Some Control and Observation Issues in Cellular Automata

Samira El Yacoubi
Théo Plénet

IMAGES-ESPACE-DEV, Univ. Perpignan Via Domitia, Perpignan, France
ESPACE-DEV, Univ. Montpellier, IRD, Montpellier, France

Sara Dridi
University of Setif, Setif, Algeria

Franco Bagnoli
Dept. Physics and Astronomy and CSDC, University of Florence
via G. Sansone 1, 50019 Sesto Fiorentino (FI), Italy also INFN, sez. Firenze

Laurent Lefèvre
Clément Raïevsky

Univ. Grenoble Alpes, Grenoble INP, Institute of Engineering Univ.
Grenoble Alpes, LCIS, 26000 Valence, France

This review article focuses on studying problems of observability and controllability of cellular automata (CAs) considered in the context of control theory, an important feature of which is the adoption of a state-space model. Our work first consists in generalizing the obtained results to systems described by CAs considered as the discrete counterpart of partial differential equations, and in exploring possible approaches to prove controllability and observability. After having introduced the notion of control and observation in cellular automata models, in a similar way to the case of discrete-time distributed parameter systems, we investigate these key concepts of control theory in the case of complex systems. For the controllability issue, the Boolean class is particularly studied and applied to the regional case, while the observability is approached in the general case and related to the reconstructibility problem for linear or nonlinear CAs.

Keywords: cellular automata; control systems; controllability; observability

1. Introduction

Controllability and observability are among the most prominent and main considered issues in control theory. They were introduced by Kalman and well studied during the second half of the last century. A wide variety of works related to controllability and observability of distributed parameters systems (DPS) has been achieved [1]. The

study of these notions on DPS via the structure of actuators (inputs) and sensors (outputs) was the subject of intense research activity [1, 2]. The traditional and most-used models in these studies are based on a set of partial differential equations (PDEs) for the description of system input–output dynamics. Whereas controllability concerns the ability to steer the processes so as to bring them toward desired profiles through specific actions, observability deals with the ability to reconstruct the initial system state, taking into account a sufficient knowledge of the system dynamics based on specified output measurements. These two major concepts have already been studied for continuous systems described by PDEs as reported in the literature [1, 3, 4]. In the case of deterministic linear systems analysis, the so-called Kalman condition [5, 6] is essential and has been widely used to obtain the main characterization results regarding the choice of actuator/sensor structures, locations, number and types (mobile or fixed). See, for example, [1, 7] and the references therein.

The aim of this paper is to summarize some recent advances in controllability and observability of systems described by cellular automata (CAs), considered as the discrete counterpart of PDEs, and explore other suitable approaches to prove controllability and observability for such systems. CAs are widely used mathematical models for studying dynamical properties of discrete systems and constitute very promising tools for describing complex natural systems in terms of local interactions [8–10].

CAs are the simplest models of spatially extended systems that may provide a good description for complex phenomena. They are discrete dynamical systems where space and time, as well as states related to physical quantities, are all discrete. Evolution is governed by a set of simple local and microscopic transition rules that may exhibit a complex behavior. In the macroscopic limit (i.e., after space and time coarse graining), CAs can reproduce the same phenomena usually modeled with PDEs. A wide range of applications in biology, chemistry, physics and ecology was successfully developed using the cellular automaton (CA) paradigm as reported in the large dedicated literature [11–13].

Since CA models were so far considered as autonomous systems, the idea was first to introduce the notion of control and observation in these models, in a similar way as was previously done for discrete-time distributed parameter systems, in order to be able to study some concepts of control theory related to inputs and outputs [14]. We then started to study the two key and most popular concepts of control theory, namely controllability and observability, in the case of complex systems modeled by CAs.

For the controllability problem, we focused on a particular case of the so-called regional controllability. In the context of distributed

parameter systems, the term regional has been used to refer to control problems in which the desired state is only defined, and may be reachable, on some portion of the domain. In many physical problems, the regional controllability is naturally chosen in order to shape a natural phenomenon just in a subregion of the whole domain. The case of Boolean CAs has been particularly examined to investigate the boundary regional controllability [15]. It consists in considering objective functions defined on a subregion of the domain and exerting control actions on the boundary of the target region. This problem has been dealt with using several tools: namely the Kalman theorem, Markov chains and graph theory [16, 17]. The extension to nonlinear CAs has also been studied in these works.

For the problem of observability, we assume that the studied system is autonomous and we apply the tools mentioned to prove observability as a dual notion of controllability. Where in controllability analysis the state of the system has to be steered to a desired value using an unknown input signal with minimal energy, in the case of observability we seek to observe through a given output signal the maximum energy of the unknown state of the system. The first results were obtained for affine CAs and a rank condition for observability was proved in [18]. This condition is reminiscent of deterministic linear systems theory. Several criteria to assess the observability and the reconstructibility of CAs were formulated according to the choice of sensor structures, locations and types (mobile or fixed). Some examples were given to illustrate the theoretic results. The nonlinear case as well as the probabilistic case is currently under investigation.

2. Controllability of Cellular Automata

2.1 Introduction to Distributed Parameters Systems

Several phenomena exhibit spatially distributed behavior and are distributed parameter systems (DPS) where the state variables depend on time and space and describe the system's behavior in terms of inputs and outputs that also depend on space and time.

