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Intrinsic Harnack inequality for local weak solutions to an anisotropic parabolic equation

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Intrinsic Harnack inequality for local weak solutions to an anisotropic parabolic equation

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Abstract

We prove a Harnack inequality for non-negative solutions of a parabolic equation having an anisotropic slow diffusion. We study the propagation of support of solutions, through an iterative technique reminiscent of De Giorgi's method and through the investigation of particular embeddings in anisotropic Sobolev spaces. At this point, we make an analysis of the natural scaling of the equation to reduce the problem to a Fokker-Planck equation and construct a self-similar Barenblatt solution thanks to finite speed of propagation. Then we exploit translation invariance to obtain positivity near the origin via a self-iteration method and deduce a sharp anisotropic expansion of positivity. This eventually yields a scale invariant Harnack inequality in an anisotropic intrinsic geometry, dictated by the powers of the diffusion coefficients. Finally we show some consequences as Hölder continuity of solutions, Liouville-type theorems and we formulate some open problems.

MSC 2020: 35K15, 35K59, 35K65, 35K92, 35R35, 60J60, 74N25.

Key Words: Anisotropic diffusion, Finite speed of propagation, Fundamental solution, Harnack inequality, Hölder continuity, Intrinsic geometry, Fokker-Planck equation.

Università degli Studi di Firenze, Dipartimento di Matematica e Informatica DIMAI "Ulisse Dini".

Contents

1 Introduction

and Main Results

Curiosity can be a very fierce sentiment

\- Joseph Conrad -

The Shadow Line

In the present work we study those properties of regularity that are owned by solutions to the model parabolic anisotropic equation

$$u_t - \sum_{i=1}^{N} \partial_i \Big(|\partial_i u|^{p_i - 2} \partial_i u \Big) = 0, \tag{1.1}$$

which is satisfied in a suitable weak meaning in $\Omega \times (0,T)$, $\Omega \subset\subset \mathbb{R}^N$ for powers $p_i > 2$, $i \in \{1, \dots, N\}$. These kind of equations raised increasing interest in the last decades as they embody an interesting feature, namely an *anisotropic diffusion* with orthotropic structure. Besides its inherent mathematical interest, the latter is useful when modeling diffusion in materials such as earth's crust or wood, where the velocity of propagation of diffusion varies according to the different orthogonal directions. Evolutionary equations of this type have been studied for more than fifty years, see for instance the paper [67] by Vishik. Moreover, equation (1.1) appears already as an example of sum of monotone operators in the monographs [49], [62] and [68]. Nevertheless, although the problem dates way back, the theory of regularity for this kind of operators is still a widely open problem, that continues to challenge us as the time passes by. The motivations for this uphill path are deep; indeed, they are related to the fact that it does not exist nowadays a unified regularity theory for singular and degenerate isotropic p-Laplacean equations. We will clarify more precisely this point when the correct intrinsic self-similar geometry of the equation (1.1) will be introduced.
From the mathematical point of view, the principal part in (1.1) arises as the Euler-Lagrange equation of the energy functional

$$\mathcal{E}(u) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i} \, dx,$$

whose integrand $\mathcal{F}(\nabla u)$ is part of a more general class of functionals, called with *non-standard growth*, that are of the type

$$\int_\Omega \mathcal{F}(\nabla u)\, dx, \qquad \frac{1}{C}(|z|^p - 1) \le \mathcal{F}(z) \le C(|z|^q + 1), \qquad z \in \mathbb{R}^N,$$

for some $p < q$, as opposed to the standard growth condition $p = q$. One can not expect the same regularity to hold true in both cases, as shown by pioneering examples in [37, 51]. The theory of regularity for solutions of the corresponding Euler-Lagrange elliptic equations is much more delicate and rich than the standard one. Since then, literature on the elliptic regularity theory grew in considerable size. Even if this has not always been the case, the general principle underlying to the theory is that most regularity results can be recovered

when the power gap $q-p$ in the non-standard growth condition is small. Since it would be impossible to collect here all the contributions, we refer to the surveys [52] and [54, Section 6], for a general overview of the subject and comprehensive bibliographic references.
While the non-standard elliptic theory matured, its parabolic counterpart became a research theme as well. The delay in development was considerable, mainly because already the isotropic problem with $p_i \equiv p \neq 2$ presented great difficulties, solved in full generality only a decade ago through the work of DiBenedetto and collaborators, see [29] and the literature therein. Nevertheless, parabolic equations with non-standard growth were considered well before, giving birth to a large amount of results on existence, well-posedness, L^∞-estimates and diffusion analysis. For an extensive bibliography on this research, we refer to [4] and for the theory of variational solutions to [52, Section 12] and the references therein.

Despite some partial results, however, the regularity theory for these parabolic anisotropic equations was largely unknown, while the case of bounded and measurable coefficients is still completely open. In the parabolic general case of non-standard growth the boundedness of spatial gradient has been investigated in [10]. More recently, Lipschitz regularity under general assumptions has been proven in [12] for the case $p_i = p$, $\forall i = 1, \dots, N$, through the use of an iterative scheme of fine energy estimates and therefore using a different method from ours. On the other hand in [34] existence of of self-similar solutions has been used to study the asymptotic behaviour of the fast diffusion counterpart of (1.1). While these results have some point in common with our study of Barenblatt solutions of (1.1), the equation in [34] falls within the framework of *fast diffusion*, presenting features which in many respects are opposite to ours. Moreover, we aim at deriving different qualitative properties of *general* non-negative solutions of (1.1): the Hölder continuity of solutions as well as the validity of a suitable (necessarily intrinsic) parabolic Harnack estimates. The latter is precisely the aim of our work.

INTRINSIC HARNACK INEQUALITY
Let $u \geq 0$ be a local weak solution to (1.1) in $\Omega \times [-T, T]$ and suppose that

$$\forall i = 1, \dots, N \quad 2 < p_i < \bar{p}\left(1 + \frac{1}{N}\right) \qquad \bar{p} := \left(\frac{1}{N}\sum_{i=1}^N \frac{1}{p_i}\right)^{-1} < N$$

and $u(0,0) > 0$. Then, there exist constants $C_1 \geq 0, C_3 \geq C_2 \geq 1$ depending only on N and the p_i's such that, letting $M = u(0,0)/C_1$ it holds

$$\frac{1}{C_3} \sup_{\mathcal{K}_\rho(M)} u(\,\cdot\,, -M^{2-\bar{p}}\,(C_2\,\rho)^{\bar{p}}) \leq u(0,0) \leq C_3 \inf_{\mathcal{K}_\rho(M)} u(\,\cdot\,, M^{2-\bar{p}}\,(C_2\,\rho)^{\bar{p}})$$

whenever $M^{2-\bar{p}}\,(C_3\,\rho)^{\bar{p}} < T$ and $\mathcal{K}_{C_3\,\rho}(M) \subseteq \Omega$, being

$$\mathcal{K}_r(M) := \prod_{i=1}^N \big\{|x_i| < M^{(p_i-\bar{p})/p_i} r^{\bar{p}/p_i}/2\big\}.$$

Let us make some comments on the statement, significance and proof of the previous theorem.

Intrinsic geometry.
A parabolic Harnack inequality for a non-homogeneous equation such as (1.1) cannot hold true with the classical statement. This was first realised for the parabolic p-Laplacean equation

$$\partial_t u = \Delta_p u \tag{1.2}$$

through an analysis of the so-called Barenblatt Fundamental solutions: a family of explicit solutions encompassing most of the features which distinguish the classical heat equation from (1.2). The correct formulation of the Harnack inequality for (1.2) was first found in [22] when $p \geq 2$, and for a fixed point $(x_o, t_o) \in \Omega_T \subset \mathbb{R}^N$ such that $u(x_o, t_o) > 0$, it has the *intrinsic* form

$$\gamma^{-1} \sup_{K_r(x_0)} u\left(\cdot\,, t_o - \left(\frac{C}{u(x_o,t_o)}\right)^{p-2} r^p\right) \leq u(x_o,t_o) \leq \gamma \inf_{K_r(x_0)} u\left(\cdot\,, t_o + \left(\frac{C}{u(x_o,t_o)}\right)^{p-2} r^p\right), \tag{1.3}$$

being $C, \gamma > 0$ constants independent of u and r, and for all radii $r \in (0,1)$ such that the intrinsic cylinders $(x_o, t_o) + \{K_{4\rho} \times (t_o - (C/u(x_o,t_o))^{p-2}(4r)^p, t_o - (C/u(x_o,t_o))^{p-2}(4r)^p)$ are contained in Ω_T.

Hence, a Harnack inequality for non-negative solutions of a nonlinear parabolic equation expresses a pointwise control on the solution in a full spatial neighbourhood of a point, in terms of its value at that point. The parabolic nature of the equation allows such a control to hold only after a positive (or negative) time delay has passed. For the heat equation this *waiting time* depends only on the size of the region where we seek for the lower bound, and it does not depend on the solution. On the other hand, for the parabolic p-Laplacean equation (1.2), the waiting time $(C/u(x_o,t_o))^{p-2}r^p$ depends on the value of the solution at the chosen point: the word *intrinsic* refers mainly to this phenomenon.

In the case of (1.2), the value of the solution at the chosen point affects just the waiting time, while for the anisotropic equation (1.1), it determines the full shape, or *geometry*, of the region where the control is available. This can be observed in the definition of the *intrinsic cubes* $\mathcal{K}_r(M)$, where r plays the role of an *anisotropic radius*, while M prescribes the *anisotropic geometry*. To justify the first statement, notice that the Lebesgue measure of $\mathcal{K}_r(M)$ is always r^N , regardless of M. Regarding the second, one can follow the well-known principle that *higher exponents give slower diffusion*, so that lower values of $M \simeq u(0,0)$ squeeze $\mathcal{K}_r(M)$ in directions of slower-than-average diffusion ($p_i - \bar{p} > 0$) and stretch it in directions of faster-than-average diffusion ($p_i - \bar{p} < 0$). This combines singular and degenerate effects that do not allow us to adopt DiBenedetto's method of *intrinsic scaling* (see for instance the guidelines [32], [65], or the original source [23]); because, roughly speaking, it results (at the present time) impossible to control in a unified fashion the ratio of powers of levels as k^{p_i} and the diameters of level sets $[u < k]$ when the behaviour is both singular (faster-than-average) and degenerate (slower-than-average).

Assumptions.
The main condition required in the Harnack inequality is $2 < \max\{p_i\} < \bar{p}(1 + 1/N)$. On one hand, assumption $p_i > 2$ for all i means that we are considering the *slow diffusion regime*. In this framework solutions of (1.2) for $p > 2$ preserve compactness of the support forward in time (as opposed to what happens for the heat equation). In the setting of the anisotropic equation (1.1), the support moves in different directions with different speeds, as we will see in Section 5.1, and plays an important role in our proof. The other condition $p_i < \bar{p}\,(1 + 1/N)$ requires that the powers p_i are not too sparse, following the above mentioned principle in problems with non-standard growth. Local boundedness holds in the larger range $p_i < \bar{p}\,(1 + 2/N)$, but we are not aware of counterexamples if this condition is violated. It would be interesting to know whether the Harnack inequality holds true also for $p_{\max} \in [\bar{p}\,(1+1/N), \bar{p}\,(1+2/N))$ but, if so, its proof likely requires different techniques than the ones employed here. The result however is not to be expected when $1 < p_i < 2N/(N + 1)$, because of the phenomenon of extinction in finite time (see discussion (3-i) of Chap VII of [23]) proven in [5]. Fast diffusion regime is outside the scopes of our work but it let us think that it may be possible that no Harnack inequality shall hold when $p_{\max} \in [\bar{p}\,(1 + 1/N), \bar{p}\,(1 + 2/N))$, because the *relative* diffusion process becomes too slow to expand the positivity.
A few comments on the constants C_i in the statement. The Harnack estimate is found by expanding positivity in comparison with a Barenblatt solution, whose construction is abstract and we do not know if a uniqueness theorem holds, up to translation and scaling. The constants depend on a lower bound on the Barenblatt solution, hence, ultimately on the choice of the latter. They are thereby undetermined from the quantitative point of view.
Finally, the number $u(0,0)$ is not a-priori well-defined for a weak solution. Its definition can be given in case the origin is supposed to be a Lebesgue point of the function. However, we will see in Section 4 that any solution to (1.1) under our assumptions possesses a semicontinuous representative, allowing us to give a meaning to $u(0,0)$. This ambiguity in the choice will then be eliminated by the a-posteriori continuity of the solution. Clearly, the theorem is meaningful only when $u(0,0) > 0$.

Idea of the proof.
As already mentioned, our first task is to build a family of Barenblatt solutions. We find all the natural scalings of (1.1) and construct a bijection between solutions of (1.1) and solutions of an anisotropic Fokker-Plank equation (see e.g. [15] for a similar approach). We then seek for a stationary solution to the latter, which is found through a fixed point argument and comparison principles. Here, the slow diffusion regime plays a pivotal role in recovering sufficient compactness to apply Shauder fixed point theorem. Indeed we use in a crucial way the controlled evolution of the support; this property is shown in Section 5.1, through an iterative De Giorgi argument (see [21]). Let us note that we rely on a weak continuity result (Lemma 6.1, point 3) of independent interest, which we were not able to find in the literature.

At this stage, the stationary solution to the Fokker-Plank equation is a rather irregular object of little use. However, exploiting its correspondence to a Barenblatt self-similar solution to (1.1) and using a self-iteration method based on comparison principles and translation invariance, we are able to prove a positive lower bound in a small neighbourhood of the origin. Here is crucial that we have a *stationary* solution to the Fokker-Planck equation. Transferring the bound to the Barenblatt solution, we find a quantitative expansion of positivity rate for it.
We then proceed in a manner reminiscent of the proof in [22] of the Harnack inequality for (1.2), namely finding a positivity set and then expanding it forward in time through comparison with Barenblatt solutions. For the first step, we actually employ a simplification described in [27], which makes use of the so-called *Clustering lemma* of [25]. We have to face two main difficulties: the intrinsic geometry of the problem, contrary to what happens in most instances of the theory, involves not only the time variables but also, and mainly, the spatial ones (in an anisotropic way). Secondly, even disregarding the geometry, the natural intrinsic cubes as per $\mathcal{K}_\rho(M)$ come from a quasi-metric rather than from a metric. To face the first difficulty we heavily rely on the natural transformations leaving (1.1) invariant; for the second one, we prove a general abstract version of the so-called Krylov-Safonov trick, of independent interest (Lemma 7.1).

