mathfrak{m}$ is such that

$$p - 1 < \mathfrak{m} < p^* - 1, \tag{4}$$

the function $h : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is nonnegative, continuous such that there exist positive exponents s and θ with

$$p - 1 < s < \mathfrak{m}, \qquad p - 1 < \theta < \frac{\mathfrak{m}p}{\mathfrak{m}+1}(< p) \tag{5}$$

and a positive constant C so that

$$h(x, u, \eta) \leq C(|u|^s + |\eta|^\theta) \quad \text{in} \quad \Omega \times \mathbb{R} \times \mathbb{R}^N. \tag{6}$$

Precisely, our main existence theorem, recently appeared in [12], is the following.

Theorem 1. *Assume* (4). *Let* $h \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N)$ *be a nonnegative function verifying* (6), *with* s *and* θ *as in* (5).

If $1 < p < 2$, *suppose that the further conditions hold*

$$\frac{2N}{N+1} < p < 2, \qquad \frac{1}{p-1} < \mathfrak{m} < p^* - 1, \qquad s, \theta > 1. \tag{7}$$

Then problem (3) *admits at least a positive solution.*

To prove Theorem 1 we need to produce suitable estimates, by modifying the standard scaling method in order to take care of possible singularity of the gradient on the boundary. As in [17], we achieve this by introducing some suitable weighted norms which have been already used in the context of second order elliptic equations (cf. [75]). Dealing with these weighted norms new problems appear since the scaling needed near the boundary is not the same one as in the interior. Therefore, first of all, we need pointwise a priori estimates or "universal bounds", in the spirit of [121], for solutions of p-Laplacian elliptic equations which involve the prototype with nonlinearity of type (a), obtained in [11] thanks to the Doubling Lemma due to Poláčik, Quittner and Souplet in [114].

After that, we perform a delicate investigation near the boundary, yielding crucial boundary estimates whose proof is one of the most tricky point since it requires a cumbersome analysis of several parameters. In addition, the proof of the theorem relays also in the construction of opportune barriers for equations with a singular right-hand side, which are well behaved with respect to suitable perturbations of the domain. The conclusion follows making use of the powerful topological method given by the Leray-Schauder degree.

Chapter 2 is devoted to prove multiplicity, that is existence of infinitely many solutions, and existence results of nontrivial weak solutions of the following non-linear elliptic problem

$$-\Delta_p u - \Delta_q u = \lambda V(x)|u|^{k-2}u + K(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N \tag{8}$$

where $1 < q < p < N$, $N \geq 3$, the parameter λ is positive, the exponent k is such that $1 < k < p^*$, the weights are nontrivial and satisfy

$$0 \leq V \in L^r(\mathbb{R}^N), \quad r = \frac{p^*}{p^* - k}, \tag{9}$$

and

$$0 \leq K \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N). \tag{10}$$

The energy functional E_λ associated to (8) is defined in $D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$, where $D^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : Du \in L^p(\mathbb{R}^N)\}$, and it is given by

$$E_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N} |Du|^p dx + \frac{1}{q}\int |Du|^q dx - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|u|^k dx - \frac{1}{p^*}\int_{\mathbb{R}^N} K|u|^{p^*} dx. \tag{11}$$

In Section 2.2 we deal with the less investigated case in which $q < k < p$, precisely we obtain the following theorem, appeared in [13].

Theorem 2. *Let $1 < q < k < p < N$. Assume that V satisfies* (9) *and furthermore $V > 0$ on some open subset $\Omega_V \subset \mathbb{R}^N$, with $|\Omega_V| > 0$.*

Let K verify (10)*. If $\|K\|_\infty$ is sufficiently small, then there exist $\lambda_*, \lambda^* > 0$, with $\lambda_* < \lambda^*$, such that, for all $\lambda \in (\lambda_*, \lambda^*)$, problem* (8) *has infinitely many weak solutions with (finite) negative energy, that is $E_\lambda(u) < 0$.*

We observe that condition $\|K\|_\infty$ sufficiently small guarantees that $\lambda_* < \lambda^*$. In particular, since $\lambda^* = C\|V\|_r^{-1} \cdot \|K\|_\infty^{(k-p)/(p^*-p)}$, for some $C = C(p, k, N) > 0$, then $\lambda^* \to \infty$ when $\|K\|_\infty \to 0$.

The proof of Theorem 2 is based on concentration compactness principle, on the use of the truncated energy functional, which needs to be introduced since the original energy functional is not bounded form below, and the conclusion follows via the theory of Krasnosel'skii genus, introduced in [85]. As a standard procedure, we have first to prove the boundedness of $(\mathrm{PS})_c$ sequences, $c \in \mathbb{R}$, for E_λ, which is obtained in Lemma 8 for all k such that $1 < k < p^*$. Then, we have to face one of the main difficulties, which consists in verifying the compactness Palais Smale condition at level c for E_λ when the values c are negative. To solve this problem, as described before, we have to prove tight convergence for the sequence $(|u_n|^{p^*})_n$. We emphasize that, due to the new condition $q < k < p$, the qualitative behavior of E_λ is completely different with respect to the case treated in [36] and in [80].

Then, in Section 2.3, we treat the case k p-superlinear and subcritical. In particular in [15] we proved the following existence result.

Theorem 3. *Assume* $1 < q \le p < N$, $p < k < p^*$ *and that* V *satisfies* (9). *Furthermore, suppose* $V > 0$ *on some open subset* $\Omega_V \subset \mathbb{R}^N$, *with* $|\Omega_V| > 0$. *Let* K *verify* (10). *There exist* $\lambda^{**} > 0$ *such that, for any* $\lambda > \lambda^{**}$, *problem* (8) *has at least one nontrivial nonnegative weak solution with positive (finite) energy.*

To prove Theorem 3 we make use of the Mountain Pass Theorem. Thus, in addition to the boundedness of every $(\mathrm{PS})_c$ sequences, $c \in \mathbb{R}$, for E_λ, we have to verify that the energy functional defined in (11) has a Mountain Pass geometry, bypassing the difficulty given also by the non homogeneity of the (p, q)-Laplacian operator. Then, the lack of compactness becomes manifest since we are not allowed to prove that the functional E_λ satisfies the $(\mathrm{PS})_c$ condition. Indeed, we have to deal with only the almost everywhere convergence of $(Du_n)_n$ in $\mathbb{R}^N$, which, nevertheless is a sufficient property to prove Theorem 3.

Finally Section 2.4 is devoted to the study of existence and multiplicity results for problem (8) under a symmetric setting, contained in [14]. In particular, we consider a group $T \subset O(N)$, where $O(N)$ is the group of orthogonal linear transformations in $\mathbb{R}^N$ and we deal with T-symmetric functions, which are functions invariant respect to every orbit of T, called T_x. We denote with $D^{1,p}_T(\mathbb{R}^N)$ the subspace of $D^{1,p}(\mathbb{R}^N)$ consisting of all T-symmetric functions and $X_T = D^{1,p}_T(\mathbb{R}^N) \cap D^{1,q}_T(\mathbb{R}^N)$.

In particular, we obtain the following theorems, where $|T| := \inf_{x \in \mathbb{R}^N,\ x \neq 0} |T_x|$.

Theorem 4. *Assume* $1 < k < q \le p < N$ *and consider a group* $T \subset O(N)$. *Let* V *and* K *be* T*-symmetric functions satisfying* (9) *and* (10). *Then, there exist* $\lambda^*_T > 0$ *such that for all* $\lambda < \lambda^*_T$ *problem* (8) *possesses at least one nontrivial solution in* X_T *with (finite) positive energy, that is* $E_\lambda(u) > 0$.

Theorem 5. *Assume* $1 < q \le p < N$ *and* $1 < k < p^*$. *Let* V *and* K *be* T*-symmetric functions satisfying* (9) *and* (10). *If*

$$K(0) = K(\infty) = 0 \quad \text{and} \quad |T| = \infty, \tag{12}$$

where $K(\infty) = \limsup_{|x| \to \infty} K(x)$, *then for all* $\lambda > 0$ *problem* (8) *possesses infinitely many solutions in* X_T *with (finite) positive energy.*

Note that Theorem 4 holds for general weights K and any $T \subset O(N)$ without requiring additional assumptions neither on K nor on $|T|$, while the multiplicity result in Theorem 5 holds true provided that K and T satisfy (12).

The proof of Theorem 4 is based on Mountain Pass Theorem. In particular, when we deal with the compactness property $(\mathrm{PS})_c$ for E_λ at positive levels c, the lack of compactness is recovered by the symmetric setting producing the validity of the compactness property for every critical levels c below a positive threshold, an analogous upper bound holds for the parameter λ.

On the other hand, the proof of Theorem 5, in which we weaken an assumption in [80], is obtained by the use of Fountain Theorem due to Bartsch [20], for which it is necessary that the Palais Smale condition holds for every $c > 0$, this is reached by virtue of assuption (12). In the proof of both theorems a key role is due to the principle of symmetric criticality by Palais in [106].

In Chapter 3 we study multiplicity results for solutions both with negative and with positive energy, of the following singular quasilinear Schrödinger equation with a nonlinearity of type (b), precisely

$$-\Delta_p u - \frac{\alpha}{2}\Delta_p(|u|^\alpha)|u|^{\alpha-2}u = \lambda V(x)|u|^{k-2}u + \beta K(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N \tag{13}$$

where $0 < \alpha < 1$, $\beta, \lambda > 0$, $N \geq 3$, $1 < k < p^*$ and now assumption (9) needs to be strengthen while (10) can be weakened, namely we assume

$$0 \leq V \in L^r(\mathbb{R}^N) \cap C(\mathbb{R}^N), \quad K \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N). \tag{14}$$

To describe in a clearer way our results, we have inserted a positive parameter β in the critical term of the nonlinearity in (13).

Furthermore, we emphasize that here the weight K can change sign.

Actually, problem (13) is critical and it exhibits a double loss of compactness, as soon as $0 < \alpha < 1$, indeed when $\alpha > 1$ the corresponding critical exponent in the nonlinearity is αp^*. For the Laplacian case $p = 2$, we refer to [1] for a detailed discussion in this direction, see also [139].

In addition, problems of type (13) are rather delicate to be treated since the corresponding Euler Lagrange functional

$$H_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N} g(u)^p|Du|^p dx - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|u|^k dx - \frac{\beta}{p^*}\int_{\mathbb{R}^N} K|u|^{p^*} dx, \tag{15}$$

include the singular term g at $t = 0$ because of $0 < \alpha < 1$, whose expression will be given in Section 3.1, which causes the fact that H_λ may be not well defined in $D^{1,p}(\mathbb{R}^N)$, so that variational methods cannot directly be applied. A way to solve this problem is to perform a change of variables, introduced in [42], in order to manage a "good" functional in $D^{1,p}(\mathbb{R}^N)$ which falls in a suitable variational setting.

The first result we obtain is a multiplicity result for solution of (13) with negative energy, but under the restriction $2 < k < p < N$ due the tricky environment produced by the change of variables, for details cfr. Section 3.1.

Theorem 6. *Let* $2 < k < p < N$, $k/p < \alpha < 1$ *and* V, K *satisfies* (14), *respectively. Then,*

(i) For any $\lambda > 0$, *there exists* $\beta_S^* > 0$ *such that for any* $0 < \beta < \beta_S^*$, *then problem* (13) *has infinitely many weak solutions with (finite) negative energy.*

(ii) For any $\beta > 0$, *there exists* $\lambda_S^* > 0$ *such that for any* $0 < \lambda < \lambda_S^*$, *then problem* (13) *has infinitely many weak solutions with (finite) negative energy.*

The proof of Theorem 6 relies on the technique used in [13]: concentration compactness principle, truncation of the energy functional and the theory of genus by Krasnosel'skii. However, we have to deal with the reformulation of problem (13) in which new delicate estimates have to be performed due to the appearance of new terms.

In the next theorem we succeed in removing the upper bound for λ and β given in Theorem 6 by restricting our attention to the symmetric setting given before Theorem 4 and by requiring the same additional assumptions on K and T as in Theorem 4. In particular, the symmetric setting allows us to improve Theorem 6 obtaining the corresponding multiplicity result not only for all λ, β positive, but also removing the lower bound 2 for k, as it evident in the statement of the following

Theorem 7. *Let* $1 < k < p < N$, $k/p < \alpha < 1$ *and* V *and* K *be* T*-symmetric functions satisfying* (14). *If* (12) *holds, then, for all* $\lambda, \beta > 0$ *problem* (13) *possesses infinitely many weak solutions in* $D_T^{1,p}(\mathbb{R}^N)$ *with (finite) negative energy.*

The proof of Theorem 7 is similar to that of Theorem 6, with the novelty that $(\mathrm{PS})_c$ property for the functional follows, by virtue of (12) and the advantageous symmetric setting, for all λ, $\beta > 0$ and $c \in \mathbb{R}$.

Finally, in the last theorem, we consider solutions with positive energy, here no additional restrictions on α, p, k appear, indeed the range for α and k is the largest possible, but nonnegativity of the weight K is assumed.

Theorem 8. *Assume* $0 < \alpha < 1$ *and* $1 < k < p^*$. *Let* V *and* K *be* T*-symmetric functions satisfying* (14) *and* (12), *with* K *nonnegative (nontrivial) in* $\mathbb{R}^N$. *Then for all* $\lambda, \beta > 0$ *problem* (13) *possesses infinitely many solutions in* $D_T^{1,p}(\mathbb{R}^N)$ *with (finite) positive energy.*

The main ingredients used in the proof of Theorem 8 is the Fountain Theorem, which requires the Palais Smale property for the functional for any positive level. This is in force, as for Theorem 5, by virtue of the crucial assumption (12).

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Chapter 1

Existence results for p-Laplacian elliptic problems with gradient terms in bounded domains

The following chapter is devoted to obtain an existence result for problem (3), recently appeared in [12], which extends previous theorems in [119] and [59].

In particular, in [119] and [59] problem (3) is considered when $\mathfrak{m}$ satisfies the stronger restriction, respect to (4), given by $\mathfrak{m} < p_* - 1$, where p_* is the Serrin exponent. Of course $p_* < p^*$. Indeed in [119], as well as in [59], this stronger restriction $\mathfrak{m} < p_* - 1$ is mainly due to the use of a Harnack type inequality due to Trudinger in [134] (cfr. [120], [121]), and in addition to a Liouville theorem for the inequality $-\Delta_p u \geq u^{\mathfrak{m}}$.

As concerns Liouville theorems, it is well known that the critical exponent of Sobolev embeddings p^* is optimal for Liouville theorems for elliptic equations of the type $-\Delta_p u = u^{\mathfrak{m}}$ in $\mathbb{R}^N$, $u \geq 0$, while p_* is optimal for Liouville theorems for elliptic inequalities of the type $-\Delta_p u \geq u^{\mathfrak{m}}$ in $\mathbb{R}^N$, $u \geq 0$. For further results in this direction we refer to Serrin and Zou [121] and Mitidieri and Pohozaev [103].

In our setting, under a weaker assumption than (6), with exponents of the nonlinearity satisfying (4) and (5), in Theorem 11, given in Subsection 1.2, we obtain a priori uniform estimates for a suitable weighted norms, taking inspiration from Barrios et al. in [17], which allow us to prove the main existence result which enlarge the range of $\mathfrak{m}$ to $\mathfrak{m} < p^* - 1$ for $p > 1$.

Actually, in the case $1 < p < 2$, we have an additional restriction due only to the application of the fixed point theorem (cfr. Subsection 1.4), indeed the a priori estimates in the weighted norm hold for every $1 < p < N$. Remaining in the case $1 < p < 2$, we mention the interesting paper by Tan et al. in [131], in which regularity results for solutions of general p-Laplacian problems with gradient terms are developed.

This chapter is divided into four sections. Section 1.1 is composed by three subsections: Subsection 1.1.1 gives some preliminary results such as pointwise a priori estimates, while Subsection 1.1.2 contains a Liouville type result for p-Laplacian equations in the halfspace, with no dependence on the gradient of the nonlinearity. Finally, the topic of Subsection 1.1.3 is about some main properties of the distance function from the boundary of an open bounded set. Preparatory lemmas are stated in Section 1.2, such as uniform estimate for the weighted norm, so that, the proof of Theorem 1, whose statement is given in the Introduction, can be performed in Section 1.4 thanks to a suitable version a degree theorem contained in Subsection 1.1.2.

1.1 Main ingredients

The following section deals with some main definitions and well known classical results we are going to use throughout Chapter 1.

1.1.1 Pointwise a priori estimates via the doubling lemma

We begin stating pointwise a priori estimates, the main ingredient in the proof of the existence Theorem 1. These type of estimates, contained in Theorem 3.1 in [11], here reported for completeness, are called "universal bounds", in the sense of Serrin and Zou in [121], because they are independent of the solutions and do not need any boundary conditions.

Theorem 9. *Let Ω be an arbitrary domain of $\mathbb{R}^N$, $N \geq 2$. Let*

$$1 < p < N, \quad p-1 < \mathfrak{m} < p^* - 1, \quad 0 \leq r < \mathfrak{m}. \tag{1.1}$$

Assume that $f : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function and that there exist $\mathfrak{m}_1 \in (0, \mathfrak{m})$, s and S with

$$0 \leq r \leq S < \min\left\{\mathfrak{m}, \frac{(\mathfrak{m}+1)(r+1)}{p} - 1\right\},$$

$$0 < \theta < \Theta := \frac{(\mathfrak{m}-S)p}{\mathfrak{m}+1}(< p),$$

such that

$$-C_1\big(1 + z^{\mathfrak{m}_1} + z^r|\eta|^\theta\big) \leq f(x, z, \eta) \leq C_1\big(1 + z^{\mathfrak{m}} + z^S|\eta|^\Theta\big) \tag{1.2}$$

for all $x \in \Omega$ *and for* $z \geq 0$ *and* $\eta \in \mathbb{R}^N$*, with* $C_1 > 0$*. Suppose that for all* $x \in \overline{\Omega}$*, if* Ω *is bounded,*

$$\lim_{z\to\infty,\Omega\ni y\to x} z^{-\mathfrak{m}} f(y, z, z^{(\mathfrak{m}+1)/p}\eta) = l(x) \in (0, \infty) \tag{1.3}$$

uniformly for η *bounded. Moreover, if* Ω *is unbounded, then assume that* (1.3) *holds also for* $x = \infty$*.*

Then there exist $C = C(p, N, f) > 0$ *(independent of* Ω *and* u*) such that for any nonnegative solution* u *of*

$$-\Delta_p u = f(x, u, Du) \quad in \;\; \Omega, \tag{1.4}$$

then holds

$$u + |Du|^{p/(\mathfrak{m}+1)} \leq C\big(1 + d(x)^{-p/(\mathfrak{m}-p+1)}\big) \quad in \;\; \Omega,$$

where $d(x) = dist(x, \partial\Omega)$*.*

Their proof is based on the well known doubling lemma, Theorem 5.1 in [114], which roughly is a way to avoid a Liouville theorem in the halfspace. Actually, the previous result extends Theorem 6.1 proved by Poláčik, Quitter and Souplet in [114] when $p = 2$ and $r = 0$ in (1.1). In particular, the authors in [114] take into account a nonlinearity depending also on the gradient only when the Laplace operator is involved, while the p-Laplacian operator is investigated with nonlinearities not depending on the gradient. As discussed in [114], an important consequence of results of this kind is that theorems that provide uniform estimates (in norm) of the solutions are substantially equivalent to Liouville theorems in the entire $\mathbb{R}^N$.

From the theorem above we obtain the following bounds in Ω for all nonnegative solutions of the equation (1.4) and for its gradient

$$\begin{gathered} u(x) \leq C\big(1 + d(x)^{-p/(\mathfrak{m}-p+1)}\big), \\ |Du(x)| \leq C\big(1 + d(x)^{-(\mathfrak{m}+1)/(\mathfrak{m}-p+1)}\big) = C\big(1 + d(x)^{-1-p/(\mathfrak{m}-p+1)}\big). \end{gathered} \tag{1.5}$$

1.1.2 A Liouville theorem in $\mathbb{R}^N_+$

A Liouville theorem in the halfspace represents another key ingredient. Actually, these type of results, under rather general assumptions, are not common in literature. We make use of the version a Liouville type theorem in the halfspace for p-Laplacian elliptic equations proved by Zou in [146], requiring some stringent conditions for nonlinearities depending on the gradient, which can be removed in the subcase when the nonlinearity depends only on u. With this in mind, we shall apply the theorem of Zou to the limit problem, in which the dependence of the nonlinearity form the gradient disappears. Precisely, for the equation

$$-\Delta_p u = g(u) \quad \text{in } \mathbb{R}^N_+, \tag{1.6}$$

when $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, the following Liouville type theorem, Theorem 1.1 in [146], holds.

Theorem 10. *Assume that $g(z)$ is continuously differentiable for $u > 0$ and that there exist positive constants $K > 0$, $\mathfrak{m} \in (p-1, p^*-1)$ and $r \in (0, p^*-1)$ such that for $z > 0$*

$$K^{-1}z^{\mathfrak{m}} \leq g(z) \leq Kz^{\mathfrak{m}}, \quad rg(z) \geq zg'(z). \tag{1.7}$$

Then equation (1.6) does not admit any nonnegative nontrivial distributional solutions $u \in W^{1,p}_0(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ vanishing on $\partial\mathbb{R}^N_+$.

Remark 1. *Of course when $g(z) = z^{\mathfrak{m}}$ with $\mathfrak{m} \in (p-1, p^*-1)$, then conditions (1.7) are trivially verified with $K = 1$ and $r = \mathfrak{m}$.*

Liouville results in the halfspace, applied to the limit problem, allow to prove uniform bounds in a suitable norm for positive solutions of (1.4) when the blow up technique is developed. Roughly, first we assume the existence of a divergence sequence of solutions of (1.4) which violates the uniform bound, that is u_n attaining their maxima on a point x_n in Ω. Then, suitable scaling arguments "centered" on x_n, produce positive solutions of the limit problem (1.6), either in $\mathbb{R}^N$, in the case in which $x_n \to x_0 \in \Omega$ up to subsequences, or in the halfspace if $x_n \to x_0 \in \partial\Omega$ up to subsequences. Thus, a Liouville theorem gives the required contradiction.

1.1.3 Properties of $d(x) = dist(x, \partial\Omega)$

First of all we refer to the euclidean distance function from the boundary, defined by $d(x) = dist(x, \partial\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a nonempty, bounded open connected set of class C^2, say a smooth bounded domain, following [44] (see also [31]). It is well known that d is a solution of the eikonal equation

$$\begin{cases} |Du| = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

precisely it is unique in the class of viscosity solutions, as discussed in [44]. For every $\xi \in \partial\Omega$ we denote by $\kappa_i(\xi)$, $i = 1, \ldots, N-1$ the principal curvatures of $\partial\Omega$ at ξ, and by $\nu = \nu(\xi)$ the outward normal unit vector of $\partial\Omega$ at ξ. Since the boundary of Ω is regular, it is possible to extend d outside $\overline{\Omega}$ in such a way that this extension, called signed distance function form $\partial\Omega$ and denoted by d^s, turns out to be of class C^2 in a tubular neighborhood of $\partial\Omega$ of the following form $A_\mu = \{x \in \mathbb{R}^N : -\delta < d^s(x) < \delta\}$, with $\delta > 0$ (cfr. Theorem 4.16 in [44]). This fact allows to define Dd e D^2d on points of $\partial\Omega$. In particular, in Lemma 4.3 in [44], by using also Lemma 3.5 in [113], a relation between the gradient of the extension of d to the entire $\mathbb{R}^N$ and the outward normal of $\partial\Omega$ is proved. Precisely, if $x \in \Omega \setminus \Sigma$, where Σ is the set of those points in Ω where d is not differentiable, then $Dd = \lambda\nu$, $\lambda \leq 0$. Furthermore, in Lemma 4.18 in [44], it is given an explicit representation of the Hessian matrix D^2d of d in $x_0 \in \partial\Omega$ with respect to the principal coordinate system (cfr. also Remark 4.19 in [44]). Finally, as observed in [66], it results that $-\Delta d(x) \sim (N-1)H(\xi)$ as $x \to \partial\Omega$, where $x \in \Omega$, $\xi \in \partial\Omega$ is the projection of x on $\partial\Omega$, while $H(\xi)$ stands for the mean curvature at ξ. Therefore, when the domain Ω is convex we have

$$-\Delta d(x) \geq 0 \tag{1.9}$$

in a tubular neighborhood of $\partial\Omega$, defined as follows.

Denote, for all positive $\delta > 0$,

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}.$$

In particular, it holds the following.

Lemma 1. *Let $\gamma \in (0,1)$. There exist $\delta > 0$ and $C_\gamma > 0$ such that*

$$-\Delta_p d(x)^\gamma \geq C_\gamma d(x)^{\gamma(p-1)-p} \quad \textit{in} \ \ \Omega_\delta. \tag{1.10}$$

Proof. Since $|Dd| = 1$ by (1.8), for $\gamma > 0$,

$$-\Delta_p d(x)^\gamma = \gamma^{p-1}\big\{(1-\gamma)(p-1) - d(x)\Delta d(x)\big\}d(x)^{\gamma(p-1)-p}.$$

Thanks to $\gamma < 1$ and the regularity of d in Ω_δ, we can choose δ so small that $|d(x)\Delta d(x)| < \varepsilon$ for all $x \in \Omega_\delta$, $\varepsilon > 0$, and $C_\gamma = (1-\gamma)(p-1) - \varepsilon > 0$. This completes the proof. □

Note that, by (1.9), when Ω is convex then (1.10) is clearly true in Ω for every $0 < \gamma < 1$. Moreover, in the special case $\Omega = B_R(0)$, with $R > 0$ and $d(x) = d_R(x) = dist(x, \partial\Omega) = R - |x|$, then $-\Delta d_R(x) = (N-1)/|x| \geq 0$ in $\Omega \setminus \{0\}$ and $d_R(x)^\gamma$ is a solution of

$$-\Delta_p d_R(x)^\gamma = \gamma^{p-1}\left[p - \gamma(p-1) - 1 + \frac{(N-1)d_R(x)}{R - d_R(x)}\right]d_R(x)^{\gamma(p-1)-p} \quad \text{in} \ \ \Omega \setminus \{0\},$$

so that, not only (1.10) holds for $0 < \gamma < 1$, but also

$$-\Delta_p d_R(x)^\gamma \geq A_\gamma\, d_R(x)^{\gamma(p-1)-p} \quad \text{in} \quad (\Omega_\delta)^c, \quad \delta = \frac{R}{N},$$

where $A_\gamma = p - \gamma(p-1) > 0$ if and only if $0 < \gamma < \frac{p}{p-1}(> 1)$.

1.2 Preliminary lemmas on suitable "barriers" and "scaled domains"

In this section we present some preliminary lemmas on scaled domains based on weighted norm which involves the distance function from the boundary of an open bounded set, whose properties are presented in Subsection 1.1.3, namely let Ω be a smooth bounded domain.

Take $\xi \in \partial\Omega$, $\mu > 0$ and define

$$\Omega^\mu = \{y \in \mathbb{R}^N : \xi + \mu y \in \Omega\}, \quad d_\mu(y) = dist(y, \partial\Omega^\mu). \tag{1.11}$$

It is clear that $d_\mu(y) = \mu^{-1} d(\xi + \mu y)$ from the fact that roughly $\Omega = \xi + \mu\Omega^\mu$, or equivalently $\Omega^\mu = \mu^{-1}(\Omega - \xi)$. Furthermore, for $d_\mu(y)$ the analogous result given in Lemma 1 holds, namely $d_\mu(y)$ satisfies

$$-\Delta_p d_\mu(x)^\gamma \geq C'_\gamma d_\mu(x)^{\gamma(p-1)-p} \quad \text{in} \quad (\Omega^\mu)_\delta, \tag{1.12}$$

with $C'_\gamma = \mu^{-p} C_\gamma$, where C_γ is given in (1.10). In what follows we make use of the function $u_\mu = u(\xi + \mu y)$ defined in Ω^μ.

Lemma 2. *Let $\delta > 0$ and $\gamma \in \big(0, 1\big)$. If $u_\mu \in L^\infty(\Omega) \cap C(\Omega)$ verifies*

$$\begin{cases} -\Delta_p u_\mu \leq C_1 d_\mu^{\gamma(p-1)-p} & \text{in } \Omega^\mu, \\ u_\mu = 0 & \text{on } \partial\Omega^\mu, \end{cases} \tag{1.13}$$

for some $C_1 > 0$, then

$$u_\mu(x) \leq C_2(C_3 + \|u_\mu\|_{L^\infty(\Omega^\mu)}) d_\mu^\gamma \quad \text{in} \quad (\Omega^\mu)_\delta, \tag{1.14}$$

for some $C_2, C_3 > 0$ only depending on δ, γ and C_γ.

Proof. We follow the argument of Lemma 6 in [17] which can be applied thanks to the considerations before the statement of Lemma 2. Let $v = Rd_\mu^\gamma$, with d_μ given in (1.11) and $R > 0$ to be specified. Then, thanks to (1.12) and (1.13), we have

$$-\Delta_p v \geq C'_\gamma R^{p-1} d_\mu^{\gamma(p-1)-p} \geq C_1 d_\mu^{\gamma(p-1)-p} \geq -\Delta_p u_\mu \quad \text{in } (\Omega^\mu)_\delta,$$

if we choose $R > [C_1(C'_\gamma)^{-1}]^{1/(p-1)}$. Furthermore, since $(\partial\Omega^\mu)_\delta = \partial\Omega^\mu \cup A_\delta^\mu$ with $A_\delta^\mu = \{y \in \Omega^\mu : d(y, \partial\Omega^\mu) = d_\mu(y) = \delta\}$, then $v \geq u_\mu$ in $(\partial\Omega^\mu)_\delta$ being $u_\mu = 0 = v$ in $\partial\Omega^\mu$ by (1.13) and the definition of v, furthermore $v = Rd_\mu^\gamma \geq u_\mu$ in A_δ^μ eventually enlarging R such that $R \geq \delta^{-\gamma}\|u_\mu\|_{L^\infty(\Omega^\mu)}$. Hence, by comparison $v \geq u_\mu$ in $(\Omega^\mu)_\delta$, and it is possible to choose R so that (1.14) hold. Thus, take for instance $R = K\delta^{-\gamma}\|u_\mu\|_{L^\infty(\Omega^\mu)} + C$, with $K \geq 1$ and $C \geq K^{-1}(C_1 C_\gamma^{-1})^{1/(p-1)}$ so that $C_2 = K\delta^{-\gamma}$ and $C_3 = C\delta^\gamma$. □

Remark 2. *It is well known that every distributional solution of*

$$\begin{cases} -\Delta_p u = f & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{1.15}$$

is of class $L^\infty(\Omega)$ as soon as $f \in C(\Omega) \cap L^\infty(\Omega)$, for details see Theorem 6 in the Appendix in [12]. Furthermore, thanks to classical results by Di Benedetto [49], *Tolksdorf* [133], *Lieberman* [92] *it holds the following $u \in C^{1,\tau}_{loc}(\Omega)$, $\tau \in (0,1)$. Finally we recall that the following Cordes-Nirenberg type estimate*

$$\|u\|_{C^{1,\tau}(\overline{B_{R/2}})} \leq C(\|f\|_{L^\infty(B_R)} + \|u\|_{L^\infty(B_R)}). \tag{1.16}$$

is valid with a positive constant $C = C(N,p)$, see [30] *and also* [43].

As in [17], we define for $\lambda \in \mathbb{R}$ and $u \in C(\Omega)$

$$\|u\|_0^{(\lambda)} = \sup_\Omega d(x)^\lambda |u(x)|,$$

while for $u \in C^1(\Omega)$, set

$$\|u\|_1^{(\lambda)} = \sup_\Omega \Big\{ d(x)^\lambda |u(x)| + d(x)^{1+\lambda}|Du(x)| \Big\}. \tag{1.17}$$

We are now ready to give the first estimation for solutions of problem (1.15) in the norm defined above. Precisely, the next result is a different version of the existence Theorem 2 in [65].

Lemma 3. *Assume $\gamma \in (0,1)$. Let $f \in C(\Omega)$, where Ω is a smooth bounded domain. If $\|f\|_0^{(p-\gamma(p-1))} < \infty$, then the problem* (1.15) *admits a unique solution. Moreover, there exists a positive constant C such that*

$$\|u\|_0^{(-\gamma)} \leq C\Big(\|f\|_0^{(p-\gamma(p-1))}\Big)^{1/(p-1)}. \tag{1.18}$$

Proof. By Lemma 1, there exist $\delta > 0$ and $C_\gamma > 0$ such that (1.10) is verified. Now, taking inspiration from Theorem 2 in [65], we construct a supersolution of

$$\begin{cases} -\Delta_p u = C d^{\gamma(p-1)-p} & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.19}$$

for a suitable constant $C > 0$. For this aim, we consider the unique solution ψ to

$$\begin{cases} -\Delta_p \psi = 1 & \text{in } \Omega, \\ \psi > 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.20}$$

which is well known to exist and to verify $\psi \in W_0^{1,p}(\Omega) \cap C^{1,\tau}(\overline{\Omega})$ for some $\tau \in (0,1)$. Moreover, thanks to Hopf's maximum principle (see [136]), we have that

$$C_1 d \leq \psi \leq C_2 d \qquad \text{in } \Omega, \tag{1.21}$$

for some positive constant C_1, C_2. We claim that $\overline{u} = \psi^\gamma$ is a supersolution to (1.19). Indeed, a calculation shows that

$$-\Delta_p \overline{u} = \gamma^{p-1} \psi^{\gamma(p-1)-p} [\psi + (p-1)(1-\gamma)|D\psi|^p].$$

We now observe that $|D\psi|^p + \psi > 0$ in $\overline{\Omega}$, being $\psi > 0$ in Ω and $D\psi \neq 0$ on $\partial\Omega$ by Hopf's principle. Since $\gamma < 1$, and taking into account (1.21), we obtain that

$$-\Delta_p \overline{u} \geq C d^{\gamma(p-1)-p} \quad \text{in } \Omega, \tag{1.22}$$

for some positive constant C such that $C \geq C_\gamma$, where C_γ is given by (1.10).

Following now Lemma 3 in [17], we choose a sequence of smooth functions $(\phi_n)_n$ verifying the following condition

$$0 \leq \phi_n \leq 1, \quad \phi_n = 1 \text{ in } \Omega \setminus \Omega_{2/n} \quad \text{and} \quad \phi_n = 0 \text{ in } \Omega_{1/n}.$$

Define $f_n = f\phi_n$ and take into account problem

$$\begin{cases} -\Delta_p u = f_n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.23}$$

Since $f_n \in C(\overline{\Omega})$, by a classical result, problem (1.23) has a unique weak solution $u_n \in W^{1,p}_{loc}(\Omega) \cap C^{1,\tau}_{loc}(\overline{\Omega})$, for some $\tau \in (0,1)$. On the other hand,

$$|f_n| \leq |f| \leq \|f\|_0^{(p-\gamma(p-1))} d^{\gamma(p-1)-p} \quad \text{in } \Omega,$$

so that the functions

$$v_+ = \left(C^{-1}\|f\|_0^{(p-\gamma(p-1))}\right)^{1/(p-1)}\overline{u} \quad \text{and} \quad v_- = -\left(C^{-1}\|f\|_0^{(p-\gamma(p-1))}\right)^{1/(p-1)}\overline{u}$$

are sub and supersolution of (1.23) respectively, also by (1.22). Using comparison, (see [115]), we obtain

$$-\left(C^{-1}\|f\|_0^{(p-\gamma(p-1))}\right)^{1/(p-1)}\overline{u} \le u_n \le \left(C^{-1}\|f\|_0^{(p-\gamma(p-1))}\right)^{1/(p-1)}\overline{u}, \tag{1.24}$$

which holds in Ω. Now, this bound together with (1.16), Ascoli-Arzelá's theorem and a standard diagonal argument allow us to obtain a subsequence, still denoted by $(u_n)_n$, and a function $u \in C(\Omega)$ such that $u_n \to u$ uniformly on compact sets of Ω. In addition, from (1.24), u verifies in Ω

$$|u| \le \left(C^{-1}\|f\|_0^{(p-\gamma(p-1))}\right)^{1/(p-1)}\psi^\gamma \le \left(C^{-1}\|f\|_0^{(p-\gamma(p-1))}\right)^{1/(p-1)}d^\gamma,$$

which shows (1.18). This ends the proof. □

Our next estimate concerns the gradient of solutions of (1.15) and it takes inspiration from Lemma 5 in [17].

Lemma 4. *There exists a constant C_0 which depends on N and p but not on Ω such that, for every $\eta \in (1,p)$ and $f \in C(\Omega)$ with $\|f\|_0^{(\frac{\eta+p(p-2)}{p-1})} < \infty$, the unique solution u of* (1.15) *verifies*

$$\|Du\|_0^{(\frac{\eta-1}{p-1})} \le C_0(\|f\|_0^{(\frac{\eta+p(p-2)}{p-1})} + \|u\|_0^{(\frac{\eta-p}{p-1})}). \tag{1.25}$$

Proof. For the existence and the uniqueness we apply Lemma 3 with $\gamma \in (0,1)$. Indeed, $f \in C(\Omega)$ and $p - \gamma(p-1) > 0$ if and only if $\gamma < p/(p-1)$ which is true since $\gamma < 1$. By Remark 3 applied with $R = 1$ we know that if v solves $-\Delta_p v = f$ in B_1 with $v = 0$ on ∂B_1, then by (1.16) there exists a constant which depends on N and p such that

$$\|Dv\|_{L^\infty(\overline{B_{1/2}})} \le C\big(\|f\|_{L^\infty(B_1)} + \|v\|_{L^\infty(B_1)}\big). \tag{1.26}$$

Define

$$u(x) = v(x/R) \quad x \in B_R,$$

so that $\|v\|_{L^\infty(B_1)} = \|u\|_{L^\infty(B_R)}$ and $R\, Du = Dv$, from which we obtain

$$\|Dv\|_{L^\infty(B_1)} = R\|Du\|_{L^\infty(B_R)}.$$

Moreover, denoting $z = x/R$,

$$\Delta_p u = div_x(|Du|^{p-2}Du) = \frac{1}{R^p} div_z(|Dv|^{p-2}Dv)$$

so that, since v is a solution of $-\Delta_p v = f$ in B_1, we have

$$\|f\|_{L^\infty(B_1)} = R^p \|f\|_{L^\infty(B_R)}.$$

Thus, u solves $-\Delta_p u = f$ in Ω and $B_R \subset\subset \Omega$ then by (1.26)

$$R\|Du\|_{L^\infty(\overline{B_{R/2}})} \leq C\big(R^p\|f\|_{L^\infty(B_R)} + \|u\|_{L^\infty(B_R)}\big). \tag{1.27}$$

Choose a point $x \in \Omega$. By applying (1.27) in the ball $B = B_{d(x)/2}$ centered at x and multiplying by $d(x)^{\frac{\eta-p}{p-1}}$ we arrive at

$$d(x)^{\frac{\eta-1}{p-1}}|Du(x)| \leq C\big(d(x)^{\frac{\eta+p(p-2)}{p-1}}\|f\|_{L^\infty(B)} + d(x)^{\frac{\eta-p}{p-1}}\|u\|_{L^\infty(B)}\big). \tag{1.28}$$

Finally, we notice that $d(x)/2 < d(y) < 3d(x)/2$ for every $y \in B$, so that, for $c > 0$,

$$d(x)^{\frac{\eta+p(p-2)}{p-1}}|f(y)| \leq c\, d(y)^{\frac{\eta+p(p-2)}{p-1}}|f(y)| \leq c\,\|f\|_0^{(\frac{\eta+p(p-2)}{p-1})},$$

$$d(x)^{\frac{\eta-p}{p-1}}|u(y)| \leq c\, d(y)^{\frac{\eta-p}{p-1}}|u(y)| \leq c\|u\|_0^{(\frac{\eta-p}{p-1})},$$

thus

$$d(x)^{\frac{\eta+p(p-2)}{p-1}}\|f\|_{L^\infty(B)} \leq c\|f\|_0^{(\frac{\eta+p(p-2)}{p-1})} \tag{1.29}$$

$$d(x)^{\frac{\eta-p}{p-1}}\|u\|_{L^\infty(B)} \leq c\|u\|_0^{(\frac{\eta-p}{p-1})}. \tag{1.30}$$

Then, using (1.29) and (1.30) in (1.28) and taking the supremum over B, inequality (1.25) is obtained. □

Remark 3. *We point out that the bound for η in the statement of Lemma 4 is required in view of the application of Lemma 3 in which the parameter γ has to be taken in $(0, 1)$, see the proof of Theorem 1.*

1.3 Uniform estimates for a weighted norm

Before proving the a priori estimate result, we need to choose properly the parameter $\sigma > 0$ such that if $p \geq 2$

$$1 - \frac{p}{\theta} < \sigma < 1 - \frac{1}{\theta}, \tag{1.31}$$

if $1 < p < 2$

$$\max\Big\{1 - \frac{2(p-1)}{\theta}, \frac{\theta - 2(p-1)}{\theta - p + 1}\Big\} < \sigma < 1 - \frac{p-1}{\theta}. \tag{1.32}$$

The intervals for σ, given in (1.31) and (1.32), are not trivial thanks to (4), (5) and the further condition (7) if $1 < p < 2$. Let

$$E_\sigma = \{u \in C^1(\Omega) : \|u\|_1^{(-\sigma)} < \infty\}, \tag{1.33}$$

where $\|\cdot\|_1^{(-\sigma)}$ is given by (1.17) when $\lambda = -\sigma$. In the next result, we prove a priori bounds for solutions of (3) by adapting to the p-Laplacian case the argument in Lemma 9 in [17].