Although DPS are more common in industry, their applications have been expanded to include biological, ecological or economic systems and are becoming extremely related to the study of complex systems. Depending on the application, various representations of DPS can be considered with different input and output structures. The usual form they take is based on a state-space representation described as a set of PDEs that provide detailed descriptions of the internal behavior of the system.

The mathematical description of DPS is usually given by three operators: A describing the dynamics, B and C determining how the controls act on the system and how it is observed.

$$\begin{cases} z'(t) = Az(t) + Bu(t); \ 0 < t < T \\ z(0) = z_0 \in D(A) \end{cases} \tag{1}$$

and augmented by the output equation:

$$y(t) = Cz(t). \tag{2}$$

Two key concepts for analyzing such systems are controllability and observability, which are studied through operators B and C. The controllability concerns the ability to steer the system from any initial state to any desired state by acting on inputs that are involved in the operator B. Whereas, the observability deals with the capability to reconstruct the initial state of the system, taking into account sufficient knowledge of the system's dynamics through certain output measurements according to the operator C.

The problem of controllability of DPS has been widely studied in recent years [1]. Various types of controllability have been considered for DPS: exact, weak or regional [19, 20]. The regional case was introduced by Zerrik et al. [4] as a special case of output controllability [19, 21]. It consists in achieving an objective only on a region ω of the spatial domain on which the governing partial differential system is considered. This regional idea appears naturally in many real-world dynamical systems, when studying a natural phenomenon only in a specific area. This concept has been widely developed and interesting results have been proved, in particular, the possibility to reach a state on an internal subregion or on a part of the boundary of the domain when some specific actions are exerted on the system, in its domain interior or on its boundaries. CA models are particularly suitable for simulating natural phenomena that are usually highly nonlinear and better described in terms of discrete units rather than by means of PDEs [11, 22–24]. The CA approach has been recently promoted for the study of control problems on spatially extended systems for which the classical approaches cannot be used. The addressed question: can we consider CAs as a possible alternative to DPS for modeling and analysis of spatially extended systems?

2.2 Discrete-Time Distributed Parameters Systems Statement by Means of Cellular Automaton Formalism

CAs are spatially extended systems that are widely used for modeling various problems ranging from physics to biology, engineering, medicine, ecology and economics. An ultimate understanding of such systems gives us the ability to control them in order to achieve desired behavior.

Definition 1. A CA is classically defined by a quadruple $\mathcal{A} = (\mathcal{L}, \mathcal{S}, \mathcal{N}, f)$ where

- $\mathcal{L}$ is a d-dimensional lattice of cells c that are arranged depending on space dimension and cell shape. In the infinite case, $\mathcal{L} = \mathbb{Z}^d$.

- $\mathcal{S}$ denotes a discrete state set. It is a finite commutative ring given by $\mathcal{S} = \{0, 1, ..., k-1\}$ in which the usual operations use modular arithmetic.

- $\mathcal{N}$ is a mapping that defines the cell's neighborhood.

The neighborhood is usually given by:

$$\mathcal{N} : \mathcal{L} \longrightarrow \mathcal{L}^n$$
$$c \longrightarrow \mathcal{N}(c) = \{c' \in \mathcal{L} \mid \|c' - c\|_i \le r\}$$

where $\|c\|_i$, $i \in \{1, \infty\}$ indicates the sum and the maximum, respectively, of the absolute value of the components of cell c (for $d = 2$, $\|c' - c\|_1 = |c'_i - c_i| + |c'_j - c_j|$ and $\|c' - c\|_\infty = \max(|c'_i - c_i|, |c'_j - c_j|)$).

- f is a transition function that can be defined by:

$$f : \mathcal{S}^n \longrightarrow \mathcal{S}$$
$$s_t(\mathcal{N}(c)) \longrightarrow s_{t+1}(c) = f(s_t(\mathcal{N}(c))$$

where $s_t(c)$ designates the c cell state at time t and $s_t(\mathcal{N}(c)) = \{s_t(c'), c' \in \mathcal{N}(c)\}$ is the state of the neighborhood.

In the linear case, $f(x_1, ..., x_n) = \sum_{i=1}^n a_i x_i$, where the sum is taken modulo k. In the affine case, $f(x_1, ..., x_n) = \sum_{i=1}^N a_i x_i + c$.

We can also have cases in which the function f is linear only for a certain set of variables. As we shall see, for control purposes, the case of peripherally linear CAs is particularly interesting, for which

$$f(x_1, ..., x_n) = a_1 x_1 + h(x_2, ..., x_{n-1}) + a_n x_n,$$

where h is an arbitrary function (linear or nonlinear).

In order to consider CAs in the context of DPS, a description in the form of a state equation is necessary.

2.2.1 New State Equation

Consider the case $\mathcal{L} = \mathbb{Z}^d$, $(d \ge 1)$ and introduce a metric over $X = \mathcal{S}^{\mathbb{Z}^d}$ as:

$$d_\delta(x, y) = \sum_{c \in \mathbb{Z}^d} \frac{\delta(x(c), y(c))}{2^{\|c\|_\infty}}$$

where $\delta : S \times S \rightarrow \{0, 1\}$ is defined by:

$$\delta(i, j) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

The set $X = S^{\mathcal{L}}$ equipped with the distance d_δ is a compact metric space and the global dynamics F

$$F : X \longrightarrow X$$
$$s \longrightarrow F(s)$$

is continuous according to the topology induced by d_δ.