Consequences of Harnack Inequality.
Following Moser's original ideas, we use the Harnack inequality to prove that local weak solutions to (1.1) are Hölder continuous. Moreover, we show that solutions to (1.1) in a strip $\mathbb{R}^N \times (-\infty, T)$ that are bounded from below and bounded from above for a single time $s < T$ are constant in the strip $\mathbb{R}^N \times (-\infty, s)$. Another interesting rigidity result can be obtained when the solution u in the whole $\mathbb{R}^N \times \mathbb{R}$ is bounded on a trajectory that goes to infinity in time (see Theorem 8.4). Finally, we show in Theorem 8.15 an Harnack estimate that holds true for all times, thus freeing time variable from being intrinsic.

The Cauchy problem and compact support propagation.
In order to find a self-similar solution to (1.1), we exploit an important correspondence between this solution and the corresponding one of a Fokker Planck equation (see Section 3.1). Solving the Cauchy problem with initial datum taken in L^2 related to the Fokker Planck equation is therefore transformed into the resolution of the Cauchy problem

$$\begin{cases} u_t - \sum_{i=1}^N \partial_i \Big(|\partial_i u|^{p_i-2} \partial_i u \Big) = 0, \quad \text{in} \quad \Omega \times (0,T), \qquad p_i > 2, \\ u(x,0) = u_0(x), \quad \text{in } L^2(\Omega). \end{cases} \tag{1.4}$$

Stated as it is, this problem suffers heavy non-uniqueness phenomena, even for smooth and compactly supported initial data (see for instance Appendix A in [33]). A smaller class where the problem is well-posed is the one of $L^{\mathbf{p}}$-solutions, that are solutions with a proper directional Sobolev integrability in the whole strip $\mathbb{R}^N \times (0,T)$. We prove this assertion in Section 5.2 and we exploit the full power of the finite speed of propagation for solutions to (1.4) in order to achieve an important consequence for the corresponding solutions to the Fokker Planck equation. Indeed, these ones have a support which not only stays compact along its evolution, but it shrinks when small initial datum and unitary mass are considered. This allows us to use a fixed point theorem on a suitable topology.

Barenblatt solutions.
One of the main byproducts of our proof is the construction of a family of self-similar Barenblatt solutions for (1.1) and the analysis on their basic properties. Self-similar solutions are by now a classical subject ([6]) and have been extensively studied in various parabolic nonlinear frameworks, see e.g. [66, Ch. 16] and the therein cited literature. Their role turned out to be pivotal in understanding the general behaviour of solutions and has often been an important stepping-stone for treating more general equations and formulating sensible statements on the general expected results: compare the classical works of Pini [60] and Hadamard [38], later generalised in the linear measurable setting by Moser [58] or, in the singular/degenerate case, the first works [22], [31] employing the Barenblatt solutions, generalised in [26].
An explicit form of Barenblatt solutions to equation (1.1) is still unknown at present, and their existence is obtained through an abstract approach. This difficulty arises because the original method of G.I. Barenblatt (see [6] and [18] for an easy proof) reduced the problem to the existence of solutions to a specific ODE, while in our case this not possible due to the lack of radial symmetry. Naturally, we cannot assume any a-priori

regularity and the method heavily relies on the identification of the natural scalings of (1.1), allowing to formulate the right notion of self-similarity (see for instance [7], [8] for the original underlying ideas).

Comparison with previous results. Local boundedness of solutions of parabolic equations as (1.1) has been first proved in [55] under the condition $p_{\max} < \bar{p}\,(1 + 2/N)$. Some early regularity results in the plane are considered in [57], and regularity for parabolic problems with non-standard growth of $p(x)$ type are contained in [1, 3, 16]. The $p(x)$ growth condition does not cover the simple equation (1.1) and we are not aware of proofs of the Hölder continuity of solutions of the latter in general dimensions (see [11, Remark 1.4] for a discussion of previous attempts), let alone of the Harnack inequality.
In the elliptic setting much more is known regarding the regularity of solutions of (1.1), or for more general non-standard equations, see [52, Sections 5 and 6] for the relevant literature. The most up-to-date result for (1.1) is in [11], where the Lipschitz regularity of its *bounded* solutions is proved *for any choice* of $p_i \geq 2$. The Harnack inequality for non-standard elliptic problems has been the object of various works: [2, 9, 20, 59, 50, 40, 39, 63] focus on isotropic equations with non-standard growth of $p(x)$-type, while [53, 48] deal with energies with Uhlenbeck structure and non-standard growth. However, none of the frameworks considered therein cover the anisotropic equation (1.1): indeed, its Euler-Lagrange equation is degenerate/singular on the union of the coordinate axes, while non-standard functionals of $p(x)$- or Uhlenbeck-type exhibit this problem only at the origin.

Structure of the work.
In Section 2 we define the functional setting which is proper to give a definition of local weak solution. Section 3 collects preliminary results, most of which are modifications of well-known theorems. The most relevant part is subsection 3.1, where we set up the geometry related to the natural scaling of the equation. Next in Section 4 we study the boundedness of solutions and the lower semicontinuity of supersolutions. Moreover Section 5 is dedicated to those solutions to the Cauchy problem (1.4) which have a suitable global integrability condition, called $L^{\mathbf{p}}$-solutions. There we carry on an analysis of the propagation of the support of solutions together with the existence of solutions to (1.4) when the initial datum is taken in L^2. In Section 6 we build the Barenblatt solutions and we study their positivity set. Finally, Section 7 contains the proof of the main theorem, split in several Propositions. A Section 8 is devolved to the consequences of the previous one and the questions left open; while all those technicalities or simple/known facts omitted along the body of the work for ease of readability, are collected in an Appendix in Section 9.

Notation

- For $N \in \mathbb{N}$ positive real numbers $\{p_i\}_{i=1,\dots,N}$ we will denote by $\mathbf{p}$ the vector $\mathbf{p} = (p_1, p_2, \dots, p_N)$. Without loss of generality, we will suppose that the p_is are ordered, i.e.

$$\min_i\{p_i\} = p_1 \leq p_2 \leq \cdots \leq p_N = \max_i\{p_i\}.$$

The symbols

$$\bar{p} = \left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}\right)^{-1}, \qquad \text{and} \qquad \bar{p}^* = \frac{N\bar{p}}{N-\bar{p}},$$

will be referred to as the *harmonic mean* of p_i's and the Sobolev exponent of the harmonic mean. For a number $i \in \{1,\dots,N\}$, $\bar{p}_i$ stands for $\bar{p}_i = \bar{p}(1+i/N)$, most notably along the work we will consider $\bar{p}_1 = \bar{p}(1+1/N)$ and $\bar{p}_2 = \bar{p}(1+2/N)$.

- The special exponents α, α_i for $i = 1,\dots,N$ are defined with

$$\alpha = \frac{N}{N(\bar{p}-2)+\bar{p}}, \qquad \text{and} \qquad \alpha_i = \left(\frac{1+2\alpha}{p_i}\right) - \alpha = \frac{1}{p_i}\left(\frac{N(\bar{p}-p_i)+\bar{p}}{N(\bar{p}-2)+\bar{p}}\right).$$

- We will use the symbols $\partial_i = \frac{\partial u}{\partial x_i}$, $\partial_t = \frac{\partial u}{\partial t}$ for the i-th weak derivative and the time derivative; the weak gradient will be denoted with Du.

- We will use the single notation $\int$ for the space variables, independently of how many variables we are considering, leaving to differentials dx or dx_j the duty to point the space variables in which we consider the integration. On the other hand we will use the notation $\int\int$ when we integrate along both space and time, and $⨍⨍_E f = |E|^{-1}\iint_E f$ as usual denotes the weighted integral. When nothing is written, the integral symbol refers to the whole N-space $\mathbb{R}^N$.

- For a function v and a number $k \in \mathbb{R}$ we define the truncations as $(v-k)_+ = \max\{(v-k),0\}$, and $(v-k)_- = \max\{-(v-k),0\}$. Finally, when we write $\sup_E v$ or $\inf_E v$ we will always refer to the essential supremum/infimum of the function in an euclidean set E.

- Given $T > 0$ and $\Omega \subset \mathbb{R}^N$ an open set, we let $\Omega_T := \Omega \times (0,T)$ and $S_T := \mathbb{R}^N \times (0,T)$. More generally we will denote $S_{a,b} = \mathbb{R}^N \times (a,b)$ for numbers $a < b$, $a,b \in \bar{\mathbb{R}}$ and $S_\infty = \mathbb{R}^N \times \mathbb{R}_+ = \mathbb{R}^N \times (0,+\infty)$. If $E \subset \mathbb{R}^N$, we will write $\Omega \subset\subset E$ to consider an open bounded subset Ω of E that is compactly contained in E. The parabolic boundary of Ω_T is the set $\Sigma_T \cup (\Omega \times \{0\})$ with $\Sigma_T := \partial\Omega \times [0,T]$ the lateral boundary.

- We will denote by $K_\rho(y)$ the cube in $\mathbb{R}^N$ of side ρ with center in $y \in \mathbb{R}^N$, and by Q_1^- the unitary backward cylinder, i.e. $Q_1^- = \prod_{i=1}^N\{|x_i| < 1/2\} \times (-1,0]$. Analogous notation is adopted for centered and forward unitary cylinders Q_1, Q_1^+. If $x_o \in \mathbb{R}^N$, the set $x_o + \mathcal{K}_\rho(\theta)$ will be referred to as the *anisotropic cube* of radius ρ, "magnitude" θ and center x_o, i.e.,

$$x_o + \mathcal{K}_\rho(\theta) = \left\{x \in \mathbb{R}^N : \ |x_i - x_{o,i}| < \theta^{\frac{p_i-\bar{p}}{p_i}}\rho^{\frac{\bar{p}}{p_i}}, \quad \forall i = 1\dots,N\right\} = \prod_{i=1}^{N}\left\{|x_i - x_{o,i}| < \theta^{\frac{p_i-\bar{p}}{p_i}}\rho^{\frac{\bar{p}}{p_i}}\right\}.$$

If either $\theta = \rho$ or $p_i = p$ for all $i = 1,\dots,N$, then $x_o + \mathcal{K}_\rho(\theta) = K_\rho(x_o)$.
For any $\rho, \theta > 0$ and $(x_o,t_o) \in \mathbb{R}^{N+1}$, we will consider the *anisotropic cylinders* as indicated below:

$$\begin{cases} \text{centered cylinders: } (x_o,t_o) + \mathcal{Q}_\rho(\theta) = (x_o + \mathcal{K}_\rho(\theta)) \times (t_o - \theta^{2-\bar{p}}\rho^{\bar{p}}, t_o + \theta^{2-\bar{p}}\rho^{\bar{p}}); \\ \text{forward cylinders: } (x_o,t_o) + \mathcal{Q}_\rho^+(\theta) = (x_o + \mathcal{K}_\rho(\theta)) \times [t_o, t_o + \theta^{2-\bar{p}}\rho^{\bar{p}}); \\ \text{backward cylinders: } (x_o,t_o) + Q_\rho^-(\theta) = (x_o + \mathcal{K}_\rho(\theta)) \times (t_o - \theta^{2-\bar{p}}\rho^{\bar{p}}, t_o]. \end{cases}$$

- A constant C is said to depend only on the data if it depends only on $\mathbf{p}$ and N.

2 Anisotropic Sobolev Spaces

A natural definition of solution

Many persons who have not studied mathematics

confuse it with arithmetic and consider it a dry and arid science.

Actually, however, this science requires great fantasy.

-Sophia Kovalevskaya-

Too much Happiness, A. Munro

In this chapter we define those function spaces that are most natural for solutions to equation (1.1), i.e. spaces of functions that have different degrees of summability along different directions. Once that the natural functional setting is prepared, we give the definition of a local weak solution in a bounded domain, local weak solution in a strip, local weak $L^{\mathbf{p}}$-solution and finally we define what we mean by *taking initial datum in* L^2. Moreover we show that our definition of solution automatically guarantees the solution to be continuous as a map $t \to u(\cdot, t) \in L^2(\Omega)$. Next we refine the study of some important inequalities on the previously defined anisotropic Sobolev spaces, both of elliptic and parabolic nature, following the lines of [33], [41], [43], [64].

2.1 Functional setting

We introduce the natural elliptic and parabolic anisotropic spaces. Their common feature is the membership of the weak directional derivative $\partial_i u$ to a different space L^{p_i} for each $i = 1, \dots, N$. We define

$$W_o^{1,\mathbf{p}}(\Omega) := \{u \in W_o^{1,1}(\Omega) |\, \partial_i u \in L^{p_i}(\Omega)\}, \quad \text{with norm} \quad ||u||_{W_o^{1,\mathbf{p}}(\Omega)} := \sum_{i=1}^N ||\partial_i u||_{L^{p_i}(\Omega)},$$

$$W_{loc}^{1,\mathbf{p}}(\Omega) := \{u \in W_{loc}^{1,1}(\Omega) |\, \partial_i u \in L_{loc}^{p_i}(\Omega)\}, \quad \text{with norm} \quad ||u||_{1,\mathbf{p}} := ||u||_{L^1(\Omega)} + \sum_{i=1}^N ||\partial_i u||_{L^{p_i}(\Omega)},$$

denoting with $W_o^{1,1}(\Omega)$ the space of functions belonging to $W^{1,1}(\Omega)$ which have zero traces on $\partial\Omega$. It is simple to verify that these are Banach spaces. In general, if V is a Banach space and V' its dual, we shall consider solutions u of evolution equations where $u \in L^p(0,T;V)$ for $p \geq 1$, and the equation holds in the space $L^{p'}(0,T;V')$. With the respective norms

$$\begin{aligned} \|f\|_{L^p} &= \left(\int_0^T \|f(s)\|_V^p \, ds \right)^{1/p}, \quad 1 < p < \infty, \\ \|f\|_{L^\infty} &= \sup\{\|f(s)\|_V \ : s \in (0,T)\}, \end{aligned} \tag{2.1}$$

each $L^p(0,T,V)$ is a Banach space and moreover if V is separable and $1 \le p < \infty$, then $L^p(0,T;V)$ is separable.