Theorem 11. *Let $p - 1 < \mathfrak{m} < p^* - 1$ and let h be a nonnegative function such that*

$$h(x, u, \eta) \leq C(1 + |u|^s + |\eta|^\theta), \quad \text{in } \Omega \times \mathbb{R} \times \mathbb{R}^N, \quad C > 0, \tag{1.34}$$

where s and θ satisfy condition (5)*. Then there exist a constant $C > 0$ such that for every positive solution u of* (3) *in E_σ, with $\sigma > 0$ satisfying either* (1.31) *or* (1.32)*, we have*

$$\|u\|_1^{(-\sigma)} \leq C.$$

Proof. Assume that the conclusion of the theorem is not true. Then there exists a sequence of positive solutions u_k of (3) in E_σ, with σ as above, such that

$$\|u_k\|_1^{(-\sigma)} = \sup_\Omega M_k(x) \to \infty,$$

where

$$M_k(x) = d(x)^{-\sigma} u_k(x) + d(x)^{1-\sigma} |Du_k(x)|. \tag{1.35}$$

Now choose points $x_k \in \Omega$ such that $M_k(x_k) \geq \sup_\Omega M_k - \frac{1}{k}$ for all k, this can be done thanks to (1.17). Let ξ_k be a projection of x_k on $\partial\Omega$ and rescale u_k by setting

$$v_k(y) = \mu_k^{-\sigma} M_k(x_k)^{-1} u_k(z), \qquad z = \xi_k + \mu_k y, \qquad y \in D^k, \tag{1.36}$$

where $\mu_k = M_k(x_k)^{-\frac{\mathfrak{m}-p+1}{p+\sigma(\mathfrak{m}-p+1)}}$ and $D^k = \{y \in \mathbb{R}^N : \xi_k + \mu_k y \in \Omega\}$ according to the definition (1.11). In particular $\mu_k \to 0$ as $k \to \infty$ since $M_k(x_k) \to \infty$ as $k \to \infty$. By (1.36) we have $(\partial/\partial y_i) v_k(y) = \mu_k^{1-\sigma} M_k(x_k)^{-1} (\partial/\partial z_i) u_k(z)$, from which we obtain

$$\Delta_p v_k = div_y(|Dv_k|^{p-2} Dv_k) = \mu_k^{p-\sigma(p-1)} M_k(x_k)^{-(p-1)} \Delta_p u_k$$

so that, since u_k is a solution of (3), we have

$$\begin{aligned}\Delta_p v_k = \mu_k^{p+\sigma(\mathfrak{m}-p+1)} M_k(x_k)^{\mathfrak{m}-p+1} v_k^{\mathfrak{m}} \\ + \mu_k^{p-\sigma(p-1)} M_k(x_k)^{-(p-1)} h(z, \mu_k^\sigma M_k(x_k) v_k, \mu_k^{\sigma-1} M_k(x_k) D v_k).\end{aligned}$$

Then, according to the choice of μ_k, v_k is a solution of

$$\begin{cases} -\Delta_p v_k(y) = v_k(y)^{\mathfrak{m}} + h_k(y, v_k, Dv_k) & \text{in } D^k, \\ v_k(y) = 0 & \text{on } \partial D^k, \end{cases} \tag{1.37}$$

with

$$\begin{aligned} h_k(y,v_k, Dv_k) &= \mu_k^{p-\sigma(p-1)} M_k(x_k)^{1-p} h\big(z, \mu_k^\sigma M_k(x_k) v_k, \mu_k^{\sigma-1} M_k(x_k) D v_k\big) \\ &= M_k(x_k)^{-\frac{\mathfrak{m}p}{p+\sigma(\mathfrak{m}-p+1)}} h\Big(z, M_k(x_k)^{\frac{p}{p+\sigma(\mathfrak{m}-p+1)}} v_k, M_k(x_k)^{\frac{\mathfrak{m}+1}{p+\sigma(\mathfrak{m}-p+1)}} D v_k\Big). \end{aligned}$$

By assumption (1.34) on h, it is trivially seen that h_k verifies

$$\begin{aligned} 0 \le h_k &\le C M_k(x_k)^{-\frac{p\mathfrak{m}}{p+\sigma(\mathfrak{m}-p+1)}} (1 + M_k(x_k)^{\frac{sp}{p+\sigma(\mathfrak{m}-p+1)}} v_k^s \\ &\qquad + M_k(x_k)^{\frac{(\mathfrak{m}+1)\theta}{p+\sigma(\mathfrak{m}-p+1)}} |Dv_k|^\theta) \\ &\le C M_k(x_k)^{-\overline{\gamma}} (1 + v_k^s + |Dv_k|^\theta), \end{aligned} \tag{1.38}$$

for some positive constant C independent of k, where

$$\overline{\gamma} = \frac{p\mathfrak{m} - \max\{sp, \theta(\mathfrak{m}+1)\}}{p + \sigma(\mathfrak{m} - p + 1)} > 0, \tag{1.39}$$

thanks to (5). Moreover, the function v_k verifies in D^k

$$\mu_k^\sigma d(\xi_k + \mu_k y)^{-\sigma} v_k(y) + \mu_k^{\sigma-1} d(\xi_k + \mu_k y)^{1-\sigma} |Dv_k(y)| = \frac{M_k(\xi_k + \mu_k y)}{M_k(x_k)},$$

which can be written as follows

$$d_k(y)^{-\sigma} v_k(y) + d_k(y)^{1-\sigma} |Dv_k(y)| = \frac{M_k(\xi_k + \mu_k y)}{M_k(x_k)} \quad \text{in } D^k, \tag{1.40}$$

by denoting $d_k(y) := d(y, \partial D^k) = \mu_k^{-1} d(\xi_k + \mu_k y)$. Furthermore, if we define $y_k = (x_k - \xi_k)/\mu_k$, we have

$$|y_k| = |x_k - \xi_k| \mu_k^{-1} = d(x_k) \mu_k^{-1}$$

we obtain by (1.40)

$$d_k(y_k)^{-\sigma} v_k(y_k) + d_k(y_k)^{1-\sigma} |Dv_k(y_k)| = 1. \tag{1.41}$$

On the other hand, equality (1.40) gives

$$d_k(y)^{-\sigma}v_k(y) + d_k(y)^{1-\sigma}|Dv_k(y)| \le 2 \quad \text{in} \quad D^k, \tag{1.42}$$

by the definition of x_k so that $M_k(\xi_k + \mu_k y) \le \sup_\Omega M_k(x) \le M_k(x_k) + \frac{1}{k}$. Next, u_k solves (1.4), with $f(x,u,\eta) = u^{\mathfrak{m}} + h(x,u,\eta)$ which satisfies (1.2) with $S = 0$ and $\theta < \Theta = p\mathfrak{m}/(\mathfrak{m}+1)$, indeed by (6) we have

$$0 \le u^{\mathfrak{m}} + h(x,u,\eta) \le u^{\mathfrak{m}} + C(1 + u^s + |\eta|^\theta) \le \begin{cases} C_1 u^{\mathfrak{m}} + C|\eta|^\theta, & if \ \ |u| \ge 1, \\ C(1 + u^{\mathfrak{m}} + |\eta|^\theta), & if \ \ |u| \le 1. \end{cases}$$

Consequently, Theorem 9 can be applied and, thanks to the estimates in (1.5), we obtain from (1.35)

$$M_k(x_k) \le Cd(x_k)^{-\sigma}\Big[1 + d(x_k)^{-p/(\mathfrak{m}-p+1)} + d(x_k)\Big],$$

where C is a positive constant independent of k. In turn,

$$\begin{aligned} d(x_k)\mu_k^{-1} &\le Cd(x_k)^{\frac{p}{p+\sigma(\mathfrak{m}-p+1)}}\Big[1 + d(x_k)^{-\frac{p}{p+\sigma(\mathfrak{m}-p+1)}} + d(x_k)^{\frac{\mathfrak{m}-p+1}{p+\sigma(\mathfrak{m}-p+1)}}\Big] \\ &= C\Big[1 + d(x_k)^{\zeta_1} + d(x_k)^{\zeta_2}\Big] \le C, \end{aligned}$$

where,

$$\zeta_1 = \frac{p}{p+\sigma(\mathfrak{m}-p+1)}, \qquad \zeta_2 = \frac{\mathfrak{m}+1}{p+\sigma(\mathfrak{m}-p+1)}.$$

In particular, $\zeta_1, \zeta_2 > 0$, thus $d(x_k)\mu_k^{-1} \le C$, from the boundedness of Ω. This bound immediately entails that, up to subsequences, $d(x_k) \to 0$ as $k \to \infty$ since $\mu_k \to 0$ and $x_k \to x_0 \in \overline{\Omega}$. Thus $|y_k| = d(x_k)\mu_k^{-1} \to d \ge 0$ as $k \to \infty$ (in particular, as noted in [17], the points ξ_k are uniquely determined at least for large k), actually $x_0 \in \partial\Omega$, being $d(x_0) = 0$. Assuming without loss of generality that the outward unit normal to $\partial\Omega$ at x_0 is $-e_N$, observing that $0 \in \partial D^k$ since $d_k(0) = d(0, \partial D^k) = \mu_k^{-1}d(\xi_k) = 0$, by $\xi_k \in \partial\Omega$, we also obtain that $D^k \to \mathbb{R}^N_+$ for $k \to \infty$.

We claim that $d > 0$. To show this, notice that from (1.42) is clearly true the following

$$v_k(y) \le 2d_k(y)^\sigma, \qquad |Dv_k| \le 2d_k(y)^{\sigma-1}, \tag{1.43}$$

then from (1.38) we have

$$0 \le h_k \le CM_k(x_k)^{-\overline{\gamma}}\Big[1 + d_k(y)^{\sigma s} + d_k(y)^{\theta(\sigma-1)}\Big] \quad \text{in} \quad D^k.$$

Consequently, again by (1.43), we obtain

$$v_k^{\mathfrak{m}} + h_k \le Cd_k(y)^{\theta(\sigma-1)}L_k(y) \quad \text{in } D^k,$$

where

$$L_k(y) := d_k(y)^{\sigma\mathfrak{m}+\theta(1-\sigma)} + M_k(x_k)^{-\overline{\gamma}}\big(d_k(y)^{\theta(1-\sigma)} + d_k(y)^{\sigma s+\theta(1-\sigma)}\big).$$

In particular, since Ω is bounded, then $d_k(y)$ is bounded in D^k and being all the exponents of d_k in $L_k(y)$ positive, we arrive to

$$0 \le L_k(y) \le C\Big(1 + M_k(x_k)^{-\overline{\gamma}}\Big),$$

in turn, $L_k(y) \le C$ since $M_k(x_k) \to \infty$ as $k \to \infty$ and thanks to (1.39). Thus, we have

$$-\Delta_p v_k \le C d_k(y)^{-\theta(1-\sigma)} \quad \text{in} \quad D^k, \tag{1.44}$$

for k large. Now to use Lemma 2, we need to divide the proof into two cases.

Case $p \ge 2$: we apply Lemma 2 with $\gamma = \frac{p-(1-\sigma)\theta}{p-1}$, this can be done by regularity of v_k and since v_k verifies (1.44) vanishing at the boundary of D^k. In particular, $\gamma \in (0,1)$ by (1.31).

Thus there exists a positive constant C such that

$$0 \le v_k(y) \le C d_k(y)^{\frac{p-(1-\sigma)\theta}{p-1}} \quad \text{when} \quad d_k(y) < \delta, \quad \text{for} \quad \delta > 0. \tag{1.45}$$

It remains to consider the case $d_k(y) \ge \delta$. First we observe that

$$\frac{p-(1-\sigma)\theta}{p-1} - \sigma = \frac{p-\theta}{p-1} + \sigma\frac{\theta - p + 1}{p-1} > 0 \tag{1.46}$$

being $\sigma > 0$ and $p - 1 < \theta < p$ by $(5)_2$.

Using (1.43) and (1.46) we have, when $d_k(y) \ge \delta$

$$v_k(y) \le 2d_k(y)^{\sigma} \le 2\delta^{\sigma - \frac{p-(1-\sigma)\theta}{p-1}} d_k(y)^{\frac{p-(1-\sigma)\theta}{p-1}} \le C d_k(y)^{\frac{p-(1-\sigma)\theta}{p-1}}. \tag{1.47}$$

Thus, conditions (1.45) and (1.47) give

$$0 \le v_k(y) \le C d_k(y)^{\frac{p-(1-\sigma)\theta}{p-1}} \quad \text{in} \quad D^k, \tag{1.48}$$

or equivalently $\|v_k\|_0^{(-\frac{p-(1-\sigma)\theta}{p-1})}$ is bounded. Furthermore, we can use Lemma 4, with $f = v_k^{\mathfrak{m}} + h_k$ in (1.15) and $\eta = (1-\sigma)\theta$ provided the following $\eta \in (1,p)$, $\|f\|_0^{(\frac{\eta+p(p-2)}{p-1})} < \infty$. The first condition is equivalent to (1.31). The latter condition follows from (1.44), noting that

$$\frac{\eta + p(p-2)}{p-1} = \frac{\theta(1-\sigma) + p(p-2)}{p-1}.$$

Then, thanks to (1.48), we have

$$\begin{aligned}\|Dv_k\|_0^{\left(\frac{(1-\sigma)\theta-1}{p-1}\right)} &\leq C_0(\|f\|_0^{\left(\frac{\theta(1-\sigma)+p(p-2)}{p-1}\right)} + \|v_k\|_0^{\left(-\frac{p-(1-\sigma)\theta}{p-1}\right)}) \\ &\leq C(\|f\|_0^{\left(\frac{\theta(1-\sigma)+p(p-2)}{p-1}\right)} + 1) \leq C,\end{aligned} \tag{1.49}$$

where in the last inequality we have used (1.44) and that $(p-2)[p-(1-\sigma)\theta] \geq 0$ by $\sigma > 0$, $\theta < p$ and $p \geq 2$.

Therefore, from (1.49) we achieve

$$|Dv_k(y)| \leq Cd_k(y)^{\frac{1-(1-\sigma)\theta}{p-1}} \quad \text{in} \quad D^k, \tag{1.50}$$

where C is also independent of k. Replacing inequalities (1.48) and (1.50) in (1.41), we deduce

$$1 \leq \; Cd_k(y_k)^{\beta_1}, \qquad \beta_1 = \frac{p-\theta+\sigma(\theta-p+1)}{p-1} > 0. \tag{1.51}$$

Case $1 < p < 2$: From (1.44), since $1 < p < 2$, we have

$$-\Delta_p v_k(y) \leq Cd_k(y)^{p-2-\theta(1-\sigma)} \quad \text{in} \quad D^k, \tag{1.52}$$

so that now we can apply Lemma 2 to (1.52), by choosing $\gamma = 2 - \frac{\theta(1-\sigma)}{p-1}$. In particular, $\gamma \in (0,1)$ is equivalent to

$$1 - \frac{2(p-1)}{\theta} < \sigma < 1 - \frac{p-1}{\theta}$$

that is in force thanks to (1.32). Thus, by regularity of v_k, there exists a positive constant C such that

$$0 \leq v_k(y) \leq Cd_k(y)^{2-\frac{\theta(1-\sigma)}{p-1}} \quad \text{when} \quad d_k(y) < \delta, \quad \text{for} \quad \delta > 0. \tag{1.53}$$

It remains to take into account the case $d_k(y) \geq \delta$. First we observe that, by (1.32) and $(5)_2$,

$$2 - \frac{\theta(1-\sigma)}{p-1} - \sigma = 2 - \frac{\theta}{p-1} + \sigma\frac{\theta-p+1}{p-1} > 0 \tag{1.54}$$

So using (1.43) and (1.54) we have, when $d_k(y) \geq \delta$

$$v_k(y) \leq 2d_k(y)^\sigma \leq 2\delta^{\sigma-2+\frac{\theta(1-\sigma)}{p-1}} d_k(y)^{2-\frac{\theta(1-\sigma)}{p-1}} \leq Cd_k(y)^{2-\frac{\theta(1-\sigma)}{p-1}}. \tag{1.55}$$

Thus, by (1.53) and (1.55), we achieve

$$0 \leq v_k(y) \leq Cd_k(y)^{2-\frac{\theta(1-\sigma)}{p-1}} \quad \text{in} \quad D^k, \tag{1.56}$$

or equivalently $\|v_k\|_0^{(-2+\frac{\theta(1-\sigma)}{p-1})}$ is bounded. Moreover, we can use Lemma 4 with $f = v_k^{\mathfrak{m}} + h_k$ in (1.15) and $\eta = (1-\sigma)\theta(p-1) - p(p-2)$ provided that $\eta \in (1,p)$ and $\|f\|_0^{(\frac{\eta+p(p-2)}{p-1})} < \infty$. The early condition requires

$$1 - \frac{p}{\theta} < \sigma < 1 - \frac{p-1}{\theta}, \tag{1.57}$$

which is in force by (1.32). The latter condition $\|f\|_0^{(\frac{\eta+p(p-2)}{p-1})} < \infty$ is a consequence of (1.44), since $\frac{\eta+p(p-2)}{p-1} = \theta(1-\sigma)$. Thus, we obtain

$$\begin{aligned} \|Dv_k\|_0^{((1-\sigma)\theta-p+1)} &\leq C_0(\|f\|_0^{((1-\sigma)\theta)} + \|v_k\|_0^{((1-\sigma)\theta-p)}) \\ &\leq C(1 + \|v_k\|_0^{((1-\sigma)\theta-p)}). \end{aligned} \tag{1.58}$$

Now we apply Lemma 3 choosing $f = v_k^{\mathfrak{m}} + h_k$ and $\gamma = p - (1-\sigma)\theta$ provided that $\gamma \in (0,1)$ and $\|f\|_0^{((p-1)(1-\sigma)\theta-p(p-2))} < \infty$. The first condition is equivalent to (1.57) which is in force thanks to (1.32). The condition $\|f\|_0^{((p-1)(1-\sigma)\theta-p(p-2))} < \infty$ is a consequence of the boundedness of Ω and (1.44), namely

$$f(y)d_k(y)^{(p-1)(1-\sigma)\theta-p(p-2)} \leq C d_k(y)^{(p-1)(1-\sigma)\theta-p(p-2)-(1-\sigma)\theta} \leq C.$$

since $(p-2)[(1-\sigma)\theta - p] > 0$ by (1.32). Thus, from (1.58) we have

$$|Dv_k(y)| \leq C d_k(y)^{p-1-\theta(1-\sigma)} \quad \text{in} \quad D^k, \tag{1.59}$$

where C is also independent of k. Replacing inequalities (1.56) and (1.59) in (1.41), we deduce

$$1 \leq C\Big(d_k(y_k)^{\beta_2} + d_k(y_k)^{\beta_3}\Big), \tag{1.60}$$

where

$$\beta_2 = 2 - \frac{\theta(1-\sigma)}{p-1} - \sigma, \quad \beta_3 = p - \theta + \sigma(\theta - 1).$$

Now, $\beta_2 > 0$ by (1.54), while $\beta_3 > 0$ by (1.32), indeed if $\theta \geq 1$ the positivity of β_2 is trivial, while if $0 < \theta < 1$ then $\beta_3 > 0$ follows from $1-(p-1)/\theta < (p-\theta)/(1-\theta)$.

Consequently, both in *Case* $p \geq 2$ and in *Case* $1 < p < 2$, since all the exponents of $d_k(y_k)$ in (1.51) and (1.60) are positive, we obtain that $d_k(y_k)$ is bounded away from zero. Observing that $0 \in D^k$ and $D^k \to \mathbb{R}^N_+$ as $k \to \infty$ we see that $|y_k| = d(x_k)\mu_k^{-1} = d_k(y_k)$ also is bounded away from zero, thus, by passing to a further subsequence $y_k \to y_0$, where $|y_0| = d > 0$ as claimed. Note that y_0 is in the interior of the halfspace $\mathbb{R}^N_+$. Finally, we can use (1.16) together with $Ascoli - Arzelá's$ theorem and a diagonal argument to obtain that $v_k \to v$ in $C^1_{loc}(\mathbb{R}^N_+)$, where by (1.41) for d instead of d_k, the function v verifies

$d^{-\sigma}v(y_0) + d^{1-\sigma}|Dv(y_0)| = 1$, for some $y_0 \in \mathbb{R}^N_+$, hence v is not trivial and, from $d(y, \partial\mathbb{R}^N_+) = y_N$ for $y \in \mathbb{R}^N_+$, we have for $0 < y_N < \delta$,

$$v(y) \le C \begin{cases} y_N^{\frac{p-(1-\sigma)\theta}{p-1}} & p \ge 2, \\ y_N^{2-\frac{\theta(1-\sigma)}{p-1}} & 1 < p < 2, \end{cases}$$

so that $v \in C(\overline{\mathbb{R}^N_+})$ in both cases with v vanishing on the boundary. From the growth condition (1.38), since v_k and Dv_k are bounded, $h_k \to 0$ as $k \to \infty$, thus, thanks to (1.37), v satisfies

$$\begin{cases} -\Delta_p v = v^{\mathfrak{m}} & \text{in } \mathbb{R}^N_+, \\ v = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

so, by the strong maximum principle for the p-Laplacian (see Theorem 5.3.1 in [115]) it results $v(y) > 0$ for all $y \in \mathbb{R}^N_+$. This contradicts the Liouville type Theorem 10 being $\mathfrak{m} < p^* - 1$ and concludes the proof of the theorem. □

1.4 Proof of Theorem 1

In the current section we prove Theorem 1, the main existence theorem of Chapter 1.

First, we introduce our setting. Fix σ verifying (1.31), (1.32) then consider the Banach space E_σ, defined in (1.33), which is an ordered Banach space with norm $\|\cdot\| = \|\cdot\|_1^{(-\sigma)}$ and let $P = \{u \in E_\sigma : u \ge 0 \,\text{in}\, \Omega\}$ be the cone of nonnegative functions whose topology is given by $\|\cdot\|_0^{(-\sigma)}$. We will assume that h is nonnegative and verifies the growth condition (6) presented in the statement of Theorem 1, which obviously is stronger than (1.34) used in Theorem 11 in order to obtain the uniform a priori estimate. We observe that, from (6) it follows for every $v \in P$ that

$$\begin{aligned} h(x, v(x), Dv(x)) &\le C\Big[\big(\|v\|_0^{(-\sigma)}\big)^s d^{\sigma s} + \big(\|Dv\|_0^{(1-\sigma)}\big)^\theta d^{-(1-\sigma)\theta}\Big] \\ &\le C d(x)^{-(1-\sigma)\theta}\Big[(\|v\|)^s d^{\sigma s+(1-\sigma)\theta} + (\|v\|)^\theta\Big] \\ &\le C(\|v\|) d(x)^{(\sigma-1)\theta} \end{aligned} \tag{1.61}$$

for every $x \in \Omega$, thanks to $\sigma s + (1-\sigma)\theta > 0$ and being Ω bounded. In particular,

$$\begin{aligned} &d(x)^{(1-\sigma)\theta}\big[v(x)^{\mathfrak{m}} + h(x, v(x), Dv(x))\big] \\ &\qquad \le d(x)^{(1-\sigma)\theta+\sigma\mathfrak{m}}\big(\|v\|_0^{(-\sigma)}\big)^{\mathfrak{m}} + C \le C_1, \end{aligned} \tag{1.62}$$

with $C_1 > 0$, thanks to the positivity of $(1-\sigma)\theta+\sigma\mathfrak{m}$. Consequently, we can apply Lemma 3 with $f = v^{\mathfrak{m}} + h$ and $\gamma = \sigma \in (0,1)$ provided that $\|f\|_0^{(p-\sigma(p-1))} < \infty$ indeed, from (1.62)

$$f(x)d(x)^{p-\sigma(p-1)} \le C_1 d(x)^{p-\sigma(p-1)-(1-\sigma)\theta} \le C$$

because trivially $\sigma > 0 > \frac{\theta-p}{\theta-p+1}$. Thus, there exist a unique solution u of

$$\begin{cases} -\Delta_p u = v^{\mathfrak{m}} + h(x,v,Dv) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

such that

$$\|u\|_0^{(-\sigma)} < \infty. \tag{1.63}$$

Furthermore, we can use Lemma 4 with $f = v^{\mathfrak{m}} + h$ in (1.15) and $\eta = p - \sigma(p-1)$. Indeed, $\eta \in (1,p)$ since $\sigma < 1$, while to obtain the second assumption of Lemma 4, that is $\|f\|_0^{(\frac{\eta+p(p-2)}{p-1})} = \|f\|_0^{(p-\sigma)} < \infty$, we observe that being $v \in P$, we have

$$v(x)^{\mathfrak{m}} d(x)^{p-\sigma} \le \left(\|v\|_0^{(-\sigma)}\right)^{\mathfrak{m}} d(x)^{\sigma(\mathfrak{m}-1)+p} \le Cd(x)^{\sigma(\mathfrak{m}-1)+p}$$

and by (1.61)

$$h(x,v(x),Dv(x))d(x)^{p-\sigma} \le \|h\|_0^{((1-\sigma)\theta)} d(x)^{\sigma(\theta-1)+p-\theta} \le Cd(x)^{\sigma(\theta-1)+p-\theta}.$$

Thus, from the positivity of $\sigma(\mathfrak{m}-1)+p$ since $\mathfrak{m} > \max\{p-1,1\}$ and $\sigma(\theta-1)+p-\theta$ by (1.31)-(1.32), we achieve that

$$\|f\|_0^{(p-\sigma)} \le C, \tag{1.64}$$

as required. Consequently, Lemma 4 gives $\|Du\|_0^{(1-\sigma)} \le C(\|f\|_0^{(p-\sigma)} + \|u\|_0^{(-\sigma)})$. In turn, by (1.63) and (1.64), we obtain

$$\|Du\|_0^{(1-\sigma)} < \infty. \tag{1.65}$$

Hence, by (1.63) and (1.65) we have $u \in E_\sigma$. In this way, we can define the operator $T : P \to P$ by means of $u = T(v)$, in particular, T maps P into P because of the maximum principle, furthermore nonnegative solutions of (3) in E_σ coincide with the fixed points of T in P.

We continue by showing a fundamental property of the operator T.

Lemma 5. *The operator $T : P \to P$ is compact.*

Proof. Proceeding as in [17], we show continuity first: let $(v_n)_n \subset P$ such that $v_n \to v$ in E_σ and denote $T(v_n) = u_n$. In particular, $v_n \to v$ and $Dv_n \to Dv$ uniformly on compact sets of Ω, so that the continuity of h implies

$$h(\cdot, v_n, Dv_n) \to h(\cdot, v, Dv) \quad \text{uniformly on compact sets of } \Omega. \tag{1.66}$$

Moreover, since v_n is bounded in E_σ, similarly as in (1.61) we also have that

$$h(\cdot, v_n, Dv_n) \leq C d^{-(1-\sigma)\theta} \quad \text{in } \Omega \tag{1.67}$$

for a constant C that does not depend on n, and the same is true for v after passing to the limit. We claim that

$$\sup_\Omega [d^\lambda |h(\cdot, v_n, Dv_n) - h(\cdot, v, Dv)|] \to 0 \quad \text{as } n \to \infty, \tag{1.68}$$

for every $\lambda > (1-\sigma)\theta$. Indeed, if we take $\epsilon > 0$ then

$$d^\lambda |h(\cdot, v_n, Dv_n) - h(\cdot, v, Dv)| \leq C d^{\lambda - \theta(1-\sigma)} \leq C \delta^{\lambda - \theta(1-\sigma)} \leq \epsilon$$

if $d \leq \delta$, by choosing a δ small enough. When $d \geq \delta$,

$$d^\lambda |h(\cdot, v_n, Dv_n) - h(\cdot, v, Dv)| \leq (\sup_\Omega d)^\lambda |h(\cdot, v_n, Dv_n) - h(\cdot, v, Dv)| \leq \epsilon$$

just by choosing $n \geq n_0$ by (1.66) since we are in a compact set. This shows (1.68). Now we can use Lemma 3, applied with $\gamma = \sigma \in (0,1)$ and

$$f = v_n^{\mathfrak{m}} - v^{\mathfrak{m}} + h(\cdot, v_n, Dv_n) - h(\cdot, v, Dv). \tag{1.69}$$

Note that, from (1.68) with $\lambda = p - \sigma(p-1)$, we have

$$\sup_\Omega [d^{p-\sigma(p-1)} |h(\cdot, v_n, Dv_n) - h(\cdot, v, Dv)|] \to 0 \quad \text{as } n \to \infty, \tag{1.70}$$

since $p - \sigma(p-1) > (1-\sigma)\theta$ by (1.31), (1.32). Furthermore, by $p - \sigma(p-1) + \sigma\mathfrak{m} > 0$ from $\mathfrak{m} > p-1$, we obtain

$$d^{p-\sigma(p-1)} |v_n^{\mathfrak{m}} - v^{\mathfrak{m}}| \leq d^{p-\sigma(p-1)+\sigma\mathfrak{m}} \Big(\|v_n - v\|_0^{(-\sigma)} \Big)^{\mathfrak{m}} \leq C \Big(\|v_n - v\|_0^{(-\sigma)} \Big)^{\mathfrak{m}}.$$

Then, taking the supremum over Ω, we achieve

$$\sup_\Omega [d^{p-\sigma(p-1)} |v_n^{\mathfrak{m}} - v^{\mathfrak{m}}|] \to 0 \quad \text{as } n \to \infty. \tag{1.71}$$

Thus, we get

$$\|T(v_n) - T(v)\|_0^{(-\sigma)} \leq C \Big[\|f\|_0^{(p-\sigma(p-1))} \Big]^{1/(p-1)}.$$

Now, taking into account (1.70) and (1.71), we have by (1.69)

$$\sup_{\Omega} d^{-\sigma}\Big|T(v_n) - T(v)\Big| \to 0 \quad \text{as } n \to \infty. \tag{1.72}$$

Moreover, we can apply Lemma 4 with f as in (1.69) and $\eta = p - \sigma(p-1) \in (1,p)$ since $\sigma < 1$. From (1.68) with $\lambda = p - \sigma$ we have

$$\sup_{\Omega}[d^{p-\sigma}|h(\cdot, v_n, Dv_n) - h(\cdot, v, Dv)|] \to 0 \quad \text{as } n \to \infty, \tag{1.73}$$

thanks to $p - \sigma > (1-\sigma)\theta$ by (1.31), (1.32). Following the same technique to get (1.71), since $p + \sigma(\mathfrak{m} - 1) > 0$ from $\mathfrak{m} > \max\{p-1, 1\}$, we arrive to

$$\sup_{\Omega}[d^{p-\sigma}|v_n^{\mathfrak{m}} - v^{\mathfrak{m}}|] \to 0 \quad \text{as } n \to \infty. \tag{1.74}$$

Therefore, we get

$$\|D(T(v_n) - T(v))\|_0^{(1-\sigma)} \leq C\Big(\|f\|_0^{(p-\sigma)} + \|T(v_n) - T(v)\|_0^{(-\sigma)}\Big).$$

Thus, by reason of (1.72), (1.73) and (1.74), we achieve

$$\sup_{\Omega} d^{1-\sigma}\Big|D\Big(T(v_n) - T(v)\Big)\Big| \to 0 \quad \text{as } n \to \infty. \tag{1.75}$$

Combining (1.72) and (1.75), the continuity of the operator T is verified. To prove compactness, let $(v_n)_n \subset P$ be bounded. Thus, (1.67) holds. By (1.16) we obtain that for every $\Omega' \subset\subset \Omega$ the $C^{1,\beta}$ norm of $T(v_n)$ in Ω' is bounded. Therefore, we may assume by passing to a subsequence that $T(v_n) \to u$ in $C^1_{loc}(\Omega)$. Following the same technique above we have

$$\sup_{\Omega}\Big\{d^{-\sigma}|T(v_n) - u| + d^{1-\sigma}|D(T(v_n) - u)|\Big\} \to 0 \quad \text{as } n \to \infty,$$

which shows compactness. The proof is so concluded. □

As last ingredients, the proof of Theorem 1 relies on the use of the following suitable version of a degree theorem by Krasnosel'skii (cfr. [84, 35] see also [46]).

Theorem 12. *Suppose that $(E, \|.\|)$ is an ordered Banach space with positive cone P, and $U \subset P$ is an open bounded set containing 0. Let $\rho > 0$ be such that $B_\rho(0) \cap P \subset U$. Assume $K : U \to P$ is compact and satisfies*

(a) for every $\mu \in [0,1)$, we have $u \neq \mu K(u)$ for every $u \in P$ with $\|u\| = \rho$;

(b) there exists $\psi \in P \setminus \{0\}$ such that $u - K(u) \neq t\psi$, for every $u \in \partial U$, for every $t \geq 0$.

Then K has a fixed point in $U \setminus B_\rho(0)$.

Finally we are in position to prove Theorem 1, whose statement is given in the Introduction.

Proof of Theorem 1. Adapting the main ideas in Theorem 2 in [17] and Lemma 3.1 in [101], it is enough to apply Theorem 12 with $E = E_\sigma$ as a Banach space with the norm $\|\cdot\| = \|\cdot\|_1^{(-\sigma)}$ and with $K = T$ a continous and compact operator, thanks to Lemma 5. Following the proof of Theorem 2 in [17], assume by contradiction $u = \mu T(u)$ for some $\mu \in [0, 1)$ and $u \in P$. This is equivalent to

$$\begin{cases} -\Delta_p u = \mu^p(u^{\mathtt{m}} + h(x, u, Du)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.76}$$

By the growth condition (6) on h and $\mu < 1$, we get that $f = \mu(u^{\mathtt{m}} + h)$ can be bounded by

$$f(x, u, Du) \leq Cd^{(\sigma-1)\theta}(\|u\|^{\mathtt{m}} + \|u\|^s + \|u\|^\theta). \tag{1.77}$$

Now, we apply Lemma 3 with $\gamma = \sigma \in (0, 1)$ and $f = \mu(u^{\mathtt{m}} + h)$, such that $\|f\|_0^{(p-\sigma(p-1))} < \infty$ indeed, by (1.77),

$$\begin{aligned} f(x, u, Du)d^{p-\sigma(p-1)} &\leq Cd^{(\sigma-1)\theta+p-\sigma(p-1)}(\|u\|^{\mathtt{m}} + \|u\|^s + \|u\|^\theta) \\ &\leq C(\|u\|^{\mathtt{m}} + \|u\|^s + \|u\|^\theta) \end{aligned}$$

from the positivity of the exponent $(\sigma - 1)\theta + p - \sigma(p - 1)$ by (1.31), (1.32). Thus, Lemma 3 gives the existence of a positive constant C such that

$$d^{-\sigma}|u(x)| \leq C\Big[\|f\|_0^{(p-\sigma(p-1))}\Big]^{\frac{1}{p-1}} \leq C\Big(\|u\|^{\mathtt{m}} + \|u\|^s + \|u\|^\theta\Big)^{\frac{1}{p-1}}. \tag{1.78}$$

On the other hand, we use Lemma 4 with $f = \mu(u^{\mathtt{m}} + h)$ in problem (1.76) and $\eta = p - \sigma(p-1)$, provided that $\eta \in (1, p)$ and $\|f\|_0^{(\frac{\eta+p(p-2)}{p-1})} < \infty$. The first condition produces $\sigma < 1$. From (1.77), we get

$$\begin{aligned} f(x, u, Du)d^{p-\sigma} &\leq Cd^{(\sigma-1)\theta+p-\sigma}(\|u\|^{\mathtt{m}} + \|u\|^s + \|u\|^\theta) \\ &\leq C(\|u\|^{\mathtt{m}} + \|u\|^s + \|u\|^\theta), \end{aligned} \tag{1.79}$$

because $(\sigma - 1)\theta + p - \sigma > 0$ thanks to (1.31),(1.32), so also the second condition is verified. Hence, Lemma 4 guarantees the existence of a positive constant C such that

$$d^{1-\sigma}|Du(x)| \leq C\Big[\|f\|_0^{(p-\sigma)} + \|u\|_0^{(-\sigma)}\Big].$$

Thus, by reason of (1.78) and (1.79), we have

$$d^{1-\sigma}|Du(x)| \le C\Big(\|u\|^{\mathfrak{m}} + \|u\|^{s} + \|u\|^{\theta} + (\|u\|^{\mathfrak{m}} + \|u\|^{s} + \|u\|^{\theta})^{\frac{1}{p-1}}\Big) \tag{1.80}$$

In turn, using (1.78) and (1.80), by the definition of $\|\cdot\|$,

$$\|u\| \le C\Big(\|u\|^{\frac{\mathfrak{m}}{p-1}} + \|u\|^{\frac{s}{p-1}} + \|u\|^{\frac{\theta}{p-1}} + \|u\|^{\mathfrak{m}} + \|u\|^{s} + \|u\|^{\theta}\Big).$$

Since $\mathfrak{m}, s, \theta > \max\{p-1, 1\}$ from (4), (5) and (7), this implies that $\|u\| > \rho$ for some positive ρ. Thus, there are no solutions of $u = \mu T(u)$ if $\|u\| = \rho$ and $\mu \in [0, 1)$, so that the proof of (a) is completed.

To check (b), we follow the idea in the proof of Lemma 3.1 in [101]. Take $\psi \in P$ to be the unique solution of the problem (1.20). We claim that there are no solutions in P of the equation $u - T(u) = t\psi$ for $t \ge 0$. For this purpose we note that the last equation is equivalent to

$$\begin{cases} -\Delta_p u = u^{\mathfrak{m}} + h(x, u, Du) + t & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.81}$$

Take $w = \left(\frac{t}{2}\right)^{\frac{1}{p-1}}\psi$, then from the nonnegativity of $h(x, u, \eta)$ and u, it follows that $-\Delta_p w = \frac{t}{2} \le t \le -\Delta_p u$ in Ω and $u = w$ on $\partial\Omega$. Using the comparison lemma (see [132]), we see that $u \ge w$ in Ω. Hence

$$\max_{\overline{\Omega}} w = \left(\frac{t}{2}\right)^{\frac{1}{p-1}} \max_{\overline{\Omega}} \psi \le \max_{\overline{\Omega}} u.$$

Taking $\psi_M = \max_{\overline{\Omega}} \psi > 0$ by (1.20), then problem (1.81) admits solution if we take into account $t \le 2\Big(\dfrac{\max_{\overline{\Omega}} u}{\psi_M}\Big)^{p-1} := C$, consequently for $t > C$ no solutions of (1.81) can exist. But, if $t \le C$, assumption (1.34) is in force and in turn, thanks to Theorem 11, the uniform estimate $\|u\| \le M$ is clearly true for some $M > 0$. Thus, Theorem 12 can be applied with $U = B_M(0) \cap P$ and the existence of a solution in P follows. This solution is positive, by virtue of maximum principle. This completes the proof. □

Chapter 2

Critical (p, q)-Laplacian problems in $\mathbb{R}^N$

In this chapter we treat multiplicity and existence results for an elliptic problem of (p, q)-Laplacian type involving a critical term, nontrivial nonnegative weights and a positive parameter λ, whose prototype is given by (8). In particular, the main results of this chapter are contained in the following recent papers [13, 14, 15].

An important example, widely studied, in which a subcase of problem (8) appears, is the study of solitary waves or solitons which are special solutions whose profile remains unchanged under the evolution in time, of the nonlinear Schrödinger equation see [29], [38] and [129], of the typical form

$$i\partial_t \psi + \Delta\psi + \Delta_q \psi - U(x)\psi + |\psi|^{\overline{k}-1}\psi = 0, \quad 2 < \overline{k} < 2^*, \tag{2.1}$$

where i is the imaginary unit and the function U is the potential. This class of equations was proposed by Derrick in [48] as a model for elementary particles. In particular, a function $\psi(x, t) = e^{-i\omega t}u(x)$ is a standing-wave solution of (2.1), where $\omega \in \mathbb{R}$ is the energy, if and only if the function u satisfies

$$-\Delta u - \Delta_q u + [U(x) - \omega]u = |u|^{\overline{k}-1}u.$$

The subcase of (2.1) when $q = 2$ and a cubic nonlinearity, $\overline{k} = 3$, is involved is called the Gross-Pitaevskii equation.

We work on unbounded domains where the loss of compactness of the Sobolev embeddings shows up and this is underlined by the presence of the critical exponent in the nonlinearity. Thus, variational techniques are more delicate to apply. In particular, one of the main common difficulties lies on the application of the second concentration compactness by Lions [94] which requires the tight convergence of a certain sequence of measures. This property, up to subsequences, follows immediately if we consider a bounded sequence $(u_n)_n$ in $D^{1,p}(\Omega)$ with Ω bounded, indeed by standard extensions theorems we may assume, without loss of generality that $(u_n)_n \subset D^{1,p}(\mathbb{R}^N)$ and $|u_n|^{p^*} \overset{*}{\rightharpoonup} \nu$. Contrarily, in the case of a bounded sequence $(u_n)_n$ in $D^{1,p}(\mathbb{R}^N)$, to obtain the tight convergence, we need to perform a deeper study.