Proposition 1. [14]

- The compact configurations set X defines the state space of the autonomous CA.
- The sequence of continuous global maps F^i defined as the ith iteration under F plays the same role as the semi-group, usually denoted by (Φ_t), generated by the operator A.

In a way similar to discrete-time DPS, the evolution of an autonomous CA starting from a given initial configuration s_0 can be defined in terms of the global dynamics by the state equation:

$$\begin{cases} s_{t+1} = Fs_t \\ s_0 \in X \end{cases}$$

In the linear case, the operator F is simply a circulant band matrix J (of width n),

$$s_{t+1} = Js_t,$$

where the matrix sum-product is taken modulo k. In the affine case,

$$s_{t+1} = Js_t + C.$$

2.2.2 Control and Observation in Cellular Automata

The CA model will be completed by control and measurement functions. For the control aspects, it is done via inputs (actuators) that have a spatial structure (number, spatial location and distribution).

Let us consider the following general hypothesis.

- $\mathcal{L} = \mathbb{Z}^d$ is a cellular domain whose elements $c = (i_1, \dots, i_d)$.
- $I_T = \{0, 1, \dots, T\}$ is a discrete time horizon.
- $\mathcal{L}_p$ is a subdomain that defines the region of the lattice $\mathcal{L}$ where the CA is excited. It contains p cells that may be connected or not.

- The control operator G that defines the way the control excites the CA through the cells of $\mathcal{L}_p$ is given by:

$$G : \mathcal{U} \longrightarrow \mathcal{S}^{\mathbb{Z}^d}$$
$$u \longrightarrow Gu$$

where $U = \ell^2(\mathcal{L}_p, \mathbb{R}) = \left\{ u : \mathcal{L}_p \rightarrow \mathbb{R} \mid \sum_{c \in \mathcal{L}_p} u^2(c) < \infty \right\}$ is the control space.

  G is an operator that transforms the physical actions u to a realization Gu in the state space. Its action on cell c is denoted as $g_c(u)$.

- The CA is then considered as a controlled system, defined by the local transition function:
$$s_{t+1}(c) = f(s_t(\mathcal{N}(c))) + g_c(u_t)$$

where again the sum is taken modulo k. Considering for simplicity the Boolean case, this means that where $g_c(u)$ is one, the actions of the f_c are reversed, and where $g_c(u)$ is zero, it is not modified.

- The corresponding state equation is:

$$\begin{cases} s_{t+1} = Fs_t + Gu_t, \, t \in I_T \\ \quad s_0 \in X. \end{cases}$$

- The observation problem can be considered by duality where an observation space and a global observation operator have to be defined.

  For $I_m = \{0, 1, ..., Tm\}$ and $O = \ell^2(\mathcal{L}_q, \mathbb{R})$, the observation space consists of all bounded measurements made in $\mathcal{L}_q \subset \mathcal{L}$ and is given by a measurement variable (output) denoted by $\theta : I_m \rightarrow O$ that defines the measurement at time t. The global observation operator H defined by:

$$H : \mathcal{S}^{\mathbb{Z}^d} \longrightarrow O$$
$$s \longrightarrow Hs$$

associates a measurement to each configuration s.

- This leads to a complete description of CAs in terms of inputs and outputs where the state equation is augmented with
$$\theta_t = Hs_t, \, t \in I_m$$

and then defines the so-called distributed CA.

- The obtained CA statement is very close to the usual discrete-time distributed parameter systems formulation augmented by the output function.

The problem that we want to address here is that of forcing the appearance of a given pattern inside a region by imposing a suitable set of values onto some specific sites that could be in the lattice or in its boundary.

The idea is to explore different formalisms and approaches, some of which are specific to PDEs, that can prove the regional controllability of CAs, focusing on Boolean CAs.

▌2.3 Regional Controllability of Cellular Automata: Kalman Condition

Let us recall the classical Kalman rank condition [5] as stated for the finite dimensional systems:

Definition 2. The controllability matrix related to equation (1) is a matrix of dimension $n \times nm$ defined by:

$$M_c = [B, AB, ..., A^{n-1}B]. \tag{3}$$

The determination of the matrix gives information about whether the system is controllable or not. We have the following theorem (Kalman condition):

Theorem 1. Equation (1) is controllable if and only if the controllability matrix is of full rank; in other words:

$$\text{rank}(M_c) = n. \tag{4}$$

This was generalized to CAs and allowed to prove the regional controllability.

Definition 3. Let us consider:

- A CA defined on a discrete lattice $\mathcal{L}$ with state set $\mathcal{S}$ and described by a transition function f

- $\omega \subset \mathcal{L}$

- s_ω the restriction to ω of the CA configuration s

- $\mathcal{S}^\omega = \{s : \omega \to \mathcal{S}\}$

The CA is said to be regionally controllable if for a given $s_d \in \mathcal{S}^\omega$ there exists a control sequence $u = (u_0, ..., u_{T-1})$ with $u_i \in \mathcal{U}$ such that:

$$s_T = s_d \text{ on } \omega$$

where s_T is the final configuration at time T and $\mathcal{U}$ is the control space.