With this stipulations, we define the anisotropic parabolic spaces

$$L^{\mathbf{p}}(0,T;W_o^{1,\mathbf{p}}(\Omega)) := \{u \in L^1(0,T;W_o^{1,1}(\Omega)) |\, \partial_i u \in L^{p_i}(0,T;L^{p_i}(\Omega))\},$$
$$L^{\mathbf{p}}_{loc}(0,T;W_o^{1,\mathbf{p}}(\Omega)) := \{u \in L^1_{loc}(0,T;W_o^{1,1}(\Omega)) |\, \partial_i u \in L^{p_i}_{loc}(0,T;L^{p_i}_{loc}(\Omega))\},$$
$$L^{\mathbf{p}}(0,T;W^{1,\mathbf{p}}(\mathbb{R}^N)) := \{u \in L^1(0,T;W^{1,1}(\mathbb{R}^N)) |\, \partial_i u \in L^{p_i}(0,T;L^{p_i}(\mathbb{R}^N))\},$$

equipped respectively with norms (2.1). These spaces are in some sense the smallest request for a function to be measurable, weakly differentiable and to satisfy the requested integrability on the directional derivatives.

2.2 Definitions of local weak solution

In this subsection we discuss the meaning of solution to the equation (1.1) and to the Cauchy Problem (1.4). The main novelty is the consideration of local weak solutions also to the Cauchy Problem, that leads us to different results on properties of solutions.

Definition 2.1. *A function*

$$u \in L^2_{loc}(0,T;L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}_{loc}(0,T;W^{1,\mathbf{p}}_{loc}(\Omega))$$

is called a local weak solution to (1.1) *in* Ω_T*, if for almost every* $0 < t_1 < t_2 < T$ *and any compact set* $K \subset\subset \Omega$ *the following integral equality holds true,*

$$\int_K u\varphi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \Big(- u\, \partial_t \varphi + \sum_{i=1}^N |\partial_i u|^{p_i - 2} \partial_i u\, \partial_i \varphi \Big)\, dxdt = 0, \tag{2.2}$$

for all test function $\varphi \in C^\infty_{loc}(0,T;C^\infty_o(K))$*. By a density and approximation argument this actually holds for any test function of the kind*

$$\varphi \in W^{1,2}_{loc}(0,T;L^2_{loc}(K)) \cap L^{\mathbf{p}}_{loc}(0,T;W_o^{1,\mathbf{p}}(K))$$

for any rectangular domain $K \subset\subset \Omega$*. Secondly, by a local weak solution to the equation* (1.1) *in* $\mathbb{R}^N \times \mathbb{R}_+$*, we mean a function*

$$u \in L^2_{loc}(\mathbb{R}_+;L^2_{loc}(\mathbb{R}^N)) \cap L^{\mathbf{p}}_{loc}(\mathbb{R}_+;W^{1,\mathbf{p}}_{loc}(\mathbb{R}^N))$$

that for each choice of $T > 0$ *and* $\Omega \subset\subset \mathbb{R}^N$ *is a weak local solution in* Ω_T *in the meaning of previous definition. Finally, by a* $L^{\mathbf{p}}$*-solution, we will understand a local weak solution* u *such that* $u \in \cap_{i=1}^N L^{p_i}(S_T)$*.*

The boundary terms are attained within the meaning of the following Lebesgue's point-limits for each $i = 1, 2$,

$$\int_K u(x,t_i)\eta(x,t_i)\, dx =: \lim_{h \downarrow 0} \int_{t_i}^{t_i + h} \int_K u(x,s)\eta(x,s)\, dxds.$$

Considered that $u \in L^1_{loc}(0,T;L^2_{loc}(\Omega))$ by definition of local weak solution, for almost every point $t \in (0,T)$ it is possible to apply Lebesgue's differentiation theorem and recover the limits above.
Nevertheless, by Proposition 2.1 equality (2.2) of Definition 2.1 is actually holding for every time $0 < t_1 < t_2 < T$ and $u \in C(0,T;L^2_{loc}(\Omega))$, so that the first two integrals in (2.2) are well- defined.

Definition 2.2. *A local weak solution* u *to* (1.1) *is said to be a local weak solution to Cauchy Problem* (1.4) *with initial datum* $u_0 \in L^2(\Omega)$ *if the initial values are attained in the following sense*

$$\lim_{h \downarrow 0} \int_\Omega \Big(\frac{1}{h} \int_0^h u(x,s)\, ds - u_0(x) \Big)^2 dx = 0. \tag{2.3}$$

We end this section by showing that Definition 2.1 can be equivalently stated by assuming a priori the continuity of the law $(0,T) \ni t \to u(\cdot,t) \in L^2_{loc}(\Omega)$.

Proposition 2.1. *If* $u \in L^2_{loc}(0,T;L^2_{loc}(\Omega))$ *is a local weak solution to* (1.1) *then* $u \in C_{loc}(0,T;L^2_{loc}(\Omega))$.

Proof. For u solution to the equation (1.1), let us fix $0<t_1<t_2<T$ and $K \subset\subset \Omega$. Let us call

$$D(t_1,t_2,K) = L^2(t_1,t_2;L^2(K)) \cap L^{\mathbf{p}}(t_1,t_2;W^{1,\mathbf{p}}(K)), \quad \text{and} \quad \mathcal{D}(t_1,t_2,K) = (D(t_1,t_2,K))',$$

and let us equip them with norms

$$\|u\|_{D(t_1,t_2,K)} = \|u\|_{L^2(t_1,t_2;L^2(K))} + \|u\|_{L^{\mathbf{p}}(t_1,t_2;W^{1,\mathbf{p}}(K))} \quad \text{and}$$

$$\|\partial_t u\|_{\mathcal{D}(t_1,t_2,K)} = \sup\left\{ \int_{t_1}^{t_2}\int_K (\partial_t u)\varphi\, dxdt \,\middle|\, ||\varphi||_{D(t_1,t_2,K)} \le 1 \right\}.$$

We interpret $\partial_t u$ as the bounded linear operator $\partial_t u : D(t_1,t_2,K) \to \mathbb{R}$ that for each $\varphi \in D(t_1,t_2,K)$ gives

$$\int_{t_1}^{t_2}\int_K (\partial_t u)\varphi\, dxdt = \int_K u\varphi\, dx\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} u(\partial_t\varphi)\, dxdt = -\sum_{i=1}^N \int_{t_1}^{t_2}\int_K |\partial_i u|^{p_i-2}\partial_i u \partial_i\varphi\, dxdt, \tag{2.4}$$

by the equation itself, with

$$\|\partial_t u\|_{\mathcal{D}(t_1,t_2;K)} = \sup\left\{ \int_{t_1}^{t_2}\int_K (\partial_t u)\varphi\, dxdt \,\middle|\, ||\varphi||_{D(t_1,t_2;K)} \le 1 \right\} \le \sum_{i=1}^N ||\partial_i u||^{p_i-1}_{L^{p_i}(t_1,t_2;L^{p_i}(K))}, \tag{2.5}$$

by Hölder inequality. Now we let a,b be numbers such that $0<a<t_1<t_2<b<T$ and we extend u to be a function defined by symmetry in (a,b) by

$$u(t_1-s,x) := u(t_1+s,x), \quad \text{and} \quad u(t_2+\tau,x) := u(t_2-\tau,x), \quad \forall s \in [0,t_1-a], \quad \tau \in [0,b-t_2],$$

therefore defining $a = \min\{(t_2-t_1)/2, t_1/2\}$ and $b = \min\{(t_2-t_1)/2, (T-t_2)/2+t_2\}$. Now we let $\eta \in C^\infty_o(a,b)$, $0 \le \eta \le 1$ be a cut-off function such that $\eta \equiv 1$ in $[t_1,t_2]$ and we define $\tilde{u} = \eta u$. This function satisfies

$$||\partial_t\eta||_{L^\infty(a,b)} \le \gamma(a,b,t_1,t_2) = \max\{(t_1-a)^{-1},(b-t_2)^{-1}\}, \quad \text{and} \quad \tilde{u}_{|(t_1,t_2)} = u, \quad \text{and} \quad \tilde{u}(a) = \tilde{u}(b) = 0,$$

and similarly to u we have

$$\int_{t_1}^{t_2}\int_K (\partial_t\tilde{u})\varphi\, dxdt = \int_{t_1}^{t_2}\int_K u\varphi(\partial_t\eta) dxdt - \sum_{i=1}^N \int_{t_1}^{t_2}\int_K \eta|\partial_i u|^{p_i-2}\partial_i u\partial_i\varphi\, dxdt, \tag{2.6}$$

so that we have the bound $\|\partial_t\tilde{u}\|_{\mathcal{D}(t_1,t_2,K)} \le C\|u\|_{D(t_1,t_2,K)}$ that implies the membership $\partial_t\tilde{u} \in \mathcal{D}(t_1,t_2,K)$. This time, in comparison with (2.5), we used also the condition $u \in L^2(t_1,t_2,L^2(K))$ on the first integral on the right hand side, together with the boundedness of the derivative $\partial_t\eta$, as soon as we have already fixed time intervals $(t_1,t_2) \subset (a,b)$. To further regularize $\tilde{u}$, we let $0 \le \phi \in C^\infty_o(\mathbb{R})$ be a *mollifier*, i.e. a function such that

$$\int_{\mathbb{R}} \phi(t)\, dt = 1, \qquad \operatorname{supp}\phi \subset (-1,1),$$

and for $n \in \mathbb{N}$ let us regularise $\tilde{u}$ with $\eta_n(t) := n\,\phi(nt)$ by setting

$$\tilde{u}_n(x,t) := \tilde{u} * \eta_n = \int_{\mathbb{R}} \tilde{u}(x,t-s)\eta_n(s)\, ds, \quad \Rightarrow \quad \tilde{u}_n \in C^\infty_o((a,b),W^{1,\mathbf{p}}_o(K)) \quad \text{and} \quad \tilde{u}_n \to^{n\to\infty}_{L^2(a,b;L^2(K))} \tilde{u}.$$

Moreover it is simple to see that $\tilde{u}_n \to \tilde{u}$ in $D(t_1,t_2,K)$, because regularisation has been performed only in time. Therefore $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $D(t_1,t_2,K)$. Usual properties of mollifiers give that

$$\partial_t(\tilde{u}_n) = (\partial_t\tilde{u}) * \eta_n, \quad \text{and} \quad ||\tilde{u}_n||_{L^2(t_1,t_2;L^2(K))} \le \|\tilde{u}\|_{L^2(t_1,t_2;L^2(K))}.$$

We show that $\partial_t \tilde{u}_n \in \mathcal{D}(t_1, t_2, K)$ by evaluating, through the properties above, the following quantity

$$\begin{aligned}\int_{t_1}^{t_2}\int_K (\partial_\tau \tilde{u}_n)(x,\tau)\varphi(x,\tau)\,dxd\tau = \\ = \int_{t_1}^{t_2}\int_K \Big(\int_{\mathbb{R}} \partial_\tau(u\eta)(x,\tau-s)\eta_n(s)\varphi(x,\tau)\,ds\Big)dxd\tau \le \Big(\int_{\mathbb{R}}\eta_n(s)ds\Big)||\partial_t\tilde{u}||_{\mathcal{D}(t_1,t_2,K)}.\end{aligned} \tag{2.7}$$

Thus, thanks to (2.5) and (2.6) also the membership $\partial_t \tilde{u}_n \in \mathcal{D}(t_1, t_2, K)$ can be deduced. Finally, thanks to the condition $\tilde{u}_n(a) = 0$, the regularity in time of $\tilde{u}_n$ and fundamental theorem of calculus, we write for $m, n \in \mathbb{N}$

$$\begin{aligned}\frac{||\tilde{u}_n(t)-\tilde{u}_m(t)||^2_{L^2(K)}}{2} &= 0 + \int_a^t \partial_t ||\tilde{u}_n(s)-\tilde{u}_m(s)||_{L^2(K)}\,ds = \\ &= \int_a^t\int_K (\partial_t\tilde{u}_n - \partial_t\tilde{u}_m)(\tilde{u}_n-\tilde{u}_m)\,dxds \\ &\le 2\|\partial_t\tilde{u}_n\|_{\mathcal{D}(t_1,t_2,K)}\,\|\tilde{u}_n-\tilde{u}_m\|_{D(t_1,t_2,K)} \to 0.\end{aligned}$$

when $m, n \to \infty$, because $\tilde{u}_n$ is a Cauchy sequence in $D(t_1, t_2, K)$. The estimate being independent of $t \in (a, b)$, we can pass to the evaluation of the supremum of the left-hand side and infer that the sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([a, b], L^2(K))$. By completeness it converges uniformly to a function $v \in C([a, b], L^2(K))$ but as already $\{\tilde{u}_n\}_{n\in\mathbb{N}} \to \tilde{u}$ in $L^2(a, b; L^2(K))$ by regularisation, we can identify $v = \tilde{u}$ and conclude the proof by remembering that $\tilde{u} = u$ on (t_1, t_2). □

Remark 2.1. *Following the lines of ([62], Prop. 1.2 Chap. III), in Proposition 2.1 we proved a little bit more. Indeed, we have shown that if $u \in D(t_1, t_2, K)$ is such that $\partial_t u \in \mathcal{D}(t_1, t_2, K)$ then $u \in C_{loc}(0, T, L^2_{loc}(\Omega))$.*

We end this section with a discussion on global extension of the equation up to time value T. Indeed, if we know that $u \in L^2(0, T; L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}(0, T; W^{1,\mathbf{p}}_{loc}(\Omega))$, which is, if a local weak solution is globally integrable in time, then Definition 2.1 is valid for all $0 \le t_1 < t_2 \le T$. This comes as an application of estimates that are global in time. By our integrability assumption, for almost every time t_2 we have that $u(\cdot, t_2) \in L^2_{loc}(\Omega)$ so that the sequence $\{u(\cdot, t_2)\}_{t_2}$ is uniformly bounded in $L^2_{loc}(\Omega)$, therefore admitting a subsequence weakly convergent $\{u(\cdot, t_2)\} \rightharpoonup \bar{u}$. The weak limit is identified by means of the equation, using the local definition itself. Indeed, by definition for each $\varphi \in C^\infty_o(\Omega)$ independent of time

$$\int_\Omega \bar{u}(x)\phi(x)\,dx \leftarrow \int_\Omega u(x,t_2)\varphi(x)\,dx = -\int_\Omega u(x,t_1)\varphi(x)\,dx - \int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^N |\partial_i u|^{p_i-2}\partial_i u\,\partial_i\varphi\,dxdt,$$

for $t_2 \uparrow T$. Again by global integrability, the energy term is continuous on time and the limit for $t_2 \uparrow T$ is achieved on the right. Thus the candidate $\bar{u}$ is our new definition of $u(\cdot, T)$.
Finally, in case $u \in L^2(0, T; L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}(0, T; W^{1,\mathbf{p}}_{loc}(\Omega))$ is globally integrable in time, the proof of Proposition 2.1 gives us that $u \in C([0, T]; L^2(\Omega))$.