Moreover, technical difficulties arise in applying usual elliptic methods for existence of weak solutions of (8), due to the fact that the operator is not homogeneous and to the presence of nontrivial weights.

In Chapter 2, we denote with X the reflexive Banach space $D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$, where $D^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : Du \in L^p(\mathbb{R}^N)\}$, endowed with the norm

$$\|u\| := \|u\|_X = \|u\|_{D^{1,p}(\mathbb{R}^N)} + \|u\|_{D^{1,q}(\mathbb{R}^N)} = \|Du\|_p + \|Du\|_q \tag{2.2}$$

and $\|\cdot\|_p$ is the L^p norm in $\mathbb{R}^N$. Furthermore, we denote by S the Sobolev's constant, i.e

$$S = \inf\left\{\frac{\|Du\|_p^p}{\|u\|_{p^*}^p} : u \in D^{1,p}(\mathbb{R}^N), u \neq 0\right\}. \tag{2.3}$$

We recall that the value S is achieved in $D^{1,p}(\mathbb{R}^N)$, for details we refer to Appendix A in [61].

While the setting which characterizes Section 2.4 is described as follows. Consider a group $T \subset O(N)$, where $O(N)$ is the group of orthogonal linear transformations in $\mathbb{R}^N$. Precisely, we assume that the weights V and K are T-symmetric (or T-invariant), where for a T-symmetric function $f : \Omega \to \mathbb{R}^N$ we mean that $f(\tau x) = f(x)$ for all $\tau \in T$ and $x \in \Omega$, with Ω an open T-symmetric (or T-invariant) subset of $\mathbb{R}^N$, that is if $x \in \Omega$, then $\tau x \in \Omega$ for all $\tau \in T$. In particular, denoting with $|T_x|$ the cardinality of a T-orbit $T_x := \{\tau x | \tau \in T\}$ with $|T_\infty| := 1$ and, necessarily, $|T_0| = 1$, then we set

$$|T| := \inf_{x \in \mathbb{R}^N,\ x \neq 0} |T_x|.$$

For example, even functions are T-symmetric functions with $T = \{id, -id\}$, thus $|T| = 2$, and radially symmetric functions are T-symmetric functions if $T = O(N)$, thus $|T| = \infty$.

We denote with $D^{1,p}_T(\mathbb{R}^N)$ the subspace of $D^{1,p}(\mathbb{R}^N)$ consisting of all T-symmetric functions, in turn $X_T = D^{1,p}_T(\mathbb{R}^N) \cap D^{1,q}_T(\mathbb{R}^N)$.

This chapter is divided into four sections. Section 2.1 is composed by four subsections: Subsections 2.1.1 and 2.1.2 concern, respectively, the analysis of the regularity of the energy functional associated to the problem and the property of boundedness of any $(\mathrm{PS})_c$ sequence for the energy functional common to all situations. Then, Subsection 2.1.3, sheds light on concentration compactness principle by Lions, up to the principle at infinity developed by Chabrowski in [34], with a special attention to the property of tightness and the role of symmetry. Finally, Sections 2.1.4 is about some main tools used in our results, such as the Theory of Genus by Krasnosel'skii, the Mountain Pass and the Fountain Theorems. The last three Sections are dedicated to the proofs of the main results for critical (p, q)-Laplacian problems in $\mathbb{R}^N$. In particular, the topic of Section 2.2 is the multiplicity result in the p-sublinear and q-superlinear case for solutions with negative energy contained in [13], while Section 2.3 consists of an existence result in the p-superlinear case via the Mountain Pass Theorem included in [15]. Finally, the existence and multiplicity results under a symmetric setting in [14] are investigated in Section 2.4.

2.1 Preliminaries and tools

This section deals with the main tools needed in the proofs of existence and multiplicity results Theorems 2-5, whose statements are given in the Introduction.

2.1.1 Regularity of the energy functional

We will start by the analysis of the energy functional associated to our problem (8). Of course, the functional E_λ defined in (11) is well defined in X, indeed if $u \in X$, by Hölder's inequality with the exponents $r = p^*/(p^* - k)$, $r' = p^*/k$, we have

$$E_\lambda(u) \le \frac{1}{p}\|u\|^p + \frac{1}{q}\|u\|^q + \frac{\lambda}{k}\|V\|_r \|u\|_{p^*}^k + \frac{1}{p^*}\|K\|_\infty \|u\|_{p^*}^{p^*} < \infty,$$

thanks to (9) and (10).

The proof of the regularity of E_λ is almost standard, but for completeness we include it. Obviously, it is enough to study the regularity of the functionals

$$\hat{J}(u) = \int_{\mathbb{R}^N} V|u|^k dx \quad \text{and} \quad \hat{H}(u) = \int_{\mathbb{R}^N} K|u|^{p^*} dx. \tag{2.4}$$

First, we analyze the regularity of $\hat{J}$.

Lemma 6. *If $V \in L^r\left(\mathbb{R}^N\right)$, then $\hat{J}$ is weakly continuous on $D^{1,p}(\mathbb{R}^N)$. Moreover, $\hat{J}$ is continuously differentiable and $\hat{J}' : D^{1,p}(\mathbb{R}^N) \to [D^{1,p}(\mathbb{R}^N)]'$ is given by*

$$\hat{J}'(u)\psi = k \int_{\mathbb{R}^N} V|u|^{k-2} u\psi dx, \tag{2.5}$$

for all $\psi \in D^{1,p}(\mathbb{R}^N)$.

Proof. Let $(u_n)_n \in D^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $D^{1,p}(\mathbb{R}^N)$, thus $u_n \rightharpoonup u$ in $L^{p^*}(\mathbb{R}^N)$ and $(u_n)_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$, in $L^{p^*}(\mathbb{R}^N)$ and also $(|u_n|^k)_n$ in $L^{p^*/k}(\mathbb{R}^N)$ since we have $\||u_n|^k\|_{p^*/k} = \|u_n\|_{p^*}^k$. Furthermore, by the compactness of the embedding, up to subsequences,

$$u_n \to u \ \text{ in } \ L^s(\omega), \quad \omega \Subset \mathbb{R}^N, \quad 1 \le s < p^*.$$

Consequently, by using an increasing sequence of compact sets whose union is $\mathbb{R}^N$ and a diagonal argument, we also have

$$u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N. \tag{2.6}$$

In turn, by Hölder's inequality, $\|V|u_n|^k\|_1 \le C\|V\|_r < \infty$ by (9) so that using Lebesgue dominated convergence Theorem we have

$$\hat{J}(u_n) = \int_{\mathbb{R}^N} V|u_n|^k dx \to \int_{\mathbb{R}^N} V|u|^k dx = \hat{J}(u),$$

namely, weak continuity holds. In order to prove $\hat{J} \in C^1$ it is enough to show that $\hat{J}$ has continuous Gâteaux derivative. Let $u, \psi \in D^{1,p}(\mathbb{R}^N)$ and $0 < |t| < 1$, it follows

$$\frac{\hat{J}(u+t\psi) - \hat{J}(u)}{t} = \int_{\mathbb{R}^N} V \frac{|u+t\psi|^k - |u|^k}{t} dx, \tag{2.7}$$

By the mean value Theorem there exists $\lambda \in (0,1)$ such that

$$\frac{\big||u+t\psi|^k - |u|^k\big|}{t} = k|u+\lambda t\psi|^{k-1}|\psi| \le k\big(|u|^{k-1}|\psi| + |\psi|^k\big).$$

We now use Hölder's inequality twice with exponents r, $p^*/(k-1)$, p^* and r, p^*/k respectively, so that

$$\int_{\mathbb{R}^N} V\big(|u|^{k-1}|\psi| + |\psi|^k\big) dx \le \|V\|_r \|\psi\|_{p^*} \left(\|u\|_{p^*}^{k-1} + \|\psi\|_{p^*}^{k-1}\right).$$

that is $V\big(|u|^{k-1}|\psi| + |\psi|^k\big) \in L^1(\mathbb{R}^N)$, thus, by letting $t \to 0$ in (2.7), thanks to the Lebesgue dominated convergence Theorem, we have that $\hat{J}$ is Gâteaux differentiable and (2.5) holds with $'$ in the Gâteaux sense.

In order to check the differentiability of $\hat{J}$, it remains to prove continuity of the Gâteaux derivative. Let $u_n \to u$ in $D^{1,p}(\mathbb{R}^N)$ then $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$, thus there exists $U \in L^{p^*}(\mathbb{R}^N)$ such that $|u_n(x)| \le U(x)$ a.e. in $\mathbb{R}^N$. For simplicity let $W(u) = V|u|^{k-2}u$ and we show that $W(u) \in L^{(p^*)'}(\mathbb{R}^N)$, indeed

$$|W(u)|^{(p^*)'} = |V|^{(p^*)'}|u|^{(k-1)(p^*)'} \le |V|^r + |u|^{p^*},$$

where in the last inequality we have applied Young's inequality with exponents $r/(p^*)'$ and $(p^*-1)/(k-1)$. Thus, for $c>0$, we have

$$|W(u_n)-W(u)|^{(p^*)'} \le c(|W(u_n)|^{(p^*)'}+|W(u)|^{(p^*)'}) \le c\big(|V|^r+|U|^{p^*}+|u|^{p^*}\big) \in L^1(\mathbb{R}^N),$$

so that by Lebesgue dominated convergence Theorem, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |W(u_n)-W(u)|^{(p^*)'}dx = 0 \tag{2.8}$$

since, $W(u_n(x)) \to W(u(x))$ a.e. in $\mathbb{R}^N$, by continuity of W.

Finally, by Hölder's inequality, for all $\psi \in D^{1,p}(\mathbb{R}^N)$, we have

$$|\big(\hat{J}'(u_n)-\hat{J}'(u)\big)\psi| \le k\int_{\mathbb{R}^N} |W(u_n)-W(u)||\psi|dx \le k\|W(u_n)-W(u)\|_{(p^*)'}\|\psi\|_{p^*},$$

consequently,

$$\|\hat{J}'(u_n)-\hat{J}'(u)\|_{[D^{1,p}(\mathbb{R}^N)]'} \le C\|W(u_n)-W(u)\|_{(p^*)'} \to 0$$

as $n\to\infty$ thanks to (2.8). Actually, we have proved that for every sequence $u_n \to u$ in $D^{1,p}(\mathbb{R}^N)$, there is a subsequence respect to which $\hat{J}'$ is sequentially continuous, from this it is an elementary exercise to conclude that $\hat{J}'$ is sequentially continuous in all of $[D^{1,p}(\mathbb{R}^N)]'$. In turn, $\hat{J} \in C^1$. □

Analogously, it holds the following.

Lemma 7. *If $K \in L^\infty(\mathbb{R}^N)$, then $\hat{H}$ is continuously differentiable in $D^{1,p}(\mathbb{R}^N)$ and its derivative $\hat{H}' : D^{1,p}(\mathbb{R}^N) \to [D^{1,p}(\mathbb{R}^N)]'$ is given by*

$$\hat{H}'(u)\psi = p^*\int_{\mathbb{R}^N} K|u|^{p^*-2}u\psi dx,$$

for all $\psi \in D^{1,p}(\mathbb{R}^N)$.

Finally, using the continuity of the embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, so that if $u_n \to u$ in X, that is $u_n \to u$ $D^{1,p}(\mathbb{R}^N)$ and in $D^{1,q}(\mathbb{R}^N)$, then

$$u_n \to u \quad \text{in } L^{p^*}(\mathbb{R}^N), \qquad Du_n \to Du \quad \text{in } L^p(\mathbb{R}^N) \text{ and in } L^q(\mathbb{R}^N).$$

Since the first two terms of E_λ are norms with exponents $p,q>1$, and thanks to Lemmas 6 and 7, then immediately $E_\lambda \in C^1(X)$, with $E'_\lambda : X \to X'$ defined by

$$\begin{aligned} E'_\lambda(u)\psi = &\int_{\mathbb{R}^N} |Du|^{p-2}DuD\psi dx + \int_{\mathbb{R}^N} |Du|^{q-2}DuD\psi dx \\ &- \lambda\int_{\mathbb{R}^N} V|u|^{k-2}u\psi dx - \int_{\mathbb{R}^N} K|u|^{p^*-2}u\psi dx. \end{aligned} \tag{2.9}$$

for all $u, \psi \in X$.

A *weak solution* of problem (8) is a function $u \in X$ such that

$$E'_\lambda(u)\psi = 0 \quad \text{for all } \psi \in X,$$

that is u is a critical point of the functional E_λ or equivalently, by (2.9), u satisfies the weak formulation of problem (8), namely

$$\begin{aligned}\int_{\mathbb{R}^N} |Du|^{p-2} DuD\psi dx + \int_{\mathbb{R}^N} |Du|^{q-2} DuD\psi dx \\ = \lambda \int_{\mathbb{R}^N} V|u|^{k-2} u\psi dx + \int_{\mathbb{R}^N} K|u|^{p^*-2} u\psi dx\end{aligned} \tag{2.10}$$

for all $\psi \in X$.

In the case of results under a symmetric setting, we restrict the energy functional E_λ in X_T, the space of T-symmetric functions. Thus, a critical point $u \in X_T$ of E_λ is not necessarily a solution of problem (8). For this reason, we have to take into account the following principle due to Palais in [106], knows in the literature as the principle of symmetric criticality, cfr. also [111], which states the following

$$\begin{aligned}&\text{for all } \varphi \in C^1(W, \mathbb{R}) \text{ and } \mathcal{G}\text{-invariant if } u \text{ is a critical} \\ &\text{point of } \varphi \text{ restricted to } \Sigma \text{ then } u \text{ is a critical point of } \varphi,\end{aligned} \tag{$\mathcal{PSC}$}$$

where $\mathcal{G}$ is a group acting on the Banach space W and $\Sigma = \{u \in W : gu = u,\ g \in \mathcal{G}\}$. Roughly speaking, this principle says that for every $\mathcal{G}$-invariant C^1 functional φ, then Σ is a natural constraint for φ. For completeness, we report Theorem 5.6.27 in [111] which guarantees the validity of condition ($\mathcal{PSC}$) in our setting.

Theorem 13. *If W is a Banach space and $\mathcal{G}$ is a compact topological group acting on W, then the ($\mathcal{PSC}$) holds.*

Remark 4. *Often Lie groups arise as subgroups of certain larger Lie groups. For example, the orthogonal group $O(N)$, which is a compact, but not connected, Lie group, is a subgroup of the general linear group of all invertible matrices. Hence, every $T \subset O(N)$ subgroup, being closed, is a compact Lie group, for further details we refer to Section 1.2 in [105]. Thus, applying to Theorem 13 with $\mathcal{G} = T$, $W = X$ and $\Sigma = X_T$, any critical point of E_λ in X_T is also a weak solution of* (8) *in X.*

From the Remark above, we get that $u \in X_T$ satisfies problem (8) if and only if (2.10) holds for all $\psi \in X$.

2.1.2 Boundedness of Palais Smale sequences

Let us briefly give the well known definitions of $(\mathrm{PS})_c$ sequence and the $(\mathrm{PS})_c$ condition for a given functional.

Definition 1. *Let Y be a Banach space and $E : Y \to \mathbb{R}$ be a differentiable functional. A sequence $(u_n)_n \subset Y$ is called a $(PS)_c$ sequence for E if $E(u_n) \to c$ and $E'(u_n) \to 0$ as $n \to \infty$. Moreover, we say that E satisfies the $(PS)_c$ condition if every $(PS)_c$ sequence for E has a converging subsequence in Y.*

In the next result, we prove a remarkable property of $(\mathrm{PS})_c$ sequences for the functional E_λ defined in (11). We point out that here the value k satisfies a very general assumption.

Lemma 8. *Assume $1 < k < p^*$, $1 < q \le p$. Let (9) and (10) be verified and let $(u_n)_n \subset X$ be a $(PS)_c$ sequence for E_λ for all $c \in \mathbb{R}$. Then $(u_n)_n$ is bounded in X.*

In particular, if $1 < k < p$ and $c < 0$, it holds

$$\|u_n\|_{p^*} \le C_* \lambda^{1/(p-k)}, \qquad C_* = \left[\frac{N(p^*-k)}{Skp^*}\|V\|_r\right]^{1/(p-k)}, \tag{2.11}$$

where S is the Sobolev's constant.

Proof. Let $(u_n)_n \subset X$ be a $(\mathrm{PS})_c$ sequence for E_λ for all $c \in \mathbb{R}$ namely, by Definition 1,

$$E_\lambda(u_n) = c + o(1), \;\; E'_\lambda(u_n) = o(1) \quad \text{as} \quad n \to \infty,$$

so that $|E'_\lambda(u_n)(u_n)| \le \|u_n\|$ for n large. Now we divide the proof in two cases.

Case $1 < k < p$: by (2.9), thanks to (2.3) and Hölder's inequality with exponents r and r' we have

$$\begin{aligned} c + o(1) + o(1)\|u_n\| &= E_\lambda(u_n) - \frac{1}{p^*}E'_\lambda(u_n)u_n \\ &\ge \left(\frac{1}{p} - \frac{1}{p^*}\right)\|Du_n\|_p^p + \left(\frac{1}{q} - \frac{1}{p^*}\right)\|Du_n\|_q^q \\ &\quad - \lambda\left(\frac{1}{k} - \frac{1}{p^*}\right)S^{-k/p}\|V\|_r\|Du_n\|_p^k \end{aligned} \tag{2.12}$$

where we have used that $V \in L^r(\mathbb{R}^N)$ and $\|u\|_{p^*}S^{1/p} \le \|Du\|_p$ for all $u \in D^{1,p}(\mathbb{R}^N)$. Consequently, writing explicitly $\|\cdot\|$ given in (2.2), we get

$$c + \|u_n\|_{D^{1,q}} - c_2\|u_n\|_{D^{1,q}}^q \ge c_1\|u_n\|_{D^{1,p}}^p - \lambda c_3\|u_n\|_{D^{1,p}}^k - \|u_n\|_{D^{1,p}}, \tag{2.13}$$

where c_1, c_2, c_3 are positive constants independent of n. From (2.13) it immediately follows that $(\|u_n\|)_n$ should be bounded, indeed, if there exists a subsequences

$(u_n)_n$ such that $\|u_n\|_{D^{1,q}} \to \infty$ and $(\|u_n\|_{D^{1,p}})_n$ bounded, then by letting $n \to \infty$ in (2.13) we obtain a contradiction since the left hand side goes to $-\infty$, being $q > 1$ while the right term is bounded. Again, up to subsequences, if $\|u_n\|_{D^{1,p}} \to \infty$ as $n \to \infty$ and $(\|u_n\|_{D^{1,q}})_n$ is bounded, then by letting $n \to \infty$ in (2.13) we obtain a contradiction since the right hand side goes to ∞, being $p > 1$ and $p > k$, while the left term is bounded. Finally if $\|u_n\|_{D^{1,p}}, \|u_n\|_{D^{1,q}} \to \infty$, up to subsequences, then the left hand side of (2.13) goes to $-\infty$ while the right goes to ∞. This last contradiction concludes the proof of the first case.

Case $p \leq k < p^*$: arguing as in (2.12), with $1/p^*$ replaced by $1/k$, since $K(x) \geq 0$ in $\mathbb{R}^N$, we obtain

$$c + \|u_n\|_{D^{1,q}} - c_2'\|u_n\|_{D^{1,q}}^q \geq c_1'\|u_n\|_{D^{1,p}}^p - \|u_n\|_{D^{1,p}}, \tag{2.14}$$

where c_1', c_2' are positive constants independent of n. From (2.14), using a similar argument as in the first case, it follows that $(\|u_n\|)_n$ should be bounded in X.

To obtain (2.11), it is enough to observe that using the boundedness of $(u_n)_n$, from (2.12), being $c < 0$, it follows, for n large, that

$$\frac{1}{N}\|Du_n\|_p^p - \lambda\left(\frac{1}{k} - \frac{1}{p^*}\right) S^{-k/p}\|V\|_r\|Du_n\|_p^k \leq 0,$$

so that

$$\|Du_n\|_p^{p-k} \leq \lambda S^{-k/p}\frac{N(p^* - k)}{kp^*}\|V\|_r,$$

which yields (2.11) by virtue of Sobolev's inequality and for $1 < k < p$.

Thus, the proof is completed. □

We end this subsection giving the following convergence result, that is Lemma 2.7 in [89], used in proving the $(\mathrm{PS})_c$ property for the energy functional.

Lemma 9. *Let Ω be an open set in $\mathbb{R}^N$ and ζ, β positive numbers. Suppose $a(x,\xi) \in C(\Omega \times \mathbb{R}^N, \mathbb{R}^N)$ such that*

1. *$\zeta|\xi|^s \leq a(x,\xi)\xi$ for all $(x,\xi) \in \Omega \times \mathbb{R}^N$;*
2. *$|a(x,\xi)| \leq \beta|\xi|^{s-1}$ for all $(x,\xi) \in \Omega \times \mathbb{R}^N$;*
3. *$(a(x,\xi) - a(x,\eta))(\xi - \eta) > 0$ for $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^N$, $\xi \neq \eta$.*

Consider $(u_n)_n, u \in W^{1,s}(\Omega)$, $s > 1$, then $Du_n \to Du$ in $L^s(\Omega)$ if and only if

$$\lim_{n\to\infty}\int_\Omega \Big(a(x, Du_n(x)) - a(x, Du(x))\Big)\Big(Du_n(x) - Du(x)\Big)dx = 0.$$

2.1.3 Concentration compactness principles and the role of symmetry

In this subsection we shed light on concentration compactness principles by Lions and on the property of tightness, which plays an important role in Lions's principles.

As soon as the papers by Lions in [94] and [96] appeared, the concentration compactness principles in the calculus of variations have been widely used by many authors to examine the behaviour of weakly convergent sequences in Sobolev's spaces in situations where the lack of compactness occurs either due to appearance of a critical Sobolev exponent or due to the unboundedness of a domain. The application of these principles helps to find level sets of a given variational functional for which the Palais Smale condition holds.

The first concentration compactness principle is known as *the locally compact case*, Lemma I.1 in [94], and the second concentration compactness principle is known as *critical or non compact case*, Lemma I.1 in [96].

Before stating the first concentration compactness lemma, we recall, for completeness, some well known notions, following [63]. Let Y be a locally compact Hausdorff space and let $M(Y,\mathbb{R})$ be the space of all finite signed Radon measures (cfr. Definitions 1.5, 1.166 and 1.55 in [63]). In this setting, we have

$$(C_0(Y))' = M(Y,\mathbb{R}),$$

where $C_0(Y)$ is the space of all continuous functions that vanish at infinity or, equivalently, it is the completion of $C_c(Y)$, i.e. the space of all functions whose support is compact, relative by the supremum norm $\|\cdot\|_\infty$. First, we recall the definition of the *(standard) convergence of measures,* also called in some works, [96], [97] and [104], *weak convergence of measures.*

Definition 2. *A sequence of measures $(\mu_n)_n \in M(Y,\mathbb{R})$ converges (standard or weakly) to a measure $\mu \in M(Y,\mathbb{R})$, that is $\mu_n \rightharpoonup \mu$, if for every $\varphi \in C_0(Y)$*

$$\int_Y \varphi d\mu_n \to \int_Y \varphi d\mu, \quad as \quad n \to \infty.$$

Equivalently, the (standard or weak) convergence of measures is the weak star convergence of measures respect to $(C_0(Y))'$.

Now, let $C_b(Y)$ be the space of real bounded functions defined in Y and we report the definition of *tight convergence of measures* in the same setting as above.

Definition 3. *A sequence of measures $(\mu_n)_n \in M(Y,\mathbb{R})$ converges tightly to a measure $\mu \in M(Y,\mathbb{R})$, that is $\mu_n \overset{*}{\rightharpoonup} \mu$, if for every $\varphi \in C_b(Y)$*

$$\int_Y \varphi d\mu_n \to \int_Y \varphi d\mu, \quad as \quad n \to \infty.$$

Equivalently, the tight convergence of measures is the weak star convergence respect to $(C_b(Y))'$.

Remark 5. *It is known that the dual spaces of $C_c(Y)$ and $C_0(Y)$ coincide, up to isomorphisms, while $(C_b(Y))'$ is larger. Thus, the tight convergence is stronger than the (standard or weak) convergence of measures, unless Y is compact, since in this case $C_0(Y) = C_c(Y) = C_b(Y)$, so that $(C_b(Y))'$ is still the set of signed Radon measures. Of course if Y is only bounded, the above equalities hold for $\overline{Y}$. If Y is unbounded, then the dual of $C_b(Y)$ is the space of regular finitely additive signed measures (see Sections 1.3.3 and 1.3.4 in [63]).*

Now we are ready to present the first concentration compactness lemma, Lemma I.1 in [94].

Lemma 10. *Let $(\rho_n)_n$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \geq 0 \text{ in } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} \rho_n dx = \Lambda,$$

where $\Lambda > 0$ is fixed. Indeed, up to a subsequence, one of the following three situations hold:

(a) (Compactness) There exists a sequence $(y_n)_n$ in $\mathbb{R}^N$ such that $\rho_n(\cdot + y_n)$ is tight that is for any $\varepsilon > 0$ there exists $0 < R_\varepsilon < \infty$ for which

$$\int_{B_{R_\varepsilon}(y_n)} \rho_n(x)dx \geq \Lambda - \varepsilon \quad \text{for all } n \in \mathbb{N} \text{ large.}$$

(b) (Vanishing) For all $R > 0$ there holds

$$\lim_{n\to\infty} \left(\sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n(x)dx \right) = 0.$$

(c) (Dichotomy) There exists $\ell \in (0, \Lambda)$ such that for any $\varepsilon > 0$ there exist $R > 0$, $(R_n)_n$, with $R < R_n \to \infty$, $(y_n)_n$ in $\mathbb{R}^N$, $n_0 \geq 1$ such that for $n \geq n_0$, it holds

$$\left| \int_{B_R(y_n)} \rho_n(x)dx - \ell \right| < \varepsilon$$

$$\left| \int_{\mathbb{R}^N \setminus B_{R_n}(y_n)} \rho_n(x)dx + \ell - \Lambda \right| < \varepsilon$$

$$\left| \int_{D_n} \rho_n(x)dx \right| < \varepsilon, \quad D_n = B_{R_n}(y_n) \setminus B_R(y_n).$$

Theorem 1.208 (Prohorov) in [63], reported below, gives a sufficient condition to obtain the tight convergence for a sequence of bounded Borel measures.

Theorem 14. *Let Y be a metric space and let $(\mu_n)_n$ be a sequence of bounded Borel measure. Assume that for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset Y$ such that*

$$\sup_n [\mu_n(Y \setminus K_\varepsilon)] \leq \varepsilon. \tag{2.15}$$

Then there exist a subsequence $(\mu_{n_k})_k \subset (\mu_n)_n$ and a Borel measure μ such that $\mu_{n_k} \overset{*}{\rightharpoonup} \mu$.

Remark 6. *From Proposition 1.202 in [63], any sequence of bounded measures admits a subsequence which converges in the sense of Definition 2. Thus, from Theorem 14, to obtain the tight convergence in the sense of Definition 3, we need, in addition to the boundedness of the sequence, that for all $\varepsilon > 0$ there exists a compact set K_ε such that* (2.15) *holds. From Lemma 10, the compactness condition (a) assert that, for the translated measures $\mu_n := \rho_n(y_n + \cdot)$ condition* (2.15) *is satisfied, therefore, the sequence of translated measures $(\mu_n)_n$ admits a subsequence which converges tightly.*

Prohorov's Theorem, as pointed out in [104], was used by Lions in [94] to characterize the property of tightness through the following definition.

Definition 4. *A sequence $(\rho_n)_n \in L^1(\mathbb{R}^N)$ is called tight if for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{\mathbb{R}^N \setminus B_R(0)} |\rho_n| dx < \varepsilon, \quad \text{for every } n \in \mathbb{N}.$$

The convergence of measures $(\rho_n dx)_n$ is called convergence tight if $(\rho_n)_n$ is a tight sequence.

The second concentration compactness principle, Lemma I.1 in [96] stated below, roughly speaking, regards a possible concentration of a weakly convergent sequence at finite points, precisely

Lemma 11. *Assume $\Omega \subset \mathbb{R}^N$ a domain, $1 \leq p < N$. Let $(u_n)_n$ be a bounded sequence in $D^{1,p}(\Omega)$ converging weakly to some u and such that $|Du_n|^p \rightharpoonup \mu$ and either $|u_n|^{p^*} \rightharpoonup \nu$ if Ω is bounded or $|u_n|^{p^*} \overset{*}{\rightharpoonup} \nu$ if Ω is unbounded, where μ, ν are bounded nonnegative measures on Ω. Then there exist some at most countable set J such that*

(i) $\nu = |u|^{p^*} + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \nu_j \geq 0,$

(ii) $\mu \geq |Du|^p + \sum_{j\in J} \mu_j \delta_{x_j}, \quad \mu_j \geq 0,$

(iii) $S\nu_j^{p/p^*} \leq \mu_j, \qquad \sum_{j\in J} \nu_j^{p/p^*} < \infty,$

where $(x_j)_{j\in J}$ *are distinct points in* Ω, δ_x *is the Dirac-mass of mass 1 concentrated at* $x \in \Omega$ *and* S *is the Sobolev's constant.*

In the unbounded case, no concentration at infinity or, equivalently, tightness can be proved either directly, by the definition, or indirectly using the first concentration compactness principle, by showing that vanishing and dichotomy cannot occur.

In some situations, the proof of the not validity of dichotomy often leads to rather cumbersome and tricky calculations. To get rid of these difficulties, Chabrowskii shows for the Laplacian case how one can avoid the use of the first concentration compactness principle by applying a version at infinity of the second principle, cfr. Proposition 2 in [34], see also Bianchi et al. in [25]. Later Ben-Naoum et al. in [22] obtain the following result to p-Laplacian, Proposition 3.3 in [22].

Proposition 1. *Let* $(u_n)_n$ *be a bounded sequence in* $D^{1,p}(\mathbb{R}^N)$ *and define*

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{p^*} dx, \qquad \mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |Du_n|^p dx. \tag{2.16}$$

Then, the quantities ν_∞ *and* μ_∞ *exist and satisfy*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty, \tag{2.17}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |Du_n|^p dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty, \tag{2.18}$$

$$S\nu_\infty^{p/p^*} \leq \mu_\infty, \tag{2.19}$$

where ν *and* μ *are as in* (i) *and* (ii) *in Lemma 11 and such that* (iii) *is valid.*

For completeness, we give here the key point of the proof of (2.17) following [22]. From the boundedness of a sequence $(u_n)_n$ in $D^{1,p}(\mathbb{R}^N)$, up to subsequence, it is guaranteed the existence of a bounded positive measures ν, μ such that $(|u_n|^{p^*}dx)_n$ and $(|Du_n|^p dx)_n$ converge weakly in the sense of measures respectively to ν and μ. Thus,

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} &= \lim_{n\to\infty} \int_{\mathbb{R}^N\cap\{|x|\leq R\}} |u_n|^{p^*} + \limsup_{n\to\infty} \int_{\mathbb{R}^N\cap\{|x|>R\}} |u_n|^{p^*} \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N\cap\{|x|\leq R\}} d\nu + \limsup_{n\to\infty} \int_{\mathbb{R}^N\cap\{|x|>R\}} |u_n|^{p^*} \to \int_{\mathbb{R}^N} d\nu + \nu_\infty, \quad R\to\infty,\end{aligned}$$

where ν_∞ is given in $(2.16)_1$, while Lemma 11 is applied in the bounded set given by $\Omega = \mathbb{R}^N \cap \{|x| \leq R\}$. Similarly, (2.18) is clearly true.

To sum up, in order to prove Palais Smale condition at a certain level set for a given variational functional, we need to reject concentration both around points and at infinity, i.e. proving that $\nu_j = \nu_\infty = 0$, for all j, in the spirit of the first concentration compactness principle by Lions and concentration compactness principle at infinity by Ben-Naoum et al., respectively.

Now we discuss the role of symmetry in the second concentration compactness principle. It is well known that the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact provided that $\Omega \subset \mathbb{R}^N$ is a bounded domain and q is subcritical, i.e. $1 < q < 2^*$. This kind of compactness plays a very important role in the study of nonlinear partial differential equations as well as in the study of nonlinear elliptic equations and it assures the Palais Smale condition. However, when spaces are invariant under the action of some noncompact groups, the compactness of the embedding breaks down. For example, the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ with $2 < q < 2^*$ is no more compact because of the action of translations. Even if Ω is bounded, the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact because of the action of dilations.

At the same time, it is well known that the presence of symmetries has usually the effect of producing additional solutions. Indeed, if the embedding is restricted on the subspace with full symmetry, it recovers the compactness. For example, the radially symmetric subspace $H_r^1(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$ is embedded compactly in $L^q(\mathbb{R}^N)$ with $2 < q < 2^*$, and for the case where Ω is an annulus, the radially symmetric subspace $H_r^1(\Omega)$ of $H_0^1(\Omega)$ is embedded compactly in $L^{2^*}(\Omega)$. Thus, the presence of the full symmetry saves the necessity of concentration compactness principle at infinity.

In the setting of the invariance by a group of orthogonal transformation of $\mathbb{R}^N$, as remarked in [97] Section 3.2 and in [98], the situation is quite easier for minimization problems. Namely, the first concentration compactness principle, under suitable symmetric condition, yields that dicothomy cannot occur. Thus we have either vanishing or compactness and, usually, the property of vanishing can be easily overcome so, in general, it remains compactness, called also tightness.

There have been a variety of papers such as [21, 79, 81] and [137] in which the author proposes a version of the concentration compactness lemma with a nonnegative nontrivial and T-symmetric weight $K \not\equiv 1$ in the critical term of the nonlinearity. In particular, in Lemma 4.3 in [137], the author requires, among other conditions, for a sequence $(u_n)_n \subset D_T^{1,p}(\mathbb{R}^N)$, that

$$K|(u_n - u)|^{p^*} \rightharpoonup \gamma, \tag{2.20}$$

with γ is a positive bounded measure. As discussed in Remark 4.6 in [137], condition (2.20) is stronger than the usual condition $|u_n - u|^{p^*} \rightharpoonup \nu$ and, defining

γ_∞ in the same way as ν_∞, we have $\gamma(x) = K(x)\nu(x)$ for every $x \in \mathbb{R}^N$ and $\gamma_\infty \leq K(\infty)\nu_\infty$. Thus γ and ν concentrate at the same points if $K > 0$. Further, if there exists $\lim_{|x|\to\infty} K(x) = K(\infty)$, then $\gamma_\infty = K(\infty)\nu_\infty$.

2.1.4 Classical theorems

Now we recall briefly the definition of the genus inspired by [6]. Let Y be a real Banach space and let

$$\Sigma = \{A \subset Y \setminus \{0\} \mid \ A \text{ closed and symmetric } u \in A \Rightarrow -u \in A\}.$$

Let $A \in \Sigma$, the genus of A, say $\gamma(A)$, is defined as the smallest integer N such that there exists $\Phi \in C\left(Y, \mathbb{R}^N \setminus \{0\}\right)$ such that Φ is odd and $\Phi(x) \neq 0$ for all $x \in A$. We set $\gamma(\emptyset) = 0$ and $\gamma(A) = \infty$ if there are no integers with the above property.

The main properties of genus are listed in the following proposition.

Proposition 2. *Let $A, B \in \Sigma$, then*

a) If there exists $f \in C(A, Y)$ odd, then $\gamma(A) \leq \gamma(f(A))$;

b) If $A \subset B$ then $\gamma(A) \leq \gamma(B)$;

c) If there exists an odd homeomorphism between A and B, then $\gamma(A) = \gamma(B)$;

d) If S^{N-1} is the unit sphere in $\mathbb{R}^N$ then $\gamma(S^{N-1}) = N$;

e) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$;

f) If $\gamma(B) < \infty$ then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;

g) If A is compact then $\gamma(A) < \infty$. Moreover, there exists a $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$, where $N_\delta(A) = \{x \in Y : d(x, A) \leq \delta\}$;

h) If W is a subspace of X with codimension k and $\gamma(A) > k$ then $A \cap W \neq \emptyset$.

Remark 7. *In particular, as emphasized by Struwe in Observation 5.5 in [128], if $A \in \Sigma$ is a finite collection of antipodal pairs u_i and $-u_i$, then $\gamma(A) = 1$.*

For completeness, we recall the classical Deformation Lemma (see [117]).

Lemma 12. *Let Y be a Banach space and consider $f \in C^1(Y, \mathbb{R})$, satisfying the $(PS)_c$ condition. If $c \in \mathbb{R}$ and N is any neighborhood of*

$$K_{c,f} := \{u \in Y : f(u) = c, f'(u) = 0\},$$

then there exist $\eta(t, u) = \eta_t(u) \in C([0, 1] \times Y, Y)$ and constants $\bar{\varepsilon} > \varepsilon > 0$ such that

1. $\eta_0(u) = u$ *for all* $u \in Y$;

2. $\eta_t(u) = u$ *for all* $u \notin f^{-1}[c - \bar{\varepsilon}, c + \bar{\varepsilon}]$;

3. $\eta_t(u) = u$ *is a homeomorphism of* Y *onto* Y, *for all* $t \in [0, 1]$;

4. $f(\eta_t(u)) \leq f(u)$ *for all* $t \in [0, 1]$ *and for all* $u \in Y$;

5. $\eta_1(f^{c+\varepsilon} \backslash N) \subset f^{c-\varepsilon}$, *where* $f^c = \{u \in Y : f(u) \leq c\}$, *for all* $c \in \mathbb{R}$;

6. *if* $K_{c,f} = \emptyset$, $\eta_1(f^{c+\varepsilon}) \subset f^{c-\varepsilon}$;

7. *if* f *is even,* η_t *is odd in* u.

Let us introduce the following version of the Mountain Pass Theorem by Ambrosetti and Rabinowitz in [7], whose statement takes into account also the Ekeland's Variational Principle.

Theorem 15. *Let* $(V, \|.\|_V)$ *be a Banach space and take into account* $F \in C^1(V)$. *We assume that*
(i) $F(0) = 0$,
(ii) There exist $\zeta, R > 0$ *such that* $F(u) \geq \zeta$ *for all* $u \in V$, *with* $\|u\|_V = R$,
(iii) There exists $v_0 \in V$ *such that* $\limsup_{t \to \infty} F(tv_0) < 0$.
Let $t_0 > 0$ *be such that* $\|t_0 v_0\|_V > R$ *and* $F(t_0 v_0) < 0$ *and let*

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} F(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C^0([0, 1], V) \,/\, \gamma(0) = 0 \text{ and } \gamma(1) = t_0 v_0\}.$$

Then, there exists a Palais Smale sequence at level c, that is a sequence $(u_n)_n \subset V$ *such that*

$$\lim_{n \to \infty} F(u_n) = c \quad \text{and} \quad \lim_{n \to \infty} F'(u_n) = 0 \quad \text{strongly in } V'.$$

We want to end this section with the Fountain Theorem, needed to prove the multiplicity result in the symmetric setting. First, we remind some well known basic definitions.

Definition 5. *Let* $\mathcal{G}$ *a topological group and* $(Y, \| \cdot \|)$ *a normed vector space. An action of* $\mathcal{G}$ *on* Y *is a continuous map* $\mathcal{G} \times Y \to Y$ *such that* $(g, y) \mapsto gy$ *and*

(1) $ey = y$, *for all* $y \in Y$;

(2) $h(gy) = (hg)y$, *for all* $y \in Y$, $h, g \in \mathcal{G}$;

(3) the map $y \mapsto gy$ is linear for every $g \in \mathcal{G}$.

The action of $\mathcal{G}$ is isometric if $\|gy\| = \|y\|$, for all $y \in Y$, $g \in \mathcal{G}$.

Definition 6. *Let $f : D_f \to \mathbb{R}$, where $D_f \subset \mathbb{R}$ and it is $\mathcal{G}$-invariant. Then f is $\mathcal{G}$-equivariant if $f(gy) = gf(y)$ for all $g \in \mathcal{G}$ and $y \in D_f$.*

The Fountain Theorem states that, under suitable assumptions, an invariant functional has infinitely many critical values. This result depends on the notion of admissible action introduced by Thomas Bartsch in [20].

Definition 7. *Assume that a compact group $\mathcal{G}$ acts diagonally on W^k, that is*

$$g(v_1, \dots, v_k) := (gv_1, \dots, gv_k)$$

where W is a finite dimensional space and W^k is the product space of W repeated k times. This action of $\mathcal{G}$ is admissible if every continuous $\mathcal{G}$-equivariant map $\partial U \to W^{k-1}$ has a zero, where U is an open bounded invariant neighborhood of 0 in W^k with $k \geq 2$.