2.3.1 Special Case of Boolean Cellular Automata

Let us consider a Boolean CA defined on a lattice $\mathcal{L}$ that is assumed to be finite and composed of N interior cells and two boundary cells where the actions will be exerted, denoted by c_l and c_r for the left and right boundary cells, respectively. We are interested in finding the suitable sequences of controls acting on the boundary of the lattice, $\left(u_l^0, u_l^1, ..., u_l^{T-1}\right)$ and $\left(u_r^0, u_r^1, ..., u_r^{T-1}\right)$, so as to steer the system from

a given initial state s_0 to a desired configuration s_d on the subregion ω at a given time T, such that $s_T(c_i) = s_d(c_i)$, $\forall\ c_i \in \omega$. The desired configuration s_d is assumed to be reachable in the evolution of the CA rule.

In the following, to make notation more clear, we will indicate with the symbol $\oplus$ the sum modulo two.

The state equation for linear Boolean CAs is

$$\begin{cases} s_{t+1} = Js_t \oplus Bu_t;\ 0 < t < T \\ \quad s_0 \in X \end{cases}$$

where

- J is the circulant band matrix, each line of which contains the coefficients $a_1, ..., a_n$ in the positions corresponding to the neighborhood $\mathcal{N}$.

- B is an $(n \times 2)$ matrix that represents the control operator.

- $u_t = [u_l^t, u_r^t]$ is the control matrix at time t consisting of a two-component vector in this particular case.

Theorem 2. Kalman condition for controllability.

A one-dimensional linear CA is regionally controllable via boundary actions if and only if:

$$\text{Rank}(M_c) = \text{Rank}(B \quad JB \quad J^2B...J^{T-1}B) = T = N - 1.$$

Where T is the time horizon, N is the size of the CA lattice and J is the Jacobian matrix.

Proof. Let s_0 be the initial configuration of $\mathcal{A}$. Assume that the CA is regionally controllable in ω_1 by acting on c_l such as $\omega_1 \subset \omega$ and $\omega_1 = \omega - \{c_l\}$, then a sequence of control $\{u_l^t, u_r^t\}$ exists such as $t = 0, ...T - 1$. By using

$$s^{t+1} = Js^t \oplus Bu^t \tag{5}$$

it follows that

$$s^1 = Js^0 \oplus Bu_l^0$$
$$s^2 = Js^1 \oplus Bu_l^1$$
$$s^2 = J(Js^0 \oplus Bu_l^0) \oplus Bu_l^1$$
$$s^2 = J^2s^0 \oplus JBu_l^0 \oplus Bu_l^1$$
$$s^3 = Js^2 \oplus Bu_l^2$$
$$s^3 = J(J^2s^0 \oplus JBu_l^0 \oplus Bu_l^1) \oplus Bu_l^2$$
$$s^3 = J^3s^0 \oplus J^2Bu_l^0 \oplus JBu_l^1 \oplus Bu_l^2.$$

Therefore:

$$s^T = J^T s^0 + \begin{pmatrix} B & JB & J^2B...J^{T-1}B \end{pmatrix} \begin{pmatrix} u_l^{T-1} & u_l^{T-2}...u_l^0 \end{pmatrix}^{tr}.$$

We define the controllability matrix M_c:

$$\begin{pmatrix} B & JB & J^2B...J^{T-1}B \end{pmatrix}.$$

We get the regional controllability when

$$\text{Rank}(M_c) = T.$$

For peripheral linear CAs, we also know that in order to be able to change any state of region $\omega_1 = \{c_1, c_2, ..., c_N\}$ from one boundary c_l (if the CA is left-linear), the time T should equal $N-1$ [25], where N is the size of the CA. Hence, we can get that:

$$\text{Rank}(M_c) = T = N - 1.$$

Now suppose $\text{Rank}(M_c) = T = N - 1$ where $\text{Rank}(M_c) = \dim(\text{Image}(M_c))$; then for each initial configuration s_0, we can associate a desired configuration s_d. Hence the proof. □

Remark 1. It is difficult to develop algorithms to decide whether a generic rule is controllable. But for linear rules or peripherally linear ones (that depend linearly on a peripheral site), this can be done iteratively: the least external site can control a site in the target region, flipping its value if wrong. And then this can be repeated for all sites to be controlled.
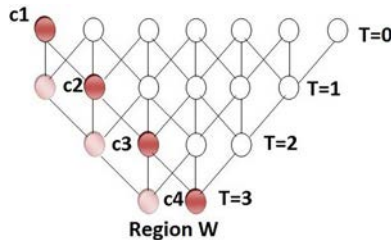

Figure 1. Control of the region ω by applying the control on one boundary c_1. The change in the initial condition on the controlled cell c_1 will be propagated to the cell c_4 at time $T = 3$.

Example 1. Consider the elementary cellular automaton (ECA) rule 150. We suppose that we act only on the left boundary cell of a region $\omega = \{c_{10}, ..., c_{30}\}$ and show that there exists a sequence of controls $u = \{u_{c_l}^0, ..., u_{c_l}^{T-1}\}$ that steers the system in ω from the initial configuration to a desired one constituted of ones, $s_d = (1, ..., 1)$ at time $T = 39$.