2.3 Anisotropic Embeddings

Next we recall the following anisotropic elliptic embedding. It consists in a Sobolev-type inequality derived for powers of the function that are different in each coordinate direction.

Proposition 2.2 (Sobolev-Troisi embedding)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a rectangular bounded domain, $\bar{p} < N$ and $\alpha_i > 0$, $i = 1, \dots, N$. If we define*

$$p^*_\alpha = \bar{p}^*\frac{\tilde{\alpha}}{N}, \qquad \tilde{\alpha} = \sum_{i=1}^N \alpha_i,$$

then there exists a constant $C = C(N, \mathbf{p}, \alpha) > 0$ such that for all $u \in W^{1,\mathbf{p}}_0(\Omega)$ it holds

$$\|u\|_{L^{p^*_\alpha}(\Omega)} \le C\prod_{i=1}^N \|\partial_i |u|^{\alpha_i}\|^{\frac{1}{\tilde{\alpha}}}_{L^{p_i}(\Omega)}. \tag{2.8}$$

Remark 2.2. *The existence and boundedness in $L^{p_i}(\Omega)$ of the directional Sobolev derivatives $\partial_i(|u|^{\alpha_i})$ is part of the hypothesis of Proposition 2.2, being otherwise (2.8) vacuously true.*

Proof. First we show the result for $u_\lambda = \max\{-\lambda, \min\{u, \lambda\}\}$, and then by a dominated convergence argument we will end the proof by taking the limit for $\lambda \to \infty$. Moreover, by a monotone convergence argument we suppose Ω bounded, being otherwise possible to approximate u by compactly supported functions.

STEP 1- *Result (2.8) holds for truncated functions u_λ.*

Let $u_\lambda \in W^{1,\mathbf{p}}_o(\Omega) \cap L^\infty(\Omega)$ and let us fix $\tilde{u}_\lambda \in AC(l^i)$, i.e. a representative of u_λ which is absolutely continuous along any line l^i parallel to the coordinate axis x_i. For $x \in \Omega$, let l^i_x be a line parallel to a coordinate axis which joins x to a point $y \in \partial\Omega$. We suppose the right hand side of (2.8) finite, being the inequality otherwise trivial. Therefore the membership $\partial_i(|\tilde{u}_\lambda|^{\alpha_i}) \in L^{p_i}(\Omega)$ implies again by AC characterisation of Sobolev functions along lines Theorem 9.5 that $|\tilde{u}_\lambda|^{\alpha_i} \in AC(l^i_x)$.

Now for powers $s_i > 1$, the functions $v_i = |\tilde{u}_\lambda|^{\alpha_i}$ and $v_i^{s_i}$ are absolutely continuous and satisfy

$$v_i^{s_i}(x) := (|\tilde{u}|^{\alpha_i})^{s_i} = s_i \int_{l^i_x} v_i^{s_i-1}(x+te_i)|\partial_i v_i|(x+te_i)\, dt \le s_i \Big(\int_{\mathbb{R}_i} v_i^{\frac{(s_i-1)p_i}{p_i-1}}(x) dt \Big)^{\frac{p_i-1}{p_i}} \Big(\int_{\mathbb{R}_i} |\partial_i v_i|^{p_i}(x)\, dt \Big)^{\frac{1}{p_i}}, \tag{2.9}$$

by the fact that $u(y) = 0$ for $\mathcal{L}^{N-1}$-a.e. $y \in \partial\Omega$, denoting with

$$\int_{\mathbb{R}_i} f(x)\, dt := \int_{\mathbb{R}} f(x + te_i)\, dt.$$

Now we take the product and successively the $(N-1)$-th root of (2.9) to get

$$\prod_{i=1}^N v_i^{\frac{s_i}{N-1}} \le C(s_i) \prod_{i=1}^N \Big(\int_{\mathbb{R}_i} v_i^{\frac{(s_i-1)p_i}{p_i-1}}\, dt \Big)^{\frac{p_i-1}{(p_i)(N-1)}} \Big(\int_{\mathbb{R}_i} |\partial_i v_i|^{p_i}\, dt \Big)^{\frac{1}{(N-1)p_i}}. \tag{2.10}$$

Furthermore we integrate (2.10) along x_j, observing that the j-th integral in (2.10) is independent of x_j, to get

$$\begin{aligned} \int_{\mathbb{R}_j} \prod_{i=1}^N v_i^{\frac{s_i}{N-1}}\, dt \le C \Big(\int_{\mathbb{R}_j} v_j^{(s_j-1)p'_j}(x)\, dt \Big)^{\frac{1}{(N-1)p'_j}} \Big(\int_{\mathbb{R}_j} |\partial_j v_j|^{p_j}\, dt \Big)^{\frac{1}{(N-1)p_j}} \times \\ \times \int_{\mathbb{R}_j} \Big[\prod_{i \ne j} \Big(\int_{\mathbb{R}_i} v_i^{(s_i-1)p'_i}\, dt \Big)^{\frac{1}{p'_i(N-1)}} \Big(\int_{\mathbb{R}_i} |\partial_i v_i|^{p_i}\, dt \Big)^{\frac{1}{(N-1)p_i}} \Big] dt, \end{aligned} \tag{2.11}$$

denoting with $p'_i = p_i/(p_i - 1)$ for all $i \in \{1, .., N\}$. In order to exchange product and integral signs, we use the Hölder inequality (9.7) with $r = 1$ and

$$\sum_{i \ne j} \frac{1}{(N-1)p'_i} + \frac{1}{(N-1)p_i} = 1,$$

to get

$$\begin{aligned} \int_{\mathbb{R}_j} \prod_{i=1}^N v_i^{\frac{s_i}{N-1}}\, dt \le & C \Big(\int_{\mathbb{R}_j} v_j^{(s_j-1)p'_j}(x)\, dt \Big)^{\frac{1}{(N-1)p'_j}} \Big(\int_{\mathbb{R}_j} |\partial_j v_j|^{p_j}\, dt \Big)^{\frac{1}{(N-1)p_j}} \times \\ & \times \prod_{i \ne j} \Big[\Big(\int_{\mathbb{R}_j} \int_{\mathbb{R}_i} v_i^{(s_i-1)p'_i}\, dt \Big)^{\frac{1}{p'_i(N-1)}} \Big(\int_{\mathbb{R}_j} \int_{\mathbb{R}_i} |\partial_i v_i|^{p_i}\, dt \Big)^{\frac{1}{(N-1)p_i}} \Big] dt. \end{aligned} \tag{2.12}$$

The whole procedure can be repeated for all $j \in \{1, .., N\}$ to obtain

$$\int_{\mathbb{R}^N} \Big(\prod_{i=1}^N v_i^{\frac{s_i}{N-1}} \Big) dx \le C \prod_{i=1}^N \Big(\int_{\mathbb{R}^N} v_i^{(s_i-1)p'_i}\, dx \Big)^{\frac{1}{p'_i(N-1)}} \Big(\int_{\mathbb{R}^N} |\partial_i v_i|^{p_i}\, dx \Big)^{\frac{1}{(N-1)p_i}}, \tag{2.13}$$

which is, by expliciting $v_i = |\tilde u_\lambda|^{\alpha_i}$,

$$\int_{\mathbb{R}^N} |\tilde u_\lambda|^{\sum \frac{\alpha_i s_i}{N-1}}\,dx \le C\prod_{i=1}^N \left(\int_{\mathbb{R}^N} |\tilde u_\lambda|^{\alpha_i(s_i-1)p_i'}\,dx\right)^{\frac{1}{p_i'(N-1)}} \left(\int_{\mathbb{R}^N} |\partial_i |\tilde u_\lambda|^{\alpha_i}|^{p_i}\,dx\right)^{\frac{1}{(N-1)p_i}}. \tag{2.14}$$

We set the powers of $|\tilde u_\lambda|$ on the integral on the left and the first one on the right as the same. This results in a system of N equations as follows, that can be solved for the only number p^*_α, that is in symbols

$$\begin{cases}\alpha_i(s_i-1)p_i/(p_i-1) = q = \sum_{i=1}^N \frac{\alpha_i s_i}{N-1}\\ \forall i = 1,..,N,\end{cases} \quad\Rightarrow\quad \begin{cases} q = p^*_\alpha = \left(\frac{N\bar p}{N-\bar p}\right)\sum_{i=1}^N \frac{\alpha_i}{N},\\ s_i = 1+\frac{\bar p\sum_i \alpha_i}{(N-\bar p)}\left(\frac{1}{\alpha_i p_i'}\right) = 1+\left(\frac{q}{\alpha_i p_i'}\right).\end{cases}$$

This proves (2.8) for u_λ, by reabsorbing on the left the first term of the product on the right-hand side of (2.14).

STEP2- *Passage to the limit*

This shows (2.8) for the dominated function $|u_\lambda| \le |u| \in W^{1,\mathbf{p}}(\Omega)$, being

$$u_\lambda = \begin{cases} \lambda & \text{almost everywhere in} \quad [u\ge\lambda],\\ u & \text{almost everywhere in} \quad [-\lambda < u < \lambda],\\ -\lambda & \text{almost everywhere in} \quad [u \le -\lambda].\end{cases}$$

Now we apply the dominated convergence theorem for $\lambda\uparrow\infty$ to take the limit in (2.8) and get

$$\|u\|_{L^{p^*_\alpha}(\Omega)} = \lim_{\lambda\uparrow\infty}\|u_\lambda\|_{L^{p^*_\alpha}(\Omega)} \le \lim_{\lambda\uparrow\infty} C\prod_{i=1}^N \|\partial_i|u_\lambda|^{\alpha_i}\|^{\frac{1}{\bar\alpha}}_{L^{p_i}(\Omega)} = C\prod_{i=1}^N \|\partial_i|u|^{\alpha_i}\|^{\frac{1}{\bar\alpha}}_{L^{p_i}(\Omega)},$$

owing last equality to the validity of the passage to the limit to each component of the product. □

Remark 2.3. *When $\alpha_i \equiv 1$ then $p^*_\alpha = \bar p^*$ and we recover the classical inequality of Troisi [64],*

$$||u||^N_{L^{\bar p^*}(\Omega)} \le C\prod_{i=1}^N ||\partial_i u||_{L^{p_i}(\Omega)}. \tag{2.15}$$

See also Remark 4 in [17] for a more general approach through Young's functions.

Next result is a parabolic anisotropic Sobolev embedding.

Theorem 2.1. *Let $\Omega\subseteq\mathbb{R}^N$ be a rectangular domain, $\bar p < N$, $\alpha_i > 0$ for $i = 1,\dots,N$ and $\sigma\in[1,p^*_\alpha]$. For any number $\theta\in[0,\bar p/\bar p^*]$ define*

$$q = q(\theta,\mathbf{p},\alpha) = \theta\,p^*_\alpha + \sigma\,(1-\theta).$$

Then there exists a constant $C = C(N,\mathbf{p},\alpha,\theta,\sigma) > 0$ such that

$$\iint_{\Omega_T} |u|^q\,dx\,dt \le C\,T^{1-\theta\,\frac{\bar p^*}{\bar p}}\left(\sup_{t\in[0,T]}\int_\Omega |u|^\sigma(x,t)\,dx\right)^{1-\theta}\prod_{i=1}^N\left(\iint_{\Omega_T}|\partial_i|u|^{\alpha_i}|^{p_i}\,dx\,dt\right)^{\frac{\theta\,\bar p^*}{N\,p_i}}, \tag{2.16}$$

for any $u\in L^1(0,T;W^{1,1}_0(\Omega))$.

Proof. For a.e. $t\in[0,T]$, we apply interpolation inequality (9.8) as

$$\int_\Omega |u|^q(x,t)\,dx \le \left(\int_\Omega |u|^\sigma(x,t)\,dx\right)^{1-\theta}\left(\int_\Omega u^{p^*_\alpha}(x,t)\,dx\right)^\theta$$

and through (2.8) deduce

$$\int_\Omega |u|^q(x,t)\,dx \le C\left(\int_\Omega |u|^\sigma(x,t)\,dx\right)^{1-\theta}\prod_{i=1}^N\left(\int_\Omega |\partial_i|u|^{\alpha_i}|^{p_i}(x,t)\,dx\right)^{\frac{\theta\,\bar p^*}{N\,p_i}}.$$

If we set

$$\gamma_0 = 1-\theta, \quad f_0(t) = \int_\Omega |u|^\sigma(x,t)\,dx$$

and for $i = 1,\ldots,N$

$$\gamma_i = \frac{\theta\,\bar p^*}{N\,p_i}, \quad r = \frac{\bar p}{\bar p^*}\frac{1}{\theta} \ge 1, \quad f_i(t) = \int_\Omega |\partial_i|u|^{\alpha_i}|^{p_i}(x,t)\,dx,$$

then it is possible to integrate in the t-variable along $(0,T)$ and use the generalised Hölder inequality (9.7) in time, to get

$$\int_0^T \prod_{i=0}^N f_i^{\gamma_i}\,dt \le T^{1-\frac{1}{r}}\sup_{t\in[0,T]} f_0^{\gamma_0}\left(\int_0^T\prod_{i=1}^N f_i^{\gamma_i\,r}\,dt\right)^{\frac{1}{r}} \le T^{1-\frac{1}{r}}\sup_{t\in[0,T]} f_0^{\gamma_0}\prod_{i=1}^N\left(\int_0^T f_i\,dt\right)^{\gamma_i},$$

as long as the exponents satisfy

$$\sum_{i=1}^N r\,\gamma_i = 1, \qquad r\ge 1, \qquad \gamma_i > 0.$$

□

Corollary 2.1. *Let* $p_i > 2 \quad \forall i = 1,..,N$ *and* $u \in L^1(0,T;W_0^{1,1}(\Omega))$. *Let us define* $\bar p_2 = \bar p\,(1+2/N)$. *If* $2<\bar p_2<\bar p^*$, *then there is a constant* $C = C(N,p_i)>0$ *such that*

$$\int_0^T\int_\Omega |u|^{\bar p_2}\,dxdt \le C\bigg(\sup_{[0,T]}\int_\Omega |u|^2dx + \int_0^T\int_\Omega\sum_{i=1}^N|\partial_i u|^{p_i}\,dxdt\bigg)^{\frac{N+\bar p}{N}}. \tag{2.17}$$