Remark 8. *The Borsuk-Ulam Theorem in [143] says that the antipodal action of $\mathcal{G} := \mathbb{Z}/2$ on $W = \mathbb{R}$ is admissible.*

Before presenting the Fountain Theorem, we describe the setting.

(A1) The compact group $\mathcal{G}$ acts isometrically on the space $M = \overline{\bigoplus_{j\in\mathbb{N}} M_j}$, which is a Banach space, where the spaces M_j are $\mathcal{G}$-invariant and there exists a finite dimensional space W such that, for every $j \in \mathbb{N}$, $M_j \simeq W$ and the action of $\mathcal{G}$ on W is admissible.

From the decomposition of the Banach space M in **(A1)**, we define Y_m and Z_m as follows

$$Y_m := \bigoplus_{j=0}^{m} M_j, \quad Z_m := \overline{\bigoplus_{j=m}^{\infty} M_j} \tag{2.21}$$

and set

$$B_m := \{u \in Y_m : \|u\| \leq \rho_m\}, \quad N_m = \{u \in Z_m : \|u\| = r_m\}$$

where $\rho_m > r_m > 0$.

Now we are ready to establish the Fountain Theorem developed by Bartsch in 1992 in the following version, Theorem 3.6 in [143].

Theorem 16. *Under assumption* **(A1)**. *Let $\varphi \in C^1(M, \mathbb{R})$ be an invariant functional. If, for every $m \in \mathbb{N}$, there exists $\rho_m > r_m > 0$ such that*

(A2) $a_m = \max_{u \in Y_m, \|u\| = \rho_m} \varphi(u) \leq 0$,

(A3) $b_m = \inf_{u \in Z_m, \|u\| = r_m} \varphi(u) \to \infty, \;\; m \to \infty$,

(A4) *φ satisfies the $(PS)_c$ condition for every $c > 0$,*

where Y_m and Z_m as in (2.21)*. Then φ has an unbounded sequence of critical values.*

Remark 9. *We will apply Theorem 16 with $\mathcal{G} = \mathbb{Z}/2$, $M = X_T$ so that, since X_T is a separable Banach space (see [24] page 44), there is a linearly independent sequence $(e_j)_j$ such that the decomposition in* **(A1)** *holds with $M_j = X_j :=$ span $\{e_j\}$. Note that X_j are trivially $\mathcal{G}$-invariant and isomorphic to $\mathbb{R}$. Thus, using Remark 8, condition* **(A1)** *is satisfied with $W = \mathbb{R}$.*

2.2 A Multiplicity result in the p-sublinear and q-superlinear case

In this section we are interested in proving a multiplicity result for nontrivial weak solutions with negative energy in X of problem (8), when $1 < q < k < p < N$, by using variational methods and concentration compactness principles.

We recall that a first serious problem on unbounded domains is the loss of compactness of the Sobolev's embeddings, which renders variational techniques more delicate.

A strategy to prove multiplicity of solutions of (8), is to apply the result of multiple critical points for the energy functional E_λ associated to (8), given by (11). In particular, we make use of the classical result by Rabinowitz in [117] for even functionals, so that 0 is a critical point and critical points occur in antipodal pairs. Under further conditions, the functional possesses additional critical points. Precisely, we apply Theorem 1.9 in [117] in which the Krasnosel'skii genus is involved with its properties. A fundamental step in the proof of Theorem 2, whose statement is given in the introduction, is to verify of a standard and crucial compactness condition $(\mathrm{PS})_c$ for E_λ, for $c < 0$. This is a delicate point, indeed for critical problems in all of $\mathbb{R}^N$ this compactness condition is often loss, as it is already discussed in the Introduction. This section is mostly based on the recent work [13].

In Section 2.2.1 we give a first proof of tightness by the first concentration compactness principle due to Lions stated in Section 2.1.3. The validity of the $(\mathrm{PS})_c$ property for the energy functional is proved in Section 2.2.2 where a second proof of tightness is performed, while the truncated functional is introduced in Section 2.2.3 and its properties are listed. Finally, the proof of Theorem 2 is developed in Section 2.2.4.

2.2.1 A proof of tightness via the first concentration compactness principle

As pointed out in Section 2.1.3, in order to apply Lemma 11, we need the property of tightness, that is $|u_n|^{p^*} \overset{*}{\rightharpoonup} \nu$, with ν bounded nonnegative measure. This property, up to subsequences, follows immediately in bounded domains. Contrarily, in the case of a bounded sequence $(u_n)_n$ in $D^{1,p}(\mathbb{R}^N)$, to obtain the tight convergence, we have in general almost three possibilities: verifying directly Definition 3, passing through the first concentration compactness principle by Lions avoiding two possible behaviours, or using a version at infinity of the second principle, namely Proposition 1. In the following lemma, taking inspiration in [130], we show the validity of tightness using the first concentration compactness principle, that is Lemma 10 in Subsection 2.1.3.

Lemma 13. *Let $1 < q < p$. Assume that V and K satisfy (9) and (10). Define, for $1 < k < p$,*

$$\overline{\lambda}^* := S^{(p^*-k)/(p^*-p)} \frac{k}{N} \left(\frac{p^*}{p^*-k}\right)^{k/p} \left(\frac{p}{p-k}\right)^{(p-k)/p} \frac{1}{\|V\|_r \|K\|_\infty^{(p-k)/(p^*-p)}},$$

where S is the Sobolev's constant. If $c < 0$ and either $p < k < p^*$ and $\lambda \in (0, \infty)$ or

$$1 < k < p \quad \text{and} \quad \lambda \in (0, \overline{\lambda}^*), \tag{2.22}$$

then every (PS)$_c$ sequence for E_λ, $(u_n)_n$, for E_λ is such that, up to subsequences,

$$\nu_n = |u_n|^{p^*} dx \overset{*}{\rightharpoonup} \nu,$$

where ν is a bounded nonnegative measure.

Proof. Let $(u_n)_n$ be a (PS)$_c$ sequence for E_λ. Thus, as $n \to \infty$,

$$\frac{1}{p}\|Du_n\|_p^p + \frac{1}{q}\|Du_n\|_q^q - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|u_n|^k dx - \frac{1}{p^*}\int_{\mathbb{R}^N} K|u_n|^{p^*} dx = c + o_n(1) \tag{2.23}$$

and

$$\|Du_n\|_p^p + \|Du_n\|_q^q - \lambda \int_{\mathbb{R}^N} V|u_n|^k dx - \int_{\mathbb{R}^N} K|u_n|^{p^*} dx = \|u_n\| o_n(1), \tag{2.24}$$

where $o_n(1) \to 0$ as $n \to \infty$ and $\|\cdot\|$ is the norm given in (2.2). Using Lemma 8, the sequence $(u_n)_n$ is bounded in X. By Banach Alaoglu Theorem, since X is a reflexive space, there exists $u \in X$ such that, up to subsequences, $u_n \rightharpoonup u$ in X, and the following hold

$$u_n \rightharpoonup u \text{ in } L^{p^*}(\mathbb{R}^N), \quad u_n \rightharpoonup u \text{ in } L^{q^*}(\mathbb{R}^N),$$

$$Du_n \rightharpoonup Du \text{ in } L^p(\mathbb{R}^N), \quad Du_n \rightharpoonup Du \text{ in } L^q(\mathbb{R}^N).$$

$$u_n \to u \text{ in } L^s(\omega), \quad \omega \Subset \mathbb{R}^N, \quad 1 \le s < p^*.$$

Consequently, by using an increasing sequence of compact sets whose union is $\mathbb{R}^N$ and a diagonal argument, we also have (2.6). Consider the auxiliary sequence of functions $(z_n)_n$, $z_n(x) \ge 0$ in $\mathbb{R}^N$ for all $n \in \mathbb{N}$, given by

$$z_n(x) = |Du_n(x)|^p + |Du_n(x)|^q + |u_n(x)|^{p^*} + \lambda V(x)|u_n(x)|^k.$$

Define $\eta_n = z_n dx$. We claim that η_n converges tightly to a bounded nonnegative measure η on $\mathbb{R}^N$, that is, $z_n \overset{*}{\rightharpoonup} \eta$. First, we prove that there is $\Lambda > 0$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} z_n(x)dx = \Lambda > 0 \quad \text{as } n \to \infty. \tag{2.25}$$

Indeed, by weak convergence, the sequences $(\|Du_n\|_p)_n$, $(\|u_n\|_{p^*})_n$, $(\|Du_n\|_q)_n$ are bounded so that, by the Bolzano-Weiestrass Theorem, up to subsequences, there exist L, M, $Q \geq 0$ such that

$$L = \lim_{n\to\infty} \|Du_n\|_p^p, \quad M = \lim_{n\to\infty} \|u_n\|_{p^*}^{p^*}, \quad Q = \lim_{n\to\infty} \|Du_n\|_q^q. \tag{2.26}$$

Actually $L, M > 0$. Indeed, using (9), (10) and Hölder's inequality with exponents r and p^*/k, we have

$$E_\lambda(u_n) \geq \frac{1}{p}\|Du_n\|_p^p - \frac{\lambda}{k}\|V\|_r\|u_n\|_{p^*}^k - \|K\|_\infty\|u_n\|_{p^*}^{p^*}.$$

Hence, if $M = 0$, then, by letting $n \to \infty$, thanks to obtain that (2.23), we arrive to $0 \leq L/p \leq c < 0$ which is a contradiction. Thus, $M > 0$ and Sobolev's inequality gives $L > 0$.

The continuity of the functional J in $L^{p^*}(\mathbb{R}^N)$, J given in Lemma 6, implies the existence of the following limit

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V|u_n|^k dx =: H. \tag{2.27}$$

Clearly $H \geq 0$. We claim that $H > 0$. Multiplying (2.23) by p^* and then subtracting (2.24), we obtain, as $n \to \infty$,

$$\left(\frac{p^*}{p}-1\right)\|Du_n\|_p^p+\left(\frac{p^*}{q}-1\right)\|Du_n\|_q^q-\lambda\left(\frac{p^*}{k}-1\right)\int_{\mathbb{R}^N} V|u_n|^k dx = cp^*+\|u_n\|o_n(1).$$

By letting $n \to \infty$, since $(u_n)_n$ is bounded in X, we get

$$\left(\frac{p^*}{p}-1\right)L + \left(\frac{p^*}{q}-1\right)Q - \lambda\left(\frac{p^*}{k}-1\right)H = cp^*,$$

since p, q, $k < p^*$, $\lambda > 0$, $L > 0$ and $Q \geq 0$, necessarily $H > 0$ being $c < 0$. Consequently, condition (2.25) holds with $\Lambda = L+Q+M+\lambda H > 0$. We can apply Lemma 10 to the sequence $(z_n)_n$. Hence, up to a subsequence, three situations can occur: *Compactness*, *Vanishing* or *Dichotomy*. In particular, thanks to Theorem 14 (cfr. Remark 6), *Compactness* is equivalent to tightness so that we have to exclude *Vanishing* and *Dichothomy* for the sequence $(z_n)_n$.

We immediately see that *Vanishing* cannot occur. Indeed from (2.25), we can assume that there exists $R_1 \in (0,\infty)$ such that $\int_{B_{R_1}(0)} z_n(x)dx \geq \Lambda/2 > 0$, in turn (b) in Lemma 10 fails.

To prove that *Dichotomy* cannot hold, we argue by contradiction and we assume that there exists $\ell \in (0,\Lambda)$ such that for all $\varepsilon > 0$, there exist $R > 0$, $\ell \in (0,\Lambda)$, $(R_n)_n$, with $R < R_n \to \infty$ and $(y_n)_n$ in $\mathbb{R}^N$ such that, for all n large, we get

$$\begin{aligned}
&\left|\int_{B_R(y_n)} z_n(x)dx - \ell\right| < \varepsilon,\\
&\left|\int_{\mathbb{R}^N\setminus B_{R_n}(y_n)} z_n(x)dx\, + \ell - \Lambda\right| < \varepsilon, \\
&\left|\int_{D_n} z_n(x)dx\right| < \varepsilon, \quad D_n = B_{R_n}(y_n)\setminus B_R(y_n).
\end{aligned} \tag{2.28}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$, $\varphi_{|B_1(0)} \equiv 1$ and $\varphi_{|B_2(0)^c} \equiv 0$. We define $u_n^1 = \varphi_n^1 u_n$ and $u_n^2 = (1-\varphi_n^2)u_n$, where

$$\varphi_n^1(x) := \varphi\left(\frac{x-y_n}{R}\right), \qquad \varphi_n^2(x) := \varphi\left(\frac{x-y_n}{R_n}\right),$$

for all $x \in \mathbb{R}^N$ and all $n \in \mathbb{N}$. Then, $\mathrm{Supp}(u_n^1) = \{x \in \mathbb{R}^N : |x-y_n| \leq 2R\}$ and $\mathrm{Supp}(u_n^2) = \{x \in \mathbb{R}^N : |x-y_n| \geq R_n\}$ are disjoint sets for every $n \in \mathbb{N}$. In addition, $\mathrm{dist}\big(\mathrm{Supp}(u_n^1), \mathrm{Supp}(u_n^2)\big) \to \infty$. In particular, it follows

$$\int_{\mathbb{R}^N} |Du_n^1|^p\,dx = \int_{B_R(y_n)} |Du_n|^p\,dx + \int_{D_n} \big[|\varphi_n^1|^p|Du_n|^p + |u_n^1|^p|D\varphi_n^1|^p\big]dx$$

and

$$\int_{\mathbb{R}^N} |Du_n^2|^p\,dx = \int_{\mathbb{R}^N\setminus B_{2R_n}(y_n)} |Du_n|^p\,dx + \int_{D_n} \big[(1-\varphi_n^2)^p|Du_n|^p + |u_n^2|^p|D\varphi_n^2|^p\big]dx.$$

So that, by (2.28) and the facts that $\|D\varphi_n^1\|_\infty \leq c/R$, $\|D\varphi_n^2\|_\infty \leq c/R_n$, this yields

$$\int_{\mathbb{R}^N} |Du_n^1|^p\,dx = \int_{B_R(y_n)} |Du_n|^p\,dx + o_\varepsilon(1),$$

$$\int_{\mathbb{R}^N} |Du_n^2|^p\,dx = \int_{\mathbb{R}^N\setminus B_{2R_n}(y_n)} |Du_n|^p\,dx + o_\varepsilon(1),$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Similar formulas hold for $\int_{\mathbb{R}^N} |Du_n^i|^q dx$, $i = 1,2$. Furthermore, by Hölder's inequality and (2.28), we get

$$\int_{\mathbb{R}^N} V|u_n^1|^k dx = \int_{B_R(y_n)} V|u_n|^k dx + o_\varepsilon(1),$$

$$\int_{\mathbb{R}^N} V|u_n^2|^k dx = \int_{\mathbb{R}^N\setminus B_{2R_n}(y_n)} V|u_n|^k dx + o_\varepsilon(1).$$

Similar formulas hold for $\int_{\mathbb{R}^N} K|u_n^i|^{p^*}dx$, $i = 1, 2$. Consequently, (2.23), (2.24), (2.28) give, respectively,

$$\sum_{i=1}^{2}\Big(\frac{1}{p}\|Du_n^i\|_p^p + \frac{1}{q}\|Du_n^i\|_q^q - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|u_n^i|^k dx \\ - \frac{1}{p^*}\int_{\mathbb{R}^N} K|u_n^i|^{p^*}dx\Big) = c + o_n(1) + o_\varepsilon(1), \tag{2.29}$$

and

$$\sum_{i=1}^{2}\Big(\|Du_n^i\|_p^p + \|Du_n^i\|_q^q - \lambda\int_{\mathbb{R}^N} V|u_n^i|^k dx - \int_{\mathbb{R}^N} K|u_n^i|^{p^*}dx\Big) \\ = \sum_{i=1}^{2}\|u_n^i\|o_n(1) + o_\varepsilon(1) \tag{2.30}$$

where we first let $n \to \infty$ and then $\varepsilon \to 0$. As above, eventually passing to subsequences, there exist nonnegative limits α_i, β_i, $i = 1, 2$, defined by

$$\alpha_i = \lim_{n\to\infty}\int_{\mathbb{R}^N} V|u_n^i|^k dx \quad \text{and} \quad \beta_i = \lim_{n\to\infty}\int_{\mathbb{R}^N} K|u_n^i|^{p^*}dx.$$

Now, replacing the values of α_i, β_i in (2.30) we obtain as $n \to \infty$

$$\sum_{i=1}^{2}\big(\|Du_n^i\|_p^p + \|Du_n^i\|_q^q\big) = \sum_{i=1}^{2}\big(\lambda\alpha_i + \beta_i + \|u_n^i\|o_n(1)\big) + o_\varepsilon(1).$$

Multiplying (2.29) by q and p, respectively, and then subtracting (2.30), both evaluated in u_n^i, we obtain

$$\sum_{i=1}^{2}\|Du_n^i\|_p^p = \sum_{i=1}^{2}\Big(\lambda\frac{p(q-k)}{k(q-p)}\int_{\mathbb{R}^N} V|u_n^i|^k + \frac{p(q-p^*)}{p^*(q-p)}\int_{\mathbb{R}^N} K|u_n^i|^{p^*}dx \\ + \|u_n^i\|o_n(1)\Big) + c\frac{qp}{q-p} + o_\varepsilon(1)$$

and

$$\sum_{i=1}^{2}\|Du_n^i\|_q^q = \sum_{i=1}^{2}\Big(\lambda\frac{q(p-k)}{k(p-q)}\int_{\mathbb{R}^N} V|u_n^i|^k - \frac{q(p^*-p)}{p^*(p-q)}\int_{\mathbb{R}^N} K|u_n^i|^{p^*}dx \\ + \|u_n^i\|o_n(1)\Big) + c\frac{pq}{p-q} + o_\varepsilon(1),$$

from which we deduce, for $n \to \infty$ e since $\|u_n^i\|$ is bounded,

$$\sum_{i=1}^{2} \|Du_n^i\|_p^p = \sum_{i=1}^{2} \left(\lambda \frac{p(q-k)}{k(q-p)} \alpha_i + \frac{p(q-p^*)}{p^*(q-p)} \beta_i \right) + c \frac{qp}{q-p} + o_n(1) + o_\varepsilon(1) \quad (2.31)$$

and

$$\sum_{i=1}^{2} \|Du_n^i\|_q^q = \sum_{i=1}^{2} \left(\lambda \frac{q(p-k)}{k(p-q)} \alpha_i - \frac{q(p^*-p)}{p^*(p-q)} \beta_i \right) + c \frac{pq}{p-q} + o_n(1) + o_\varepsilon(1), \quad (2.32)$$

as $\varepsilon \to 0$. In particular, since $q < p$ and using that the left hand side is nonnegative, then (2.31) and (2.32) give, respectively,

$$c \leq \left(\frac{1}{q} - \frac{1}{p^*} \right) (\beta_1 + \beta_2) - \lambda \left(\frac{1}{k} - \frac{1}{q} \right) (\alpha_1 + \alpha_2), \quad (2.33)$$

and

$$c \geq \frac{\beta_1 + \beta_2}{N} - \lambda \left(\frac{1}{k} - \frac{1}{p} \right) (\alpha_1 + \alpha_2). \quad (2.34)$$

If $q < p < k < p^*$, then (2.33) is trivial, while (2.34) cannot occur since $c < 0$ but the right hand side is positive being $p < k$. This contradiction proves that, in this case, *Compactness* holds.

We claim that inequality (2.34) cannot occur also when $1 < k < p$, so that we have covered both cases $q < k < p$ and $1 < k \leq q < p$. At this aim note that, from (2.28), it follows either $\alpha_1 = 0$ or $\alpha_2 = 0$ depending whether $(y_n)_n$ is unbounded or not. Indeed, if $(y_n)_n$ is unbounded then $\mathrm{Supp}(u_n^1)$ reduces to the empty set when $n \to \infty$, consequently, from

$$\int_{B_R(y_n)} V|u_n|^k dx \leq \|V\|_{L^r(B_R(y_n))} \|u_n\|_{p^*}^k \leq C \|V\|_{L^r(B_R(y_n))}, \quad (2.35)$$

where C is the constant obtained from the boundedness of the $(\mathrm{PS})_c$ sequence and thanks to the continuity of the embedding of $D^{1,p}(\mathbb{R}^N)$ in $L^{p^*}(\mathbb{R}^N)$, then, thanks to (9) we can apply Lebesgue dominated convergence Theorem to the function $\chi_{B_R(y_n)} V^r$, obtaining

$$\begin{aligned} \alpha_1 &= \lim_{n\to\infty} \int_{\mathbb{R}^N} V|u_n^1|^k dx = \lim_{n\to\infty} \int_{B_R(y_n)} V|u_n|^k dx \\ &\leq C \lim_{n\to\infty} \|V\|_{L^r(B_R(y_n))} = 0 \end{aligned}$$

by virtue of $|B_R(y_n)| \to \emptyset$ when $n \to \infty$ since $y_n \to \infty$.

On the other hand, if $(y_n)_n$ is bounded, then arguing as above and noting that in this case $\mathrm{Supp}(u_n^2)$ becomes the empty set for $n \to \infty$, we get $\alpha_2 = 0$.

First, consider the case $\alpha_2 = 0$, of course $\alpha_1 > 0$ since

$$\alpha_1 + \alpha_2 + o(1) = \lim_{n\to\infty} \int_{\mathbb{R}^N} V|u_n|^k dx = H > 0.$$

From (2.30) with $i = 2$, by the definition of β_2 and Sobolev's inequality we get, as in [130],

$$\beta_2 + o_\varepsilon(1) = \lim_{n\to\infty} \left\{ \|Du_n^2\|_p^p + \|Du_n^2\|_q^q \right\} \geq \lim_{n\to\infty} \|Du_n^2\|_p^p$$
$$\geq S \lim_{n\to\infty} \|u_n^2\|_{p^*}^p \geq S\beta_2^{(N-p)/N} \|K\|_\infty^{-(N-p)/N},$$

yielding

$$\beta_2 \geq S^{N/p} \|K\|_\infty^{-(N-p)/p}. \tag{2.36}$$

Inserting (2.36) in (2.34) and using that $\beta_1 \geq 0$, we have

$$c \geq \frac{S^{N/p}}{N\|K\|_\infty^{(N-p)/p}} - \lambda\alpha_1 \left(\frac{1}{k} - \frac{1}{p} \right) \tag{2.37}$$

which is a contradiction since $c < 0$ while the right hand side is nonnegative if λ satisfies $(2.22)_2$ thanks to

$$\alpha_1 \leq \alpha_1 + \alpha_2 \leq \|V\|_r \lim_{n\to\infty} \|u_n\|_{p^*}^k \leq \lambda^{k/(p-k)} \|V\|_r^{p/(p-k)} \left[\frac{N(p^* - k)}{Sp^*k} \right]^{k/(p-k)},$$

where we have used Sobolev's inequality and (2.11). In the case $\alpha_1 = 0$, we can repeat the argument above to reach the required contradiction. The proof of the claim is so concluded, in other words, *Compactness* is clearly true also in case (2.22).

Consequently, the first concentration compactness principle guarantees that there exists a sequence $(y_n)_n$ in $\mathbb{R}^N$ such that $z_n(\cdot + y_n)$ is tight in the sense of Lemma 10, that is for arbitrary $\varepsilon > 0$ there exists $R = R(\varepsilon) \in (0, \infty)$ with

$$\int_{\mathbb{R}^N \setminus B_R(y_n)} z_n(x) dx < \varepsilon, \tag{2.38}$$

so that

$$\int_{\mathbb{R}^N \setminus B_R(y_n)} V|u_n|^k dx < \varepsilon \tag{2.39}$$

from the definition of $(z_n)_n$. It must be that $(y_n)_n$ is a bounded sequence otherwise if $y_n \to \infty$ then

$$\lim_{n\to\infty} \int_{|x-y_n|<R} V|u_n|^k dx = 0,$$

thus, combining the above limit with (2.39), we arrive to

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V|u_n|^k dx = 0,$$

contradicting $H > 0$ in (2.27).

Hence, we can replace y_n by 0 in (2.38) to obtain the tightness of z_n. Moreover, since

$$\int_{\mathbb{R}^N\setminus B_R} |u_n|^{p^*} dx \le \int_{\mathbb{R}^N\setminus B_R} z_n(x)dx < \varepsilon$$

where B_R is the ball centered at the origin and radius R, we obtain the tightness of $|u_n|^{p^*}$. Finally, we define for all $n \in \mathbb{N}$ the measure $\nu_n = |u_n|^{p^*}dx$ on $\mathbb{R}^N$ which is nonnegative, bounded since $M > 0$, and such that verifies all the assumptions of Theorem 14 thus, it admits a subsequence which converges tightly (cfr. Remark 6) to ν, a bounded non negative measure on $\mathbb{R}^N$, that is $\nu_n \overset{*}{\rightharpoonup} \nu$ as claimed. The proof is so complete. $\square$

2.2.2 On the Palais Smale property for negative levels

Now we establish sufficient conditions for the energy functional E_λ, defined in (11), to satisfy the Palais Smale property. In particular, the proof of following lemma is one of the most delicate parts in obtaining the final multiplicity theorem.

Lemma 14. *Let $1 < k < p$. If $c < 0$ then there exists $\hat{\lambda}^* > 0$ such that E_λ satisfies (PS)$_c$ condition for all $\lambda \in (0, \hat{\lambda}^*]$, where $\hat{\lambda}^*$ is defined as follows*

$$\hat{\lambda}^* = S^{(p^*-k)/(p^*-p)} \frac{kp^*}{N(p^*-k)} \cdot \frac{1}{\|V\|_r \cdot \|K\|_\infty^{(p-k)/(p^*-p)}}. \tag{2.40}$$

Proof. Let $(u_n)_n$ be a (PS)$_c$ sequence for E_λ, clearly $(u_n)_n$ is bounded in X by Lemma 8. Furthermore, since $\hat{\lambda}^* < \overline{\lambda}^*$, then Lemma 13 implies that there exists $u \in X$ such that, up to subsequences, we get

(I) $u_n \rightharpoonup u$ in X,

(II) Since $Du_n \rightharpoonup Du$ in $L^p(\mathbb{R}^N)$ and $Du_n \rightharpoonup Du$ in $L^q(\mathbb{R}^N)$, then the sequence of measures $(|Du_n|^p dx + |Du_n|^q dx)_n$ is bounded, thus we have

$$|Du_n|^p dx + |Du_n|^q dx \rightharpoonup \mu,$$

(III) Analogously,

$$|u_n|^{p^*} dx \rightharpoonup \nu,$$

where μ, ν are bounded nonnegative measures on $\mathbb{R}^N$. As in Proposition 1 applied to $(u_n)_n$, there exist at most countable set J, a family $(x_j)_{j\in J}$ of distinct points in $\mathbb{R}^N$ and two families $(\nu_j)_{j\in J}$, $(\mu_j)_{j\in J} \in]0,\infty[$ such that (2.17) and (2.18) hold where δ_x is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^N$, with ν_∞, μ_∞ defined in (2.16) and ν_j, μ_j satisfy

$$\nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j} \tag{2.41}$$

$$\mu \geq |Du|^p + |Du|^q + \sum_{j\in J} \mu_j \delta_{x_j}$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^N$, with ν_j and μ_j satisfying

$$S\nu_j^{p/p^*} \leq \mu_j, \qquad \sum_{j\in J} \nu_j^{p/p^*} < \infty. \tag{2.42}$$

Take a standard cut-off function $\psi \in C_c^\infty(\mathbb{R}^N)$, such that $0 \leq \psi \leq 1$ in $\mathbb{R}^N$, $\psi = 0$ for $|x| > 1$, $\psi = 1$ for $|x| \leq 1/2$. For each index $j \in J$ and each $0 < \varepsilon < 1$, define

$$\psi_\varepsilon(x) := \psi\left(\frac{x - x_j}{\varepsilon}\right).$$

Since $E'_\lambda(u_n)\psi \to 0$ for all $\psi \in X$ being $(u_n)_n$ a $(\mathrm{PS})_c$ sequence, choosing $\psi = \psi_\varepsilon u_n$ in (2.9) we have, as $n \to \infty$

$$\begin{aligned} \int_{\mathbb{R}^N} u_n(|Du_n|^{p-2} + |Du_n|^{q-2}) Du_n D\psi_\varepsilon dx &= \lambda \int_{\mathbb{R}^N} V|u_n|^k \psi_\varepsilon dx \\ + \int_{\mathbb{R}^N} K|u_n|^{p^*}\psi_\varepsilon dx - \int_{\mathbb{R}^N} &(|Du_n|^p + |Du_n|^q)\,\psi_\varepsilon dx + o(1). \end{aligned} \tag{2.43}$$

Since $u_n \rightharpoonup u$ in $D^{1,p}(\mathbb{R}^N)$, by (I), using the weak continuity of $\hat{J}$ proved in Lemma 6, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V|u_n|^k \psi_\varepsilon dx = \int_{\mathbb{R}^N} V|u|^k \psi_\varepsilon dx. \tag{2.44}$$

Consequently, using (II), (III) and (2.44) in (2.43), we obtain

$$\begin{aligned} &\lim_{n\to\infty} \left(\int_{\mathbb{R}^N} u_n |Du_n|^{p-2} Du_n D\psi_\varepsilon dx + \int_{\mathbb{R}^N} u_n |Du_n|^{q-2} Du_n D\psi_\varepsilon dx \right) \\ &\quad = \lambda \int_{\mathbb{R}^N} V|u|^k \psi_\varepsilon dx + \int_{\mathbb{R}^N} K\psi_\varepsilon d\nu - \int_{\mathbb{R}^N} \psi_\varepsilon d\mu. \end{aligned} \tag{2.45}$$

From Hölder's inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} u_n|Du_n|^{p-2}Du_nD\psi_\varepsilon dx\right| \le \|Du_n\|_p^{p-1}\left(\int_{\mathbb{R}^N}|u_n|^p|D\psi_\varepsilon|^p dx\right)^{1/p}\\ &\le \|u_n\|^{p-1}\left(\int_{B_\varepsilon(x_j)}|u_n|^p|D\psi_\varepsilon|^p dx\right)^{1/p}.\end{aligned} \tag{2.46}$$

Furthermore, since $D^{1,p}(\omega) \hookrightarrow\hookrightarrow L^p(\omega)$ for ω bounded set in $\mathbb{R}^N$, being $p < p^*$, then taking for instance $\omega = \overline{B}_\varepsilon(x_j)$, we have, up to subsequences, $u_n \to u$ in $L^p(\omega)$ so that there exists $w_2 \in L^p(\omega)$ such that $|u_n(x)| \le w_2(x)$ a.e. in ω. Thus, $|u_n(x)D\psi_\varepsilon(x)| \le Cw_2(x)$ a.e. in ω, as well as in $\mathbb{R}^N$, and in turn, Lebesgue dominated convergence Theorem gives

$$|u_nD\psi_\varepsilon| \to |uD\psi_\varepsilon| \text{ in } L^p(\mathbb{R}^N). \tag{2.47}$$

Consequently, passing to the limit for $n \to \infty$ in (2.46), using the boundedness of $(u_n)_n$, Hölder's inequality with exponents $N/(N-p)$ and N/p, we obtain, thanks to (2.47),

$$\begin{aligned}\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} u_n|Du_n|^{p-2}Du_nD\psi_\varepsilon dx\right| &\le C\left(\int_{B_\varepsilon(x_j)}|u|^p|D\psi_\varepsilon|^p dx\right)^{1/p}\\ &\le C\left(\int_{B_\varepsilon(x_j)}|u|^{p^*}dx\right)^{1/p^*}\left(\int_{B_\varepsilon(x_j)}|D\psi_\varepsilon|^N dx\right)^{1/N}\\ &\le C\left(\int_{B_\varepsilon(x_j)}|u|^{p^*}dx\right)^{1/p^*},\end{aligned} \tag{2.48}$$

where in the last inequality we have used the properties of ψ_ε. Similarly, by replacing p with q, we gain

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} u_n|Du_n|^{q-2}Du_nD\psi_\varepsilon dx\right| \le C\left(\int_{B_\varepsilon(x_j)}|u|^{q^*}dx\right)^{1/q^*}.$$

In turn, by letting $\varepsilon \to 0$ and then $n \to \infty$, being $u \in L^{p^*}(\mathbb{R}^N) \cap L^{q^*}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} u_n|Du_n|^{p-2}Du_nD\psi_\varepsilon dx \to 0, \qquad \int_{\mathbb{R}^N} u_n|Du_n|^{q-2}Du_nD\psi_\varepsilon dx \to 0, \tag{2.49}$$

and, arguing as in (2.35), since $V|u|^k \in L^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} V|u_n|^k\psi_\varepsilon dx \le \int_{B_\varepsilon(x_j)} V|u_n|^k dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Hence, from (2.45), if $\varepsilon \to 0$ we deduce

$$K(x_j)\nu_j = \mu_j. \tag{2.50}$$

Now combining (2.50) and (2.42), we have

$$S \leq K(x_j)\nu_j^{p/N}. \tag{2.51}$$

The inequality above establishes that the concentration of the measure ν can occur only at points where $K(x_j) > 0$. Consequently, from $(2.42)_1$ and (2.50) the measure μ can concentrate at points in which the measure ν can. Hence the set $X_J := \{x_j : j \in J\}$ does not contain the points x_j which are zeros for K.

Let $J_1 := \{j \in J : K(x_j) > 0\}$, we claim that $J_1 = \emptyset$ and we proceed by contradiction. From (2.51), it follows,

$$\nu_j \geq \left(\frac{S}{K(x_j)}\right)^{N/p} \geq \left(\frac{S}{\|K\|_\infty}\right)^{N/p}, \qquad j \in J_1. \tag{2.52}$$

To prove the claim, we show that (2.52) cannot occur for λ belonging to a suitable interval. Indeed, if (2.52) is valid, then $|J_1| < \infty$ being ν a bounded measure, indeed, from (2.17), we get, thanks to (2.52),

$$\infty > \int_{\mathbb{R}^N} d\nu = \|u\|_{p^*}^{p^*} + \int_{\{x_j\}} \sum_{j\in J_1} \nu_j \delta_{x_j} dx \geq \|u\|_{p^*}^{p^*} + \left(\frac{S}{\|K\|_\infty}\right)^{N/p} |J_1|. \tag{2.53}$$

On the other hand, $q < p$ forces $1/q - 1/p^* > 1/N$ and using that $0 \leq \psi_\varepsilon \leq 1$, thanks to (2.11), we get

$$\begin{aligned} 0 > c + o(1)\|u_n\| &\geq \frac{1}{N}\int_{\mathbb{R}^N} \big(|Du_n|^p + |Du_n|^q\big)\psi_\varepsilon dx - \lambda\frac{p^*-k}{kp^*}\|V\|_r\|u_n\|_{p^*}^k \\ &\geq \frac{1}{N}\int_{B_\varepsilon(x_j)} \big(|Du_n|^p + |Du_n|^q\big)dx \, - (C_*)^k\,\frac{p^*-k}{kp^*}\|V\|_r\lambda^{p/(p-k)}, \end{aligned}$$

so that, letting $n \to \infty$ and using (II), (2.50) and (2.52), we arrive to

$$0 > c \geq \frac{1}{N}\mu_j - C\lambda^{p/(p-k)} \geq \frac{1}{N}S^{N/p}\|K\|_\infty^{(p-N)/p} - C\lambda^{p/(p-k)}, \tag{2.54}$$

where

$$C = \left(\frac{N}{S}\right)^{k/(p-k)} \left(\frac{\|V\|_r(p^*-k)}{kp^*}\right)^{p/(p-k)}.$$

If $\lambda \in (0, \hat{\lambda}^*]$, then (2.54) produces the required contradiction, so that $J_1 = \emptyset$, concluding the proof of the claim.

It remains to show that the concentration of ν cannot occur at infinity. It is clear that ν_∞ and μ_∞ defined in (2.16) both exist and are finite.

Let $R > 0$, take another cut off function $\psi_R \in C^\infty(\mathbb{R}^N)$ such that $0 \le \psi_R \le 1$ in $\mathbb{R}^N$, $\psi_R(x) = 0$ for $|x| < R$ and $\psi_R(x) = 1$ for $|x| > 2R$. Then, from $E'_\lambda(u_n)\psi \to 0$ for all $\psi \in X$ as $n \to \infty$ being $(u_n)_n$ a $(\mathrm{PS})_c$ sequence, choosing $\psi = \psi_R u_n$ in (2.9), we get

$$\begin{aligned}\int_{R<|x|<2R} u_n\big([Du_n|^{p-2} + |Du_n|^{q-2}\big)Du_n D\psi_R dx = \lambda \int_{|x|>R} V|u_n|^k \psi_R dx \\ + \int_{|x|>R} K|u_n|^{p^*}\psi_R dx - \int_{|x|>R}\big(|Du_n|^p + |Du_n|^q\big)\psi_R dx + o(1),\end{aligned} \tag{2.55}$$

as $n \to \infty$. Similarly to the proof of (2.48), we have

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} u_n|Du_n|^{p-2}Du_n D\psi_R dx\right| \le C\left(\int_{R<|x|<2R}|u|^{p^*}dx\right)^{1/p^*},$$

and

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} u_n|Du_n|^{q-2}Du_n D\psi_R dx\right| \le C\left(\int_{R<|x|<2R}|u|^{q^*}dx\right)^{1/q^*},$$

so that, using that $u \in L^{p^*}(\mathbb{R}^N) \cap L^{q^*}(\mathbb{R}^N)$, from (2.55) we obtain

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left\{\lambda\int_{\mathbb{R}^N} V|u_n|^k\psi_R dx + \int_{\mathbb{R}^N} K|u_n|^{p^*}\psi_R dx\right\} = \mu_\infty. \tag{2.56}$$

Furthermore, we have

$$\begin{aligned}&\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N} V|u_n|^k\psi_R dx \\ &\qquad \le \lim_{R\to\infty}\limsup_{n\to\infty}\|V\|_{L^r(|x|>R)}\|u_n\|^k_{L^{p^*}(|x|>R)} = 0,\end{aligned} \tag{2.57}$$

being $(u_n)_n$ bounded in $L^{p^*}(\mathbb{R}^N)$ and by definition of ν_∞, we gain

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left\{\int_{\mathbb{R}^N} K|u_n|^{p^*}\psi_R dx\right\} \le \|K\|_\infty \nu_\infty. \tag{2.58}$$

Thanks to (2.19), (2.56), (2.57) and (2.58) we have so obtained

$$\|K\|_\infty\nu_\infty \ge \mu_\infty \ge S\nu_\infty^{p/p^*}.$$

Reasoning as above, we deduce that concentration at infinity cannot occur if we take $\lambda \in (0, \hat{\lambda}^*]$.

Consequently
$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}dx=\int_{\mathbb{R}^N}|u|^{p^*}dx.$$
Furthermore, since $u_n(x)\to u(x)$ a.e. in $\mathbb{R}^N$ from (2.6), then Brezis Lieb Lemma in [26], implies
$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|^{p^*}dx=0,$$
thus
$$\lim_{n\to\infty}\int_{\mathbb{R}^N}K|u_n|^{p^*-1}|u_n-u|dx=0, \tag{2.59}$$
since $(u_n)_n$ is bounded in X and
$$0\le\int_{\mathbb{R}^N}K|u_n|^{p^*-1}|u_n-u|dx\le\|K\|_\infty\|u_n\|_{p^*}^{p^*-1}\left(\int_{\mathbb{R}^N}|u_n-u|^{p^*}dx\right)^{1/p^*}.$$
A similar argument shows that
$$\lim_{n\to\infty}\int_{\mathbb{R}^N}V|u_n|^{k-1}|u_n-u|dx=0. \tag{2.60}$$
Now we define
$$<A_p(u),\varphi>=\int_{\mathbb{R}^N}|Du|^{p-2}DuD\varphi dx,$$
for all $u,\varphi\in X$. Using (2.9) with $\psi=u_n-u$, we have
$$\lim_{n\to\infty}\Bigg[<A_p(u_n),u_n-u>+<A_q(u_n),u_n-u> \\ -\lambda\int_{\mathbb{R}^N}V|u_n|^{k-2}u_n(u_n-u)dx-\int_{\mathbb{R}^N}K|u_n|^{p^*-2}u_n(u_n-u)dx\Bigg]=0,$$
so that, by (2.59) and (2.60),
$$\lim_{n\to\infty}\big[<A_p(u_n),u_n-u>+<A_q(u_n),u_n-u>\big]=0. \tag{2.61}$$
Using the monotonicity of A_q, see [67], we have
$$<A_p(u_n)+A_q(u),u_n-u>\le<A_p(u_n)+A_q(u_n),u_n-u>,$$
thus, applying the limsup to both terms and using (2.61) we get
$$\limsup_{n\to\infty}\big[<A_p(u_n),u_n-u>+<A_q(u),u_n-u>\big]\le 0 \tag{2.62}$$

Since $u_n \rightharpoonup u$ in $D^{1,q}(\mathbb{R}^N)$, then $< A_q(u), u_n - u >\to 0$ as $n \to \infty$, in turn (2.62) gives

$$\limsup_{n\to\infty} < A_p(u_n), u_n - u >\leq 0. \tag{2.63}$$

On the other hand, using the monotonicity of A_p and the definition of weak convergence, we obtain, thanks to (2.63),

$$0 \leq \limsup_{n\to\infty} < A_p(u_n) - A_p(u), u_n - u >= \limsup_{n\to\infty} < A_p(u_n), u_n - u >\leq 0$$

which gives

$$\lim_{n\to\infty} < A_p(u_n) - A_p(u), u_n - u >= 0. \tag{2.64}$$

The same argument holds for A_q. Thus, by virtue of Lemma 9 applied with $a(x,\xi) = |\xi|^{p-2}\xi$, condition (2.64) is equivalent to

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |D(u_n - u)|^p dx = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |D(u_n - u)|^q dx = 0,$$

that is the strong convergence in $L^p(\mathbb{R}^N)$ and in $L^q(\mathbb{R}^N)$ of the sequence $(Du_n)_n$. In turn, by Sobolev Gagliardo Niremberg's inequality, we obtain the required property, namely E_λ satisfies $(\mathrm{PS})_c$ condition for every $c < 0$. This completes the proof. □

2.2.3 The role of the truncated functional E_∞

In what follows, we will define E_∞, the truncated functional of E_λ, whose main feature is to be bounded from below, indeed this property fails for E_λ. With the help of E_∞, which satisfies the $(\mathrm{PS})_c$ condition, we are able to obtain the desired behavior of E_λ when $\|u\|$ is small.