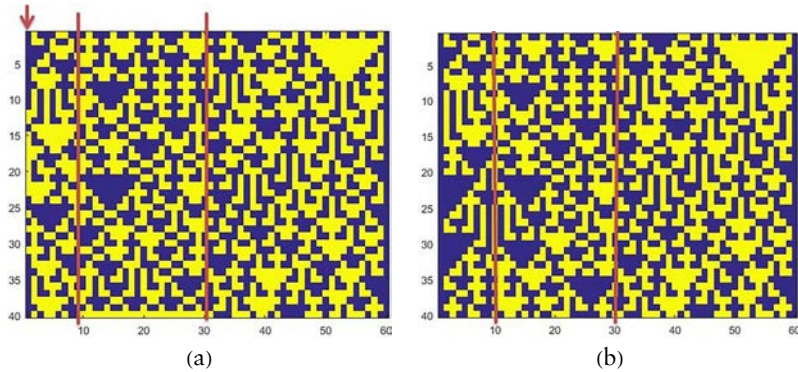(a)   (b)

Figure 2. The evolution of CA rule 150 (a) with and (b) without the application of the left boundary controls.

┃ 2.4 Alternative Methods to Kalman Condition for Regional Controllability of Cellular Automata

The notion of controllability was identified by Kalman as one of the central properties determining system behavior. His simple rank condition is prevalent in the analysis of linear systems. In this section, we aim at exploring new approaches to prove the regional controllability of CAs in more general cases.

2.4.1 Markov Chains Approach for Controllability

The first method comes from the idea that CAs and Markov chain modeling are of great interest when merged and applied in practical situations. The evolution of all possible configurations of a probabilistic CA can be written as a Markov chain. Since deterministic CAs are limit cases of probabilistic ones, they also can be seen as particular Markov chains. A Markov chain such that for some t, $M^j > 0$ for all i, j is said to be regular, and this implies that any configuration can be reached by any configuration in time t. The evolution of a controlled CA can be seen as a Markov chain where the states are the possible configurations of region ω. Two CA configurations restricted to the target region ω, $s_0 \mid_\omega$ and $s_1 \mid_\omega$ are related to each other if there exists a boundary control (l, r) such that $p_{s_0, s_1} = 1$, where p_{s_0, s_1} denotes the probability for jumping from s_0 to s_1 in one step. In that way, the so-called transition matrix can be constructed in order to describe and analyze a Markov chain.

Example 2. Consider the CA rule 150 and its transition matrix and associated graph, as shown in Figure 3.
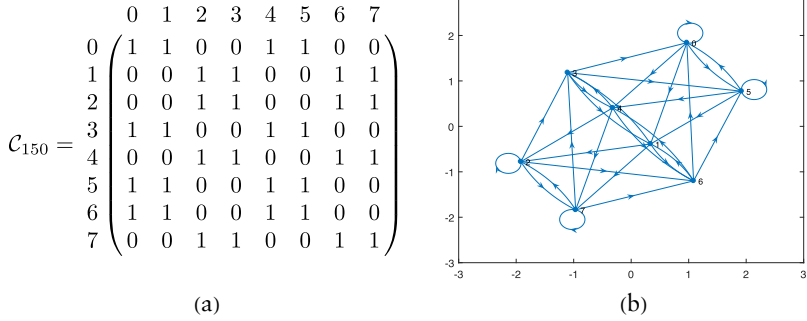
$$
C_{150} = \begin{array}{c}
\\
0\\
1\\
2\\
3\\
4\\
5\\
6\\
7
\end{array}
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\\
\left(\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{array}\right)
\end{array}
$$

(a)   (b)

Figure 3. (a) The transition matrix C_{150} for $|\omega| = 3$. (b) Graph of C_{150}.

Theorem 3. A CA (linear or nonlinear) is regionally controllable if and only if there exists a power of matrix C^T such that all the components are strictly positive.

2.4.2 Graph Theory Approach for Regional Controllability of Cellular Automata

Markov chains can be described using directed graphs where the nodes represent the different possible states and the edges represent the probability of the system moving from one state to the other in the next time instance. The graph theory seems to be a good and appropriate tool to study the problem of regional controllability of CAs. We give the important characterization result in terms of graph theory.

Theorem 4. A CA is regionally controllable if and only if there exists a t such that the graph associated to the transformation matrix C^t contains a Hamiltonian circuit; that is, a circuit of a graph $G = (V, AR)$ is a simple directed path of G that includes every vertex exactly once [17].

3. Observability and Reconstructibility of Cellular Automata

3.1 Introduction to Observability and Reconstructibility

Observability, as defined by Kalman [5, 6], allows us to determine if a system is observable by one or more sensors, that is, if it is possible to reconstruct the state of the system based on the measurements of these sensors. This criterion makes it possible to choose and place efficiently the different sensors needed for the observation. Once the sensors are placed so that they can observe the system, it is necessary to build the observer, an entity that estimates the state of the system

from the measurements of the sensors. As the observability criterion is verified, we know that such an observer exists.