Proof. It is sufficient to consider (2.16) with $\alpha_i\equiv 1$ for all $i=1,\ldots,N$, $\sigma = 2$, $\theta = \bar p/\bar p^*$ and consequently $q = \bar p_2 = \bar p(1+2/N)$. Next by estimating roughly from above each single integral term of (2.16) on the right by the whole integral sum on the right of (2.17) we get the claim because $(1-\theta)+\sum \bar p/(Np_i) = \bar p/N+1$. □

Remark 2.4. *In the isotropic case, previous Theorem 2.1 ensures an analogous local summability estimate without the assumption that* u *vanishes outside* Ω, *just by adding an* L^1 *term to the right-hand side. Unfortunately, this is no longer true in the anisotropic setting. Indeed, let us suppose that* u *does not vanish on* $\partial\Omega$, $\alpha_i\equiv 1$ *and let*

$$\eta\in C_o^\infty(2\Omega), \quad 0\le\eta\le 1, \quad \eta(\Omega)\equiv 1.$$

Let us apply Theorem 2.1 to $(u\eta)$ *to get*

$$\begin{aligned}\int\int_{\Omega_T} |u|^q\,dxdt &\le CT^{1-\theta\bar p^*/\bar p}\bigg(\sup_{t\in[0,T]}\int_{2\Omega}|u\eta|^\sigma(x,t)\,dx\bigg)^{1-\theta}\prod_{i=1}^N\bigg(\int_0^T\int_{2\Omega}|\partial_i u\eta|^{p_i}\,dxdt\bigg)^{\frac{\theta\bar p^*}{Np_i}}\\ &\le CT^{1-\theta\bar p^*/\bar p}\bigg(\sup_{t\in[0,T]}\int_{2\Omega}|u\eta|^\sigma(x,t)\,dx\bigg)^{1-\theta}\prod_{i=1}^N\bigg[\int_0^T\int_{2\Omega}\Big(|\partial_i u|^{p_i}+|u|^{p_i}|\partial_i\eta|^{p_i}\Big)\,dxdt\bigg]^{\frac{\theta\bar p^*}{Np_i}}.\end{aligned} \tag{2.18}$$

We observe that last integral on the right is not necessarily bounded: the function u *may be not included in* $L^{p_N}(\Omega)$ *(see for instance the counter-examples in [41], [43]). More precisely, under the previous assumptions on the parameters, it may happen that* $L^{\mathbf p}(0,T;W^{1,\mathbf p}_{\rm loc}(\mathbb R^N))\cap L^\infty(0,T;L^\sigma_{\rm loc}(\mathbb R^N))$ *fails to be contained in* $L^q(0,T;L^q_{\rm loc}(\mathbb R^N))$ *for* q *given above. In order to remove the zero boundary condition, one is either forced to assume a-priori a suitable degree of summability, or to further constrain the location of the* p_i*s.*

The following result deals with the problem mentioned in the previous remark when $\alpha_i\equiv 1$.

Theorem 2.2. *Let $1 \le p_1 \le \cdots \le p_N$, $\bar p < N$ and for any $\sigma \in [1, \bar p^*]$ let the critical parabolic exponent be*

$$\bar p_\sigma = \bar p \left(1 + \frac{\sigma}{N}\right). \tag{2.19}$$

Then the embedding

$$L^{p_N}(0,T; L^{p_N}_{\rm loc}(\mathbb{R}^N)) \cap L^{\mathbf{p}}(0,T; W^{1,\mathbf{p}}_{\rm loc}(\mathbb{R}^N)) \cap L^\infty(0,T; L^\sigma_{\rm loc}(\mathbb{R}^N)) \subset L^{\bar p_\sigma}(0,T; L^{\bar p_\sigma}_{\rm loc}(\mathbb{R}^N)) \tag{2.20}$$

holds true. Moreover, under the assumption

$$\bar p_\sigma > p_N = \max\{p_1, \dots, p_N\} \tag{2.21}$$

we directly have

$$L^{\mathbf{p}}(0,T; W^{1,\mathbf{p}}_{\rm loc}(\mathbb{R}^N)) \cap L^\infty(0,T; L^\sigma_{\rm loc}(\mathbb{R}^N)) \subset L^{\bar p_\sigma}(0,T; L^{\bar p_\sigma}_{\rm loc}(\mathbb{R}^N)).$$

Proof. By ordering of the p_is and Hölder's inequality we have

$$L^{p_N}(0,T; L^{p_N}_{\rm loc}(\mathbb{R}^N)) \subset L^{p_i}(0,T; L^{p_i}_{\rm loc}(\mathbb{R}^N)), \quad \forall i = 1, \dots, N.$$

Indeed, for all $K \subset\subset \mathbb{R}^N$,

$$\int_0^T \int_K |u|^{p_i}\, dxdt \le (T|K|)^{\frac{p_N - p_i}{p_N}} \left(\int_0^T \int_K |u|^{p_N}\, dxdt\right)^{\frac{p_i}{p_N}}. \tag{2.22}$$

Let $K \subset K'$ be two arbitraty compact sets in $\mathbb{R}^N$ and $\eta \in C^\infty_c(\mathbb{R}^N)$ a cuf-off function between them, i.e.

$$\eta_{|K} \equiv 1, \quad \eta_{|\{\mathbb{R}^N - K'\}} = 0, \quad 0 < \eta < 1, \quad |D\eta| \le C(\eta).$$

Let us apply previous Theorem 2.1 with $\alpha_i \equiv 1$, $\theta = \bar p/\bar p^*$ to $(u\,\eta)$. With the notations of our statement, it holds $q = \bar p_\sigma$ and we get

$$\int_0^T \int_K |u|^{\bar p_\sigma}\, dx\, dt \le \left(\sup_{t\in[0,T]} \int_{K'} |u\,\eta|^\sigma(x,t)\, dx\right)^{1-\theta} \prod_{i=1}^N \left(\int_0^T \int_{K'} |\partial_i(u\,\eta)|^{p_i}\, dx\, dt\right)^{\frac{\bar p}{N\, p_i}}.$$

The first factor on the right is finite by assumption, while for the other terms of the product on the right we use (2.22) to get the following estimate

$$\int_0^T \int_{K'} |\partial_i(u\,\eta)|^{p_i}\, dx\, dt \le C \int_0^T \int_{K'} |\partial_i u|^{p_i} + |u|^{p_i}\, dx\, dt < \infty,$$

and the proof of embedding (2.20) is completed. Let us prove the second embedding under assumption (2.21). On vectors $\mathbf{q} \in \mathbb{R}^N$ we consider the component-wise partial ordering

$$(q_1, \dots, q_N) \ge (r_1, \dots, r_N) \quad \Leftrightarrow \quad q_i \ge r_i \quad \text{for all } i = 1, \dots, N.$$

By abuse of notation we will say that, for $\lambda \in \mathbb{R}$ it holds $\mathbf{q} \ge \lambda$ to mean $\mathbf{q} \ge (\lambda, \dots, \lambda)$. Define by recursion the following sequence $\{\mathbf{p}^n\}_{n\in\mathbb{N}} \in \mathbb{R}^N$:

$$\begin{cases} \mathbf{p}^1 = (1, \dots, 1) \\ (\mathbf{p}^{n+1})_i = \min\{p_i, q^n\}, \end{cases} \qquad q^n := \frac{N + \sigma}{\sum_1^N 1/(\mathbf{p}^n)_i},$$

where we write $(\mathbf{p}^n)_i$ for the i-th component o the vector $\mathbf{p}^n$ and we observe that q^n is the critical parabolic exponent (2.19) for the vector $\mathbf{p}^n$. We claim that if (2.21) holds, then there exists $\bar n \in \mathbb{N}$ such that

$$q^{\bar n} \ge p_N. \tag{2.23}$$

Let us postpone the proof of (2.23) momentarily and show how this implies $u \in L^{p_N}(0,T;L^{p_N}_{\rm loc}(\mathbb{R}^N))$. Consider the formula that we obtain from the previous discussion relatively to the vector $\mathbf{p}^n$,

$$\int_0^T \int_K |u|^{q^n}\,dxdt \le \left(\sup_{t\in[0,T]}\int_{K'} |u\,\eta|^\sigma(x,t)\,dx\right)^{1-\theta} \prod_{i=1}^N \left(C\int_0^T\int_{K'} |\partial_i u|^{(\mathbf{p}^n)_i} + |u|^{(\mathbf{p}^n)_i}\,dx\,dt\right)^{\frac{\mathbf{p}^n}{N\,(\mathbf{p}^n)_i}}. \tag{2.24}$$

For $n=1$ inequality (2.24) ensures that $u \in L^{q^1}(0,T;L^{q^1}_{\rm loc}(\mathbb{R}^N))$. Suppose that $u \in L^{q^n}(0,T;L^{q^n}_{\rm loc}(\mathbb{R}^N))$ for some $q_n \le p_N$. Then, since $\mathbf{p}^{n+1} \le \mathbf{p}$, Hölder's inequality ensures $u \in L^{\mathbf{p}^{n+1}}(0,T;W^{1,\mathbf{p}^{n+1}}_{\rm loc}(\mathbb{R}^N))$ and the embedding (2.24) with vector $\mathbf{p}^{n+1}$ implies $u \in L^{q^{n+1}}(0,T;L^{q^{n+1}}_{\rm loc}(\mathbb{R}^N))$. At each step, the embedding improves the integrability of u. Therefore in a finite number $\bar n$ of steps the claim (2.23) implies that $\mathbf{p}^{\bar n} = \mathbf{p} = (p_1,\dots,p_N)$ and we get $u \in L^{\bar p_\sigma}(0,T;L^{\bar p_\sigma}_{\rm loc}(\mathbb{R}^N))$ by the claim. Clearly, this process also proves the second stated embedding.

\- *Proof of the Claim* (2.23)-

Since q^n is a fixed multiple of the harmonic mean of $\mathbf{p}^n$ and $q^1 = (N+\sigma)/N \ge 1 =: q^0$, it follows from

$$q^n \ge q^{n-1} \quad\Rightarrow\quad \mathbf{p}^{n+1} \ge \mathbf{p}^n \quad\Rightarrow\quad q^{n+1} \ge q^n$$

that $\{q^n\}$ is non-decreasing. Suppose there exists the smallest integer $1 \le h < N$ such that $q^n \le p_{h+1}$ for all $n \ge 0$, and let $q = \lim_n q^n \le p_{h+1}$. Then we infer

$$q^{n+1} = \frac{N+\sigma}{\sum_1^h \frac{1}{p_i} + \frac{N-h}{q^n}} \Rightarrow \quad q = \frac{N+\sigma}{\sum_1^h \frac{1}{p_i} + \frac{N-h}{q}} \quad\Leftrightarrow\quad q = r_h,$$

where we defined for all $k = 1,\dots,N$

$$r_k := \frac{k+\sigma}{\sum_1^k \frac{1}{p_i}}.$$

Notice that

$$r_k \le p_{k+1} \quad\Leftrightarrow\quad \frac{k+\sigma}{p_{k+1}} \le \sum_{i=1}^k \frac{1}{p_i}$$

so that, adding $1/p_{k+1}$ to both sides, rearranging and using the monotonicity of the p_i, we get

$$r_k \le p_{k+1} \quad\Rightarrow\quad r_{k+1} \le p_{k+2}.$$

Since $q = r_h \le p_{h+1}$ by assumption, we eventually get by induction $r_{N-1} \le p_N$, which is equivalent to say

$$p_N \ge \frac{N+\sigma}{\sum_1^N \frac{1}{p_i}} = \bar p\left(1+\frac{\sigma}{N}\right) = \bar p_\sigma.$$

This contradicts (2.21), proving the claim (2.23) and the theorem. □

3 Basic properties of solutions

Poi piovve dentro a l'alta fantasia.

-Dante Alighieri-

Canto XVII Purgatorio, Divina Commedia.

In this section we make an analysis of the scaling properties of solutions to (1.1) and establish an important correspondence of the latter with the solutions to a Fokker-Planck equation. The parametric scaling that we employ is just one choice amongst many others which constitute the group of transformations of (1.1), as G.I. Barenblatt defined for general equations in [7], [8]. This approach leads us to the definition of a self-similar solution, a *Barenblatt Fundamental Solution*, and to the description of the self-similar geometry where the equation can be naturally read.
Furthermore, we determine some particular and some more general energy estimates; we study conservation of L^1 and L^2 mass, and we state a De Giorgi-type Lemma, which can be thought of as a sort of measure-theoretical principle of maximum. Indeed, loosely speaking it states that if the relative measure of the set where u is greater than a certain value is small enough, then u is above the half of that value in at least half of the set itself.
On a successive step, we make a detailed analysis of the local clustering (following [25]) of solutions to (1.1), that will be useful to find a lower bound for the solution u in an undetermined region of Ω_T, that will be the starting point for further expansion of positivity.
Finally we prove two comparison principles, one of local nature and the other of global nature, that can be both read either for solutions to (1.1), or for solutions to the Fokker Planck equation.

3.1 Scaling properties of solutions.

Proposition 3.1. *Let u be a solution to the equation (1.1) in Ω_T, let $M, \rho > 0$ be two chosen parameters, and define the parametric transformation*

$$T_{\rho,M}(x,t) = \bigg(M^{\frac{p_i - \bar{p}}{p_i}} \rho^{\frac{\bar{p}}{p_i}} x_i, M^{2-\bar{p}} \rho^{\bar{p}} t \bigg). \tag{3.1}$$

Then the transformed function

$$\mathcal{T}(u)(x,t) = M^{-1} u \bigg(T_{\rho,M}(x,t) \bigg) = M^{-1} u \bigg(M^{\frac{p_i - \bar{p}}{p_i}} \rho^{\frac{\bar{p}}{p_i}} x_i, M^{2-\bar{p}} \rho^{\bar{p}} t \bigg) \tag{3.2}$$

is still a solution to (1.1) in $T^{-1}_{\rho,M}(\Omega_T)$.