First, by Hölder's and Sobolev's inequalities we have, for all $u \in X$,

$$E_\lambda(u) \geq \frac{1}{p}\|u\|^p_{D^{1,p}} - \lambda c_1 \|u\|^k_{D^{1,p}} - c_2 \|u\|^{p^*}_{D^{1,p}}.$$

where $c_1 = S^{-k/p}\|V\|_r/k$ and $c_2 = S^{-p^*/p}\|K\|_\infty/p^*$.

Define, for all $1 < k < p$, the function $h(t) = t^p/p - \lambda c_1 t^k - c_2 t^{p^*}$ in $\mathbb{R}^+_0$ and write

$$h(t) = t^k \hat{h}(t), \qquad \hat{h}(t) := -\lambda c_1 + \frac{1}{p} t^{p-k} - c_2 t^{p^*-k}, \tag{2.65}$$

in turn $\hat{h}(0) < 0$, $\hat{h}(t) \to -\infty$ as $t \to \infty$ and $\hat{h}'(t) > 0$ for $t > 0$ close to 0. Thus, there exists $T > 0$ such that

$$\hat{h}'(T) = 0, \qquad T = \left[\frac{p-k}{c_2 p(p^*-k)}\right]^{1/(p^*-p)}.$$

If we have $\hat{h}(T) > 0$, then there exist T_0 and T_1, with $0 < T_0 < T < T_1$ such that $\hat{h}(T_0) = \hat{h}(T_1) = 0$. Let

$$\lambda^* = \frac{k(p^*-p)}{p(p^*-k)} \left(\frac{p^*(p-k)}{p(p^*-k)}\right)^{(p-k)/(p^*-p)} \cdot \frac{S^{(p^*-k)/(p^*-p)}}{\|V\|_r \cdot \|K\|_\infty^{(p-k)/(p^*-p)}}, \tag{2.66}$$

then the following is clearly true if $\lambda < \lambda^*$,

$$\hat{h}(T) = (p^*-p)\left(\frac{p-k}{c_2}\right)^{(p-k)/(p^*-p)} \left(\frac{1}{p(p^*-k)}\right)^{(p^*-k)/(p^*-p)} - \lambda c_1 > 0,$$

so that, since $h(T_0) = h(T_1) = 0$, being $h(t) = t^k \hat{h}(t)$, we have

$$h(t) > 0 \text{ in } (T_0, T_1), \qquad h(t) \leq 0, \text{ in } [0, T_0] \cup [T_1, \infty),$$

cfr. Figure 2.1.

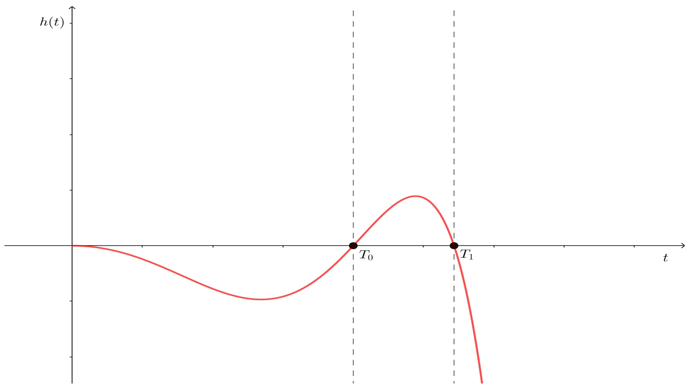

Figure 2.1: $h(t)$

In particular, we have $\lambda^* < \hat{\lambda}^*$, with $\hat{\lambda}^*$ given in (2.40), since

$$\frac{kp^*}{N(p^*-k)} > \frac{k(p^*-p)}{p(p^*-k)} \left(\frac{p^*(p-k)}{p(p^*-k)}\right)^{(p-k)/(p^*-p)}$$

which is equivalent to

$$1 > \frac{N(p^*-p)}{p^* p} \left(\frac{p^*(p-k)}{p(p^*-k)}\right)^{(p-k)/(p^*-p)},$$

but $N(p^* - p)/p^*p = 1$, hence the above inequality reduces to the inequality $p^*(p-k)/p(p^*-k) < 1$ which trivially holds, being $p^* > p$.

Next, take a cutoff function $\tau \in C^\infty(\mathbb{R}_0^+)$, nonincreasing and such that $\tau(t) = 1$ if $0 \le t \le T_0$ and $\tau(t) = 0$ if $t \ge T_1$. We consider, for all $1 < k < p$, the truncated functional

$$E_\infty(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{q}\|Du\|_q^q - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|u|^k dx - \frac{\tau\left(\|u\|_{D^{1,p}}\right)}{p^*}\int_{\mathbb{R}^N} K|u|^{p^*} dx$$

and define

$$\overline{h}(t) = \frac{1}{p}t^p - \lambda c_1 t^k - c_2 t^{p^*}\tau(t), \quad t \in \mathbb{R}_0^+,$$

then $\overline{h}(t) \to \infty$ as $t \to \infty$ and $\overline{h}(t) \ge h(t)$ for all $t \ge 0$ so that

$$\begin{aligned} \overline{h}(t) = h(t) \text{ in } (0, T_0), \ \ \overline{h}(T_0) = \overline{h}(T_0) = 0, \\ \overline{h}(t) \ge h(t) > 0 \text{ in } (T_0, T_1), \ \ \overline{h}(T_1) > 0, \end{aligned} \tag{2.67}$$

furthermore, $\overline{h}(t) > 0$ in (T_1, ∞) since, for $t \ge T_1$, it holds $\overline{h}(t) = t^k\kappa(t)$ with $\kappa(t) = \frac{1}{p}t^{p-k} - \lambda c_1$ which is strictly increasing and positive in (T_1, ∞), cfr. Figure 2.2.

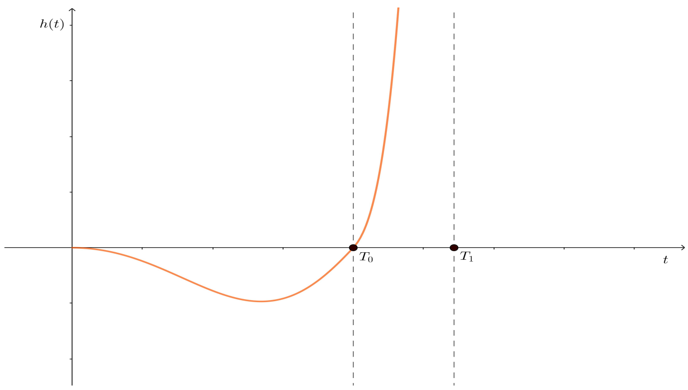

Figure 2.2: $\overline{h}(t)$

Thus, $E_\infty(u) \ge \overline{h}\left(\|u\|_{D^{1,p}}\right)$ for all $u \in X$ and

$$E_\lambda(u) = E_\infty(u) \quad \text{if} \quad 0 \le \|u\|_{D^{1,p}} \le T_0. \tag{2.68}$$

Furthermore, by the regularity both of τ and of E_λ we get $E_\infty(u) \in C^1(X, \mathbb{R})$.

Lemma 15. *Let E_∞ be the truncated functional of E_λ.*

(a) If $E_\infty(u) < 0$, then $\|u\|_{D^{1,p}} < T_0$ and $E_\lambda(v) = E_\infty(v)$ for all v in a small enough neighborhood of u.

(b) For all $\lambda \in (0, \lambda^*)$, $E_\infty(u)$ satisfies the (PS)$_c$ condition for $c < 0$.

Proof. We prove (a) by contradiction. If $\|u\|_{D^{1,p}} \in [T_0, \infty)$, by the above analysis we see that

$$E_\infty(u) \geq \overline{h}\left(\|u\|_{D^{1,p}}\right) \geq 0.$$

This contradicts $E_\infty(u) < 0$, thus $\|u\|_{D^{1,p}} < T_0$ and the last part of (a) is a consequence of the continuity of E_∞ and $(2.67)_1$.

About claim (b), if $c < 0$ and $(u_n)_n \subset X$ is a (PS)$_c$ sequence for E_∞, then we may assume that $E_\infty(u_n) < 0$ and $E'_\infty(u_n) \to 0$ as $n \to \infty$. By (a), we have $\|u_n\|_{D^{1,p}} < T_0$, so that $E_\infty(u_n) = E_\lambda(u_n)$ and $E'_\infty(u_n) = E'_\lambda(u_n)$. By Lemma 14, since $\lambda^* < \hat{\lambda}^*$, E_λ satisfies (PS)$_c$ condition for $c < 0$, thus there is a convergent subsequence $(u_n)_n$ in X. In other words, E_∞ satisfies (PS)$_c$ condition for every $c < 0$. The proof is complete. □

2.2.4 Proof of Theorem 2

We now come to the main subject of the current chapter, the proof of the existence Theorem 2, whose statement is given in the Introduction.

Proof of Theorem 2. Let $K_c = K_{c,E_\infty} = \{u \in X : E_\infty(u) = c,\ E'_\infty(u) = 0\}$ and take $m \in \mathbb{N}^+$. For $1 \leq j \leq m$ define

$$c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} E_\infty(u)$$

where

$$\Sigma_j = \{A \subset X \setminus \{0\} : A \text{ is closed in } X,\ -A = A,\ \gamma(A) \geq j\}.$$

We claim that

$$-\infty < c_j < 0 \quad \text{for all } j \geq 1. \tag{2.69}$$

To reach the claim it is enough to prove that for all $j \in \mathbb{N}$, there is an $\varepsilon_j = \varepsilon(j) > 0$ such that

$$\gamma(E_\infty^{-\varepsilon_j}) \geq j, \ \text{ where } \ E_\infty^a = \{u \in X : E_\infty(u) \leq a\} \quad \text{with} \quad a \in \mathbb{R}. \tag{2.70}$$

Let $\Omega \subset \mathbb{R}^N$, $|\Omega| > 0$, be a bounded open set in which $V > 0$, eventually $\Omega \subset \Omega_V$ where Ω_V is given in the hypothesis. Extending functions u in $D_0^{1,p}(\Omega)$ by 0 outside Ω, where $D_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm $\|u\|_{D_0^{1,p}(\Omega)} = \|Du\|_{L^p(\Omega)}$, then $u \in D^{1,p}(\mathbb{R}^N)$ and we can assume that $D_0^{1,p}(\Omega) \subset X$. Let W_j be a j-dimensional subspace of $D_0^{1,p}(\Omega)$. For every $v \in W_j$ with $\|v\|_{D_0^{1,p}(\Omega)} = 1$, from the assumptions of V it is easy to see that there exists a $d_j > 0$ such that

$$\int_\Omega V|v|^k dx \geq d_j. \tag{2.71}$$

Since W_j is a finite-dimensional space, all the norms in W_j are equivalent. Thus, we can define

$$\begin{aligned} a_j &= \sup\left\{\|Dv\|_q^q : v \in W_j, \|v\|_{D_0^{1,p}(\Omega)} = 1\right\} < \infty, \\ b_j &= \sup\left\{\|v\|_{p^*}^{p^*} : v \in W_j, \|v\|_{D_0^{1,p}(\Omega)} = 1\right\} < \infty. \end{aligned} \tag{2.72}$$

On the other hand, for $t \in (0, T_0)$, by (2.68) and since $K(x) \geq 0$ in $\mathbb{R}^N$, we arrive to

$$E_\infty(tv) = E_\lambda(tv) = \frac{1}{p}t^p + \frac{t^q}{q}\|Dv\|_q^q - \frac{\lambda t^k}{k}\int_\Omega V|v|^k dx - \frac{t^{p^*}}{p^*}\int_\Omega K|v|^{p^*} dx,$$

for every $v \in W_j$ with $\|v\|_{D_0^{1,p}(\Omega)} = 1$. Now we obtain, thanks to (2.71) and (2.72),

$$E_\infty(tv) \leq t^q\left(\frac{a_j}{q} - \frac{\lambda d_j}{k}t^{k-q} + \frac{1}{p}t^{p-q}\right), \quad t \in (0, T_0).$$

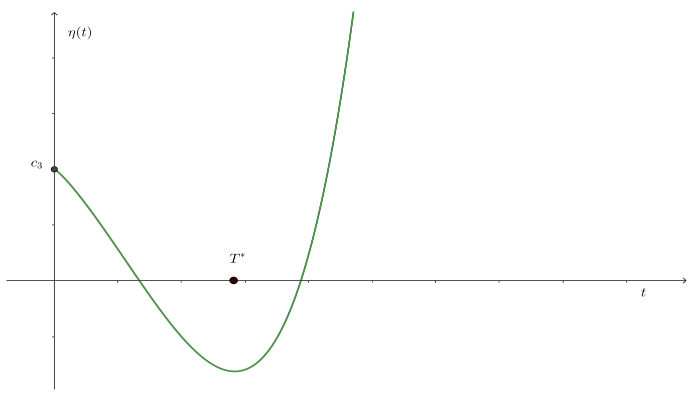

Figure 2.3: $\eta(t)$

Let

$$\eta(t) = c_3 - \lambda\frac{c_4}{k-q}t^{k-q} + \frac{c_5}{p-q}t^{p-q},$$

with positive constants given by $c_3 = a_j/q$, $c_4 = d_j(k-q)/k$ and $c_5 = (p-q)/p$, cfr. Figure 2.3.

We prove first that there exists $T^* > 0$ such that for all $\lambda > 0$

$$\eta'(T^*) = 0, \quad \eta(t) \geq \eta(T^*) \quad \text{in} \quad (0, T^*). \tag{2.73}$$

First, we observe that $\eta(0) = c_3 > 0$, $\eta(t) \to \infty$ when $t \to \infty$ and $\eta'(t) < 0$ when $t \to 0^+$, since $\eta'(t) = t^{k-q-1}\big(-\lambda c_4 + c_5 t^{p-k}\big)$ and $p > k$. Moreover, from $\eta'(t)/t^{k-q-1}$ is strictly increasing, we deduce that there exists a unique $T^* > 0$ such that $\eta'(T^*) = 0$, with T^* minimum for η, precisely $T^* = (\lambda c_4/c_5)^{1/(p-k)}$ and

$$\eta(T^*) = c_3 - \lambda^{(p-q)/(p-k)} \left(\frac{c_4^{p-q}}{c_5^{k-q}}\right)^{1/(p-k)} \frac{p-k}{(k-q)(p-q)}.$$

In particular, $\eta(T^*) < 0$ if and only if $\lambda > \lambda_*$, where

$$\begin{aligned} \lambda_* &= \frac{c_5^{(k-q)/(p-q)}}{c_4} \left(\frac{c_3(k-q)(p-q)}{p-k}\right)^{(p-k)/(p-q)} \\ &= \left(\frac{a_j}{q(p-k)}\right)^{(p-k)/(p-q)} p^{(q-k)/(p-q)} \frac{k(p-q)}{d_j(k-q)^{(k-q)/(p-q)}}. \end{aligned}$$

It holds $\lambda_* < \lambda^*$, with λ^* defined in (2.66), if

$$\begin{aligned} \|K\|_\infty <&p^* \left(\frac{S}{p^*-k}\right)^{(p^*-k)/(p-k)} \left(\frac{q}{a_j}\right)^{(p^*-p)/(p-q)} \left(1 - \frac{k}{p}\right)^{(p^*-q)/(p-q)} \\ &\cdot \left(\frac{d_j(p^*-p)}{(p-q)\|V\|_r}\right)^{(p^*-p)/(p-k)} (k-q)^{(k-q)(p^*-p)/(p-q)(p-k)}, \end{aligned} \tag{2.74}$$

say for $\|K\|_\infty$ sufficiently small.

Finally, we prove that $T^* \in (0, T_0)$ if $\lambda < \lambda^*$. From $h(t) \leq E_\lambda(tv) \leq t^q \eta(t)$ for all $t > 0$ and $v \in W_j$ with $\|v\|_{D_0^{1,p}} = 1$, we deduce $h(T^*) < 0$ so that $T^* \in (0, T_0)$ or $T^* > T_1$. Assume by contradiction that $T^* > T_1$, then $T^* > T$ or equivalently, using the explicit values of T^* and T,

$$\lambda > \left[S^{p^*/p} \frac{p^*(p-k)}{\|K\|_\infty p(p^*-k)}\right]^{(p-k)/(p^*-p)} \frac{k(p-q)}{d_j(k-q)p}.$$

Since $\lambda < \lambda^*$, then

$$S^{-k/p} \frac{\|V\|_r}{d_j} < \frac{k-q}{p-q} \cdot \frac{p^*-p}{p^*-k} < 1, \tag{2.75}$$

but, by (2.71), we have $d_j \leq \|V\|_r \|v\|_{p^*}^k \leq S^{-k/p}\|V\|_r$ being $\|v\|_{D_0^{1,p}} = 1$, so that (2.75) produces the required contradiction since $S^{-k/p}\|V\|_r/d_j \geq 1$. Consequently, $T^* \in (0, T_0)$.

Thus, (2.73) is verified for $T^* \in (0, T_0)$ and with $\eta(T^*) < 0$ if $\lambda \in (\lambda_*, \lambda^*)$, consequently

$$E_\infty(T^* v) \leq -\varepsilon_j < 0, \qquad \varepsilon_j = -(T^*)^q \eta(T^*).$$

Denote $S_{T^*} = \left\{ v \in X : \|v\|_{D_0^{1,p}(\Omega)} = T^* \right\}$, then $S_{T^*} \cap W_j \subset E_\infty^{-\varepsilon_j}$. By Proposition 2,

$$\gamma(E_\infty^{-\varepsilon_j}) \geq \gamma(S_{T^*} \cap W_j) = j,$$

which proves (2.70). Consequently, $E_\infty^{-\varepsilon_j} \in \Sigma_j$, in turn

$$c_j \leq \sup_{u \in E_\infty^{-\varepsilon_j}} E_\infty(u) \leq -\varepsilon_j < 0.$$

Furthermore, E_∞ is bounded from below, hence $c_j > -\infty$ (that is why we take into account E_∞ instead of E_λ), thus the proof of claim (2.69) is concluded.

By [40] and [117], it follows from (2.69) that c_j, $j \in \mathbb{N}$, is a critical value for E_∞. Then, from Lemma 15, we see that E_∞ satisfies the $(\mathrm{PS})_{c_j}$ condition for all $c_j < 0$ and this implies that K_{c_j} is a compact set, hence $\gamma(K_{c_j}) < \infty$ by virtue of Proposition 2.

We claim that, if for some $j \in \mathbb{N}$ there is an $i \geq 1$ such that if

$$c = c_j = c_{j+1} = \cdots = c_{j+i}, \quad \text{then} \quad \gamma(K_c) \geq i + 1. \tag{2.76}$$

In particular, as a consequence of Remark 7 (cfr. Lemma 5.6 Chapter II in [128]), if $\gamma(K_c) > 1$ then K_c is infinite.

The proof is almost standard, but for completeness we enclose it. We proceed by contradiction. If $\gamma(K_c) \leq i$, there exists a closed and symmetric set U with $K_c \subset U$ and $\gamma(U) \leq i$, since $c < 0$, we can also assume that the closed set $U \subset E_\infty^0$. Using Lemma 12-(5), there is an odd homeomorphism $\eta\colon [0,1] \times X \to X$ such that $\eta_1(E_\infty^{c+\delta} \backslash U) \subset E_\infty^{c-\delta}$ for some $\delta \in (0, -c)$. From definition of $c = c_{j+i}$, there exists an $A \in \Sigma_{j+i}$ for which $\sup_{u \in A} E_\infty(u) < c + \delta$. Thus, from (2) and (5) of Proposition 12, respectively, we get

$$\eta_1(A \backslash U) \subset \eta_1(E_\infty^{c+\delta} \backslash U) \subset E_\infty^{c-\delta},$$

which means

$$\sup_{u \in \eta_1(A \backslash U)} E_\infty(u) \leq c - \delta. \tag{2.77}$$

But Proposition 2-(a),(b) and (f) being $\gamma(U) < \infty$ and since $A \backslash U$ is closed, reveals that

$$\gamma(\overline{\eta_1(A \backslash U)}) \geq \gamma(\eta_1(A \backslash U)) \geq \gamma(A \backslash U) = \gamma(\overline{A \backslash U}) \geq \gamma(A) - \gamma(U) \geq j.$$

Hence $\overline{\eta_1(A\backslash U)} \in \Sigma_j$, so that by definition of c_j and thanks to (2.77),

$$c = c_j \leq \sup_{u \in \overline{\eta_1(A\backslash U)}} E_\infty(u) = \sup_{u \in \eta_1(A\backslash U)} E_\infty(u) \leq c - \delta.$$

This contradiction proves claim (2.76).

To complete the proof, we observe that for all $j \in \mathbb{N}^+$, we have

$$\Sigma_{j+1} \subset \Sigma_j \quad \text{and} \quad c_j \leq c_{j+1} < 0.$$

If all c_j are distinct, then $\gamma(K_{c_j}) \geq 1$, so that $K_{c_j} \neq \emptyset$ and thus $(c_j)_j$ is a sequence of distinct negative critical values of E_∞, thus a sequence of solutions with negative energy is obtained, as required.

If for some j_0, there exists an $i \geq 1$ such that

$$c = c_{j_0} = c_{j_0+1} = \cdots = c_{j_0+i},$$

from (2.76) we have $\gamma(K_{c_{j_0}}) \geq i + 1 > 1$, which shows that $K_{c_{j_0}}$ has infinitely many distinct elements. Also in this case we arrive to a sequence of solutions with negative energy.

By Lemma 15, then $E_\lambda(u) = E_\infty(u)$ for every $u \in X$ such that $E_\infty(u) < 0$, so that the functional E_λ, being even, possesses at least m pairs of critical nonzero points of with negative critical values. Therefore, problem (8) has at least $2m$ weak nontrivial solutions with negative energy. This completes the proof. □

2.3 An Existence result in the p-superlinear and subcritical case

In this section, whose main results are contained in [15], we prove existence of nontrivial weak solutions in X of the nonlinear elliptic problem (8) when the parameters satisfies the following ranges

$$1 < q \leq p < N, \quad p < k < p^*. \tag{2.78}$$

In our situation, we have to face with the loss of compactness, in the sense that $(\mathrm{PS})_c$ fails at certain levels and, hence, not all Palais Smale sequences for the functional contain some convergent subsequences.

Note that the presence of weights in our problem (8) produces several difficulties, even to prove that the solution obtained is non trivial, we cannot use results such as Lemma I.1 in [95], Proposition 2.5 in [53] and Lemma 2.8 in [62], which are classical tools in this framework when weights are not involved. Regularity results for solutions of problem (8), in the subcase of the p-Laplacian, are developed in [116].

We mention here some papers in which the concentration compactness principle is combined with the Mountain Pass Theorem in order to obtain existence results. Precisely, [88] for the Laplacian, [64], [76], [130], [125] for the p-Laplacian, [38] for the (p, q)-Laplacian but for a subcritical nonlinearity and [4] for more general operators.

A classical approach to prove existence of solutions of (8), is to construct them as critical points of the energy functional (11) via the Mountain Pass Theorem of Ambrosetti and Rabinowitz, see [7] and [117]. This is exactly the technique used to prove our main existence result, one of the hardest part in its proof is devoted to a careful analysis of the behavior of Palais Smale sequences to understand the consequences of spreading or concentration of mass. In particular, while in the case $1 < q < k < p$, treated in Subsection 2.2, the principal obstacle to this careful analysis can be found in proving tightness for $1 < k < p$, in this section the main difficulty relies on the fact that assumption (2.78) does not allow to prove the validity of $(\mathrm{PS})_c$ condition for the functional E_λ defined in (11), a well known crucial property to obtain Theorem 2. Thus, it is not possible to apply the same technique used in Section 2.2, but we had to take different directions that give us a weaker condition which nevertheless guarantees the existence of a weak solution using Mountain Pass Theorem. Indeed, after the proof of tightness, developed in Subsection 2.3.1, which is valid for every Palais Smale sequences at any level c below a certain positive threshold, using arguments derived from [36], we conclude that the nonnegative critical point for E_λ, obtained by the Mountain Pass Theorem, is the non trivial one.

In particular, in Subsection 2.3.2 we verify the validity of the Mountain Pass geometry as well as the proof of almost everywhere convergence of the gradient. Finally, the proof of Theorem 3 is developed in Subsection 2.3.3.

2.3.1 Tightness in the p-superlinear case

As discussed in the Introduction, we do not manage to get the $(\mathrm{PS})_c$ property for E_λ, defined in (11), thus the direct proof of tight convergence is crucial and it is given in the following Lemma.

Lemma 16. *Assume* $1 < q \le p < k < p^*$ *and let* $\lambda \in (0, \infty)$. *Suppose that* V *and* K *satisfy* (9) *and* (10) *respectively. If*

$$0 < c < \overline{c} := \frac{S^{N/p}}{N\|K\|_\infty^{(N-p)/p}}, \tag{2.79}$$

then every $(PS)_c$ *sequence,* $(u_n)_n \subset X$, *for* E_λ *is such that, up to subsequences,*

$$\nu_n = |u_n|^{p^*} dx \overset{*}{\rightharpoonup} \nu,$$

where ν *is a bounded nonnegative measure.*

Remark 10. *The case* $c < 0$ *of Lemma 16 is treated in Lemma 13 in Subsection 2.2.1, whose proof is inspired on an argument based on Swanson and Yu [130].*

Proof of Lemma 16. We follow the proof of Lemma 13 in Subsection 2.2.1, but adapted to the new case. Let $(u_n)_n \subset X$ be a $(\mathrm{PS})_c$ sequence. Then we can repeat word on word the proof of Lemma 13 up to the setting of $L, M, Q \ge 0$ in (2.26). We have to prove again that there exists $\Lambda > 0$ such that (2.25) holds.

As a consequence of (2.23), being $c > 0$, necessarily $L + Q > 0$.

The continuity of the functional J in $L^{p^*}(\mathbb{R}^N)$, whose J is given in Lemma 6, implies (2.27). Clearly $H \ge 0$ by (9), we claim that $H > 0$. If $H = 0$, then (2.24) would imply,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K|u_n|^{p^*} dx = L + Q. \tag{2.80}$$

Consequently, from (2.23) being $q \le p$, we get

$$c = \frac{L}{p} + \frac{Q}{q} - \frac{L+Q}{p^*} > \frac{L+Q}{N}, \tag{2.81}$$

equivalently, since c satisfies (2.79),

$$L + Q < cN < \frac{S^{N/p}}{\|K\|_\infty^{(N-p)/p}}. \tag{2.82}$$

Since K is non trivial and verifies (10), we have

$$\|u_n\|_{p^*}^{p^*} \geq \frac{1}{\|K\|_\infty} \int_{\mathbb{R}^N} K|u_n|^{p^*} dx,$$

so that

$$\|u_n\|_{p^*}^{p} \geq \|K\|_\infty^{-p/p^*} \left(\int_{\mathbb{R}^N} K|u_n|^{p^*} dx \right)^{p/p^*} \tag{2.83}$$

Furthermore, using Sobolev's inequality, (2.80) and (2.83), we gain

$$\begin{aligned} L + Q \geq L \geq S \lim_{n\to\infty} \|u_n\|_{p^*}^{p} \\ \geq S\|K\|_\infty^{-p/p^*} \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} K|u_n|^{p^*} dx \right)^{p/p^*} = S\|K\|_\infty^{-p/p^*} (L+Q)^{p/p^*}, \end{aligned}$$

which is equivalent to $L + Q \geq S^{N/p}\|K\|_\infty^{-(N-p)/p}$ contradicting (2.82). Thus, $H > 0$.

Consequently, condition (2.25) holds with $\Lambda = L + Q + M + \lambda H > 0$, then we can apply Lemma 10 to the sequence $(\rho_n)_n = (z_n)_n$ as in Lemma 13 whose proof is still valid up to (2.37), that is $c \geq \overline{c}$, which is a contradiction since $p < k < p^*$ and c satisfies (2.79).

Then we arrive to the conclusion of the Lemma as in the proof of Lemma 13 but with $M \geq 0$. The proof is complete.

□

2.3.2 Mountain Pass geometry and the lack of the Palais Smale property for the energy functional

In Subsection 2.1.4 we have introduced the Mountain Pass Theorem. In this subsection we first check that the functional E_λ, defined in (11), satisfies a Mountain Pass geometry through the following lemma.

Lemma 17. *Assume* $1 < q \leq p < k < p^*$*.Then, the functional* E_λ *verifies the hypotheses of the Mountain Pass Theorem.*

Proof. From the observations in Subsection 2.1.1, $E_\lambda \in C^1(X)$ and, clearly condition (i) of Theorem 15 is satisfied since $E_\lambda(0) = 0$. Using Hölder's inequality with exponents r and r', Sobolev's inequality, since $q \leq p$ and $K \in L^\infty(\mathbb{R}^N)$, we get

$$E_\lambda(u) \geq c_1\|u\|^p - \lambda c_2\|u\|^k - c_3\|u\|^{p^*} = \|u\|^p(c_1 - \lambda c_2\|u\|^{k-p} - c_3\|u\|^{p^*-p}),$$

where $c_1 = 1/p$, $c_2 = S^{-k/p}\|V\|_r/k$ and $c_3 = S^{-p^*/p}\|K\|_\infty/p^*$ are positive constants. Let us define $h(t) = t^p(c_1 - \lambda c_2 t^{k-p} - c_3 t^{p^*-p})$, for $t \geq 0$. It is easy to see that, since $p < k < p^*$, there exists $t_1 > 0$ such that $h(t) > 0$ for all $t \in (0, t_1]$. Therefore, there exist $\zeta, R > 0$ with R small enough, so that $E_\lambda(u) \geq \zeta > 0$ whenever $\|u\| = R$. Thus, condition (ii) of Theorem 15 is satisfied.

Now, let $u \in X \setminus \{0\}$ such that $u \geq 0$. Then, for any $t > 0$, we have

$$E_\lambda(tu) = \frac{t^p}{p}\|Du\|_p^p + \frac{t^q}{q}\|Du\|_q^q - \frac{t^k}{k}\int_{\mathbb{R}^N} Vu^k dx - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} Ku^{p^*} dx. \tag{2.84}$$

Since $p < k < p^*$ and $K \geq 0$ nontrivial, we get $E_\lambda(tu) \to -\infty$ as $t \to \infty$. Thus, let $t_u > 0$ be such that $E_\lambda(tu) < 0$ for all $t \geq t_u$ and $\|t_u u\| > R$. So the proof of (iii) is concluded.

Consider

$$\Gamma_u := \{\gamma \in C^0([0,1], X) \,/\, \gamma(0) = 0 \text{ and } \gamma(1) = t_u u\},$$

and we define c_u as in (2.119). Then the hypotheses of Theorem 15 are satisfied. This ends the proof. □

In what follows we make use of the following standard inequality.

Remark 11. *For all $x, y \in \mathbb{R}^N$ with $|x| + |y| \neq 0$, there exists a constant $C(s) > 0$ such that*

$$< |x|^{s-2}x - |y|^{s-2}y, x - y > \geq C(s)\begin{cases} \frac{|x-y|^2}{(|x|+|y|)^{2-s}}, & \text{if } 1 \leq s < 2, \\ |x-y|^s, & \text{if } s \geq 2. \end{cases}$$

Note that differently from Section 2.2, condition $p < k < p^*$ does not allow us to prove that the functional E_λ satisfies the $(\mathrm{PS})_c$ condition. In particular, the main problem lies in the proof that the atomic part of the measure ν in (2.41) is zero, how it happens if $1 < k < p$. For this reason, in the following Lemma we have to face both the case when the atomic part of ν is zero and when it is not zero. This latter case produces the convergence of $(Du_n)_n$ in L^p and L^q of a subset of $\mathbb{R}^N$, not in the entire $\mathbb{R}^N$ as it occurs in Section 2.2. Consequently, only convergence a.e. of $(Du_n)_n$ arise but not strong convergence in L^p and L^q. Thus, we have to deal with the almost everywhere convergence to prove the existence of a solution of (8).

Lemma 18. *Assume $1 < q \leq p < k < p^*$. Let $(u_n)_n \subset X$ be a $(PS)_c$ sequence with c satisfying (2.79). Then, there exists a nonnegative function $u \in X$ such that, up to subsequence,*

$$Du_n(x) \to Du(x) \quad a.e. \text{ in } \mathbb{R}^N. \tag{2.85}$$

Proof. Let $(u_n)_n$ be a $(\mathrm{PS})_c$ sequence with c satisfying (2.79). Now we follow word for word Lemma 14 up to the finiteness of J_1 proved in (2.53).

Thus, from (2.53), there exist a finite natural number $s = |J_1|$ of indices $i \in J_1$ such that (2.52) holds for all x_i with $i = 1, \ldots, s$.

First, we take into account the case in which $s > 0$. Take a standard cut-off function $\psi \in C_c^\infty(\mathbb{R}^N)$, such that $0 \le \psi \le 1$ in $\mathbb{R}^N$, $\psi = 0$ for $|x| > 1$, $\psi = 1$ for $|x| \le 1/2$ and consider $\varepsilon_0 > 0$ such that

$$\{x_1, \ldots, x_s\} \subset B_{1/2\varepsilon_0}, \quad B_{\varepsilon_0}(x_i) \cap B_{\varepsilon_0}(x_j) \quad \text{for} \quad i \neq j.$$

Define

$$\Psi_\varepsilon(x) := \psi(\varepsilon x) - \sum_{i=1}^{s} \psi\left(\frac{x - x_i}{\varepsilon}\right)$$

for all $0 < \varepsilon < \varepsilon_0$. Thus,

$$\Psi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in \bigcup_{i=1}^{s} B_{\varepsilon/2}(x_i), \\ 1, & \text{if } x \in A_\varepsilon, \end{cases}$$

where $A_\varepsilon := B_{1/2\varepsilon} \setminus \bigcup_{i=1}^{s} B_\varepsilon(x_i)$. Now, let

$$P_n := < |Du_n|^{p-2} Du_n - |Du|^{p-2} Du + |Du_n|^{q-2} Du_n - |Du|^{q-2} Du, Du_n - Du > .$$

Using Remark 11, we immediately get $P_n \ge 0$. Fix $\delta, \varepsilon > 0$ with $0 < \varepsilon < \delta < \varepsilon_0$. Then, since $A_\delta \subset A_\varepsilon$, we get $\Psi_\varepsilon \equiv 1$ in A_δ by the definition of Ψ_ε, so that

$$\int_{A_\delta} P_n dx = \int_{A_\delta} P_n \Psi_\varepsilon dx \le \int_{A_\varepsilon} P_n \Psi_\varepsilon dx \le \int_{\mathbb{R}^N} P_n \Psi_\varepsilon dx.$$

Thus, by (2.9), we gain

$$\begin{aligned} \int_{A_\delta} P_n dx \le & \int_{\mathbb{R}^N} (|Du_n|^p \Psi_\varepsilon - |Du_n|^{p-2} \Psi_\varepsilon Du_n Du) dx \\ & + \int_{\mathbb{R}^N} (|Du|^p \Psi_\varepsilon - |Du|^{p-2} \Psi_\varepsilon Du_n Du + |Du_n|^q \Psi_\varepsilon) dx \\ & + \int_{\mathbb{R}^N} (|Du|^q \Psi_\varepsilon - |Du_n|^{q-2} \Psi_\varepsilon Du_n Du - |Du|^{q-2} \Psi_\varepsilon Du_n Du) dx \end{aligned} \tag{2.86}$$

that is

$$\begin{aligned}
\int_{A_\delta} P_n dx \leq\ & E'_\lambda(u_n)(u_n\Psi_\varepsilon) - E'_\lambda(u_n)(u\Psi_\varepsilon) \\
& - \int_{\mathbb{R}^N} |Du_n|^{p-2}Du_nD\Psi_\varepsilon u_n dx - \int_{\mathbb{R}^N} |Du_n|^{q-2}Du_nD\Psi_\varepsilon u_n dx \\
& + \int_{\mathbb{R}^N} |Du_n|^{p-2}Du_nD\Psi_\varepsilon u\, dx + \int_{\mathbb{R}^N} |Du_n|^{q-2}Du_nD\Psi_\varepsilon u\, dx \\
& - \lambda\int_{\mathbb{R}^N} V|u_n|^{k-2}u_nu\Psi_\varepsilon dx - \int_{\mathbb{R}^N} K|u_n|^{p^*-2}u_nu\Psi_\varepsilon dx \\
& + \int_{\mathbb{R}^N} |Du|^p\Psi_\varepsilon dx + \int_{\mathbb{R}^N} |Du|^q\Psi_\varepsilon dx - \int_{\mathbb{R}^N} |Du|^{p-2}\Psi_\varepsilon Du_nDu\, dx \\
& - \int_{\mathbb{R}^N} |Du|^{q-2}\Psi_\varepsilon Du_nDu\, dx + \lambda\int_{\mathbb{R}^N} V|u_n|^k\Psi_\varepsilon dx + \int_{\mathbb{R}^N} K|u_n|^{p^*}\Psi_\varepsilon dx.
\end{aligned} \tag{2.87}$$

Being $(u_n)_n \subset X$ a $(\mathrm{PS})_c$ sequence for E_λ, from Lemma 8 $(u_n)_n$ is bounded, thus we obtain

$$E'_\lambda(u_n)(u_n\Psi_\varepsilon),\ E'_\lambda(u_n)(u\Psi_\varepsilon) \to 0, \tag{2.88}$$

as $n \to \infty$. Then, similarly to the proof of (2.49) in Lemma 14, using the properties of Ψ_ε and the boundedness of $(u_n)_n$, we have

$$\int_{\mathbb{R}^N} |Du_n|^{p-2}Du_nD\Psi_\varepsilon u_n dx \to 0, \qquad \int_{\mathbb{R}^N} |Du_n|^{q-2}Du_nD\Psi_\varepsilon u_n dx \to 0, \tag{2.89}$$

and

$$\int_{\mathbb{R}^N} |Du_n|^{p-2}Du_nD\Psi_\varepsilon u\, dx \to 0, \qquad \int_{\mathbb{R}^N} |Du_n|^{q-2}Du_nD\Psi_\varepsilon u\, dx \to 0, \tag{2.90}$$

as $n \to \infty$ and $\varepsilon \to 0$. Thus, using (2.88)-(2.90), (2.87) becomes

$$\begin{aligned}
\int_{A_\delta} P_n dx \leq \lambda\Bigg[&\int_{\mathbb{R}^N} V|u_n|^k\Psi_\varepsilon dx - \int_{\mathbb{R}^N} V|u_n|^{k-2}u_nu\Psi_\varepsilon dx\Bigg] \\
& + \int_{\mathbb{R}^N} K|u_n|^{p^*}\Psi_\varepsilon dx - \int_{\mathbb{R}^N} K|u_n|^{p^*-2}u_nu\Psi_\varepsilon dx \\
& + \int_{\mathbb{R}^N} |Du|^p\Psi_\varepsilon dx - \int_{\mathbb{R}^N} |Du|^{p-2}\Psi_\varepsilon Du_nDu\, dx \\
& + \int_{\mathbb{R}^N} |Du|^q\Psi_\varepsilon dx - \int_{\mathbb{R}^N} |Du|^{q-2}\Psi_\varepsilon Du_nDu\, dx + o_n(1),
\end{aligned} \tag{2.91}$$

as $n \to \infty$. We proceed by defining the following functional

$$f(v) := \int_{\mathbb{R}^N} (|Du|^{p-2} + |Du|^{q-2})DuDv\Psi_\varepsilon dx$$

for every $v \in X$. Since f is clearly bounded in X, recalling that $u \in X$, we have

$$\int_{\mathbb{R}^N} (|Du|^{p-2}+|Du|^{q-2}) Du Du_n \Psi_\varepsilon dx \to \int_{\mathbb{R}^N} (|Du|^{p-2}+|Du|^{q-2})|Du|^2 \Psi_\varepsilon dx, \quad (2.92)$$

as $n \to \infty$. Hence, from (2.92), (2.91) turns into

$$\begin{aligned} \int_{A_\delta} P_n dx \leq \lambda \left[\int_{\mathbb{R}^N} V|u_n|^k \Psi_\varepsilon dx - \int_{\mathbb{R}^N} V|u_n|^{k-2} u_n u \Psi_\varepsilon dx \right] \\ + \int_{\mathbb{R}^N} K|u_n|^{p^*} \Psi_\varepsilon dx - \int_{\mathbb{R}^N} K|u_n|^{p^*-2} u_n u \Psi_\varepsilon dx + o_n(1). \end{aligned} \quad (2.93)$$

Again by the boundedness of $(u_n)_n$ in $L^{p^*}(\mathbb{R}^N)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$ by (2.6), thus, being

$$\| |u_n|^{p^*-2} u_n \|_{(p^*)'} = \|u_n\|_{p^*}^{p^*-1}, \quad \| |u_n|^k \|_{p^*/k} = \|u_n\|_{p^*}^k, \quad \| |u_n|^{k-2} u_n \|_{p^*/(k-1)} = \|u_n\|_{p^*}^{k-1},$$

applying Lemma 20, we get

$$|u_n|^{p^*-2} u_n \rightharpoonup |u|^{p^*-2} u \text{ in } L^{(p^*)'}(\mathbb{R}^N), \qquad |u_n|^k \rightharpoonup |u|^k \text{ in } L^{p^*/k}(\mathbb{R}^N),$$
$$|u_n|^{k-2} u_n \rightharpoonup |u|^{k-2} u \text{ in } L^{p^*/(k-1)}(\mathbb{R}^N).$$

Therefore, we gain

$$\int_{\mathbb{R}^N} V|u_n|^k \Psi_\varepsilon dx, \int_{\mathbb{R}^N} V|u_n|^{k-2} u_n u \Psi_\varepsilon dx \to \int_{\mathbb{R}^N} V|u|^k \Psi_\varepsilon dx \qquad (2.94)$$

as $n \to \infty$, where in the second integral we have used that $V \in L^r(\mathbb{R}^N)$ and the facts that $|u_n|^{k-2} u_n \in L^{p^*/(k-1)}(\mathbb{R}^N)$ and $u \in L^{p^*}(\mathbb{R}^N)$. Moreover, we have

$$\int_{\mathbb{R}^N} K|u_n|^{p^*-2} u_n u \Psi_\varepsilon dx \to \int_{\mathbb{R}^N} K|u|^{p^*} \Psi_\varepsilon dx, \qquad (2.95)$$

as $n \to \infty$. Moreover, from (I), it follows that

$$\int_{\mathbb{R}^N} K|u_n|^{p^*} \Psi_\varepsilon dx \to \int_{\mathbb{R}^N} K|u|^{p^*} \Psi_\varepsilon dx, \qquad (2.96)$$

as $n \to \infty$. Now, replacing (2.94)-(2.96) in (2.93), we arrive to

$$\lim_{n \to \infty} \int_{A_\delta} P_n \leq 0,$$

thus, since $P_n \geq 0$, we can see

$$\lim_{n\to\infty}\int_{A_\delta} P_n = 0,$$

for every $\delta > 0$ such that $\varepsilon < \delta < \varepsilon_0$. From Remark 11 applied first with p and then with q, we have that P_n can be divided in two nonnegative terms, thus, we get

$$\lim_{n\to\infty}\int_{A_\delta} < |Du_n|^{p-2}Du_n - |Du|^{p-2}Du, Du_n - Du > dx = 0, \tag{2.97}$$

and

$$\lim_{n\to\infty}\int_{A_\delta} < |Du_n|^{q-2}Du_n - |Du|^{q-2}Du, Du_n - Du > dx = 0. \tag{2.98}$$

Now we consider (2.97), since the same conclusions follow for (2.98) when p is replaced with q. By virtue of Lemma 9 applied with $a(x,\xi) = |\xi|^{t-2}\xi$ first with $t = p$ then with $t = q$, we obtain

$$Du_n \to Du \quad \text{in } L^p(A_\delta) \cap L^q(A_\delta).$$

Thus, up to subsequences, we get

$$Du_n(x) \to Du(x) \quad \text{a.e. in } A_\delta.$$

Taking $0 < \varepsilon_m < \delta_m < \varepsilon_0$ and letting $\varepsilon_m, \delta_m \to 0$ as $m \to \infty$, it holds $A_{\delta_m} \to \mathbb{R}^N$ as $m \to \infty$. Then (2.85) is satisfied in the case in which $s > 0$.