The estimated state of the system, denoted $\hat{z}$, is constructed from the y measurements taken by the sensors and from the dynamics of the physical system. Its mathematical description is similar to the augmented definition (equations (1) and (2)) of the physical system except for the addition of an output error feedback via the observer operator L. The estimated state dynamics satisfy therefore

$$\begin{cases} \dot{\hat{z}}(t) = A\hat{z}(t) + L\big(y(t) - \hat{y}(t)\big) + Bu(t); \ 0 < t < T \\ \hat{y}(t) = C\hat{z}(t) \\ \hat{z}(0) = \hat{z}_0 \in D(A). \end{cases} \qquad (6)$$

The L operator represents the process by which the estimated state converges toward the state of the system; the way it is constructed influences the performance or the robustness of the observer.

In most studies, observability, and by extension all the aspects related to observation, are studied through controllability by the principle of duality that binds one to the other [5]. In this section, we will pay particular attention to the use of observation for monitoring and estimation of initial state for complex systems modeled by CAs, specifically *autonomous* CAs, that is, those that are not controlled ($\forall\, t \in [0, T]$, $u(t) = 0$). In the following, we present separately our study on CA monitoring and CA initial state estimation. The first consists of determining the current state of the system, while the second focuses on finding out the initial state. These two notions are called, respectively, *reconstructibility* and *observability* [26]. The latter is stronger than the former because if the initial state can be estimated, then it is possible to compute the current state from this initial state. Reconstructibility is rarely addressed in the study of linear continuous-time invariant (LTI) systems or continuous-time DPS because of its equivalence to observability (which is not valid any longer for discrete-time systems; see [27]). It is, however, starting to be investigated in the case of the Boolean control network [26, 28].

▎ 3.2 Observability and Reconstructibility of Cellular Automata

CAs are mathematical models where time, space and state are discrete. This total discretization prevents the direct use of the observability and controllability results of continuous-time control theory, or even discrete-time-only control theory. Indeed, the existence and uniqueness of an inverse requires the state space to be a field (in the mathematical sense). To build a finite (or Galois) field from only a finite number of possible state values requires this number to be prime. On the other hand, the finiteness of the state values makes it

possible to use discrete mathematics results such as, for instance, those from graph theory. The behavior of the system can be represented in the form of a graph, called a configuration graph, that links the different possible states of the system [29]. This discrete approach to observability and controllability is recent in control theory, but not in computer science nor in applied mathematics. There are numerous methods of state reconstruction, but they are not based on observability and reconstructibility criteria.

From Section 2.2.2 given the global dynamics (F) and observation (H) operators, a sequence of output Θ around a time horizon T can be reconstructed from the initial state, through some extended observability (or Gramian) operator:

$$\Theta_T : s_0 \mapsto \{\theta_0, \theta_1, ..., \theta_{T-1}\} = \{Hs_0, HFs_0, ..., HF^{T-1}s_0\} \qquad (7)$$

where Θ_T represents the T first outputs of the initial configuration $s_0 \in X$. Plenet et al. [29] proposed a definition for the observability and reconstructibility that can both be mathematically linked to the property of injectivity of the output sequence Θ_T.

Definition 4. Observability and reconstructibility.

Let A be a CA with a global transition function F and an output sequence Θ_T; then the two following propositions hold:

- A is observable
  $\iff \forall s_0', s_0'' \in X, \Theta_T(s_0') = \Theta_T(s_0'') \Longrightarrow s_0' = s_0''$.

- A is reconstructible
  $\iff \forall s_0', s_0'' \in X, \Theta_T(s_0') = \Theta_T(s_0'') \Longrightarrow F^T(s_0') = F^T(s_0'')$.

From Definition 4, it is clear that both can be used to reconstruct a configuration from the measurements, as every output sequence is the result of only one configuration. Moreover, each proposition deals with a specific configuration: the observability deals with the initial configuration (s_0), whereas the reconstructibility deals with the current configuration ($s_t = F^T(s_0)$). In addition, observability is stronger than reconstructibility, as s_T can be calculated from s_0; thus the observability must be assessed before the reconstructibility.

The observability of CAs is scarcely studied, even for systems with stationary sensors. The following sections present two methods to determine CA observability: a function-based method and a configuration-based method. In the first, the algebraic properties of the transition function (local or global) are studied to derive observability properties. In the second, the discrete aspect of CAs is combined with the notion of relationship between the automata configurations to get information about the observability of the CA. The function-based method has the advantage of a linear complexity with

the number of cells, since each cell has a finite number of states. Conversely, the configuration-based method has exponential complexity, since the number of configurations depends on the number of cells and the number of states (e.g., a Boolean CA with N cells has 2^N possible configurations).

3.2.1 Observability of Cellular Automata: Kalman Condition

This section presents a function-based method to determine observability for *affine* CAs through the affine property of the transition function. It relies on the Kalman condition presented in Theorem 1, and on the controllability/observability duality. A CA is said to be affine if its F transition function is affine; that is, it uses only addition and multiplication by constants. The transition function can then be expressed with a linear map and a constant vector.

Definition 5. Affine CAs.