Proof. If generally $\mathcal{T}(u) = M^{-1}u\Big(L_i x_i, Tt\Big)$ for a number $L > 0$, then

$$\partial_t\Big(\mathcal{T}u\Big) = M^{-1}T\Big(\partial_t u(L_i x_i, Tt)\Big), \quad \partial_i\Big(\mathcal{T}u\Big) = M^{-1}L_i\partial_i u(L_i x_i, Tt),$$

thus using the equation for u, the function $\mathcal{T}u$ satisfies

$$\partial_t\Big(\mathcal{T}u\Big) = M^{-1}T\partial_t\Big(u(L_i x_i, Tt)\Big) = \sum_{i=1}^{N} L_i^{p_i} M^{1-p_i}\partial_i\Big(|\partial_i u|^{p_i-2}\partial_i u\Big) = \sum_{i=1}^{N}\partial_i\Big(|\partial_i\mathcal{T}u|^{p_i-2}\partial_i\mathcal{T}u\Big).$$

We let

$$L_i = M^{\frac{p_i-\bar{p}}{p_i}} A^{\frac{1}{p_i}},$$

to restore the homogeneity in the equation. In order to let $\mathcal{T}u$ satisfy the same equation we need

$$M^{-1}T = M^{1-\bar{p}}A, \quad \Rightarrow \quad A = TM^{\bar{p}-2},$$

and so the transformation is

$$\mathcal{T}u = M^{-1}u\Big(M^{\frac{p_i-\bar{p}}{p_i}} T^{\frac{1}{p_i}} M^{\frac{\bar{p}-2}{p_i}} x_i, Tt\Big) = M^{-1}u\Big(\Big[M^{p_i-2}T\Big]^{\frac{1}{p_i}} x_i, Tt\Big).$$

By taking $T = M^{2-\bar{p}}\rho^{\bar{p}}$ we obtain the statement. □

Remark 3.1. *The proof shows that* (3.2) *is not the only invariant: we may as well consider for instance* $T = 1$ *to get the function*

$$v(x,t) = M^{-1}u\Big(M^{\frac{p_i-2}{p_i}} x_i,\, t\Big).$$

Definition 3.1. *We define the intrinsic anisotropic cube just by letting act the transformation* (3.1) *on the space variables,*

$$\mathcal{K}_\rho(M) = \prod_{i=1}^{N}\Big\{|x_i| < M^{\frac{p_i-\bar{p}}{p_i}}\rho^{\frac{\bar{p}}{p_i}}\Big\}, \tag{3.3}$$

and similarly the intrinsic anisotropic cylinders

$$\mathcal{Q}_\rho^-(M) := T_{\rho,M}(Q_1^-) = \prod_{i=1}^{N}\Big\{|x_i| < M^{\frac{p_i-\bar{p}}{p_i}}\rho^{\frac{\bar{p}}{p_i}}\Big\} \times \Big(-M^{2-\bar{p}}\rho^{\bar{p}}, 0\Big]. \tag{3.4}$$

The following property can be readily checked:

$$T_{\rho,\theta}(K_R) = \mathcal{K}_{R\rho}(R\theta), \qquad \text{and} \qquad T_{\rho,\theta}(Q_R^-) = \mathcal{Q}_{R\rho}^{R\theta}. \tag{3.5}$$

Remark 3.2. *If u solves* (1.1) *in* $Q_\rho^-(M)$, *then* $\mathcal{T}u$ *solves* (1.1) *in* Q_1^-. *Vice-versa if u solves* (1.1) *in* Q_1^- *then* $\mathcal{T}^{-1}(u) = Mu\Big(M^{\frac{\bar{p}-p_i}{p_i}}\rho^{-\frac{\bar{p}}{p_i}} x_i, M^{\bar{p}-2}\rho^{-\bar{p}}t\Big)$ *solves* (1.1) *in* $Q_\rho^-(M)$.

Proposition 3.2. *Let u be a local weak solution to equation* (1.1). *The parametric transformations $\mathcal{T}u$ that preserve the L^1 norm of u are the corresponding to* (3.2) *for* $M = \rho^{-N}$, *i.e.*

$$\mathcal{T}_\rho u = \rho^N u\Big(\rho^{N(\frac{\bar{p}-p_i}{p_i})+\frac{\bar{p}}{p_i}} x_i, \rho^{N(\bar{p}-2)+\bar{p}}t\Big). \tag{3.6}$$

Proof. The proof consists just in a change of variables:

$$\int_{T_{\rho,M}(K_1)} \mathcal{T}u(x,t)dx = M^{-1}\int_{K_1} u(y,s)dy \prod_{i=1}^{N}\Big(M^{\frac{p_i-\bar{p}}{p_i}}\rho^{\frac{\bar{p}}{p_i}}\Big)^{-1}, \tag{3.7}$$

and since $\prod M^{\frac{p_i-\bar{p}}{p_i}}\rho^{\frac{\bar{p}}{p_i}} = \rho^N$ we obtain the statement. □

Remark 3.3. *We observe that in the previous proposition we used an important geometric property of the anisotropic cubes: for each $M, \rho > 0$ the total volume of the anisotropic cube is independent from the p_is*

$$|\mathcal{K}_\rho(M)| = \rho^N = \rho^N |K_1|.$$

Definition 3.2. *We say that a solution u to* (1.1) *is a self-similar solution if it satisfies $\mathcal{T}_\rho u = u$ for each $\rho > 0$.*

Now we consider the following continuous transformation

$$\Phi(u) = w(x,t) = e^{\alpha t} u(e^{\alpha_i t} x_i, e^t), \quad \text{and its inverse} \quad \Phi^{-1}(w) = u(x,s) = s^{-\alpha} w(s^{-\alpha_i} x_i, \log(s)), \tag{3.8}$$

for

$$\alpha = \frac{N}{N(\bar{p}-2)+\bar{p}}, \qquad \text{and} \qquad \alpha_i = \left(\frac{1+2\alpha}{p_i}\right) - \alpha. \tag{3.9}$$

This continuous transformation brings formally solutions to the equation (1.1) in $S_{1,\infty} = \mathbb{R}^N \times (1, +\infty)$, into solutions of the anisotropic Fokker-Planck type equation

$$w_t = \sum_{i=1}^N \partial_i [(|\partial_i w|^{p_i - 2} \partial_i w) + \alpha_i y_i w], \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}_+ =: S_\infty. \tag{3.10}$$

For each fixed time $t = \log(\rho^{-[N(\bar{p}-2)+\bar{p}]})$, $\rho > 0$, we have that the transformation Φ is a parametric transformation $\mathcal{T}_\rho$ in a cube of unitary volume, preserving the L^1-norm, as an easy calculation shows:

$$\Phi_\rho(u) = \rho^{\alpha[N(\bar{p}-2)+\bar{p}]}\, u(\rho^{\alpha_i [N(\bar{p}-2)+\bar{p}]}\, x_i,\, 1) = \rho^N u(\rho^{[N(\bar{p}-p_i)+\bar{p}]/p_i}\, x_i,\, 1). \tag{3.11}$$

This function is of the form (3.6), when computed at the time $t = \rho^{-[N(\bar{p}-2)+\bar{p}]}$. Next we show the following characterisation of L^1-norm preserving self-similar solutions.

Proposition 3.3. *Self-similar solutions to* (1.1) *in $S_{1,\infty}$ preserving the L^1 norm, correspond to stationary solutions to the Fokker-Planck equation* (3.10)*, and vice-versa.*

Proof. Firstly, we consider a self-similar solution $u = \mathcal{T} u$ to equation (1.1) in $S_{1,\infty}$ and we show that $\Phi u = w$ is a stationary solution to (3.10). We evaluate through Φ^{-1}

$$\mathcal{T}_\rho u = \rho^N u\left(\rho^{N(\frac{\bar{p}-p_i}{p_i}) + \frac{\bar{p}}{p_i}} x_i, \rho^{N(\bar{p}-2)+\bar{p}} t\right) = t^{-\alpha} w\left(x_i, \log\left(\rho^{N(\bar{p}-2)+\bar{p}} t\right)\right) = \Phi^{-1} u,$$

$$u = \Phi^{-1} w = t^{-\alpha} w\left(t^{-\alpha_i} x_i, \log(t)\right),$$

so that

$$\mathcal{T}_\rho u(\cdot, 1) = u(\cdot, 1), \quad \forall \rho > 0 \qquad \Longleftrightarrow \qquad w\left(x_i, \log\left(\rho^{N(\bar{p}-2)+\bar{p}}\right)\right) = w\left(x_i, 1\right) \qquad \forall \rho > 0.$$

So w is independent of time, and it is a solution to the stationary Fokker-Planck equation (3.10). Similarly, any stationary solution to the Fokker-Planck equation corresponds to a self-similar solution to (1.1). Choosing $e^t = \rho^{N(\bar{p}-2)+\bar{p}} e^s$ in the equality $w(x_i, t) = e^{\alpha t} u(e^{\alpha_i t} x_i, e^t) = e^{\alpha s} u(e^{\alpha_i s} x_i, e^s) = w(x_i, s)$ verifies last assertion. □

Definition 3.3. *We will refer in the following to a self-similar solution to* (1.1) *in $S_{1,\infty}$ which preserves the L^1-norm, or equivalently to a solution to* (1.1) *corresponding to a stationary solution to the Fokker-Planck equation* (3.10)*, as a Barenblatt Fundamental solution, and we will denote it with $\mathcal{B}$, in analogy with the literature regarding the p-Laplacean*[1]*.*

[1] Indeed the epithet *Fundamental* does not mean that solutions are represented by an integral convolution of $\mathcal{B}$, but when $p \to 2$ the classic $\mathcal{B}$ function approaches the heat kernel. The original solution for the p-Laplacean equation can be found in [6].

3.2 Energy Estimates

Next we provide the energy estimates of (1.1), that have to be read as a topological embedding only when the right hand side is finite. This last condition may be deduced from Theorem 2.2 whenever $\max\{p_1,\dots,p_N\} < \bar{p}(1+2/N)$, being a local solution a priori in $L^{p_N}(0,T;L^{p_N}_{\text{loc}}(\mathbb{R}^N))$.
As all arguments are of local nature, we will restrict our computation on the equation (1.1) with $\Omega_T = Q_1^-$. The estimates that we are about to derive are invariant under the transformation $\mathcal{T}u$. Then, an application of transformation (3.1) will provide the corresponding estimate in general anisotropic intrinsic cylinder $\mathcal{Q}_\rho^-(M)$.

Lemma 3.1. *Let u be a local weak solution to equation (1.1) in Q_1^-. Then there exists a constant $\gamma > 0$ depending only on the data, such that if we denote with π_i the projection into the i-th coordinate, for each function of the form*

$$C_o^\infty(Q_1) \ni \eta = \prod_{i=1}^N \eta_i^{p_i}(x_i,t) \quad \textit{for} \quad \eta_i \in C_o^\infty\Big(\pi_i(K_1)\times(-1,0]\Big), \tag{3.12}$$

we have for all $-1<s<t<0$ the estimate

$$\begin{aligned}&\int_{K_1}(u-k)_\pm^2\eta dx\Big|_s^t + \sum_{i=1}^N\int_s^t\int_{K_1}|\partial_i(u-k)_\pm\eta|^{p_i}\,dxd\tau\\ &\quad\le \gamma\Big\{\int_s^t\int_{K_1}(u-k)_\pm^2\partial_\tau\eta\,dxd\tau + \sum_{i=1}^N\int_s^t\int_{K_1}|(u-k)_\pm|^{p_i}\hat{\eta}_i|\partial_i\eta_i|^{p_i}\,dxd\tau\Big\}, \quad \textit{being} \quad \hat{\eta}_i = \frac{\eta}{\eta_i^{p_i}}.\end{aligned} \tag{3.13}$$

Proof. Let u_h be the Steklov average of u by formula (9.4) and let us choose in the formulation (9.13) the test function $\varphi := \pm\eta\,(u_h-k)_\pm \in W^{1,2}_{loc}(-1,0;W_o^{1,\mathbf{p}}(K_1))$, for η as in (3.12). This leads us to the equation

$$\int_{t_1}^{t_2}\int_{K_1}\partial_t u_h\,\varphi\,dx\,dt + \sum_{i=1}^N\int_{t_1}^{t_2}\int_{K_1}(|\partial_i u|^{p_i-2}\partial_i u)_h\,\partial_i\varphi\,dx = 0, \tag{3.14}$$

for all $-1<t_1<t_2<0$. The first integral term is estimated, for the chosen test function, as

$$\begin{aligned}\int_{t_1}^{t_2}\int_{K_1}\partial_t u_h\,\varphi\,dx\,dt &= \int_{t_1}^{t_2}\int_{K_1}\partial_t\left(\pm\frac{(u_h-k)_\pm^2}{2}\right)\eta\,dx\,dt\\ &= \int_{K_1}\frac{(u_h-k)_\pm^2}{2}\,\eta\,dx\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{K_1}\frac{(u_h-k)_\pm^2}{2}\,\partial_t\eta\,dx\,dt.\end{aligned} \tag{3.15}$$

Here we let $h\downarrow 0$ to recover the estimate (3.14) without Steklov averages. The limit is achieved for the double integrals thanks to Dominated Convergence Theorem, while the terms that are yet evaluated on times t_1,t_2, well-defined thanks to property $(a2)$ of Proposition 9.6, converge by property (d) of Proposition 9.6. In order to perform this limit operation we use in a crucial way the statement of Proposition 2.1, that $u\in C(-1,0;L^2(\Omega))$. Then let us set $\hat{\eta}_i = \eta/\eta_i^{p_i}$, which does not depend on x_i, to estimate in the set $[(u-k)_\pm>0]\cap Q_1$ the quantities

$$\sum_{i=1}^N\int_{t_1}^{t_2}\int_{K_1}(|\partial_i u|^{p_i-2}\partial_i u)_h\,\partial_i(\pm\eta(u_h-k)_\pm)\,dx \to \sum_{i=1}^N\int_{t_1}^{t_2}\int_{K_1}|\partial_i u|^{p_i-2}\partial_i u\cdot\partial_i(\pm\eta\,(u-k)_\pm) \ge$$

$$\ge \int_{t_1}^{t_2}\int_{K_1}\Big\{\sum_{i=1}^N|\partial_i(u-k)_\pm\eta|^{p_i} - (u-k)_\pm\sum_{i=1}^N|\partial_i(u-k)_\pm|^{p_i-1}\,|\partial_i\eta|\Big\}\,dx\,.$$

The last term is estimated through Young inequality by

$$\begin{aligned}(u-k)_\pm\,|\partial_i(u-k)_\pm|^{p_i-1}|\partial_i\eta| &= p_i(u-k)_\pm|\partial_i(u-k)_\pm|^{p_i-1}\,\eta_i^{p_i-1}\,|\partial_i\eta_i|\,\hat{\eta}_i\\ &\le \hat{\eta}_i\left(\varepsilon\,\eta_i^{p_i}\,|\partial_i(u-k)_\pm|^{p_i} + C(\varepsilon)(u-k)_\pm^{p_i}\,|\partial_i\eta_i|^{p_i}\right).\end{aligned} \tag{3.16}$$

Using the definition of $\hat{\eta}_i$ we get

$$\hat{\eta}_i\, \eta_i^{p_i} = \eta, \qquad \hat{\eta}_i\, |\partial_i \eta_i|^{p_i} = |\partial_i \eta^{\frac{1}{p_i}}|^{p_i}, \tag{3.17}$$

therefore taking $0 < \varepsilon < 1/2$ small enough we get in conclusion

$$\begin{aligned} \sum_{i=1}^N \int_{t_1}^{t_2} \int_{K_1} |\partial_i u|^{p_i-2} \partial_i u\, \partial_i \eta (u-k)_\pm\, dx \geq \\ \geq \frac{1}{2} \sum_{i=1}^N \int_{t_1}^{t_2} \int_{K_1} |\partial_i (u-k)_\pm|^{p_i}\, \eta\, dx\, dt - C \int_{t_1}^{t_2} \sum_{i=1}^N \int_{K_1} (u-k)_\pm^{p_i}\, |\partial_i \eta^{\frac{1}{p_i}}|^{p_i}\, dx\, dt. \end{aligned} \tag{3.18}$$

and the claim follows inserting this last inequality and (3.15) into (3.14). □

In the following Lemma we show that the energy inequalities (3.13) are invariant by transformations that bring solutions to (1.1) into solutions to (1.1).