As regards the case where $s = 0$, that is $\nu_j = 0$ for all $j \in J$, it is enough to take $\Psi_\varepsilon(x) := \psi(\varepsilon x)$ and $A_\delta := B_{1/2\delta}$ and repeat the argument above. This completes the proof. □

From now on we denote, for each $\lambda > 0$,

$$c_\lambda := \inf_{u\in X\setminus\{0\}} \max_{t\geq 0} E_\lambda(tu). \tag{2.99}$$

Remark 12. *Obviously, $c_\lambda \geq c_u$, where c_u is defined in* (2.119)*, since $E_\lambda(tu) < 0$ for $u \in X \setminus \{0\}$ and t large by the structure of E_λ. Actually, $c_u = c_\lambda$ (see also Theorem 4.2 in [143]).*

In the following lemma we point out the relationship between $\overline{c}$ and c_λ defined in (2.79) and (2.99), respectively.

Lemma 19. *There exists $\lambda^{**} > 0$ such that for all $\lambda > \lambda^{**}$ it holds*

$$0 < c_\lambda < \overline{c}.$$

Proof. Clearly, from the definition of c_u given in (2.119), then $c_u \geq \zeta$ and, by Remark 12, we have $c_\lambda = c_u > 0$ for all $u \in X \setminus \{0\}$. We emphasize that ζ might depend on λ, but it is always positive. We take the open set Ω_V where V is positive. Let $u_0 \in X \setminus \{0\}$ with support in Ω_V such that $u_0 \geq 0$. Since

$$\begin{aligned} E_\lambda(tu_0) &= \frac{t^p}{p}\|Du_0\|_p^p + \frac{t^q}{q}\|Du_0\|_q^q - \lambda\frac{t^k}{k}\int_{\mathbb{R}^N} Vu_0^k\,dx - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} Ku_0^{p^*}\,dx \\ &\leq \frac{t^p}{p}\|Du_0\|_p^p + \frac{t^q}{q}\|Du_0\|_q^q - \lambda\frac{t^k}{k}\int_{\mathbb{R}^N} Vu_0^k\,dx, \end{aligned}$$

for all $t \geq 0$, we have $E_\lambda(tu_0) \to -\infty$ as $t \to \infty$ and $E_\lambda(tu_0) \to 0^+$ as $t \to 0^+$. Thus, there exists $t_\lambda > 0$ such that

$$\max_{t\geq 0} E_\lambda(tu_0) = E_\lambda(t_\lambda u_0).$$

In particular, we get

$$\begin{aligned} 0 = \frac{d}{dt}\Big[E_\lambda(tu_0)\Big]_{t=t_\lambda} =& t_\lambda^{p-1}\|Du_0\|_p^p + t_\lambda^{q-1}\|Du_0\|_q^q - \lambda t_\lambda^{k-1}\int_{\mathbb{R}^N} Vu_0^k\,dx \\ &- t_\lambda^{p^*-1}\int_{\mathbb{R}^N} Ku_0^{p^*}\,dx, \end{aligned}$$

or, equivalently,

$$\lambda\int_{\mathbb{R}^N} Vu_0^k\,dx = \frac{\|Du_0\|_p^p}{t_\lambda^{k-p}} + \frac{\|Du_0\|_q^q}{t_\lambda^{k-q}} - t_\lambda^{p^*-k}\int_{\mathbb{R}^N} Ku_0^{p^*}dx, \tag{2.100}$$

for every $\lambda > 0$. Since the support of u_0 is contained in Ω_V, the left hand side of (2.100) is positive and it goes to ∞ if $\lambda \to \infty$. Thus, also the right hand side of (2.100) must go to ∞ if $\lambda \to \infty$. Hence, being $q \leq p < k < p^*$, necessarily $t_\lambda \to 0^+$ as $\lambda \to \infty$. From $E_\lambda(t_\lambda u_0) \to 0^+$ as $t_\lambda \to 0^+$ or equivalently when $\lambda \to \infty$, we can conclude that there exists $\lambda^{**} > 0$ such that

$$\max_{t\geq 0} E_\lambda(tu_0) = E_\lambda(t_\lambda u_0) < \frac{S^{N/p}}{N\|K\|_\infty^{(N-p)/p}} = \overline{c}.$$

By the definition of $\overline{c}$, we get $c_\lambda < \overline{c}$ for all $\lambda > \lambda^{**}$. This ends the proof. $\square$

2.3.3 Proof of Theorem 3

Before proving Theorem 3, whose statement is given in the Introduction, we recall a result following from Brezis Lieb Lemma, that is Theorem 1 in [26], combined with the Banach Alaoglu's Theorem (see Remark (iii) of [26]).

Lemma 20. *Let $1 < p < \infty$ and let $(u_n)_n \subset L^p(\mathbb{R}^N)$ be a bounded sequence converging to $u \in L^p(\mathbb{R}^N)$ almost everywhere. Then, $u_n \rightharpoonup u$ (weakly) in $L^p(\mathbb{R}^N)$.*

Note that condition $u \in L^p(\mathbb{R}^N)$ follows by Fatou's Lemma.

We are now ready to prove the existence result, that is Theorem 3, which covers the p-superlinear subcritical case.

Proof of Theorem 3. From Lemma 17 the energy functional E_λ, defined in (11), satisfies the assumptions of Lemma 15, thus there exists a Palais Smale sequence $(u_n)_n \subset X$ for E_λ at level $c_u = c_\lambda$, as pointed out in Remark 12, that is

$$E_\lambda(u_n) \to c_\lambda, \qquad E'_\lambda(u_n)\varphi \to 0 \quad \text{as} \quad n \to \infty \tag{2.101}$$

for all $\varphi \in X$. By Lemma 19, there exists $\lambda^{**} > 0$ such that $0 < c_\lambda < \bar{c}$ for every $\lambda > \lambda^{**}$. Furthermore, according to Lemmas 8 and 18, there exists a nonnegative function $u \in X$ such that

$$u_n \rightharpoonup u \quad \text{in } X, \qquad u_n \to u \quad \text{a.e. in } \mathbb{R}^N, \qquad Du_n \to Du \quad \text{a.e. in } \mathbb{R}^N.$$

Using Lemma 20, we obtain the following convergences

$$\begin{aligned} &|Du_n|^{p-2}Du_n \rightharpoonup |Du|^{p-2}Du \quad \text{in } [L^{p'}(\mathbb{R}^N)]^N, \\ &|Du_n|^{q-2}Du_n \rightharpoonup |Du|^{q-2}Du \quad \text{in } [L^{q'}(\mathbb{R}^N)]^N, \\ &u_n^{p^*-1} \rightharpoonup u^{p^*-1} \quad \text{in } L^{(p^*)'}(\mathbb{R}^N), \qquad u_n^{k-1} \rightharpoonup u^{k-1} \quad \text{in } L^{p^*/(k-1)}(\mathbb{R}^N). \end{aligned} \tag{2.102}$$

Then, from (2.102), for every $\varphi \in X$ we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |Du_n|^{p-2}Du_n D\varphi\, dx \to \int_{\mathbb{R}^N} |Du|^{p-2}DuD\varphi\, dx \\ &\int_{\mathbb{R}^N} |Du_n|^{q-2}Du_n D\varphi\, dx \to \int_{\mathbb{R}^N} |Du|^{q-2}DuD\varphi\, dx \\ &\int_{\mathbb{R}^N} Ku_n^{p^*-1}\varphi\, dx \to \int_{\mathbb{R}^N} Ku^{p^*-1}\varphi\, dx, \ \int_{\mathbb{R}^N} Vu_n^{k-1}\varphi\, dx \to \int_{\mathbb{R}^N} Vu^{k-1}\varphi\, dx. \end{aligned} \tag{2.103}$$

Thus, letting $n \to \infty$ in $(2.101)_2$ and by (2.103), we have $E'_\lambda(u)\varphi = 0$ for all $\varphi \in X$, that is u is a solution of (8). We know that $u \geq 0$ by (2.6) and $u_n \geq 0$. We claim that $u \not\equiv 0$. Recalling the definition of L and Q in (2.26), as in (2.81), we get

$$c_\lambda = \frac{L}{p} + \frac{Q}{q} - \frac{L+Q}{p^*} = \frac{L}{N} + Q\left(\frac{1}{q} - \frac{1}{p^*}\right) \geq \frac{L+Q}{N}$$

that is

$$c_\lambda N \geq L + Q. \tag{2.104}$$

As in Lemma 16, being $(u_n)_n$ a $(\mathrm{PS})_{c_\lambda}$ sequence, necessarily $L + Q > 0$ and we obtain

$$S(L+Q)^{p/p^*} \leq \|K\|_\infty^{p/p^*}(L+Q),$$

that is

$$L + Q \geq \frac{S^{N/p}}{\|K\|_\infty^{(N-p)/p}}. \tag{2.105}$$

Thus, from (2.105) in (2.104), we gain

$$c_\lambda \geq \frac{S^{N/p}}{N\|K\|_\infty^{(N-p)/p}} = \bar{c}$$

which is a contradiction, since $c_\lambda < \bar{c}$. The proof is so concluded. □

2.4 Existence and multiplicity results in a symmetric setting

In the current section we are interested in nontrivial weak solutions of the same nonlinear elliptic problem analyzed in the previous two sections, but in the space of T-symmetric functions, namely

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda V(x)|u|^{k-2}u + K(x)|u|^{p^*-2}u, & \text{in } \mathbb{R}^N \\ u \in X_T. \end{cases} \tag{2.106}$$

The results of this section are contained in [14]. The energy functional associated to problem (2.106) and given in (11), now is defined in X_T, namely $E_\lambda : X_T \to \mathbb{R}$. To find critical points of E_λ, by standard variational methods, we first find levels of E_λ for which the Palais Smale condition holds in X_T. According to the principle of symmetric criticality, see Section 1.1, any critical point of E_λ in X_T is also a T-symmetric solution of (8) in X.

As discussed in the Introduction, the "double" lack of compactness, due to the entire space $\mathbb{R}^N$ and also to the fact that $D^{1,p}(\mathbb{R}^N)$ is not compactly embedded in $L^{p^*}(\mathbb{R}^N)$, can be recovered in a symmetric setting, as discussed by Lions in [97], where the author explains how the invariance of functionals by symmetries fits in the situation of minimization problems and the concentration compactness principle.

In this section, inspired by [25], we restrict our attention to the symmetric setting described at the end of Section 2 and we prove that, if V and K are T-symmetric with $|T| < \infty$, then the Palais Smale condition is valid for E_λ at levels c below a certain threshold with λ sufficiently small. Consequently, using Mountain Pass Theorem of Ambrosetti and Rabinowitz, see [7] and [117], we show an existence result. Besides, if $|T| = \infty$ and additional conditions on the weight K hold, the energy functional E_λ satisfies the Palais Smale condition for all $c \in \mathbb{R}$ so that Fountain Theorem can be applied to obtain the existence of infinitely many solutions for (2.106), as well as T-symmetric solutions of (8), with positive energy.

Note that, the symmetric assumptions on the weights permit to have T-symmetric solutions, for details see the proof of Theorem 1 in [81].

In closing, we point out that Theorem 4 extends previous results contained in Sections 3 and 4 in [25] for Laplacian, while Theorem 5 extends Theorem 1.2 in [80] and Theorem 3 in [79] for the p-Laplacian.

This section is divided into three subsections. In Subsection 2.4.1 we prove some standard results concerning $(\mathrm{PS})_c$ sequences for E_λ but using some helpful properties of the symmetric setting, then in Subsection 2.4.2 we verify the Mountain Pass geometry and, consequently, we prove the existence results, Theorems 4. Finally, the multiplicity results, Theorem 5, is developed in Section 2.4.3.

2.4.1 On the Palais Smale property for positive levels

In this section we will focus on the validity of the $(\mathrm{PS})_c$ condition for the energy functional E_λ, the point where the usefulness of symmetry becomes manifest. Indeed in Section 2.2 we prove such a property only for negative levels, while in the following Lemma we prove it for levels in an left half line with positive end.

Lemma 21. *Let* $1 < k < p$ *and*

$$K(0) > 0, \qquad K(\infty) > 0, \qquad |T| < \infty \tag{2.107}$$

Suppose

$$\lambda < \overline{\lambda}_T^* := \left(\frac{S^{N/p}}{NC_*}\mathfrak{C}\right)^{(p-k)/p} = \frac{pp^* S^{[N(p^*-k)]/pp^*}}{N\|V\|_r} \cdot \frac{\mathfrak{C}^{(p-k)/p}}{p^*-k} \cdot \left(\frac{k}{p-k}\right)^{(p-k)/p}, \tag{2.108}$$

where

$$\mathfrak{C} = \min\left\{\frac{|T|}{\|K\|_\infty^{(N-p)/p}}, \frac{1}{K(0)^{(N-p)/p}}, \frac{1}{K(\infty)^{(N-p)/p}}\right\}. \tag{2.109}$$

Assume that V *and* K *are* T*-symmetric and verifies respectively* (9) *and* (10)*. If*

$$c < \overline{c}_\lambda^T := \frac{S^{N/p}}{N}\mathfrak{C} - \lambda^{p/(p-k)} C_* = C_*((\overline{\lambda}_T^*)^{p/(p-k)} - \lambda^{p/(p-k)}), \tag{2.110}$$

where

$$C_* = \left(\frac{N}{S}\right)^{k/(p-k)} \left(\frac{p^*-k}{pp^*}\|V\|_r\right)^{p/(p-k)} \frac{p-k}{k}. \tag{2.111}$$

Then, the functional E_λ *satisfies* $(PS)_c$ *condition in* X_T *with* c *as in* (2.110)*.*

Proof. Let $(u_n)_n \subset X_T$ be a $(\mathrm{PS})_c$ sequence for E_λ, clearly $(u_n)_n$ is bounded in X by Lemma 8, thus it is bounded also in X_T and, by Banach-Alaoglu's Theorem, there exists $u \in X_T$ such that, up to subsequences, we get

(I) $u_n \rightharpoonup u$ in X_T,

and (II), (III) as in the proof of Lemma 14 which we follow until (2.53).

Since $q \le p$ forces $1/q - 1/p^* \ge 1/N$, we have

$$E_\lambda(u_n) - \frac{1}{p^*}E'_\lambda(u_n)u_n \ge \frac{1}{N}\int_{\mathbb{R}^N}\left(|Du_n|^p + |Du_n|^q\right)dx - \lambda\frac{p^*-k}{kp^*}\int_{\mathbb{R}^N} V|u_n|^k. \tag{2.112}$$

First, we consider points $x_j \neq 0$. In particular, since the functions u_n are T-symmetric, then also ν and μ have to be T-symmetric measures. This means that if $x_j \neq 0$ is a singular point of ν, so is τx_j for each $\tau \in T$ and the mass of ν

concentrated at τx_j is the same for each $\tau \in T$. Thus, letting $n \to \infty$ in (2.112), since $(u_n)_n$ is a $(\mathrm{PS})_c$ sequence by Lemma 8 and using (II), we have

$$\begin{aligned} c + o(1) &\geq \frac{1}{N} \int_{\mathbb{R}^N} |Du|^p dx + \frac{|T|}{N} \int_{\mathbb{R}^N} \sum_{j \in J_1} \mu_j \delta_{x_j} dx - \lambda \frac{p^* - k}{kp^*} \|V\|_r \|u\|_{p^*}^k \\ &\geq \frac{S}{N} \|u\|_{p^*}^p + \frac{|T| S^{N/p}}{N \|K\|_\infty^{(N-p)/p}} - \lambda \frac{p^* - k}{kp^*} \|V\|_r \|u\|_{p^*}^k, \end{aligned}$$

where we have used Sobolev's inequality, (2.50) and (2.52). Let $f(t) = c_1 t^p - \lambda c_2 t^k$, with $c_1 = S/N$ and $c_2 = (p^* - k)\|V\|_r/(kp^*)$. The function f attains its absolute minimum, for $t > 0$, at the point $t_0 = [(\lambda k c_2)/(c_1 p)]^{1/(p-k)}$, so that

$$f(t) \geq f(t_0) = -\lambda^{p/(p-k)} C_*,$$

where C_* given in (2.111). Thus, it holds

$$c + o(1) \geq \frac{|T| S^{N/p}}{N \|K\|_\infty^{(N-p)/p}} - \lambda^{p/(p-k)} C_*, \tag{2.113}$$

which contradicts (2.110). Note that the right hand side of (2.113) is nonnegative since (2.108) is true.

On the other hand, if $x_j = 0$, similarly as before, since $|T_0| = 1$, we get

$$c \geq \frac{S^{N/p}}{N K(0)^{(N-p)/p}} - \lambda^{p/(p-k)} C_*.$$

which again contradicts (2.110), so that $J_1 = \emptyset$, concluding the proof of the claim.

It remains to show that the concentration of ν cannot occur at infinity. It is clear that ν_∞ and μ_∞ defined in (2.16) both exist and are finite. Given $\varepsilon > 0$, we find $R_0 = R_0(\varepsilon) > 0$ such that for every $R \geq R_0$ we have

$$\int_{|x|>R} K |u_n|^{p^*} dx \leq \int_{|x|>R} (K(\infty) + \varepsilon) |u_n|^{p^*} dx. \tag{2.114}$$

Consequently, since $(u_n)_n \subset L^{p^*}(\mathbb{R}^N)$ and ε arbitrarily small, we deduce

$$0 \leq \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x|>R} K |u_n|^{p^*} dx \leq K(\infty) \nu_\infty. \tag{2.115}$$

On the other hand, following the proof of Lemma 14 in Subsection 2.2.2 we get

$$\mu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x|>R} K |u_n|^{p^*} dx. \tag{2.116}$$

In turn,

$$K(\infty)\nu_\infty \geq \mu_\infty. \tag{2.117}$$

Now, following the same idea as before but using $|T_\infty| = 1$, (2.42) in Lemma 14 and (2.117), we obtain

$$c + o(1) \geq \frac{S^{N/p}}{NK(\infty)^{(N-p)/p}} - \lambda^{p/(p-k)} C_*,$$

which contradicts (2.110), thus concentration at infinity cannot occur.

Consequently,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} dx = \int_{\mathbb{R}^N} |u|^{p^*} dx.$$

From (I), $u_n \rightharpoonup u$ in $L^{p^*}(\mathbb{R}^N)$, thus, by the compactness of the embedding,

$$u_n \to u \text{ in } L^s(\omega), \quad \omega \Subset \mathbb{R}^N, \quad 1 \leq s < p^*.$$

Consequently, by using an increasing sequence of compact sets whose union is $\mathbb{R}^N$ and a diagonal argument, we also have (2.6).

From (2.6), then Brezis Lieb Lemma in [26], implies

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u|^{p^*} dx = 0.$$

Thanks to the monotonicity of the following operator

$$< A_p(u), \varphi >= \int_{\mathbb{R}^N} |Du|^{p-2} Du D\varphi dx,$$

for all $u, \varphi \in X_T$, and by virtue of Lemma 9 applied first with $a(x,\xi) = |\xi|^{p-2}\xi$ and then with $a(x,\xi) = |\xi|^{q-2}\xi$, following all steps in Lemma 14, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |D(u_n - u)|^p dx = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |D(u_n - u)|^q dx = 0.$$

In turn, by Sobolev Gagliardo Niremberg's inequality, we obtain the required property, namely E_λ satisfies $(\text{PS})_c$ condition in X_T for every $c < \bar{c}_\lambda^T$. The proof is complete. □

Remark 13. *We point out that the property of T-symmetry of the limit function u of the Palais Smale sequence $(u_n)_n$ can be easily deduced from (2.6) if $(u_n)_n \subset X_T$, cfr. [81] where Banach-Alaoglu's Theorem is applied in the entire X.*

In the following result we study the effect of the validity of assumption (12) for the Palais Smale property for the energy functional E_λ.

Corollary 1. *Let the assumptions of Lemma 21 hold except for* (2.107) *and assume* $1 < k < p^*$. *If* (12) *holds, then the functional* E_λ *satisfies (PS)*$_c$ *condition in* X_T *for every* $c \in \mathbb{R}$.

Proof. We only give a sketch of the proof because it is analogous to that of Lemma 21. Let $(u_n)_n \subset X_T$ be a (PS)$_c$ sequence for E_λ. Now we follow the proof of Lemma 21 word by word up to (2.53). Assuming that $x_j \neq 0$ is a singular point of ν that is $\nu_j = \nu(x_j) > 0$, so is τx_j, for each $\tau \in T$ and, being ν is T-symmetric, we obtain that the mass of ν concentrated at τx_j is the same for each $\tau \in T$, namely

$$\nu(\tau x_j) = \nu_j > 0 \text{ for all } \tau \in T.$$

As $|T| = \infty$, the sum in (iii) of Lemma 11 is infinite, which is a contradiction.

On the other hand, by (2.50) and (12) we have $\mu_0 = 0$. Thus, thanks to $0 \leq S\nu_j^{p/p^*} \leq \mu_j$ for all j, we get $\nu_0 = 0$. The next step of the proof consists in showing that concentration of ν cannot occur at infinity. Proceeding as in Lemma 21, we have (2.114) and (2.115) from which, since $K(\infty) = 0$ is valid by (12), we obtain

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} K|u_n|^{p^*} dx = 0.$$

Thus, from (2.116), we have $\mu_\infty = 0$ and, from (2.19), we obtain $\nu_\infty = 0$. Therefore, the functional E_λ satisfies (PS)$_c$ condition in X_T for every $c \in \mathbb{R}$. This completes the proof. □

Remark 14. *If* (2.107) *and* (12) *fail, then Lemma 21 continues to be valid with* (2.109) *properly modified, precisely for* $|T| < \infty$ *with either*

$$\mathfrak{C} = \min\{|T| \cdot \|K\|_\infty^{(p-N)/p}, \max\{K(0), K(\infty)\}^{(p-N)/p}\} \quad \textit{if} \quad K(0)^2 + K(\infty)^2 > 0$$

or

$$\mathfrak{C} = |T| \cdot \|K\|_\infty^{(p-N)/p} \quad \textit{if} \quad K(0) = K(\infty) = 0;$$

while for $|T| = \infty$ *with*

$$\mathfrak{C} = \max\{K(0), K(\infty)\}^{(p-N)/p} \quad \textit{if} \quad K(0)^2 + K(\infty)^2 > 0.$$

2.4.2 Proof of Theorem 4

Before proving our existence result under a symmetric setting, Theorem 4, whose statement is given in the Introduction, we check that the functional E_λ, defined in (11), satisfies the Mountain Pass geometry.

Lemma 22. *Let* $1 < k < p$ *and* λ^* *defined in* (2.66)*, that is*

$$\lambda^* = \frac{k(p^*-p)}{p(p^*-k)}\left(\frac{p^*(p-k)}{p(p^*-k)}\right)^{(p-k)/(p^*-p)} \cdot \frac{S^{(p^*-k)/(p^*-p)}}{\|V\|_r \cdot \|K\|_\infty^{(p-k)/(p^*-p)}}.$$

Assume either $1 < k < q$ *and* $\lambda < \lambda^*$ *or* $q < k < p$ *and* $\lambda > 0$.

Then, the functional E_λ *verifies the hypotheses of the Mountain Pass Theorem in* X_T.

Proof. Clearly $E_\lambda \in C^1(X_T)$ with $E_\lambda(0) = 0$ so that condition (i) of Theorem 15 is satisfied. For the proof of condition (iii), we refer to Lemma 17 since $k, q, p < p^*$. Following the proof of Lemma 17, we gain

$$E_\lambda(u) \geq c_1\|u\|^p + c_2\|u\|^q - \lambda c_3\|u\|^k - c_4\|u\|^{p^*}, \tag{2.118}$$

where $c_1 = 1/p$, $c_2 = 1/q$, $c_3 = S^{-k/p}\|V\|_r/k$ and $c_4 = S^{-p^*/p}\|K\|_\infty/p^*$ are positive constants. Now we divide the proof in two cases.

Case $q < k < p$. From (2.118), we obtain

$$E_\lambda(u) \geq c_2\|u\|^q - \lambda c_3\|u\|^k - c_4\|u\|^{p^*} = \|u\|^q(c_2 - \lambda c_3\|u\|^{k-q} - c_4\|u\|^{p^*-q}).$$

Let $g(t) = t^q(c_2 - \lambda c_3 t^{k-q} - c_4 t^{p^*-q})$, for every $t \geq 0$. Since $q < k < p^*$ and $c_2 > 0$, there exists $t_1 > 0$ such that $g(t) > 0$ for all $t \in (0, t_1]$. Thus, there exist $\zeta, R > 0$ with R small enough, so that $E_\lambda(u) \geq \zeta > 0$ for every u so that $\|u\| = R$. Thus, condition (ii) of Theorem 15 is satisfied.

Case $1 < k < q$. From (2.118), we obtain

$$E_\lambda(u) \geq c_1\|u\|^p - \lambda c_3\|u\|^k - c_4\|u\|^{p^*} = \|u\|^k(c_1\|u\|^{p-k} - \lambda c_3 - c_4\|u\|^{p^*-k}).$$

Arguing as in Subsection 2.2.3 with the functions h and $\hat{h}$ defined in (2.65), there exists $T_0, T_1 > 0$ such that $h(t) > 0$ for all $t \in (T_0, T_1)$. Therefore, there exist $\zeta, R > 0$ with $R \in (T_0, T_1)$, so that $E_\lambda(u) \geq \zeta > 0$ whenever $\|u\| = R$. Thus, condition (ii) of Theorem 15 is satisfied.

Hence, in both cases we can consider

$$\Gamma_u := \{\gamma \in C^0([0,1], X_T) \,/\, \gamma(0) = 0 \text{ and } \gamma(1) = t_u u\},$$

and

$$c_u^T := \inf_{\gamma\in\Gamma_u} \sup_{t\in[0,1]} E_\lambda(\gamma(t)). \tag{2.119}$$

Then the hypotheses of Theorem 15 are satisfied. This ends the proof. □

From now on we denote, for each $\lambda > 0$,

$$c_\lambda^T := \inf_{u \in X_T \setminus \{0\}} \max_{t \geq 0} E_\lambda(tu). \tag{2.120}$$

The observation in Remark 12 remains to be valid.

Lemma 23. *Let* $\overline{c}_\lambda^T$ *and* c_λ^T *defined in* (2.110) *and* (2.120), *respectively. Suppose* $1 < k < q$. *Then, there exists* $\lambda_T^* > 0$ *such that*

$$0 < c_\lambda^T < \overline{c}_\lambda^T < \overline{c}^T := \frac{S^{N/p}}{N}\mathfrak{C}$$

for all $\lambda < \lambda_T^*$, *with* $\mathfrak{C}$ *given in* (2.109).

Proof. Trivially $\overline{c}_\lambda^T < \overline{c}^T$ being $\lambda > 0$. Clearly, from the definition of c_u^T given in (2.119), then $c_u^T \geq \zeta$ and, by Remark 12, we have $c_\lambda^T = c_u^T > 0$ for all $u \in X_T \setminus \{0\}$. We emphasize that ζ might depend on λ, but it is always positive. We take the open set Ω_V where V is positive. Let $u_0 \in X_T \setminus \{0\}$ with support in Ω_V such that $u_0 \geq 0$. Replacing $u = u_0$ in (2.84) we have $E_\lambda(tu_0) \to -\infty$ as $t \to \infty$.

From (2.84), following the idea in the proof of Lemma 17, for $\lambda < \lambda^*$ given in (2.66) there exists $t_\lambda = T > 0$ such that

$$\max_{t \geq 0} E_\lambda(tu_0) = E_\lambda(t_\lambda u_0).$$

In particular, we get

$$\begin{aligned} 0 = \frac{d}{dt}\Big[E_\lambda(tu_0)\Big]_{t=t_\lambda} =& t_\lambda^{p-1}\|Du_0\|_p^p + t_\lambda^{q-1}\|Du_0\|_q^q \\ & - \lambda t_\lambda^{k-1}\int_{\mathbb{R}^N} V u_0^k \, dx - t_\lambda^{p^*-1}\int_{\mathbb{R}^N} K u_0^{p^*} \, dx, \end{aligned}$$

or, equivalently,

$$\lambda \int_{\mathbb{R}^N} V u_0^k \, dx = t_\lambda^{p-k}\|Du_0\|_p^p + t_\lambda^{q-k}\|Du_0\|_q^q - t_\lambda^{p^*-k}\int_{\mathbb{R}^N} K u_0^{p^*} dx, \tag{2.121}$$

for every $\lambda > 0$. Since the support of u_0 is contained in Ω_V, the left hand side of (2.121) is positive and it goes to 0^+ as $\lambda \to 0^+$. Thus, also the right hand side of (2.121) must go to 0^+ if $\lambda \to 0^+$. Hence, being $1 < k < q \leq p < p^*$, necessarily $t_\lambda \to 0^+$ as $\lambda \to 0^+$. From $E_\lambda(t_\lambda u_0) \to 0^+$ as $t_\lambda \to 0^+$ or equivalently when $\lambda \to 0^+$, we can conclude that there exists $\lambda_T^* > 0$, with $\lambda_T^* < \min\{\overline{\lambda}_T^*, \lambda^*\}$ given in (2.108) and (2.66) such that for all $\lambda < \lambda_T^*$

$$\max_{t \geq 0} E_\lambda(tu_0) = E_\lambda(t_\lambda u_0) < \overline{c}_\lambda^T, \tag{2.122}$$

being $\overline{c}_\lambda^T > 0$ for all $\lambda < \overline{\lambda}_T^*$. Furthermore, by the definition of c_λ^T, we get $c_\lambda^T < \overline{c}_\lambda^T$ for all $\lambda < \lambda_T^*$. This completes the proof. □

Remark 15. *The case $q < k < p$ is not covered in Lemma 23 because of some considerable difficulties. Indeed, for the validity of* (2.122)*, we need in* (2.121) *that $t_\lambda \to 0$. This last condition, when $q < k < p$, occurs only if $\lambda \to \infty$. Unfortunately, condition* (2.108)*, which is necessary to the positivity of $\bar{c}_\lambda^T$ required in* (2.110)*, forces λ to be small.*

Finally, we are ready to prove Theorem 4 by using Mountain Pass Theorem.

Proof of Theorem 4. From Lemma 22 the energy functional E_λ, defined in (11), satisfies the assumptions of Theorem 15, thus there exists a Palais Smale sequence $(u_n)_n \subset X_T$ for E_λ at level c_λ^T, that is

$$E_\lambda(u_n) \to c_\lambda^T, \qquad E'_\lambda(u_n)\varphi \to 0 \quad \text{as} \quad n \to \infty,$$

for all $\varphi \in X_T$. By Lemma 23, there exists $\lambda_T^* > 0$ such that $0 < c_\lambda^T < \bar{c}^T$ for every $\lambda < \lambda_T^*$. Furthermore, according to Lemma 21, there exists a nonnegative function $u \in X_T$ such that $u_n \to u$ in $D_T^{1,p}(\mathbb{R}^N)$ that is u is a critical point of E_λ in X_T. By Remark 4, the function u is a critical point of E_λ in X. In particular, $u \geq 0$ follows from $u_n \geq 0$ and a pointwise convergence in (2.6). Moreover, $u \not\equiv 0$ since $E_\lambda(u) = c_\lambda^T > 0$. The proof of Theorem 4 is so concluded.

□

2.4.3 Proof of Theorem 5

In this Subsection we give the proof of the multiplicity results under a symmetric setting, Theorem 5, whose statement is given in the Introduction. In particular, the proof is based on the use of Fountain Theorem, that is Theorem 16.

Proof of Theorem 5. It is enough to apply Theorem 16 with $\mathcal{G} = \mathbb{Z}/2$ and $M = X_T$. As outlined in Remark 9, assumption **(A1)** is verified. Obviously E_λ is even by definition and $E_\lambda \in C^1(X_T)$. By Corollary 1, the functional E_λ satisfies the $(\text{PS})_c$ condition for every $c \in \mathbb{R}$, so assumption **(A4)** of Theorem 16 holds. Since $0 \not\equiv K \geq 0$ in $\mathbb{R}^N$ and $K \in C(\mathbb{R}^N)$, there exists an open subset Ω_K of $\mathbb{R}^N$ with $K > 0$ in Ω_K. By the T-symmetry of K, then Ω_K is T-symmetric, thus we can define $D_T^{1,p}(\Omega_K) \cap D_T^{1,q}(\Omega_K)$. By extending functions in $D_T^{1,p}(\Omega_K) \cap D_T^{1,q}(\Omega_K)$ by 0 outside Ω_K we can assume $D_T^{1,p}(\Omega_K) \cap D_T^{1,q}(\Omega_K) \subset X_T$. Assume Y_m be an increasing sequence of subspaces of $D_T^{1,p}(\Omega_K) \cap D_T^{1,q}(\Omega_K)$ with $dim(Y_m) = m$, $m \in \mathbb{N}$. Thus, there exists a constant $\varepsilon_m > 0$ such that for all $v \in Y_m$ with $\|v\| = 1$ we have

$$\int_{\mathbb{R}^N} K|v|^{p^*} dx \geq \varepsilon_m. \tag{2.123}$$

On the other hand,

$$E_\lambda(u) \leq \frac{1}{p}\|u\|^p + \frac{1}{q}\|u\|^q - \frac{1}{p^*}\int_{\mathbb{R}^N} K|u|^{p^*} dx. \tag{2.124}$$

Therefore, if $u \in Y_m$, $u \neq 0$, writing $u = \rho_m v$ with $\rho_m = \|u\|$ and $v \in Y_m$ such that $\|v\| = 1$, from (2.123) and (2.124) we get

$$E_\lambda(u) \leq \frac{1}{p}\rho_m^p + \frac{1}{q}\rho_m^q - \frac{\varepsilon_m}{p^*}\rho_m^{p^*} \leq 0$$

for sufficiently large ρ_m, since $q \leq p < p^*$. This proves **(A2)** of Theorem 16.

It remains to prove **(A3)**. To this aim, define

$$\beta_m := \sup_{u \in Z_m, \|u\|=1} \left(\int_{\mathbb{R}^N} K|u|^{p^*} dx \right)^{1/p^*}, \tag{2.125}$$

and

$$\vartheta_m := \sup_{u \in Z_m, \|u\|=1} \left(\int_{\mathbb{R}^N} V|u|^k dx \right)^{1/k},$$

where Z_m is given in (2.21). It is clear that $0 \leq \beta_{m+1} \leq \beta_m$, because holds that $dimZ_{m+1} < dimZ_m$, so that $\beta_m \to \beta_0 \geq 0$. Then, for every $m \geq 1$ there exists a $u_m \in Z_m$ such that $\|u_m\| = 1$ and

$$\left(\int_{\mathbb{R}^N} K|u_m|^{p^*} dx \right)^{1/p^*} \geq \frac{\beta_0}{2}. \tag{2.126}$$

Since X_T is a reflexive and separable Banach space, Banach-Alaoglu's Theorem gives the existence of $u \in X_T$ such that, up to subsequences, $u_m \rightharpoonup u$ in X_T, with $u \in \bigcap_{j=1}^\infty Z_m$. By the Second Intersection Theorem in [77], since $(Z_m)_m$ is a decreasing sequence of bounded, closed non-empty sets whose diameter converges to 0 from its definition, we have that $\bigcap_{j=1}^\infty Z_m$ has exactly one point, that is $u = 0$, in other words $u_m \rightharpoonup 0$ in X_T. Thus, $|u_m|^{p^*} dx \rightharpoonup 0$ and $|Du_m|^p dx \rightharpoonup 0$. Applying Proposition 1, there exists ν, ν_∞, μ, μ_∞ such that (2.17) and (2.18) hold. An argument similar to the one used in proving Corollary 1 shows that the concentration of ν cannot occur at any $x \neq 0$, at 0 and ∞, thus $u_m \to 0$ in $L^{p^*}(\mathbb{R}^N)$. Consequently, from (10) we have

$$\int_{\mathbb{R}^N} K|u_m|^{p^*} dx \to 0, \text{ as } m \to \infty,$$

which implies, thanks to (2.126), that $\beta_0 = 0$ and $\beta_m \to 0$ as $m \to \infty$. From $u_m \rightharpoonup 0$ in X_T, by the weak continuity of the functional defined in $(2.4)_1$, we get $\vartheta_m \to 0$ as $m \to \infty$. Now we divide the proof in two cases:

Case $1 < k < p$. For every $u \in Z_m$, since $\lambda > 0$, by Hölder's inequality with exponents $r = p^*/(p^* - k)$ and $r' = p^*/k$, Sobolev's inequality, we get

$$E_\lambda(u) \geq \frac{1}{p}\|u\|_{D^{1,p}}^p - \frac{\lambda C}{k}\|u\|_{D^{1,p}}^k - \frac{1}{p^*}\int_{\mathbb{R}^N} K|u|^{p^*} dx, \tag{2.127}$$

where $C = \|V\|_r S^{-kp/p^*}$. Thanks to the definition of β_m in (2.125) we can write

$$\int_{\mathbb{R}^N} K|u|^{p^*}dx = \|u\|^{p^*} \int_{\mathbb{R}^N} K\frac{|u|^{p^*}}{\|u\|^{p^*}}dx \le \|u\|^{p^*}\beta_m^{p^*} \tag{2.128}$$

thus, by (2.127) and (2.128), we have

$$E_\lambda(u) \ge \frac{1}{p}\|u\|^p_{D^{1,p}} - \frac{\lambda C}{k}\|u\|^k_{D^{1,p}} - \frac{\beta_m^{p^*}}{p^*}\|u\|^{p^*}. \tag{2.129}$$

Taking $\|u\|_{D^{1,p}}$ sufficiently large, say $\|u\|_{D^{1,p}} \ge R$ for R large, then

$$\frac{\lambda C}{k}\|u\|^k_{D^{1,p}} \le \frac{1}{2p}\|u\|^p_{D^{1,p}}. \tag{2.130}$$

Now, we choose

$$r_m = \left(\frac{p^*}{p2^{p+2}\beta_m^{p^*}}\right)^{1/(p^*-p)},$$

then $r_m \to \infty$ as $m \to \infty$, since $\beta_m \to 0$, as $m \to \infty$. If we take $\|u\| = r_m > 0$ by the definition of $\|\cdot\|$ either $\|u\|_{D^{1,p}}$ or $\|u\|_{D^{1,q}}$ is not less than $r_m/2$. Without loss of generality, let $\|u\|_{D^{1,p}} \ge r_m/2$ and using (2.130) with $R = r_m/2$ in the inequality (2.129) we obtain

$$E_\lambda(u) \ge \frac{1}{p}\|u\|^p_{D^{1,p}} - \frac{1}{2p}\|u\|^p_{D^{1,p}} - \frac{\beta_m^{p^*}}{p^*}\|u\|^{p^*} \ge \frac{1}{2p}\frac{\|u\|^p}{2^p} - \frac{\beta_m^{p^*}}{p^*}\|u\|^{p^*} = \frac{1}{p2^{p+2}}r_m^p \to \infty,$$

as $m \to \infty$.