If a CA and an output operator are affine, then F (respectively H) can be written in the form of a linear map plus a constant, which can in turn be written as a matrix A and a constant $\eta = F(0)$ (respectively C and $\gamma = H(0)$). The evolution of the CA can therefore be written as:

$$\begin{cases} s_{t+1} = F = As_t + \eta \\ \theta_t = Cs_t + \gamma \\ s_0 \in X. \end{cases} \tag{8}$$

Definition 6. The output sequence Θ_T is also an affine map composed with O_T the observability matrix (the dual of the controllability matrix M_c presented in Definition 2) and Γ_T a constant vector

$$\Theta_T = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \dots \\ \theta_{T-1} \end{bmatrix} = \underbrace{\begin{bmatrix} C0 \\ CA \\ \dots \\ CA^{T-1} \end{bmatrix}}_{O_T} s_0 + \underbrace{\begin{bmatrix} \gamma \\ CJ_0\eta + \gamma \\ \dots \\ CJ_{T-2}\eta + \gamma \end{bmatrix}}_{\Gamma_T} = O_T s_0 + \Gamma_T \tag{9}$$

with $J_t = \sum_{k=0}^{t} A^k$.

The Kalman rank condition for the observability of an affine CA is the following:

Theorem 5. Kalman rank condition.

An affine CA F, observed by an affine output operator H, is *observable* under a time horizon T, if and only if O_T is full rank.

This Kalman rank condition is usually used for continuous and discrete LTI systems and was extended to affine CAs [18, 29]. A

corollary is also given, allowing the reconstruction of the initial state from the measurements by inverting (or pseudo inverting) the full rank matrix O_T.

Corollary 1. If the Kalman criterion is verified, then it is possible to recover the initial state by inverting the observability matrix:

$$x_0 = O_T^\dagger (Y_T - \Gamma_T). \tag{10}$$

Theorem 5 and Corollary 1 are applicable for an observable system but can be extended in the case of a reconstructible system with the following theorem and corollary.

Theorem 6. Reconstructibility condition.

An affine CA F, observed by an affine output operator H, is *reconstructible* if and only if there exists a finite time horizon T such that:

$$\ker O_T \subset \ker A^T. \tag{11}$$

Corollary 2. If the reconstructibility criterion is verified, then it is possible to find a matrix R such that:

$$x_T = R(Y_T - \Gamma) + J_{T-1}\eta. \tag{12}$$

These two theorems and corollaries allow us to assess the observability and reconstructibility only for affine CAs. This class of automata is in fact quite small. For instance, among Wolfram's 256 elementary CA, only 16 are affine: the eight additive rules (0, 60, 90, 102, 150, 170, 204, 240) and their complementary rules (255, 195, 165, 153, 105, 85, 51, 15). Therefore, a second method has been proposed to investigate observability and reconstructibility of CA control systems. It is presented hereafter.

3.2.2 Observability of Cellular Automata: Configuration Relation

This *configuration-based* method makes use of a binary relation representation of the F, H and Θ operators, respectively denoted $\mathcal{R}_F$, $\mathcal{R}_H$ and $\mathcal{R}_\Theta$.

Definition 7. The transition binary relation $\mathcal{R}_F$ associated to the global transition function F is defined as:

$$\mathcal{R}_F = \{(s_0, s_1) \mid s_0, s_1 \in X \text{ and } s_1 = F(s_0)\}. \tag{13}$$

The binary relation can be represented in the form of a matrix called a logical matrix.

Definition 8. The logical matrix M associated to a binary relation $\mathcal{R} \in X \times Y$ is defined as:

$$M_{i,j} = \begin{cases} 1 & (x_i, y_j) \in \mathcal{R} \\ 0 & \text{else.} \end{cases} \tag{14}$$

This binary relation representation allows us to assess the observability and reconstructibility of all kinds of CAs, independently of their algebraic nature. From Definition 4, it is clear that the observability and the reconstructibility are related to the injectivity of Θ_T. This property can be studied through the relation $\mathcal{R}_\Theta$ for the observability and $\mathcal{R}_{\Theta_T F^{-T}}$ (constructed from $\mathcal{R}_\Theta$ and the converse of $\mathcal{R}_{FT}$) for the reconstructibility. This results in the following theorem using the logical matrices of such relations:

Theorem 7. A CA is observable (reconstructible) if the logical matrix related to the binary relation $\mathcal{R}_\Theta$ ($\mathcal{R}_{\Theta_T F^{-T}}$) has at most one nonzero element per column.

These logical matrices have very few nonzero elements, making it possible to use sparse matrices to represent them numerically. This reduces not only the memory allocation but also the algorithmic complexity for the assessment of observability and reconstructibility.

The main problem with this method is the exponential increase in the number of configurations in relation with the number of cells. Even when the algorithms used to evaluate observability are optimized, the exponential complexity coming from the number of configurations makes this method less efficient when the particular case of affine CAs is investigated.

3.3 Observation of Distributed Parameters Systems through Mobile Sensors

In the case of "classic" observation problems, only the sensors' types are chosen according to the natures of the output variables to be measured and the state variables to be observed (e.g., a displacement, velocity or force for a mechanical system). But for spatially distributed systems, the position of the sensors also becomes a crucial problem in the design of the observer. The placement of sensors remains, even today, a major concern in many areas such as water networks, forest fire detection and telemonitoring of human physiological data. Sensor placement is often related to the wireless sensor network (WSN) research area [30, 31], which studies strategies to place sensors to get optimal information about an observed system.