Lemma 3.2. *Let u be a local weak solution to equation (1.1) in $(x_o, t_o) + \mathcal{Q}_\rho(M)$. Then there exists a constant $\gamma > 0$ depending only on the data such that if we denote with π_i the projection into the i-th coordinate, for each function of the form*

$$C_o^\infty(((x_o, t_o) + \mathcal{Q}_\rho(M)) \ni \eta = \prod_{i=1}^N \eta_i^{p_i}(x_i, t) \quad \textit{for} \quad \eta_i \in C_o^\infty(\pi_i(x_o + \mathcal{K}_\rho(M)) \times (t_o - \rho^{\bar{p}} M^{2-\bar{p}}, t_o]$$

we have, denoting with $\hat{\eta}_i = \eta/\eta_i^{p_i}$, the following estimates for all times $t_o - \rho^{\bar{p}} M^{2-\bar{p}} < s < t < t_o$,

$$\begin{aligned} \int_{\mathcal{K}_\rho(M)} (u-k)_\pm^2 \eta(x,\tau) dx \Big|_{\tau=s}^{\tau=t} + \sum_{i=1}^N \int_s^t \int_{\mathcal{K}_\rho(M)} |\partial_i (u-k)_\pm|^{p_i} \eta(x,\tau)\, dx d\tau \\ \leq \gamma \Big\{ \int_s^t \int_{\mathcal{K}_\rho(M)} (u-k)_\pm^2 \partial_\tau \eta(x,\tau)\, dx d\tau + \sum_{i=1}^N \int_s^t \int_{\mathcal{K}_\rho(M)} |(u-k)_\pm|^{p_i} \hat{\eta}_i |\partial_i \eta_i|^{p_i}\, dx d\tau \Big\}. \end{aligned} \tag{3.19}$$

Proof. The function $\mathcal{T}(u)$, specified by (3.2), solves (1.1) in Q_1^-. Then (3.13) are valid for $\mathcal{T}u$ giving for all $-1 < s < t < 0$, every $\bar{k} \in \mathbb{R}$ the energy estimates (3.13). Now by the change of variables (3.1)-(3.2) again on each integral and by redefining $k/M = \bar{k}$ we have that $[\mathcal{T}u > \bar{k}] = [u > k]$ for any $\bar{k} \in \mathbb{R}$, we arrive to the desired estimate. Here we show some insights.
Let $\mathcal{T}u$ satisfy the energy estimates

$$\begin{aligned} \int_{K_1} (\mathcal{T}u - \bar{k})_\pm^2 \eta\, dy \Big|_{s_1}^{s_2} + \sum_{i=1}^N \int_{s_1}^{s_2} \int_{K_1} |\partial_i (\mathcal{T}u - \bar{k})_\pm|^{p_i} \eta\, dy ds \leq \\ \gamma \int_{s_1}^{s_2} \int_{K_1} |(\mathcal{T}u - \bar{k})_\pm|^2 \partial_s \eta\, dy ds + \sum_{i=1}^N \int_{s_1}^{s_2} \int_{K_1} |(\mathcal{T}u - \bar{k})_\pm|^{p_i} \hat{\eta}_i |\partial_i \eta_i|^{p_i}\, dy ds. \end{aligned} \tag{3.20}$$

Now let us consider the change of variables

$$u(x,t) = \mathcal{T}^{-1}(\mathcal{T}u)(y(x), s(t)) = M \mathcal{T}u\Big(M^{\frac{\bar{p}-p_i}{p_i}} \rho^{-\frac{\bar{p}}{p_i}} x_i, M^{\bar{p}-2} \rho^{-\bar{p}} t \Big),$$

with the simple stipulations

$$\begin{cases} x_i = M^{\frac{p_i - \bar{p}}{p_i}} \rho^{\frac{\bar{p}}{p_i}} y_i \\ t = M^{2-\bar{p}} \rho^{\bar{p}} s, \end{cases} \qquad \text{and} \qquad \begin{cases} \partial_{x_i} u(x,s) = (M/\rho)^{\bar{p}/p_i} \partial_{y_i} \mathcal{T}u(y,s)), \\ \partial_t u(x,t) = M^{\bar{p}-1} \rho^{-\bar{p}} \partial_s \mathcal{T}u(y,s)). \end{cases}$$

We observe that $|dx(y)| = \prod |dx_i/dy_i| = \rho^{-N}$ and we evaluate each integral in (3.20) by

$$\int_{K_1} (\mathcal{T}u - \bar{k})_{\pm}^2 \eta \, dy \Big|_{s_1}^{s_2} = \int_{K_\rho(M)} (M^{-1}u(x,t) - \bar{k})_{\pm}^2 \eta(y(x), s(t)) \, (\rho^{-N} dx) \Big|_{t_1 := M^{2-\bar{p}}\rho^{\bar{p}} s_1}^{t_2 := M^{2-\bar{p}}\rho^{\bar{p}} s_2}$$
$$= M^{-2}\rho^{-N} \int_{K_\rho(M)} (u(y,t) - \bar{k}/M)_{\pm}^2 \eta \, dx \Big|_{t_1}^{t_2},$$

and similarly

$$\int_{s_1}^{s_2} \int_{K_1} |\partial_{y_i} (\mathcal{T}u - \bar{k})_{\pm}|^{p_i} \eta \, dyds = M^{-2}\rho^{-N} \int_{t_1}^{t_2} \int_{K_\rho(M)} |\partial_{x_i} (u - \bar{k}/M)_{\pm}|^{p_i} \eta \, dxdt.$$

$$\int_{s_1}^{s_2} \int_{K_1} |(\mathcal{T}u - \bar{k})_{\pm}|^2 \partial_s \eta \, dyds = M^{-2}\rho^{-N} \int_{t_1}^{t_2} \int_{K_\rho(M)} |(u(y,t) - \bar{k}/M)_{\pm}^2 \partial_t \eta \, dxdt.$$

$$\int_{s_1}^{s_2} \int_{K_1} |(\mathcal{T}u - \bar{k})_{\pm}|^{p_i} \hat{\eta}_i |\partial_{y_i} \eta_i|^{p_i} \, dyds = M^{-2}\rho^{-N} \int_{t_1}^{t_2} \int_{K_\rho(M)} |u(x,t) - \bar{k}/M)_{\pm}|^{p_i} \hat{\eta}_i |\partial_{x_i} \eta_i|^{p_i} \, dxdt.$$

□

A useful variant of the energy inequalities (3.13) is the following one. This time we prove them for general intrinsic anisotropic cubes $\mathcal{K}_\rho(M)$, in order to leave greater generality to function F on duty.

Lemma 3.3. *Let $F \in C^2(\mathbb{R})$ be such that for some $M > 0$ and every $t \in \mathbb{R}$,*

$$|F'| \le M, \quad 0 < F''(t) < M, \quad F'(t)^{p_i}/F''(t)^{p_i - 1} \le M t^{p_i}, \qquad \forall i \in \{1, \dots, N\}. \tag{3.21}$$

If u is a local weak solution to (1.1) *and η is of the form* (3.12), *compactly supported in some intrinsic cube $\mathcal{K}_\rho(M)$, and independent of t, then we have for any $0 < t_1 < t_2 < T$ the estimate*

$$\begin{aligned} \int_{\mathcal{K}_\rho(M)} F(u(x,t)) \, \eta(x) \, dx \Big|_{t_1}^{t_2} + \sum_{i=1}^N \int_{t_1}^{t_2} \int_{\mathcal{K}_\rho(M)} F''(u) \, \eta \, |\partial_i u|^{p_i} \, dx \, dt \le \\ \gamma \sum_{i=1}^N \int_{t_1}^{t_2} \int_{\mathcal{K}_\rho(M)} |F'(u)|^{p_i} F''(u)^{1-p_i} |\partial_i \eta^{\frac{1}{p_i}}|^{p_i} \, dx \, dt, \end{aligned} \tag{3.22}$$

for a constant $\gamma > 0$ depending only on the data.

Proof. Along the proof we will omit the set of integration $\mathcal{K}_\rho(M)$ as the functions considered are compactly supported in such a set. Let u_h be the Steklov average of u and let us test the equation (9.13) with $\varphi = F'(u_h)\,\eta$, which is readily checked to be admissible since $F' \in \mathrm{Lip}(\mathbb{R})$ is bounded and

$$|\partial_i \varphi| \le M |\partial_i \eta| + M |\partial_i u_h| \, \eta.$$

Notice that $|F(s)| \le M|s|$, hence $F(u_h(\cdot, \tau)) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ since $u_h(\cdot, \tau) \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$, so that we can compute

$$\int_{t_1}^{t_2} \int \partial_t u_h \varphi \, dx \, dt = \int_{t_1}^{t_2} \int \partial_t \big(F(u_h)\,\eta\big) \, dx \, dt = \int F(u_h(x,t_2))\,\eta(x) \, dx - \int F(u_h(x,t_1))\,\eta(x) \, dx$$

for any $T > t_2 > t_1 > 0$. Therefore we use the continuity of F, F', F'' and the boundedness assumptions (3.21) to let $h \downarrow 0$ in the resulting equation

$$\int F(u_h)(x,t)\eta(x) \, dx \Big|_{t_1}^{t_2} + \sum_{i=1}^N \int_{t_1}^{t_2} \int \bigg\{ \Big(|\partial_i u|^{p_i - 2} \partial_i u \Big)_h \Big[F''(u_h)(\partial_i u_h)\eta + F'(u_h)\partial_i \eta \Big] \bigg\} \, dxdt = 0,$$

an we end up with the following estimate

$$\int F(u(x,t))\,\eta(x)\,dx\bigg|_{t_1}^{t_2} + \sum_{i=1}^{N}\int_{t_1}^{t_2}\int F''(u)\,\eta\,|\partial_i u|^{p_i}\,dx\,dt \le \sum_{i=1}^{N}\int_{t_1}^{t_2}\int |\partial_i u|^{p_i-1}\,|F'(u)|\,|\partial_i\eta|\,dx\,dt. \quad (3.23)$$

Proceeding as in (3.16) and making use of (3.17) we can bound the right-hand side as

$$\begin{aligned}|\partial_i u|^{p_i-1}\,|F'(u)|\,|\partial_i\eta| &= \hat{\eta}_i\Big(|\partial_i u|^{p_i-1}\,F''(u)^{1-\frac{1}{p_i}}\,\eta_i^{p_i-1}\,\frac{|F'(u)|}{F''(u)^{1-\frac{1}{p_i}}}|\partial_i\eta_i^{p_i}|\Big)\\ &\le \frac{1}{2}F''(u)\,\eta\,|\partial_i u|^{p_i} + C|F'(u)|^{p_i}F''(u)^{1-p_i}|\partial_i\eta^{\frac{1}{p_i}}|^{p_i},\end{aligned}$$

which, inserted into (3.23) gives us (3.22). □

3.3 Conservation of Mass in L^1 and L^2

Let u be a local weak solution to the Cauchy problem (1.4) and suppose it has furthermore the better integrability $u \in \cap_{i=1}^N L^{p_i}(S_T)$. Then also its directional derivatives are bounded in $L^{p_i}(S_T)$ and the $L^2(\mathbb{R}^N)$ norm of the function stays bounded in time. This is the content of next Proposition.