Case $p < k < p^*$. Let $u \in Z_m$. Using

$$\int_{\mathbb{R}^N} V|u|^k dx = \|u\|^k \int_{\mathbb{R}^N} V\frac{|u|^k}{\|u\|^k}dx \le \|u\|^k\vartheta_m^k.$$

and (2.128), we get

$$E_\lambda(u) \ge \frac{1}{p}\|u\|^p_{D^{1,p}} - \frac{\lambda\vartheta_m^k}{k}\|u\|^k - \frac{\beta_m^{p^*}}{p^*}\|u\|^{p^*}. \tag{2.131}$$

Let

$$r_m = \min\left\{\left(\frac{\varepsilon}{\beta_m^{p^*}}\right)^{1/(p^*-p)}, \left(\frac{\varepsilon}{\vartheta_m^k}\right)^{1/(k-p)}\right\}, \tag{2.132}$$

where $\varepsilon > 0$ is chosen such that

$$\left(\frac{\lambda}{k} + \frac{1}{p^*}\right)\varepsilon < \frac{1}{2^{p+1}p}. \tag{2.133}$$

Since $\beta_m, \vartheta_m \to 0$ as $m \to \infty$, it follows that $r_m \to \infty$ as $m \to \infty$. Take $\|u\| = r_m > 0$, then without loss of generality we can assume that $\|u\|_{D^{1,p}} \geq r_m/2$, thus, from (2.131), we obtain

$$
\begin{aligned}
E_\lambda(u) &\geq r_m^p \left[\frac{1}{2^p p} - \frac{\lambda \vartheta_m^k}{k} r_m^{k-p} - \frac{\beta_m^{p^*}}{p^*} r_m^{p^*-p} \right] \\
&\geq r_m^p \left[\frac{1}{2^p p} - \left(\frac{\lambda}{k} + \frac{1}{p^*} \right) \varepsilon \right] \geq \frac{1}{2^{p+1} p} r_m^p \to \infty, \qquad \text{as} \quad m \to \infty,
\end{aligned}
$$

where in the last two inequalities we have use (2.132) and (2.133), respectively.

Thus, we choose $\rho_m > r_m$ in both cases, so that condition **(A3)** is verified and applying Theorem 16, the energy functional E_λ has unbounded sequence of critical values in X_T. By Remark 4, E_λ has unbounded sequence of critical values in X. The proof of Theorem 5 is so concluded. □

Chapter 3

Singular quasilinear Schrödinger equations with critical exponent in $\mathbb{R}^N$

In this chapter we investigate multiplicity results for nontrivial weak solutions in $D^{1,p}(\mathbb{R}^N)$ of the following singular quasilinear Schrödinger problem involving a critical term given by (13), namely

$$-\Delta_p u - \frac{\alpha}{2}\Delta_p(|u|^\alpha)|u|^{\alpha-2}u = \lambda V(x)|u|^{k-2}u + \beta K(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N$$

when $0 < \alpha < 1$. It is clear that any solution of (13) is also a solution of

$$-\left[1+\frac{\alpha^p}{2}|u|^{p(\alpha-1)}\right]\Delta_p u = \kappa|u|^{p(\alpha-1)-2}u|Du|^p + V(x)|u|^{k-2}u + K(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N,$$

with $\kappa = \alpha^p(\alpha-1)(p-1)/2$. For this reason, as it is emphasized in the Introduction, problem (13) can be seen as a p-Laplacian problem with a diffusion and with a nonlinearity which is a combination between a nonlinearities depending on the gradient and one of type (b).

The case $\alpha = 2$ has been studied extensively recently, we mainly refer to [100, 123, 126] for existence results, while the authors in [140] obtain multiplicity results for problem (13) with $p = 2$ provided that the critical or the subcritical terms in the nonlinearity are small enough. Actually, the critical exponent is 22^* since, for $\alpha > 1$, the critical exponent is exactly $\alpha 2^*$, cfr. [139].

The main difference between the cases $\alpha > 1$ and $0 < \alpha < 1$ is that in the first case the term $\Delta(|u|^\alpha)|u|^{\alpha-2}u$ is degenerate at $u = 0$, while in the latter it becomes singular at $u = 0$. In literature, the most studied case is the degenerate one.

More in detail, the case $0 < \alpha < 1$ is treated in bounded domains in [90] and in $\mathbb{R}^N$ in [138, 139], while the case $1 < \alpha \neq 2$ can be found in $\mathbb{R}^N$ in [99, 1, 3, 144], see also the references therein.

Moving to the p-Laplacian case, to best of our knowledge, only papers dealing with degenerate case, $\alpha > 1$, can be found in literature, we mention [124, 142, 47]. Thus our results are a first attempt, in investigating the singular case.

The main problem which appears in dealing with this type of problems relays on the fact that there is no natural functions space for the associated energy functional to be well defined. Thus an appropriate change of variables allows to study the problem in the "right" functions space. Consequently, in order to manage the new functional, we need to investigate all the properties of the function,singular at 0, involved in the change of variables.

This chapter is divided into three Sections. In Section 3.1 we describe in detail the change of variables which guarantees the well posedness of the energy functional, then the validity of the $(\mathrm{PS})_c$ property is shown in Section 3.2. The truncated functional is described in Section 3.3 together with its properties which are crucial in the proof of Theorem 6, developed in Section 3.4. Finally, Theorems 7 and 8, characterized by the symmetric setting, are proved in Section 3.5.

3.1 Reformulation of the problem

The preliminary results used in the multiplicity theorems of Chapter 3 are most stated in Section 2.1, except the adjustment of the variational setting.

In particular, the Euler Lagrange functional related to problem (13) is defined in (15), namely

$$H_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N} g(u)^p |Du|^p dx - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|u|^k dx - \frac{\beta}{p^*}\int_{\mathbb{R}^N} K|u|^{p^*} dx,$$

where

$$g(t) = \left[1 + \frac{\alpha^p}{2}|t|^{p(\alpha-1)}\right]^{1/p}, \quad t \in \mathbb{R}\setminus\{0\}, \qquad 0<\alpha<1. \tag{3.1}$$

Due to the appearance of the singular term g, since $\lim_{t\to 0} g(t) = \infty$, the functional H_λ may be not well defined in $D^{1,p}(\mathbb{R}^N)$, so we can not apply variational methods to deal directly with (15). For example, from [122], if we consider

$$u_e(x) = |x|^{(p-N)/2p}, \qquad x \in \mathbb{R}^N$$

then $u_e \in D^{1,p}(\mathbb{R}^N)$ but $|Du_e|^{\alpha-1+p} \notin L^1(\mathbb{R}^N)$. To overcome this difficulty, following the idea in [138] and [142], we make a change of variables, as follows

$$v = G(u) = \int_0^u g(z)dz. \tag{3.2}$$

In particular, the function g is an even function in $\mathbb{R} \setminus \{0\}$ and $t = 0$ is a singular point. Moreover, g is decreasing in $\mathbb{R}^+$ and increasing in $\mathbb{R}^-$, moreover it holds

$$\lim_{t\to 0} g(t) = \infty, \qquad \lim_{|t|\to\infty} g(t) = 1. \tag{3.3}$$

For any $\alpha > 0$ and $t \in \mathbb{R}$, we have

$$|G(t)| \leq \int_0^{|t|} \left(1 + \frac{\alpha}{2^{1/p}} y^{\alpha-1}\right) dy = |t| + \frac{1}{2^{1/p}}|t|^\alpha$$

Thus, G is well defined in $\mathbb{R}$. Moreover, G is a strictly increasing function, being $g \geq 1$, and such that $G(0) = 0$ and $\lim_{|t|\to\infty} G(t) = \infty$ and it is a continuous function so that we can define G^{-1}, an invertible, odd and C^1 function .

Thus, after the change of variables described above, the energy functional H_λ can be written by the following functional

$$F_\lambda(v) := \frac{1}{p}\int_{\mathbb{R}^N} |Dv|^p dx - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|G^{-1}(v)|^k dx - \frac{\beta}{p^*}\int_{\mathbb{R}^N} K|G^{-1}(v)|^{p^*} dx, \tag{3.4}$$

for $v \in D^{1,p}(\mathbb{R}^N)$. The proof of the regularity of F_λ takes the following steps, starting by the properties of g and G. Especially, the following properties hold.

Lemma 24. *Let $0 < \alpha < 1$. Then, it holds*

- **a)** $\lim_{s\to 0} \frac{|G^{-1}(s)|^\alpha}{|s|} = 2^{1/p}$;
- **b)** $\lim_{s\to\infty} \frac{|G^{-1}(s)|}{|s|} = 1$;
- **c)** $|G^{-1}(s)| \leq |s|$, *for every* $s \in \mathbb{R}$;
- **d)** $\alpha - 1 < \frac{g'(t)t}{g(t)} \leq 0$, *for every* $t \in \mathbb{R}$;
- **e)** $\alpha|s| < |G^{-1}(s)g(G^{-1}(s))| \leq |s|$, *for every* $s \in \mathbb{R}$;
- **f)** $|G^{-1}(s)| > |G^{-1}(1)s|$, *for every* $|s| \geq 1$;
- **g)** $\frac{\wp-1}{g(t)} \leq \frac{(\wp-1)g(t)-tg'(t)}{g^2(t)} < \wp - \alpha$ *for every* $t \in \mathbb{R}$ *and* $\wp \geq 1$.

Proof. Since G^{-1} is odd, we only consider the case $s \geq 0$. To prove **a)** and **b)** it is enough to use the change variables $s = G(t)$ and the properties of G. Indeed, Hospital's rule gives

$$\lim_{t\to 0^+} \frac{t^\alpha}{G(t)} = \alpha \lim_{t\to 0^+} \frac{t^{\alpha-1}}{g(t)} = \alpha 2^{1/p} \lim_{t\to 0^+} \frac{t^{\alpha-1}}{\alpha t^{\alpha-1}} = 2^{1/p},$$

where we have used also that

$$g(t) \sim \frac{\alpha}{2^{1/p}} t^{\alpha-1}, \qquad \text{as } t \to 0 \tag{3.5}$$

being $0 < \alpha < 1$. While

$$\lim_{t\to\infty} \frac{t}{G(t)} = \lim_{t\to\infty} \frac{1}{g(t)} = 1,$$

by (3.3) and since $0 < \alpha < 1$. Now, by $t = G^{-1}(s)$ we get easily **a)** and **b)**. Condition **c)** follows since $g(t) > 1$ for all $t > 0$, thus $G(t) = \int_0^t g(z)dz \geq t$ for every $t \geq 0$. To prove **d)**, since g is decreasing in $\mathbb{R}^+$ and positive, we have

$$0 \geq \frac{g'(t)t}{g(t)} = (\alpha - 1) \cdot \frac{\alpha^p t^{p(\alpha-1)}}{2 + \alpha^p t^{p(\alpha-1)}} \geq \alpha - 1,$$

being $0 < \alpha < 1$. To get **e)**, multiply **d)** by $g(t) > 0$ and integrate so that

$$(\alpha - 1)G(t) < \int_0^t \big\{[g(z)z]' - g(z)\big\}dz = g(t)t - G(t), \quad t > 0.$$

In turn, inequality **e)** follows taking $s = G(t)$. Now, to obtain **f)**, take

$$\left(\frac{G^{-1}(s)}{s}\right)' = \frac{s(G^{-1}(s))' - G^{-1}(s)}{s^2} = \frac{s - g(G^{-1}(s))G^{-1}(s)}{s^2 g(G^{-1}(s))} > 0,$$

from **e)**. Thus, by the strict monotonicity of the function $G^{-1}(s)/s$ in $\mathbb{R}^+$, we obtain $G^{-1}(s) > G^{-1}(1)s$ for $s \geq 1$, yielding **f)** by virtue of symmetry. Lastly, to prove **g)**, it is enough to multiply **d)** by $-1/g(t)$ and then add $(\wp - 1)/g(t)$ so that

$$\frac{\wp - 1}{g(t)} \leq \frac{(\wp - 1)g(t) - tg'(t)}{g^2(t)} < \frac{\wp - \alpha}{g(t)},$$

yielding **g)** thanks to $\alpha < 1 \leq \wp$ and $g \geq 1$. □

Remark 16. *By Lemma 24-***a)***, we get*

$$G^{-1}(s) \sim 2^{1/\alpha p}|s|^{1/\alpha}, \quad \text{as } s \to 0. \tag{3.6}$$

From (3.5) *and* (3.6), *we obtain*

$$G^{-1}(s)g(G^{-1}(s)) \sim 2^{1/\alpha p}s^{1/\alpha} \cdot \frac{\alpha}{2^{1/\alpha p}}s^{(\alpha-1)/\alpha} = \alpha s \to 0,$$

as $s \to 0^+$. *Thus we can assume* $G^{-1}(0)g(G^{-1}(0)) = 0$ *and also by Hospital's rule*

$$[G^{-1}(s)g(G^{-1}(s))]'_{|s=0} = \lim_{s\to 0^+} \frac{G^{-1}(s)g(G^{-1}(s))}{s} = \alpha.$$

On the other hand, for $s \neq 0$, *we have*

$$[G^{-1}(s)g(G^{-1}(s))]' = 1 + \frac{G^{-1}(s)g'(G^{-1}(s))}{g(G^{-1}(s))} \in (\alpha, 1) \tag{3.7}$$

*thanks to Lemma 24-***d)**. *Moreover, for any* $v \in D^{1,p}(\mathbb{R}^N)$

$$D[G^{-1}(v)g(G^{-1}(v))] = \left[1 + \frac{G^{-1}(v)g'(G^{-1}(v))}{g(G^{-1}(v))}\right]Dv \tag{3.8}$$

so, by using (3.7), *we obtain*

$$\alpha|Dv| \le |D[G^{-1}(v)g(G^{-1}(v))]| \le |Dv|. \tag{3.9}$$

In addition, since $g \ge 1$, *then we have*

$$|DG^{-1}(v)| = \frac{|Dv|}{g(G^{-1}(v))} \le |Dv|. \tag{3.10}$$

Thus, for any $v \in D^{1,p}(\mathbb{R}^N)$ *we have* $G^{-1}(v)g(G^{-1}(v)), G^{-1}(v) \in D^{1,p}(\mathbb{R}^N)$. *Besides, for* $\alpha > 0$ *and* $p > 1$ *using* (3.10), (3.7) *and* (3.8), *we have*

$$\begin{aligned}\alpha|DG^{-1}(v)|^p &\le \alpha|Dv|^p = |Dv|^{p-2}\alpha Dv\cdot Dv\\ &\le |Dv|^{p-2}\left[1 + \frac{G^{-1}(v)g'(G^{-1}(v))}{g(G^{-1}(v))}\right]Dv\cdot Dv \\ &= |Dv|^{p-2}D[G^{-1}(v)g(G^{-1}(v))]\cdot Dv.\end{aligned} \tag{3.11}$$

A key ingredient in our discussion is disclosed in the next crucial lemma.

Lemma 25. *Assume* $v_n \rightharpoonup v$ *in* $D^{1,p}(\mathbb{R}^N)$, *then*

$$G^{-1}(v_n) \rightharpoonup G^{-1}(v), \quad in \quad D^{1,p}(\mathbb{R}^N) \tag{3.12}$$

Proof. Clearly, $(v_n)_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$, so that by (3.10) we have that $G^{-1}(v_n) \in D^{1,p}(\mathbb{R}^N)$ is bounded. Using Eberlain Smulian's Theorem, there exists a subsequence $(G^{-1}(v_{n_k}))_k$ and $w \in D^{1,p}(\mathbb{R}^N)$ such that $G^{-1}(v_{n_k}) \rightharpoonup w$ in $D^{1,p}(\mathbb{R}^N)$ and, by a diagonal argument, there is a subsequence $(G^{-1}(v_{n_{k_j}}))_j$ such that $G^{-1}(v_{n_{k_j}}) \to w$ a.e. in $\mathbb{R}^N$. Using the same diagonal argument on $(v_n)_n$, we get $v_n \to v$ a.e. in $\mathbb{R}^N$. Since $G^{-1} \in C^\infty$, we have $G^{-1}(v_n) \to G^{-1}(v)$ a.e. in $\mathbb{R}^N$, so $G^{-1}(v) = w$ a.e. in $\mathbb{R}^N$. We have so obtained that

($\mathcal{W}$) for every sequence $(v_n)_n$ in $D^{1,p}(\mathbb{R}^N)$ with $v_n \rightharpoonup v$, there is a subsequence $(u_{n_k})_k$ such that $G^{-1}(v_{n_k}) \rightharpoonup G^{-1}(v)$ in $D^{1,p}(\mathbb{R}^N)$.

From this we immediately conclude the validity of (3.12), indeed if this is not true, then there exists $\bar{\phi} \in \left[D^{1,p}(\mathbb{R}^N)\right]'$ such that

$$\lim_{n\to\infty} \left[\bar{\phi}(G^{-1}(v_n)) - \bar{\phi}(G^{-1}(v))\right] \neq 0.$$

In other words, there exists $\varepsilon_0 > 0$ and $(v_{n_k})_k \in D^{1,p}(\mathbb{R}^N)$ such that

$$\left|\bar{\phi}(G^{-1}(v_{n_k})) - \bar{\phi}(G^{-1}(v))\right| \geq \varepsilon_0,$$

but this contradiction concludes the proof of (3.12) since, $(v_{n_k})_k$ is a sequence with $v_{n_k} \rightharpoonup v$ such that it does not satisfy ($\mathcal{W}$). □

In order to prove the regularity of F_λ, we need to analyze the regularity of the following functionals

$$\overline{J}(v) = \int_{\mathbb{R}^N} V|G^{-1}(v)|^k dx \quad \text{and} \quad \overline{H}(v) = \int_{\mathbb{R}^N} K|G^{-1}(v)|^{p^*} dx. \tag{3.13}$$

In particular, the following holds.

Lemma 26. *If $V \in L^r\left(\mathbb{R}^N\right)$, then $\overline{J}(v)$ is weakly continuous on $D^{1,p}(\mathbb{R}^N)$. Moreover, $\overline{J}$ is continuously differentiable and $' : D^{1,p}(\mathbb{R}^N) \to [D^{1,p}(\mathbb{R}^N)]'$ is given by (2.5), for all $\psi \in D^{1,p}(\mathbb{R}^N)$.*

Proof. Our argument is similar to Lemma 2.2 in [140].

For any $v \in D^{1,p}(\mathbb{R}^N)$, by Remark 16, also $G^{-1}(v) \in D^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$, so by Hölder inequality with exponents $r = p^*/(p^* - k)$ and $r' = p^*/k$ we have

$$\|V|G^{-1}(v)|^k\|_1 \leq \|V\|_r \|G^{-1}(v)\|_{p^*}^k \tag{3.14}$$

This implies that $\overline{J}$ is well defined. Let $(v_n)_n \in D^{1,p}(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $D^{1,p}(\mathbb{R}^N)$, thus by Lemma 25, also $G^{-1}(v_n) \rightharpoonup G^{-1}(v)$ in $D^{1,p}(\mathbb{R}^N)$ so that

$G^{-1}(v_n) \rightharpoonup G^{-1}(v)$ in $L^{p^*}(\mathbb{R}^N)$. Then, $(G^{-1}(v_n))_n$ is bounded in $L^{p^*}(\mathbb{R}^N)$ and $(|G^{-1}(v_n)|^k)_n$ is bounded in $L^{p^*/k}(\mathbb{R}^N)$. Furthermore, by the compactness of the embedding,

$$G^{-1}(v_n) \to G^{-1}(v) \text{ in } L^s(\omega), \quad \omega \Subset \mathbb{R}^N, \quad 1 \le s < p^*.$$

Consequently, by using an increasing sequence of compact sets whose union is $\mathbb{R}^N$ and a diagonal argument, we also have

$$G^{-1}(v_n(x)) \to G^{-1}(v(x)) \text{ a.e. } x \in \mathbb{R}^N.$$

In turn, by (3.14) and using Lebesgue dominated convergence Theorem we have

$$\overline{J}(v_n) = \int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k dx \to \int_{\mathbb{R}^N} V|G^{-1}(v)|^k dx = \overline{J}(v),$$

namely, weak continuity holds.

In order to prove $\overline{J} \in C^1$ it is enough to show that $\overline{J}$ has continuous Gâteaux derivative on $D^{1,p}(\mathbb{R}^N)$.

For simplicity let $\mathfrak{G} = G^{-1}$, then consider $v, \psi \in D^{1,p}(\mathbb{R}^N)$ and $0 < |t| < 1$, so that

$$\frac{\overline{J}(u+t\psi) - \overline{J}(u)}{t} = \int_{\mathbb{R}^N} V \frac{|\mathfrak{G}(v+t\psi)|^k - |\mathfrak{G}(v)|^k}{t}\, dx. \tag{3.15}$$

Using the mean value theorem, there exists $\lambda \in (0,1)$ such that

$$\begin{aligned}\frac{\big||\mathfrak{G}(v+t\psi)|^k - |\mathfrak{G}(v)|^k\big|}{|t|} &= k|\mathfrak{G}(v+t\lambda\psi)|^{k-1}|\mathfrak{G}'(v+t\lambda\psi)||\psi| \\ &\le c|v+t\lambda\psi|^{k-1}|\psi| \le c\left(|v|^{k-1}|\psi| + |\psi|^k\right),\end{aligned}$$

with $c > 0$, thanks to Lemma 24-**(c)**, the fact that $0 \le \mathfrak{G}'(s) = \left(G^{-1}\right)'(s) \le 1$ being $g \ge 1$ and, at the end, the elementary inequality $(a+b)^r \le C(a^r + b^r)$, for $r > 0$, $C > 0$ and $a, b > 0$.

By applying Hölder's inequality twice with exponents r, p^*, $p^*/(k-1)$ and r, p^*/k, we get

$$\int_{\mathbb{R}^N} V(x)\left(|v|^{k-1}|\psi| + |\psi|^k\right) dx = \|V\|_r \|\psi\|_{p^*}\left(\|v\|_{p^*}^{k-1} + \|\psi\|_{p^*}^{k-1}\right).$$

The right hand side of the above inequality is in $L^1(\mathbb{R}^N)$. Consequently, by letting $t \to 0$ in (3.15), from the Lebesgue dominated convergence theorem, $\overline{J}$ is Gâteaux differentiable and

$$\overline{J}'(v)\psi = k\int_{\mathbb{R}^N} V(x)|\mathfrak{G}(v)|^{k-2}\mathfrak{G}(v)\mathfrak{G}'(v)\psi dx,$$

that is (2.5). Now it remains to prove continuity of Gâteaux derivative. Assume that $v_n \to v$ in $D^{1,p}(\mathbb{R}^N)$ then $\mathfrak{G}(v_n) \to \mathfrak{G}(v)$ in $D^{1,p}(\mathbb{R}^N)$. By continuity of the embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, we have $\mathfrak{G}(v_n) \to \mathfrak{G}(v)$ in $L^{p^*}(\mathbb{R}^N)$. Let us define $W(v) = kV|\mathfrak{G}(v)|^{k-2}\mathfrak{G}(v)\mathfrak{G}'(v)$. We claim that $W \in L^{(p^*)'}(\mathbb{R}^N)$ arguing similarly as in [85], page 30, where in our case $p_1 = p^*$ and $p_2 = (p^*)'$. From the smoothness of $\mathfrak{G}$ and Lemma 24-**c)**, we have

$$|W(v)| \le k|V\mathfrak{G}(v)^{k-2}\mathfrak{G}(v)| \le cV|v|^{k-1},$$

where $c > 0$. Now, following essentially the proof of Lemma 1 in [13], we can conclude that $\overline{J}'$ is sequentially continuous in $[D^{1,p}(\mathbb{R}^N)]'$. Hence $\overline{J} \in C^1$. □

Similarly, it holds the following.

Lemma 27. *If K satisfies $(14)_2$, then $\overline{H}$ is continuously differentiable in $D^{1,p}(\mathbb{R}^N)$ and its derivative $\overline{H}' : D^{1,p}(\mathbb{R}^N) \to [D^{1,p}(\mathbb{R}^N)]'$ is given by*

$$\overline{H}'(v)\psi = p^* \int_{\mathbb{R}^N} K \frac{|G^{-1}(v)|^{p^*-2}G^{-1}(v)}{g(G^{-1}(v))}\psi dx,$$

for all $v, \psi \in D^{1,p}(\mathbb{R}^N)$.

Using the continuity of the embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, letting $v_n \to v$ in $D^{1,p}(\mathbb{R}^N)$, then by Lemma 25, also $G^{-1}(v_n) \to G^{-1}(v)$ in $D^{1,p}(\mathbb{R}^N)$ and

$$G^{-1}(v_n) \to G^{-1}(v) \quad \text{in } L^{p^*}(\mathbb{R}^N), \qquad DG^{-1}(v_n) \to DG^{-1}(v) \quad \text{in } L^p(\mathbb{R}^N).$$

By Lemmas 6 and 7, then $F_\lambda \in C^1(D^{1,p}(\mathbb{R}^N))$ and $F'_\lambda : D^{1,p}(\mathbb{R}^N) \to (D^{1,p}(\mathbb{R}^N))'$ is given by

$$\begin{aligned} F'_\lambda(v)\psi = \int_{\mathbb{R}^N} |Dv|^{p-2}DvD\psi dx - \lambda \int_{\mathbb{R}^N} V\frac{|G^{-1}(v)|^{k-2}G^{-1}(v)}{g(G^{-1}(v))}\psi dx \\ - \beta \int_{\mathbb{R}^N} K\frac{|G^{-1}(v)|^{p^*-2}G^{-1}(v)}{g(G^{-1}(v))}\psi dx. \end{aligned} \tag{3.16}$$

for all $v, \psi \in D^{1,p}(\mathbb{R}^N)$.

Thus, $v \in D^{1,p}(\mathbb{R}^N)$ is a (weak) *solution* of problem (13) if

$$F'_\lambda(v)\psi = 0 \quad \text{ for all } \psi \in D^{1,p}(\mathbb{R}^N),$$

Clearly, (weak) *solutions* of (13) are exactly critical points of the Euler–Lagrange functional H_λ, or equivalently F_λ, associated with (13). Moreover, every critical point of F_λ correspond to a solution of the following equation

$$-\Delta_p v = \lambda V\frac{|G^{-1}(v)|^{k-2}G^{-1}(v)}{g(G^{-1}(v))} + \beta K\frac{|G^{-1}(v)|^{p^*-2}G^{-1}(v)}{g(G^{-1}(v))}.$$

3.2 On Palais Smale sequences

This section is devoted to the study on properties of $(\mathrm{PS})_c$ sequences for the functional F_λ defined in (3.4). We start from the boundedness of every $(\mathrm{PS})_c$ sequences, as in Lemma 8 in Subsection 2.1.2, in which a useful inequality holds only in the case $1 < k < p$.

Lemma 28. *Assume $1 < k < p^*$. Let (14) be verified and let $(v_n)_n \subset D^{1,p}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for F_λ for all $c \in \mathbb{R}$. Then $(v_n)_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$.*

In particular, if $1 < k < p$ and $c < 0$, it holds

$$\|v_n\|_{p^*} \leq C_* \lambda^{1/(p-k)}, \qquad C_* = \left[\frac{N(p^*-k)}{Skp^*}\|V\|_r\right]^{1/(p-k)}, \tag{3.17}$$

where S is the Sobolev's constant.

Proof. The proof is similar to the one of Lemma 8, but we have to consider the change of variables. Let $(v_n)_n \subset D^{1,p}(\mathbb{R}^N)$ be a $(\mathrm{PS})_c$ sequence of F_λ for all $c \in \mathbb{R}$ that is, using Definition 1,

$$F_\lambda(v_n) = c + o(1), \qquad F'_\lambda(v_n)\psi = o(1)\|\psi\| \quad \text{as} \quad n \to \infty,$$

for every $\psi \in D^{1,p}(\mathbb{R}^N)$. Now take $\psi = G^{-1}(v_n)g(G^{-1}(v_n))$ as a test function, since $\psi \in D^{1,p}(\mathbb{R}^N)$ thanks to Remark 16, and using (3.9), we have

$$\begin{aligned} o(1)\|\psi\| &= F'_\lambda(v_n)(G^{-1}(v_n)g(G^{-1}(v_n))) \\ &\leq \|Dv_n\|_p^p - \lambda \int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k dx - \beta \int_{\mathbb{R}^N} K|G^{-1}(v_n)|^{p^*} dx \end{aligned} \tag{3.18}$$

Now we disjoin the proof in two cases.

Case $1 < k < p$: using (3.18), thanks to Lemma 24-**c)**, Sobolev's and Hölder's inequalities with exponents r and r' we get

$$\begin{aligned} c + o(1) + o(1)\|v_n\| &= F_\lambda(v_n) - \frac{1}{p^*}F'_\lambda(v_n)(G^{-1}(v_n)g(G^{-1}(v_n))) \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*}\right)\|Dv_n\|_p^p - \lambda\left(\frac{1}{k} - \frac{1}{p^*}\right) S^{-k/p}\|V\|_r\|Dv_n\|_p^k \end{aligned} \tag{3.19}$$

where we have used that $V \in L^r(\mathbb{R}^N)$ and $\|v\|_{p^*}S^{1/p} \leq \|Dv\|_p$ for all $v \in D^{1,p}(\mathbb{R}^N)$. Thus, since $k < p < p^*$, following Lemma 4 in [13], we conclude that $\|Dv_n\|_p$ should be bounded.

Case $p \le k < p^*$: arguing as in (3.19), with $1/p^*$ replaced by $1/k$, since $K(x) \ge 0$ in $\mathbb{R}^N$, we obtain

$$
\begin{aligned}
c + o(1) + o(1)\|v_n\| &= F_\lambda(v_n) - \frac{1}{k} F'_\lambda(v_n)(G^{-1}(v_n) g(G^{-1}(v_n))) \\
&\ge \left(\frac{1}{p} - \frac{1}{k}\right) \|Dv_n\|_p^p - \beta \left(\frac{1}{p^*} - \frac{1}{k}\right) \int_{\mathbb{R}^N} K|v_n|^{p^*} dx \ge \left(\frac{1}{p} - \frac{1}{k}\right) \|Dv_n\|_p^p.
\end{aligned}
$$

The conclusion follows from Lemma 4 in [13], as well as the proof of inequality (3.17). □

In what follows we make use of the lemma below.

Lemma 29. *Let g, G be defined respectively in* (3.1), (3.2)*. Then, for any $a, b \in \mathbb{R}$ and $\wp > 2$, it holds*

$$
\begin{aligned}
&\left| \frac{|G^{-1}(a)|^{\wp-2} G^{-1}(a)}{g(G^{-1}(a))} - \frac{|G^{-1}(b)|^{\wp-2} G^{-1}(b)}{g(G^{-1}(b))} \right| |a - b| \\
&\qquad \le (\wp - \alpha) \Big[\big| G^{-1}(a) - G^{-1}(b) \big|^{\wp-1} |a - b| \qquad (3.20) \\
&\qquad\qquad + |G^{-1}(b)|^{\wp-2} \big| G^{-1}(a) - G^{-1}(b) \big| |a - b| \Big].
\end{aligned}
$$

Proof. Applying Lagrange Theorem to the function $|t|^{\wp-2} t / g(t)$ in the interval $[G^{-1}(a), G^{-1}(b)]$, we have

$$
\begin{aligned}
&\left| \frac{|G^{-1}(a)|^{\wp-2} G^{-1}(a)}{g(G^{-1}(a))} - \frac{|G^{-1}(b)|^{\wp-2} G^{-1}(b)}{g(G^{-1}(b))} \right| \\
&\qquad = |w|^{\wp-2} \left| \frac{(\wp - 1) g(w) - w g'(w)}{g^2(w)} \right| \big| G^{-1}(a) - G^{-1}(b) \big|,
\end{aligned} \tag{3.21}
$$

where $w = \vartheta G^{-1}(a) + (1 - \vartheta) G^{-1}(b)$ for a certain $\vartheta \in (0, 1)$. Then, it follows

$$
\begin{aligned}
&|w|^{\wp-2} \cdot \big| G^{-1}(a) - G^{-1}(b) \big| \\
&\qquad \le \left| \vartheta \Big(G^{-1}(a) - G^{-1}(b) \Big) + G^{-1}(b) \right|^{\wp-2} \big| G^{-1}(a) - G^{-1}(b) \big| \qquad (3.22) \\
&\qquad \le \big| G^{-1}(a) - G^{-1}(b) \big|^{\wp-1} + \big| G^{-1}(b) \big|^{\wp-2} \big| G^{-1}(a) - G^{-1}(b) \big|
\end{aligned}
$$

Now, using Lemma 25-**g)** and (3.22) in (3.21), we get (3.20). □

The following lemma guarantees the validity of $(\mathrm{PS})_c$ condition for the functional F_λ. We point out that in the proof no assumptions on the sign of the weight K are needed.

Unfortunately, here it appears the restriction $k > 2$, which forces $p > 2$, due to application of Hölder's inequality which requires $p^*/(k - 2) > 1$.

Lemma 30. *Suppose* (14)*. Let* $2 < k < p$ *and* $c < 0$*. Then*

(i) For any $\lambda > 0$*, there exists* $\hat{\beta}^*_S > 0$ *defined as follows*

$$\hat{\beta}^*_S = \frac{\alpha}{\|K\|_\infty}\left(\frac{kp^*}{\lambda N \|V\|_r (p^* - k)}\right)^{p^2/(N-p)(p-k)} S^{(p^*-k)/(p-k)} \tag{3.23}$$

such that for every $\beta \in (0, \hat{\beta}^*_S]$*, then* F_λ *satisfies (PS)*$_c$ *condition.*

(ii) For any $\beta > 0$*, there exists* $\hat{\lambda}^*_S > 0$ *defined as follows*

$$\hat{\lambda}^*_S = S^{(p^*-k)/(p^*-p)} \frac{kp^*}{N(p^* - k)} \cdot \frac{1}{\|V\|_r} \cdot \left(\frac{\alpha}{\beta \|K\|_\infty}\right)^{(p-k)/(p^*-p)}, \tag{3.24}$$

such that for every $\lambda \in (0, \hat{\lambda}^*_S]$*, then* F_λ *satisfies (PS)*$_c$ *condition.*

Proof. We partially refer to the proof of Lemma 8 in [13], taking into account the change of variables. Let $(v_n)_n$ be a (PS)$_c$ sequence, by Lemma 28, then $(v_n)_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$ and by Banach-Alaoglu's Theorem, there exists $v \in D^{1,p}(\mathbb{R}^N)$ such that, up to subsequences, we get $v_n \rightharpoonup v$ in $D^{1,p}(\mathbb{R}^N)$. By Lemma 25, follows that $G^{-1}(v_n) \rightharpoonup G^{-1}(v)$ in $D^{1,p}(\mathbb{R}^N)$ and thus $(G^{-1}(v_n))_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$. Applying in Proposition 1, there exist μ, ν, ν_∞, μ_∞ bounded nonnegative measures on $\mathbb{R}^N$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{p^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty, \tag{3.25}$$

and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |DG^{-1}(v_n)|^{p} dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty. \tag{3.26}$$

Moreover, there exists at most countable set J, a family $(x_j)_{j\in J}$ of distinct points in $\mathbb{R}^N$ and two families $(\nu_j)_{j\in J}$, $(\mu_j)_{j\in J} \in]0, \infty[$ so that

$$\nu = |G^{-1}(v)|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \nu_j \geq 0, \qquad \mu \geq |DG^{-1}(v)|^p + \sum_{j\in J} \mu_j \delta_{x_j}, \quad \mu_j \geq 0,$$

and

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |G^{-1}(v_n)|^{p^*} dx, \quad \mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |DG^{-1}(v_n)|^{p} dx,$$

satisfying

$$S\nu_j^{p/p^*} \leq \mu_j, \qquad S\nu_\infty^{p/p^*} \leq \mu_\infty. \tag{3.27}$$

Take a standard cut-off function $\psi \in C_c^\infty(\mathbb{R}^N)$, such that $0 \le \psi \le 1$ in $\mathbb{R}^N$, $\psi = 0$ for $|x| > 1$, $\psi = 1$ for $|x| \le 1/2$. For each index $j \in J$ and each $0 < \varepsilon < 1$, define

$$\psi_\varepsilon(x) := \psi\left(\frac{x - x_j}{\varepsilon}\right).$$

Since $F'_\lambda(v_n)\phi \to 0$ for all $\phi \in D^{1,p}(\mathbb{R}^N)$ being $(v_n)_n$ a $(\text{PS})_c$ sequence, choosing $\phi = G^{-1}(v_n)g(G^{-1}(v_n))\psi_\varepsilon$ in (3.16) we have, as $n \to \infty$

$$\begin{aligned} o(1)\|v_n\| &= F'_\lambda(v_n)(G^{-1}(v_n)g(G^{-1}(v_n))\psi_\varepsilon) \\ &= \int_{\mathbb{R}^N} |Dv_n|^{p-2}Dv_n \cdot D[G^{-1}(v_n)g(G^{-1}(v_n))]\psi_\varepsilon dx \\ &\quad + \int_{\mathbb{R}^N} G^{-1}(v_n)g(G^{-1}(v_n))|Dv_n|^{p-2}Dv_n \cdot D\psi_\varepsilon dx \\ &\quad - \lambda\int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k\psi_\varepsilon dx - \beta\int_{\mathbb{R}^N} K|G^{-1}(v_n)|^{p^*}\psi_\varepsilon dx \\ &\ge \alpha\int_{\mathbb{R}^N} |DG^{-1}(v_n)|^p\psi_\varepsilon dx \\ &\quad + \int_{\mathbb{R}^N} G^{-1}(v_n)g(G^{-1}(v_n))|Dv_n|^{p-2}Dv_n \cdot D\psi_\varepsilon dx \\ &\quad - \lambda\int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k\psi_\varepsilon dx - \beta\int_{\mathbb{R}^N} K|G^{-1}(v_n)|^{p^*}\psi_\varepsilon dx. \end{aligned} \tag{3.28}$$

where in the last inequality we have used (3.11).