The main difference with our approach, based on control theory and observation, is that WSNs do not seek to reconstruct the state of the system using its dynamics. Sensor placement is carried out using different algorithmic methods such as genetic algorithms [32] and Bayesian optimization. In these studies, the sensors are generally considered fixed. This reduces their performance for the observation of spatially distributed systems. Indeed, the use of mobile sensors makes it possible to obtain observability, whereas stationary sensors would

have failed [29]. However, this poses new problems such as the calculation of sensor trajectories, which is still under investigation. Some researchers are directly studying the problem of observability with mobile sensors. Demetriou et al. [34, 35] and Hussein et al. [36] are considering networks of mobile actuators and/or sensors. Related problems occurred when mobile sensors (or actuators) are considered, such as obstacle avoidance or formation coordination to get maximum coverage. Such problems are intensively investigated, independently from the observation problem. For instance, among the many different existing methods to handle collision avoidance for mobile sensors, we can cite the predictive control approach [37] or the potential field approach [38].

As the sensors are moving in space, the cells they observe change through time. The output operator C therefore becomes time dependent (it may then be denoted $C(t)$). The observability criteria become time dependent and it is therefore necessary to evaluate the observability for all the different sensor trajectories. The two observability conditions presented in Section 3 can both be assessed with mobile sensors but for only one trajectory at a time. They can be used as a constraint in constrained trajectory planning to find a mobile sensor trajectory that respects observability and other constraints external to the observability problem, such as obstacle avoidance. This trajectory planning can be carried out either prior to system deployment or in real time.

4. Discussion

Controllability and observability are two fundamental properties that characterize the behavior of a given system and determine the relationships between state and input and output variables. As the real physical systems become more and more complex, the need to design controllers and observers for complex nonlinear systems is real. The study of controllability of linear systems was first performed in detail by Kalman and his collaborators in [6]. The first part of this paper discussed an analog of the Kalman controllability rank condition for systems described by cellular automata (CAs) considered as the discrete counterpart of partial differential equations (PDEs). It also explored some alternative approaches to this condition that could work for linear and nonlinear CAs. The paper focused on the special case of regional controllability of Boolean one-dimensional CAs. A necessary and sufficient condition for regional controllability of CAs was proved through original approaches such as Markov chains and graph theory.

This work can be extended in many tracks that can explore in particular the case of non-Boolean CAs and the highly nonlinear CAs.

The case of large lattice size is also to be considered. The examples considered concerned only a very limited number of cells because of the enormous computational cost generated in terms of spatial and temporal complexity. The control obtained is generally not unique at this stage, and the problem of optimality will be addressed later. A first problem of regional controllability in minimum time is currently under study.

Another perspective concerns the investigation of more adequate algorithms for the calculation of preimages that could be of low complexity, depending on the dimensions of the lattice.

For the observability issue, two methods for assessing observability and reconstructibility were presented. The first, using the Kalman condition, only applies to affine CAs, while the second allows us to study nonlinear CAs, but its exponential complexity makes it difficult to use. Several approaches can be considered to reduce the algorithmic complexity linked to the calculation of observability and reconstructibility. The first would be a new function-based method but for nonlinear CAs. Previous strategies already used in the case of linear continuous-time invariant (LTI) systems, such as the use of linearization or the use of nonlinear Jacobians, could be considered. The second method to make observability and reconstructibility computation tractable on CAs with a large number of cells is to use multiple observers. Each of these smaller and simpler observers is then in charge of the observation of a part of the whole cellular automaton (CA). The limited number of possible configurations for each observed region of the CA implies a dimensional reduction, making it tractable to determine observability or reconstructibility on these parts of the CA. For example, with a Boolean controller of 1000 cells, there are 2^{1000} configurations. If this automata is observed by 10 observers of 100 cells, then we must evaluate 10 times the observability for 2^{100} configurations. However, a new notion of observability will still have to be defined to account for these distributed observers in order to ensure that the local observability of the local observers can be combined to obtain the observability of the whole system.

The use of a mobile sensor network not only ensures observability where static sensors would have failed, but also makes the topology of the observation network dynamic and allows the sensors to focus on a specific part of the system. However, the use of mobile sensors coupled with the distributed observers requires strong coordination of the sensors. These two criteria make it extremely complex to calculate trajectories under constraints using the classical methods of control theory. One approach for improvement would be to use a notion already well studied in computer science: the multi-agent systems (MAS) paradigm. The agents (here observers and sensors) are

autonomous and communicate with each other. Each agent would calculate its own trajectory according to the constraints (observability and other) and the information exchanged between the agents. In the same way that the distributed observer manages to reduce the complexity of the observability calculation by distributing the task between several entities, the use of MAS produces the same effect but with respect to the trajectory calculation. The organization of the network and the communications between agents will have to be carefully designed to ensure global observability of the system based on local computations by the agents.

It should be noted that the controllability and observability problems were considered separately in this paper. For controllability issues, we assumed that the state of the system can be measured at each time, and for observability problems, the system is supposed to be autonomous. The global objective of considering a complete state equation with control and observation will be investigated in future work.

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