By the fact that $G^{-1}(v_n) \rightharpoonup G^{-1}(v)$ in $D^{1,p}(\mathbb{R}^N)$, and by the weak continuity of $\overline{J}$, given in (3.13), proved in Lemma 26, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k\psi_\varepsilon dx = \int_{\mathbb{R}^N} V|G^{-1}(v)|^k\psi_\varepsilon dx. \tag{3.29}$$

Consequently, using (3.25), (3.26) and (3.29), we obtain from (3.28)

$$\begin{aligned} &\lim_{n\to\infty}\left(\int_{\mathbb{R}^N} G^{-1}(v_n)g(G^{-1}(v_n))|Dv_n|^{p-2}Dv_n \cdot D\psi_\varepsilon dx\right) \\ &\qquad \le \lambda\int_{\mathbb{R}^N} V|G^{-1}(v)|^k\psi_\varepsilon dx + \beta\int_{\mathbb{R}^N} K\psi_\varepsilon d\nu - \alpha\int_{\mathbb{R}^N}\psi_\varepsilon d\mu. \end{aligned} \tag{3.30}$$

By using Lemma 24-**e)**, we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} G^{-1}(v_n)g(G^{-1}(v_n))|Dv_n|^{p-2}Dv_n\cdot D\psi_\varepsilon dx\right|\\ &\le \int_{\mathbb{R}^N}|Dv_n|^{p-1}|v_n||D\psi_\varepsilon|dx \le \|Dv_n\|_p^{p-1}\left(\int_{\mathbb{R}^N}|v_n|^p|D\psi_\varepsilon|^p dx\right)^{1/p}\\ &= \|v_n\|^{p-1}\left(\int_{B_\varepsilon(x_j)}|v_n|^p|D\psi_\varepsilon|^p dx\right)^{1/p}.\end{aligned} \tag{3.31}$$

where we have used Hölder's inequality. Arguing as above and using the compactness of the immersion $D^{1,p}(\omega) \hookrightarrow\hookrightarrow L^p(\omega)$ for $\omega = \overline{B}_\varepsilon(x_j)$, since $p < p^*$, then, up to subsequences, $v_n \to v$ in $L^p(\omega)$ so that there exists $w_2 \in L^p(\omega)$ such that $|v_n(x)| \le w_2(x)$ a.e. in ω. and $|v_n(x)D\psi_\varepsilon(x)| \le Cw_2(x)$ a.e. in ω, as well as in $\mathbb{R}^N$. In turn, Lebesgue Theorem gives $|v_nD\psi_\varepsilon| \to |vD\psi_\varepsilon|$ in $L^p(\mathbb{R}^N)$. Passing to the limit for $n \to \infty$ in (3.31), using the boundedness of $(v_n)_n$ in $D^{1,p}(\mathbb{R}^N)$, Hölder's inequality with exponents $N/(N-p)$ and N/p, we get

$$\begin{aligned}\lim_{n\to\infty}&\left|\int_{\mathbb{R}^N} G^{-1}(v_n)g(G^{-1}(v_n))|Dv_n|^{p-2}Dv_nD\psi_\varepsilon dx\right|\\ &\le C\left(\int_{B_\varepsilon(x_j)}|v|^p|D\psi_\varepsilon|^p dx\right)^{1/p}\\ &\le C\text{meas}(B_\varepsilon(x_j))^{1/N}\left(\int_{B_\varepsilon(x_j)}|v|^{p^*}dx\right)^{1/p^*},\end{aligned} \tag{3.32}$$

where in the last inequality we have used the properties of $D\psi_\varepsilon$. In turn, by letting $\varepsilon \to 0$ and then $n \to \infty$, being $v \in L^{p^*}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} G^{-1}(v_n)g(G^{-1}(v_n))|Dv_n|^{p-2}Dv_nD\psi_\varepsilon dx \to 0. \tag{3.33}$$

Moreover, using the properties of ψ_ε, and the boundedness of $(|G^{-1}(v_n)|^k)_n$ in $L^{p^*/k}(\mathbb{R}^N)$, as $\varepsilon \to 0$, we have

$$\int_{\mathbb{R}^N} V|G^{-1}(v)|^k\psi_\varepsilon dx \le \int_{B_\varepsilon(x_j)} V|G^{-1}(v)|^k dx \le C\|V\|_{L^r(B_\varepsilon(x_j))} \to 0. \tag{3.34}$$

Hence, from (3.30), if $\varepsilon \to 0$ we deduce

$$\beta K(x_j)\nu_j \ge \alpha\mu_j. \tag{3.35}$$

This latter inequality establishes that concentration of the measure μ cannot occur at points in which $K(x_j) \le 0$. Indeed, in this case, from (3.35), follows that $\mu_j = 0$ and $\nu_j = 0$ from $(3.27)_1$. Thus, the measures ν and μ cannot concentrate when K is nonpositive. Consequently, setting $X_J := \{x_j : j \in J\}$, it does not contain any points x_j in which K is non positive. Define $J_2 := \{j \in J : K(x_j) > 0\}$, we claim that $J_2 = \emptyset$. Combining $(3.27)_1$ and (3.35), we arrive to

$$\nu_j \ge \left(\frac{\alpha S}{\beta K(x_j)}\right)^{N/p} \ge \left(\frac{\alpha S}{\beta \|K\|_\infty}\right)^{N/p}, \qquad j \in J_2. \tag{3.36}$$

To prove the claim, we show that (3.36) cannot occur for λ or β belonging to a suitable interval.

As in [13], assumption (3.36) forces that $|J_2| < \infty$. Now, being $(v_n)_n$ a $(\mathrm{PS})_c$ sequence, choosing again $G^{-1}(v_n)g(G^{-1}(v_n))\psi_\varepsilon$ as a test function in (3.16), then using (3.9), (3.33) and that $0 \le \psi_\varepsilon \le 1$, we have, for $n \to \infty$,

$$\begin{aligned} 0 > c + o(1)\|v_n\| &= F_\lambda(v_n) - \frac{1}{p^*}F'_\lambda(v_n)(G^{-1}(v_n)g(G^{-1}(v_n)))\psi_\varepsilon \\ &\ge \frac{1}{p}\int_{\mathbb{R}^N} |Dv_n|^p dx - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} |Dv_n|^{p-1}\big|D[G^{-1}(v_n)g(G^{-1}(v_n))\big|\psi_\varepsilon dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} G^{-1}(v_n)g(G^{-1}(v_n))|Dv_n|^{p-2}Dv_n \cdot D\psi_\varepsilon dx \\ &\quad + \frac{\lambda}{p^*}\int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k\psi_\varepsilon dx \\ &\ge \frac{1}{N}\int_{\mathbb{R}^N} |Dv_n|^p\psi_\varepsilon dx - \lambda\frac{p^*-k}{kp^*}\int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k\psi_\varepsilon dx \\ &\ge \frac{1}{N}\int_{\mathbb{R}^N} |Dv_n|^p\psi_\varepsilon dx - \lambda\frac{p^*-k}{kp^*}\|V\|_r\|v_n\|_{p^*}^k, \end{aligned} \tag{3.37}$$

where in the last inequality we used Lemma 24-**c)** and Hölder's inequality. Now, thanks to (3.10) and (3.17), from (3.37), we get

$$\begin{aligned} 0 > c + o(1)\|v_n\| &= F_\lambda(v_n) - \frac{1}{p^*}F'_\lambda(v_n)(G^{-1}(v_n)g(G^{-1}(v_n)))\psi_\varepsilon \\ &\ge \frac{1}{N}\int_{B_{\varepsilon/2}(x_j)} |DG^{-1}(v_n)|^p dx - (C_*)^k\,\frac{p^*-k}{kp^*}\|V\|_r\lambda^{p/(p-k)}, \end{aligned}$$

where C_* is given in (3.17), so that, letting $n \to \infty$, $\varepsilon \to 0$ and using (3.27) and (3.36), we arrive to

$$0 > c \ge \frac{1}{N}\mu_j - C\lambda^{p/(p-k)} \ge c_1\big(\beta\|K\|_\infty\big)^{(p-N)/p} - c_2\big(\|V\|_r\lambda\big)^{p/(p-k)},$$

where

$$c_1 = \frac{S^{N/p}}{N}\alpha^{(N-p)/p}, \qquad c_2 = \left(\frac{N}{S}\right)^{k/(p-k)} \left(\frac{p^* - k}{kp^*}\right)^{p/(p-k)}.$$

To obtain the required contradiction we need to have

$$c_1 \geq c_2\big(\beta\|K\|_\infty\big)^{(N-p)/p}\big(\|V\|_r\lambda\big)^{p/(p-k)}. \tag{3.38}$$

Consequently, since $k < p < N$, if we choose any $\beta > 0$, then there exists $\hat{\lambda}_S^*$, defined in (3.24), such that for every $\lambda \in (0, \hat{\lambda}_S^*]$, inequality (3.38) is verified. Similarly, for any $\lambda > 0$ fixed, there exists $\hat{\beta}_S^*$, defined in (3.23), such that for every $\beta \in (0, \hat{\beta}_S^*]$, inequality (3.38) holds. Thus $J_2 = \emptyset$, concluding the proof of the claim.

On the other hand, following the idea of Chabrowski in [34] and Ben-Naoum et. al in [22], also a possible concentration at infinity is refused.

Indeed, take another cut off function $\psi_R \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \psi_R \leq 1$ in $\mathbb{R}^N$, with $\psi_R(x) = 0$ for $|x| < R$ and $\psi_R(x) = 1$ for $|x| > 2R$. Then, from $F'_\lambda(v_n)\phi \to 0$ for all $\phi \in D^{1,p}(\mathbb{R}^N)$ as $n \to \infty$, being $(v_n)_n$ a $(\mathrm{PS})_c$ sequence, choosing $\phi = G^{-1}(v_n)g(G^{-1}(v_n))\psi_R$ in (3.16), we have, as in (3.28), as $n \to \infty$

$$\begin{aligned} o(1)\|v_n\| &= F'_\lambda(v_n)G^{-1}(v_n)g(G^{-1}(v_n))\psi_R \\ &\geq \alpha \int_{\mathbb{R}^N} |DG^{-1}(v_n)|^p \psi_R dx \\ &\quad + \int_{\mathbb{R}^N} G^{-1}(v_n)g(G^{-1}(v_n))|Dv_n|^{p-2}Dv_n \cdot D\psi_R dx \\ &\quad - \lambda \int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k \psi_R dx - \beta \int_{\mathbb{R}^N} K|G^{-1}(v_n)|^{p^*}\psi_R dx. \end{aligned} \tag{3.39}$$

Similarly to the proof of (3.32), we have

$$\begin{aligned} \limsup_{n\to\infty} &\left| \int_{\mathbb{R}^N} G^{-1}(v_n)g(G^{-1}(v_n))|Dv_n|^{p-2}Dv_n \cdot D\psi_R dx \right| \\ &\leq C\left(\int_{R<|x|<2R} |v|^{p^*} dx\right)^{1/p^*} \to 0, \quad R \to \infty, \end{aligned}$$

being $v \in L^{p^*}(\mathbb{R}^N)$. So that from (3.39) we obtain

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left\{\lambda\int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k\psi_R dx + \beta\int_{\mathbb{R}^N} K|G^{-1}(v_n)|^{p^*}\psi_R dx\right\} \geq \alpha\mu_\infty. \tag{3.40}$$

Furthermore, as for (3.34), we have

$$\begin{aligned} \lim_{R\to\infty}\limsup_{n\to\infty} &\int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k\psi_R dx \\ &\leq \lim_{R\to\infty}\limsup_{n\to\infty} \|V\|_{L^r(|x|>R)}\|G^{-1}(v_n)\|^k_{L^{p^*}(|x|>R)} = 0, \end{aligned} \tag{3.41}$$

being $(|G^{-1}(v_n)|^k)_n$ bounded in $L^{p^*/k}(\mathbb{R}^N)$. Thus, by $(14)_2$ and the definition of ν_∞, we gain

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left\{\int_{\mathbb{R}^N} K|G^{-1}(v_n)|^{p^*}\psi_R dx\right\}\le \|K\|_\infty\nu_\infty. \tag{3.42}$$

Using (3.41) and (3.42) in (3.40), thanks to (2.19), we have so obtained

$$\frac{\beta\|K\|_\infty\nu_\infty}{\alpha}\ge\mu_\infty\ge S\nu_\infty^{p/p^*}\quad\text{that is}\quad \nu_\infty\ge\left(\frac{\alpha S}{\beta\|K\|_\infty}\right)^{N/p}.$$

Reasoning as above, either for β fixed and $\lambda\in(0,\hat{\lambda}_S^*]$ or taking $\lambda>0$ and $\beta\in(0,\hat{\beta}_S^*]$ the concentration at infinity cannot occur, that is $\nu_\infty=\mu_\infty=0$. Consequently (3.25) gives

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|G^{-1}(v_n)|^{p^*}dx=\int_{\mathbb{R}^N}|G^{-1}(v)|^{p^*}dx.$$

Furthermore, since $G^{-1}(v_n)(x)\to G^{-1}(v)(x)$ a.e. in $\mathbb{R}^N$ from (2.6), then Brezis Lieb Lemma in [26], implies

$$\lim_{n\to\infty}\|G^{-1}(v_n)-G^{-1}(v)\|_{p^*}=0. \tag{3.43}$$

Using (3.16) with v_n-v as test function we have,

$$\begin{aligned}
&o(1)\|v_n-v_m\|=[F'_\lambda(v_n)-F'_\lambda(v)](v_n-v)\\
&=\int_{\mathbb{R}^N}\left[|Dv_n|^{p-2}Dv_n-|Dv|^{p-2}Dv\right]D(v_n-v)dx\\
&\quad-\lambda\int_{\mathbb{R}^N}\left[\frac{|G^{-1}(v_n)|^{k-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}-\frac{|G^{-1}(v)|^{k-2}G^{-1}(v)}{g(G^{-1}(v))}\right]V(v_n-v)dx\\
&\quad-\beta\int_{\mathbb{R}^N}\left[\frac{|G^{-1}(v_n)|^{p^*-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}-\frac{|G^{-1}(v)|^{p^*-2}G^{-1}(v)}{g(G^{-1}(v))}\right]K(v_n-v)dx
\end{aligned} \tag{3.44}$$

Applying Lemma 29 with $a=v_n$, $b=v$, $\wp=p^*$ and Hölder's inequality we have, if $n\to\infty$

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left|\frac{|G^{-1}(v_n)|^{p^*-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}-\frac{|G^{-1}(v)|^{p^*-2}G^{-1}(v)}{g(G^{-1}(v))}\right||K||v_n-v|dx\\
&\qquad\le\|K\|_\infty(p^*-\alpha)\int_{\mathbb{R}^N}\bigg[\left|G^{-1}(v_n)-G^{-1}(v)\right|^{p^*-1}|v_n-v|\\
&\qquad\quad+|G^{-1}(v)|^{p^*-2}\left|G^{-1}(v_n)-G^{-1}(v)\right||v_n-v|\bigg]\,dx\\
&\qquad\le\|K\|_\infty(p^*-\alpha)\bigg[\|G^{-1}(v_n)-G^{-1}(v)\|_{p^*}^{(p^*-1)/p^*}\|v_n-v\|_{p^*}\\
&\qquad\quad+\|G^{-1}(v)\|_{p^*}^{(p^*-2)/p^*}\|G^{-1}(v_n)-G^{-1}(v)\|_{p^*}\|(v_n-v)\|_{p^*}\bigg]\to0,
\end{aligned} \tag{3.45}$$

by (3.43). Moreover, using again Lemma 29 with $a = v_n$, $b = v$, $\wp = k > 2$ and Hölder's inequality we have taking $n \to \infty$

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left| \frac{|G^{-1}(v_n)|^{k-2} G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{|G^{-1}(v)|^{k-2} G^{-1}(v)}{g(G^{-1}(v))} \right| |V||v_n - v| dx \\
&\leq (k - \alpha) \int_{\mathbb{R}^N} \Big[|V| \big| G^{-1}(v_n) - G^{-1}(v) \big|^{k-1} |v_n - v| \\
&\qquad + |V||G^{-1}(v)|^{k-2} \big| G^{-1}(v_n) - G^{-1}(v) \big| |v_n - v| \Big] dx \qquad (3.46) \\
&\leq (k - \alpha) \Big[\|V\|_r \|G^{-1}(v_n) - G^{-1}(v)\|_{p^*}^{(k-1)/p^*} \|v_n - v\|_{p^*} \\
&\qquad + \|V\|_r \|G^{-1}(v)\|_{p^*}^{(k-2)/p^*} \|G^{-1}(v_n) - G^{-1}(v)\|_{p^*} \|(v_n - v)\|_{p^*} \Big] \to 0,
\end{aligned}$$

So that, from (3.45) and (3.46) in (3.44), we get

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \big[|Dv_n|^{p-2} Dv_n - |Dv|^{p-2} Dv \big] \cdot (Dv_n - Dv) dx = 0,$$

using the standard inequality $|\xi - \zeta|^p \leq c\big(\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\big) \cdot \big(\xi - \zeta\big)$ for $\xi, \zeta \in \mathbb{R}^N$ and $p \geq 2$, we arrive to

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |D(v_n - v)|^p dx = 0,$$

that is the strong convergence in $L^p(\mathbb{R}^N)$ of the sequence $(Dv_n)_n$. Finally, by Sobolev Gagliardo Niremberg's inequality, we obtain the required property, namely F_λ satisfies $(\mathrm{PS})_c$ condition for every $c < 0$. The proof is now complete. □

In the proofs of Theorems 7 and 8 to get the Palais Smale property for the energy functional F_λ we make use of the following corollary in which assumption (12) has a crucial role, as in Corollary 1 in Chapter 2.

Corollary 2. *Let $1 < p < N$ and $1 < k < p^*$. If* (12) *holds, then the functional F_λ satisfies (PS)$_c$ condition in $D_T^{1,p}(\mathbb{R}^N)$ for every $c \in \mathbb{R}$.*

We do not enclose the proof since it is essentially the same as for Corollary 1 in [14], taking into account the change of variables $v = G(u)$.

3.3 The truncated functional F_∞

In this section properties of the truncated functional F_∞ of F_λ, for all $1 < k < p$, are discusses, such as its boundedness from below, differently than F_λ.

Taking Lemma 24-**c)**, Hölder's and Sobolev's inequalities we have, by (3.4), for all $v \in D^{1,p}(\mathbb{R}^N)$,

$$F_\lambda(v) \geq \frac{1}{p}\|v\|^p_{D^{1,p}} - \lambda c_1\|v\|^k_{D^{1,p}} - \beta c_2\|v\|^{p^*}_{D^{1,p}}.$$

where $c_1 = S^{-k/p}\|V\|_r/k$ and $c_2 = S^{-p^*/p}\|K\|_\infty/p^*$ are positive constants.

Define $h(t) = t^p/p - \lambda c_1 t^k - \beta c_2 t^{p^*}$ in $\mathbb{R}^+_0$. Following Subsection 2.2.3, write $h(t) = t^k\hat{h}(t)$, where $\hat{h}(t) := -\lambda c_1 + \frac{1}{p}t^{p-k} - c_2\beta t^{p^*-k}$. Since $\hat{h}(0) < 0$, $\hat{h}(t) \to -\infty$ as $t \to \infty$ and $\hat{h}'(t) > 0$ for $t > 0$ close to 0, then, there exists $T > 0$ such that

$$\hat{h}'(T) = 0, \qquad T = \left[\frac{p-k}{\beta c_2 p(p^* - k)}\right]^{1/(p^*-p)}.$$

$$\hat{h}(T) = (p^* - p)\left(\frac{p-k}{\beta c_2}\right)^{(p-k)/(p^*-p)}\left(\frac{1}{p(p^*-k)}\right)^{(p^*-k)/(p^*-p)} - \lambda c_1,$$

For β fixed, if $\lambda < \overline{\lambda}^*_S$, where $\overline{\lambda}^*_S$ is defined as follows

$$\overline{\lambda}^*_S = \frac{\overline{C}}{\|V\|_r \cdot (\beta\|K\|_\infty)^{(p-k)/(p^*-p)}}, \tag{3.47}$$

or for λ fixed, if $\beta < \overline{\beta}^*_S$, with $\overline{\beta}^*_S$ defined below

$$\overline{\beta}^*_S = \frac{1}{\|K\|_\infty} \cdot \left(\frac{\overline{C}}{\|V\|_r\lambda}\right)^{(p^*-p)/(p-k)}, \tag{3.48}$$

where

$$\overline{C} = S^{(p^*-k)/(p^*-p)}\frac{k(p^*-p)}{p(p^*-k)}\left(\frac{p^*(p-k)}{p(p^*-k)}\right)^{(p-k)/(p^*-p)}$$

then we get $\hat{h}(T) > 0$ and so $h(T) > 0$. Thus, there occur T_0 and T_1, with $0 < T_0 < T < T_1$ such that $h(T_0) = h(T_1) = 0$ and

$$h(t) > 0 \text{ in } (T_0, T_1), \qquad h(t) \leq 0, \;\text{ in } [0, T_0] \cup [T_1, \infty),$$

see cfr. Figure 2.1.

Next, take a cutoff function $\tau \in C^\infty(\mathbb{R}^+_0)$, nonincreasing and such that

$$\tau(t) = 1 \text{ if } 0 \leq t \leq T_0 \quad \text{and} \quad \tau(t) = 0 \text{ if } t \geq T_1.$$

and define the truncated functional

$$F_\infty(v) = \frac{1}{p}\|Dv\|_p^p - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|G^{-1}(v)|^k dx - \beta\frac{\tau\left(\|v\|_{D^{1,p}}\right)}{p^*}\int_{\mathbb{R}^N} K|G^{-1}(v)|^{p^*} dx. \quad (3.49)$$

Now take

$$\overline{h}(t) = \frac{1}{p}t^p - \lambda c_1 t^k - \beta c_2 t^{p^*}\tau(t), \quad t \in \mathbb{R}_0^+,$$

following [13], it holds $\overline{h}(t) \to \infty$ as $t \to \infty$ and $\overline{h}(t) \geq h(t)$ for all $t \geq 0$ so that

$$\overline{h}(t) = h(t) \text{ in } (0, T_0),\ \ \overline{h}(T_0) = h(T_0) = 0,\ \ \overline{h}(t) \geq h(t) > 0 \text{ in } (T_0, T_1),\ \ \overline{h}(T_1) > 0,$$

moreover, $\overline{h}(t) > 0$ in (T_1, ∞) since, for $t \geq T_1$, we have $\overline{h}(t) = t^k\kappa(t)$ with $\kappa(t) = \frac{1}{p}t^{p-k} - \lambda c_1$ which is a strictly increasing and positive function in (T_1, ∞), cfr. Figure 2.2.

Furthermore, we have $F_\infty(v) \geq \overline{h}\left(\|v\|_{D^{1,p}}\right)$ for all $v \in D^{1,p}(\mathbb{R}^N)$ and

$$F_\lambda(v) = F_\infty(v) \quad \text{if} \quad 0 \leq \|v\|_{D^{1,p}} \leq T_0. \quad (3.50)$$

Furthermore, by the regularity both of τ and of F_λ, we have $F_\infty \in C^1(D^{1,p}(\mathbb{R}^N), \mathbb{R})$.

Before proving the following lemma which states the validity of the $(\text{PS})_c$ condition for the truncated energy functional F_∞, define $\lambda_S^* := \min\{\hat{\lambda}_S^*, \overline{\lambda}_S^*\}$ and $\beta_S^* := \min\{\hat{\beta}_S^*, \overline{\beta}_S^*\}$, where $\hat{\lambda}_S^*, \hat{\beta}_S^*, \overline{\lambda}_S^*, \overline{\beta}_S^*$ are defined respectively in (3.24), (3.23), (3.47), (3.48).

Lemma 31. *Let F_∞ be the truncated functional of F_λ.*

(a) If $F_\infty(v) < 0$, then $\|v\|_{D^{1,p}} < T_0$ and $F_\lambda(u) = F_\infty(u)$ for all u in a small enough neighborhood of v.

(b) For all $\lambda > 0$, there exists $\beta_S^* > 0$ such that if $\beta \in (0, \beta_S^*)$, then F_∞ satisfies the $(PS)_c$ condition for $c < 0$.

(c) For all $\beta > 0$, there exists $\lambda_S^* > 0$ such that if $\lambda \in (0, \lambda_S^*)$, then F_∞ satisfies the $(PS)_c$ condition for $c < 0$.

We do not include the proof of the lemma above. It is actually enough to follow proof of Lemma 9 in [13], taking into account the change of variables $v = G(u)$.

Remark 17. *The existence of a positive maximum for the function h, yielding the presence of T_0, T_1 zeros for h used to define the cutoff function τ, is essentially guaranteed by the fact that λ or β is small enough because of the validity of Lemma 30. Obviously, as we will see in Section 3.5, the truncated energy functional can be defined even if h is nonpositive.*

3.4 Proof of Theorem 6

This Section is dedicated to the proof of the multiplicity result for negative energy, that is Theorem 6, whose statement is given in the Introduction, which is based on the theory of genus by Krasnosel'skii contained in Subsection 2.1.4.

Proof of Theorem 6. Define $K_c = K_{c,F_\infty} = \{u \in X : F_\infty(u) = c,\ F'_\infty(u) = 0\}$ and take $m \in \mathbb{N}^+$. For $1 \le j \le m$ let

$$c_j = \inf_{A\in\Sigma_j} \sup_{u\in A} F_\infty(u)$$

with

$$\Sigma_j = \{A \subset X\setminus\{0\} : A \text{ is closed in } X,\ -A = A,\ \gamma(A) \ge j\}.$$

As in Subsection 2.2.4, our claim consists in proving that $-\infty < c_j < 0$ for all $j \ge 1$, to do that it is enough to prove that for all $j \in \mathbb{N}$, there exists an $\varepsilon_j = \varepsilon(j) > 0$ such that

$$\gamma(F_\infty^{-\varepsilon_j}) \ge j, \text{ where } F_\infty^a = \{u \in X : F_\infty(u) \le a\} \quad \text{with} \quad a \in \mathbb{R}. \tag{3.51}$$

Let $\Omega_V \subset \mathbb{R}^N$, $|\Omega_V| > 0$, be a bounded open set where $V > 0$. Extending functions u in $D_0^{1,p}(\Omega_V)$ by 0 outside Ω_V, where $D_0^{1,p}(\Omega_V)$ is the closure of $C_0^\infty(\Omega_V)$ in the norm $\|u\|_{D_0^{1,p}(\Omega_V)} = \|Du\|_{L^p(\Omega_V)}$. Take W_j a j-dimensional subspace of $D_0^{1,p}(\Omega_V)$, thus all the norms in W_j are equivalent. For every $v \in W_j$ with $v \ne 0$, we write $v = r_j w$ with $w \in W_j$ and $\|w\|_{D_0^{1,p}(\Omega_V)} = 1$, from the assumptions on V, there exists a $d_j > 0$ such that

$$\int_{\Omega_V} V|w|^{k/\alpha}dx \ge d_j.$$

By Lemma 24-**a)**, for $\varepsilon > 0$ sufficiently small there exists $\sigma = \sigma(\varepsilon) > 0$ such that for every $|t| \le \sigma$, then

$$|G^{-1}(t)| \ge \frac{1}{2^{1/k}}|t|^{1/\alpha}. \tag{3.52}$$

On the other hand, for $r_j \in (0, T_0)$, so that $\|v\| < T_0$, by (3.50) and (3.52) we arrive to

$$\begin{aligned}
F_\infty(v) = F_\lambda(v) &= \frac{1}{p}\int_{\Omega_V} |Dv|^p dx - \frac{\lambda}{k}\int_{\Omega_V} V|G^{-1}(v)|^k dx - \frac{\beta}{p^*}\int_{\Omega_V} K|G^{-1}(v)|^{p^*} dx \\
&= \frac{1}{p} r_j^p - \frac{\lambda}{k}\int_{\Omega_V} V\left(\frac{1}{2}|v|^{k/\alpha} + |G^{-1}(v)|^k - \frac{1}{2}|v|^{k/\alpha}\right) dx - \frac{\beta}{p^*}\int_{\Omega_V} K|G^{-1}(v)|^{p^*} dx \\
&\le \frac{1}{p} r_j^p - \frac{\lambda}{2k} d_j r_j^{k/\alpha} - \frac{\lambda}{k}\int_{\{|v|<\sigma\}\cap\Omega_V} V\left(|G^{-1}(v)|^k - \frac{1}{2}|v|^{k/\alpha}\right) dx \\
&\quad - \frac{\lambda}{k}\int_{\{|v|\ge\sigma\}\cap\Omega_V} V\left(|G^{-1}(v)|^k - \frac{1}{2}|v|^{k/\alpha}\right) dx - \frac{\beta}{p^*}\int_{\Omega_V} K|G^{-1}(v)|^{p^*} dx \\
&\le \frac{1}{p} r_j^p - \frac{\lambda}{2k} d_j r_j^{k/\alpha} - \frac{\lambda}{k}\int_{\{|v|\ge\sigma\}\cap\Omega_V} V|G^{-1}(v)|^k dx \\
&\quad + \frac{\lambda}{2k}\int_{\{|v|\ge\sigma\}\cap\Omega_V} V|v|^{k/\alpha} - \frac{\beta}{p^*}\int_{\Omega_V} K|G^{-1}(v)|^{p^*} dx \\
&\le \frac{1}{p} r_j^p - \frac{\lambda}{2k} d_j r_j^{k/\alpha} + \frac{\lambda}{2k}\int_{\{|v|\ge\sigma\}\cap\Omega_V} V|v|^{k/\alpha - p^* + p^*} + \frac{\beta}{p^*}\|K\|_\infty \int_{\Omega_V} |v|^{p^*} dx \\
&\le \frac{1}{p} r_j^p - \frac{\lambda}{2k} d_j r_j^{k/\alpha} + \frac{\lambda}{2k}\sigma^{k/\alpha - p^*}\int_{\Omega_V} V|v|^{p^*} + \frac{\beta}{p^*}\|K\|_\infty \int_{\Omega_V} |v|^{p^*} dx \\
&\le \frac{1}{p} r_j^p - \frac{\lambda}{2k} d_j r_j^{k/\alpha} + \left(\frac{\lambda}{2k}\sigma^{k/\alpha - p^*}\|V\|_{L^\infty(\Omega_V)} + \frac{\beta}{p^*}\|K\|_\infty\right)\int_{\Omega_V} |v|^{p^*} dx \\
&\le r_j^{k/\alpha}\left[\frac{1}{p} r_j^{p-k/\alpha} - \frac{\lambda}{2k} d_j + C r_j^{p^* - k/\alpha}\right],
\end{aligned}$$

with

$$C = S^{-1}\left(\frac{\lambda}{2k}\sigma^{k/\alpha - p^*}\|V\|_{L^\infty(\Omega_V)} + \frac{\beta}{p^*}\|K\|_\infty\right),$$

and by virtue of Lemma 24-**c)**, the fact that $V \in C(\mathbb{R}^N)$, $\alpha > k/p^*$ and by $S\|v\|_{p^*} \le \|v\| = r_j$.

Consequently, for every $v \in W_j$, $v \neq 0$, we can choose $r_j \in (0, T_0)$ sufficiently small so that, since $\alpha > k/p > k/p^*$, we obtain

$$F_\infty(v) \le -\varepsilon_j < 0.$$

Letting $S_{r_j} = \left\{v \in D^{1,p}(\mathbb{R}^N) : \|v\|_{D_0^{1,p}(\Omega_V)} = r_j\right\}$, then $S_{r_j} \cap W_j \subset F_\infty^{-\varepsilon_j}$. By Proposition 2,

$$\gamma(F_\infty^{-\varepsilon_j}) \ge \gamma(S_{r_j} \cap W_j) = \gamma(S^{j-1}) = j,$$

which proves claim (3.51). Thus, from $F_\infty^{-\varepsilon_j} \in \Sigma_j$, we obtain

$$c_j \le \sup_{u \in F_\infty^{-\varepsilon_j}} F_\infty(v) \le -\varepsilon_j < 0.$$

Furthermore, since F_∞ is bounded from below, we get $c_j > -\infty$, thus the proof of claim (3.51) is concluded.

The last part of the proof follows as in [13].

□

3.5 Proofs of Theorems 7 and 8

In this section we restrict our attention on the symmetric setting and we prove Theorems 7 and 8, whose statements are given in the Introduction, for solutions with negative energy and positive energy, respectively. In particular, the symmetric setting allows us to improve Theorem 6 obtaining the corresponding multiplicity result but with $1 < k < p < N$ and for all λ, β positive, as it evident in the statement of Theorem 7.

Being in a symmetric setting, Remark 13 is valid in $D_T^{1,p}(\mathbb{R}^N)$ and we need to apply the principle of symmetric criticality due to Palais, described in Subsection 2.1.1, which states that v is critical point of F_λ, if it is a critical point of the same functional restricted on $D_T^{1,p}(\mathbb{R}^N)$. For further details we refer to [14].

We now come to the proof of the multiplicity results with negative energy in a symmetric setting, that is Theorem 7.

Proof of Theorem 7. It is enough to apply the same methods used in the proof of Theorem 6, but now assumption (12) allows us to remove the bounds from above for λ and β and the lower bound 2 for k and p.

First observe that the $(\mathrm{PS})_c$ condition for the functional F_λ follows, by virtue of (12), if $1 < k < p < N$, for all λ, $\beta > 0$ and $c \in \mathbb{R}$ from Corollary 2, instead of Lemma 30.

Concerning the construction of the truncated energy functional F_∞ in (3.49) according to the definition in (3.49), we start observing that now the function $h(t) = t^p/p - \lambda c_1 t^k - \beta c_2 t^{p^*}$ introduced in Section 3.3, which is negative and strictly decreasing in a right neighborhood of 0, could remain always negative. Indeed, differently from Lemma 30, λ and β can be as large as we want.

Thus, in order to connect h for small t, with $t^p/p - \lambda c_1 t^k$ for large t, or equivalently to build the truncated function $\bar{h}$, defined in Section 2.2.3, we need to consider two points $0 < P_0 < P_1$, not necessarily zeros of h, in the definition of the cutoff function. Precisely let $\tau_1 \in C^\infty(\mathbb{R}_0^+)$, nonincreasing and such that

$$\tau_1(t) = 1 \text{ if } 0 \le t \le P_0 \quad \text{and} \quad \tau_1(t) = 0 \text{ if } t \ge P_1.$$

For the case P_0, P_1 zeros of h we refer to Figures 2.1 and 2.2 with $T_0 = P_0$ and $T_1 = P_1$, while for the case $h(P_0)$, $h(P_1) < 0$, possible behaviours of h and $\bar{h}$ are contained in Figure 3.1 and 3.2, respectively.

In light of this, conditions **(b)**-**(c)** in Lemma 31 become the same and assure that

$$F_\infty(v) = \frac{1}{p}\|Dv\|_p^p - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|G^{-1}(v)|^k dx - \beta\frac{\tau_1\left(\|u\|_{D^{1,p}}\right)}{p^*}\int_{\mathbb{R}^N} K|G^{-1}(v)|^{p^*} dx.$$

satisfies the $(\mathrm{PS})_c$ condition for any $\lambda, \beta > 0$ and $c < 0$.

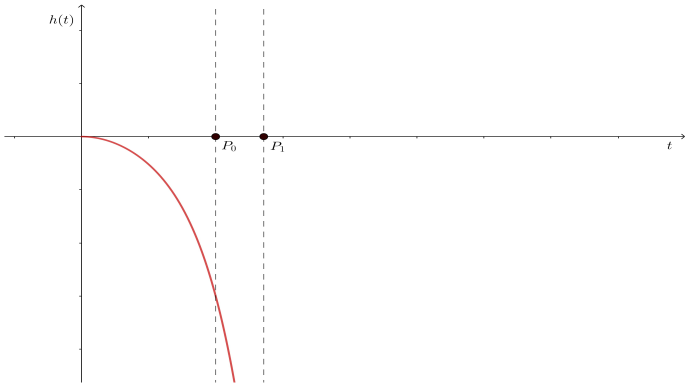

Figure 3.1: $h(t)$

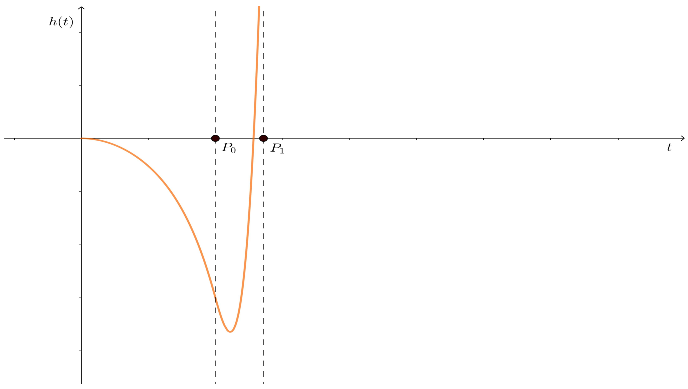

Figure 3.2: $\overline{h}(t)$

The final part of the proof follows from the proof of Theorem 6 contained in Section 3.4. □

Now we show the proof of our multiplicity result for solutions with positive energy in the symmetric setting, namely Theorem 8, whose statement is given in the Introduction. The proof is based on the use of Fountain Theorem, that is Theorem 16.

Proof of Theorem 8. We apply Theorem 16 with $\mathcal{G} = \mathbb{Z}/2$, $M = D_T^{1,p}(\mathbb{R}^N)$. Since $D_T^{1,p}(\mathbb{R}^N)$ is a separable Banach space, using the same argumentation in Remark 9 in Chapter 2, assumption **(A1)** is verified.

The functional $F_\lambda \in C^1(D_T^{1,p}(\mathbb{R}^{\mathbb{N}}))$ is even by definition and, from Corollary 2, the functional F_λ satisfies the $(\mathrm{PS})_c$ condition for every $c \in \mathbb{R}$, so assumption **(A4)** of Theorem 16 is valid.

Since $0 \not\equiv K \geq 0$ in $\mathbb{R}^N$ and $K \in C(\mathbb{R}^N)$, there exists an open subset Ω_K of $\mathbb{R}^N$ with $K > 0$ in Ω_K. By the T-symmetry of K, then Ω_K is T-symmetric, thus we can define $D_T^{1,p}(\Omega_K)$. By extending functions in $D_T^{1,p}(\Omega_K)$ by 0 outside Ω_K we can assume $D_T^{1,p}(\Omega_K) \subset D_T^{1,p}(\mathbb{R}^N)$. Assume $(Y_m)_m$ be an increasing sequence of subspaces of $D_T^{1,p}(\Omega_K)$ with $dim(Y_m) = m$. Thus there exists a constant $\varepsilon_m > 0$ such that for all $z \in Y_m$ with $\|z\| = 1$ we have

$$\int_{\mathbb{R}^N} K|z|^{p^*} dx = \int_{\Omega_K} K|z|^{p^*} dx \geq \varepsilon_m. \tag{3.53}$$

On the other hand, if $v \in Y_m$, $v \neq 0$, then we can write $v = \rho_m \omega$ with $\omega \in Y_m$ such that $\|\omega\| = 1$, so that $\rho_m = \|v\|$. By Lemma 24-**b)**, there exists $M > 0$ large enough, such that $|G^{-1}(t)| \geq p^{-1/p^*}|t|$ for $|t| \geq M$, so that, thanks to Lemma 24-**c)**, (3.53) and since $K, V \geq 0$, the following holds

$$\begin{aligned}
F_\lambda(v) &= \frac{1}{p}\rho_m^p - \frac{\lambda}{k}\int_{\Omega_K} V|G^{-1}(v)|^k dx \\
&\quad - \frac{\beta}{p^*}\int_{\Omega_K} K\left[\frac{1}{p}|v|^{p^*} + |G^{-1}(v)|^{p^*} - \frac{1}{p}|v|^{p^*}\right] dx \\
&= \frac{1}{p}\rho_m^p - \frac{\lambda}{k}\int_{\Omega_K} V|G^{-1}(v)|^k dx - \frac{\beta}{pp^*}\int_{\Omega_K} K|v|^{p^*} dx \\
&\quad - \frac{\beta}{p^*}\int_{\{|v|>M\}\cap\Omega_K} K\left[|G^{-1}(v)|^{p^*} - \frac{1}{p}|v|^{p^*}\right] dx \\
&\quad - \frac{\beta}{p^*}\int_{\{|v|\leq M\}\cap\Omega_K} K|G^{-1}(v)|^{p^*} dx + \frac{\beta}{pp^*}\int_{\{|v|\leq M\}\cap\Omega_K} K|v|^{p^*} dx \\
&\leq \frac{1}{p}\rho_m^p - \frac{\lambda}{k}\int_{\Omega_K} V|G^{-1}(v)|^k dx - \frac{\beta}{pp^*}\int_{\Omega_K} K|v|^{p^*} dx \\
&\quad + \frac{\beta}{pp^*}\int_{\{|v|\leq M\}\cap\Omega_K} K|v|^{p^*} dx \\
&\leq \frac{1}{p}\rho_m^p + \frac{\lambda}{k}\int_{\Omega_K} V|v|^k dx - \frac{\beta}{pp^*}\int_{\Omega_K} K|v|^{p^*} dx \\
&\quad + \frac{\beta}{pp^*}\|K\|_\infty M^{p^*-k}\int_{\Omega_K} |v|^k dx,
\end{aligned}$$

that is

$$F_\lambda(v) \leq \frac{1}{p}\rho_m^p + C\rho_m^k - \frac{\beta\varepsilon_m}{pp^*}\rho_m^{p^*} \leq 0$$

for sufficiently large ρ_m, since $k, p < p^*$. This proves **(A2)** of Theorem 16.

Condition **(A3)** follows exactly as in [14], taking into account the properties of v. Then applying Theorem 16, the energy functional F_λ has unbounded sequence of critical values in $D_T^{1,p}(\mathbb{R}^{\mathbb{N}})$. Thus F_λ has unbounded sequence of critical values in $D_T^{1,p}(\mathbb{R}^{\mathbb{N}})$. Theorem 8 is so proved. □

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