Proposition 3.4. *Let $u \in \cap_{i=1}^N L^{p_i}(S_T)$ solve the Cauchy problem (1.4) in S_T, with initial datum u_0 taken in $L^2(\mathbb{R}^N)$. Then we have*

$$\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}(S_T)} \le \|u_0\|_2, \quad \text{and} \quad \|u(\cdot,t)\|_2 \le \|u_0\|_2 \quad \forall t\in[0,T), \quad (3.24)$$

and therefore the memberships $u \in L^\infty(0,T;L^2(\mathbb{R}^N))$, $u \in L^{\mathbf{p}}(0,T;W^{1,\mathbf{p}}(\mathbb{R}^N)$ hold. Furthermore, for any $0<t_1<t_2<T$, $\psi \in \mathrm{Lip}([0,T];\mathbb{R})$, $\psi \ge 0$ and $k\in\mathbb{R}$ it holds

$$\int_{\mathbb{R}^N}(u-k)_+^2(x,t)\,\psi(t)\,dx\bigg|_{t_1}^{t_2} + \frac{1}{C}\sum_{i=1}^{N}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}|\partial_i(u-k)_+|^{p_i}\,\psi\,dx\,dt \le \int_{t_1}^{t_2}\int_{\mathbb{R}^N}(u-k)_+^2\,|\psi_t|\,dx\,dt. \quad (3.25)$$

Proof. To get (3.24) choose arbitrarily $R>0$ and let $0\le\eta_R\le 1$ be as in (3.12) but independent on t, being the product of $\eta_{R,i}(x_i)$ in separate variables, and such that

$$\mathrm{supp}(\eta_{R,i}) \subseteq [-2R,2R], \qquad \eta_{R,i}\equiv 1 \text{ on } [-R,R], \qquad |\partial_i\eta_{R,i}| \le \frac{C}{R}. \quad (3.26)$$

With such η_R, apply the (3.13) to both u and $-u$ for $k=0$, adding the corresponding inequalities[2]. Being u solution to the Cauchy problem, we use (2.3) to let $t_1\to 0$ and obtain the estimate

$$\int_{\mathbb{R}^N} u^2(x,t_2)\eta_R\,dx + \sum_{i=1}^{N}\int_0^{t_2}\int_{\mathbb{R}^N}|\partial_i u|^{p_i}\eta_R\,dx \le \int_{\mathbb{R}^N}u_0^2\,dx + \sum_{i=1}^{N}\frac{C}{R^{p_i}}\int_0^{t_2}\int_{\mathbb{R}^N}|u|^{p_i}\,dx\,dt.$$

We let $R\to+\infty$, apply Fatou's lemma on the left-hand side, and observe that the last integral on the right-hand side vanishes because $u\in\cap L^{p_i}(S_T)$, therefore proving (3.24). To prove (3.25), let us write (3.13) in with $\varphi=\eta_R\,\psi$ of separate variables, to get

$$\begin{aligned}&\int_{\mathbb{R}^N}(u-k)_+^2(x,t_2)\,\eta_R(x)\,\psi(t_2)\,dx + \frac{1}{C}\sum_{i=1}^{N}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\big|\partial_i\big((u-k)_+\big)\big|^{p_i}\;\eta_R\,\psi dx\,dt \le\\ &\int_{\mathbb{R}^N}(u-k)_+^2(x,t_1)\,\eta_R(x)\,\psi(t_1)\,dx + \int_{t_1}^{t_2}\int_{\mathbb{R}^N}(u-k)_+^2\,\eta_R\,|\psi_t|\,dx\,dt + C\sum_{i=1}^{N}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}(u-k)_+^{p_i}\,|\partial_i\eta_{R,i}\,\psi|\,dx\,dt.\end{aligned}$$

Letting $R\to+\infty$ cancels the last term on the right as before, while we apply Fatou on the terms on the left hand-side and dominated convergence to the right, to obtain (3.25). □

[2]Here use that $u_+^2+u_-^2=u^2$ and also with the exponent p_i.

Proposition 3.5. *Let $u \in \cap_{i=1}^N L^{p_i}(S_T)$ solve the Cauchy problem* (1.4) *in S_T, with $u_0 \in L^1(\mathbb{R}^N)$ initial datum taken within the meaning of $L^2(\mathbb{R}^N)$. Then it holds*

$$\int_{\mathbb{R}^N} |u(x,t)|\, dx \le \int_{\mathbb{R}^N} |u_0|\, dx, \qquad \forall t \in [0,T). \tag{3.27}$$

Proof. Let η be as in (3.26) and define, for $\alpha > 0$ to be determined later and $\epsilon \in (0,1)$, the function

$$F_\epsilon(s) = \int_0^s \frac{\tau}{(|\tau|^\alpha + \epsilon)^{\frac{1}{\alpha}}}\, d\tau,$$

so that

$$F'_\epsilon(s) = \frac{s}{(|s|^\alpha + \epsilon)^{\frac{1}{\alpha}}}, \qquad F''_\epsilon(s) = \frac{\epsilon}{(|s|^\alpha + \epsilon)^{\frac{1}{\alpha}+1}} > 0.$$

We then choose

$$\alpha = (\max\{p_1, \dots, p_N\} - 1)^{-1} > 0,$$

so that $p_i - 1 - \frac{1}{\alpha} \le 0$ for all $i = 1, \dots, N$. With this choice, all the assumptions of the Lemma 3.3 are satisfied with $M > 1$ big enough and $\epsilon \in (M^{-\alpha}, 1)$ and estimate (3.22) implies, by our choice,

$$\begin{aligned}\int_{\mathbb{R}^N} F_\epsilon(u(x,t))\, \eta(x,t)\, dx\Big|_{t_1}^{t_2} &\le C \sum_{i=1}^N \epsilon^{1-p_i} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |u|^{p_i} (|u|^\alpha + \epsilon)^{p_i - 1 - \frac{1}{\alpha}} |\partial_i \eta^{\frac{1}{p_i}}|^{p_i}\, dx\, dt \\ &\le C_\epsilon \sum_{i=1}^N \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |u|^{p_i} |\partial_i \eta^{\frac{1}{p_i}}|^{p_i}\, dx\, dt,\end{aligned}$$

for any $\epsilon \in\ (0,1)$ and $C_\epsilon = C(\Lambda, \mathbf{p}, \epsilon) > 0$. We use the fact that F is 1-Lipschitz, i.e. that $F_\epsilon(s) \le |s|$, to let $t_1 \to 0$ in the previous estimate. Moreover as $0 \le \eta \le 1$ and $|\partial_i \eta^{\frac{1}{p_i}}| \le \frac{C}{R}$, we get

$$\int_{\mathbb{R}^N} F_\epsilon(u(x,t_2))\, \eta(x,t_2)\, dx \le \int_{\mathbb{R}^N} |u_0|\, dx + \sum_{i=1}^N \frac{C_\epsilon}{R^{p_i}} \int_0^{t_2} \int_{\mathbb{R}^N} |u|^{p_i}\, dx\, dt,$$

for all $\epsilon \in\ (0,1)$ and $R \ge 1$. Let first $R \to +\infty$ to cancel out the last term thanks to the hypothesis $u \in \cap L^{p_i}(S_T)$ obtaining through Fatou's Lemma

$$\int_{\mathbb{R}^N} F_\epsilon(u(x,t_2))\, dx \le \int_{\mathbb{R}^N} |u_0|\, dx, \qquad \forall t_2 \in [0,T),$$

and since $0 \le F_\epsilon(s) \nearrow |s|$, we obtain the conclusion by monotone convergence. □

Remark 3.4. *Results* (3.24) *and* (3.27) *above are valid until time T, i.e. if u is a local weak solution to the Cauchy problem* (1.4) *and $u \in \cap_{i=1}^N L^{p_i}(S_T)$ then*

$$||u(x,T)||_{L^1(\mathbb{R}^N)} \le ||u_0||_{L^1(\mathbb{R}^N)}, \quad ||u(x,T)||_{L^2(\mathbb{R}^N)} \le ||u_0||_{L^2(\mathbb{R}^N)}, \quad \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}(S_T)} \le \|u_0\|_2.$$

As our definition of solution u is local, this doesn't come as a natural fact, because each formulation of solution and therefore of energy estimates includes a strictly contained interval of $[0,T]$. This comes as a consequence of the global integrability condition $u \in \cap_{i=1}^N L^{p_i}(S_T)$ and the final discussion on Section 2.2.

We give here another approach to this extension. By the results of existence and uniqueness of next sections 3.6 and 5.2, we can solve the Cauchy problem for initial datum $u_0 \in L^2(S_T)$ and obtain a solution $\tilde{u}$ which extends u in S_T, thanks to the uniqueness implied by Proposition 3.7. Then, making use of estimates (3.24) *and* (3.27) *in S_{T+1} we can treat the value $T \to u(x,T)$ as an internal value. This provides an alternative statement of Propositions 3.4 and 3.5 where the estimate holds until and included time T.*

3.4 A De Giorgi Type-Lemma

We introduce in this Section a very useful De Giorgy-type Lemma, which again can be demonstrated at ease for unitary cylinders, and then reinterpreted in the intrinsic geometry dictated by the equation by (3.6). The proof of this Lemma is nowadays a classic in the context of degenerate and singular equations, and it is proven for instance in ([27], Lemma 4.1) for the case of functions bounded from above. Therefore here we prove it just for functions bounded from below by zero.

Lemma 3.4. *Let u be a local weak solution to* (1.1) *such that $||u||_{L^\infty(Q_1^-)} \le 1$. Then for every choice of $0 < a \le 1$ there exist a number $\nu_a > 0$ depending only on a itself and p_i, N but not on u such that if*

$$|[u \ge a] \cap Q_1^-| \le \nu_a |Q_1^-| \tag{3.28}$$

then

$$u \le \frac{3}{2}a, \qquad a.e. \ in \quad \mathcal{Q}^-_{1/2} = \prod_{i=1}^N \{|x_i| < (1/2)^{\frac{\bar{p}}{p_i}}\} \times (-(1/2)^{\bar{p}}, 0]. \tag{3.29}$$

If moreover u is nonnegative in Q_1^- then

$$|[u \le a] \cap Q_1^-| \le \nu_a |Q_1^-|, \qquad \Rightarrow \qquad \inf_{\mathcal{Q}^-_{1/2}} u \ge \frac{1}{2}a\,. \tag{3.30}$$

Proof. We show only the first implication, i.e. (3.28)-(3.29), the other being similar. Let us define, for $n \in \mathbb{N}$,

$$\rho_n = \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right), \qquad k_n = a\left(\frac{3}{2} - \frac{1}{2^n}\right), \qquad Q_n = K_n \times (-\rho_n^{\bar{p}},\, 0] = \prod_{i=1}^N \{|x_i| < \rho_n^{\frac{\bar{p}}{p_i}}\} \times (-\rho_n^{\bar{p}}, 0]. \tag{3.31}$$

We will apply (3.13) to $(u-k_n)_+$. We take the cut-off functions of the form $\eta_n(x,t) = \prod_{i=1}^N \eta_{i,n}^{p_i}(x_i)\phi_n(t)$, with $\eta_{i,n} \in C_o^\infty(\pi_i(K_n))$, $\phi_n \in C_o^\infty((-\rho_n^{\bar{p}}, 0])$ such that

$$\eta_{i,n}(x_i) = \begin{cases} 1, & x_i \in \pi_i(K_{n+1}) = \{|x_i| < \rho_{n+1}^{\frac{\bar{p}}{p_i}}\}, \\ 0, & x_i \in \left\{\mathbb{R}^N - \pi_i(K_n)\right\} = \{|x_i| < \rho_n^{\frac{\bar{p}}{p_i}}\}^c \end{cases} \quad \text{and} \quad |\partial_i \eta_{i,n}| \le 2^{n+1}\gamma \tag{3.32}$$

and

$$\phi_n(t) = \begin{cases} 1, & t \ge -\rho_{n+1}^{\bar{p}}, \\ 0, & t \le -\rho_n^{\bar{p}} \end{cases} \qquad \text{with} \qquad |(\phi_n)_t| \le 2^{n+1}\gamma. \tag{3.33}$$

The energy inequality (3.13), with these stipulations, yields

$$\begin{aligned} &\sup_{(-\rho_n^{\bar{p}},0]} \int_{K_n} (u-k_n)_+^2 \eta\, dx + \sum_{i=1}^N \int\int_{Q_n} |\partial_i (u-k_n)_+|^{p_i} \eta\, dxdt \le \\ &\gamma\Big\{ \int\int_{Q_n} (u-k_n)_+^2 \eta_t\, dxdt + \sum_{i=1}^N \int\int_{Q_n} |(u-k_n)_+|^{p_i} \hat{\eta}_i |\partial_i \eta_i|^{p_i}\, dxdt \Big\} \le \\ &\qquad \le \gamma 2^{n+1} |[u > k_n] \cap Q_n|. \end{aligned} \tag{3.34}$$

Now, combining the embedding (2.17), previous estimates and a touch of Hölder estimates, we have

$$\begin{aligned} \left(\frac{a}{2^{n+1}}\right)^{\bar{p}_2} |A_{n+1}| &\le \int\int_{Q_{n+1}} |(u-k_{n+1})_+|^{\bar{p}_2}\, dxdt \le \int\int_{Q_n} |(u-k_n)_+|^{\bar{p}_2} \eta\, dxdt \le \\ &\gamma\Big(\sup_{(-\rho_n^{\bar{p}},0]} \int_{K_n} (u-k_n)_+^2 dx + \sum_{i=1}^N \int\int_{Q_n} |\partial_i (u-k_n)_+|^{p_i}\, dxdt \Big)^{\frac{N+\bar{p}}{N}} \\ &\le \gamma 2^{(n+1)(\frac{N+\bar{p}}{N})} |A_n|^{1+\alpha}, \qquad \alpha = \bar{p}/N, \end{aligned} \tag{3.35}$$

being $A_n = [u > k_n] \cap Q_n$. Now, by observing that $|Q_n| \le 2^{n+\bar{p}}|Q_{n+1}|$ we get the iterative estimate

$$\left(\frac{a}{2^{n+1}}\right)^{\bar{p}_2} \frac{|A_{n+1}|}{|Q_{n+1}|} \le \gamma 2^{n(\frac{N+\bar{p}}{N})} \frac{|A_n|^{1+\alpha}}{|Q_n|^{1+\alpha}} |Q_n|^{\frac{\bar{p}}{N}}, \tag{3.36}$$

so that by denoting by $Y_n = |A_n|/|Q_n|$ we get for a constant $\gamma = \gamma(N, p_i) > 1$ the recursive inequality

$$Y_{n+1} \le \gamma a^{-\bar{p}_2} 2^{(N+\bar{p})n} Y_n^{1+\alpha}. \tag{3.37}$$

Thus by applying Lemma 9.3 with $\beta_i \equiv \bar{p}/N$ for all $i = 1, \dots, N$, if

$$Y_0 \le \left(\frac{\gamma}{a^{\bar{p}_2}}\right)^{-\frac{N}{\bar{p}}} 2^{-\frac{(N+\bar{p})N^2}{\bar{p}^2}} =: \nu_a \tag{3.38}$$

we obtain the thesis (3.29) as $n \to \infty$. □

Corollary 3.1. *Let u be a local weak solution to* (1.1) *in such that $||u||_{L^\infty(\Omega_T)} \le M$. Let $(x,t) \in \mathbb{R}^{N+1}$ be a point and for $\rho > 0$ let $(x,t) + \mathcal{Q}_\rho^-(M) \subset \Omega_T$. Then for every choice of $0 < a \le 1$ there exists a number $\nu_a > 0$ depending only on the data and a, but not on u, ρ such that if*

$$|[u \ge aM] \cap (x,t) + \mathcal{Q}_\rho^-(M)| \le \nu_a |\mathcal{Q}_\rho^-(M)|, \tag{3.39}$$

then

$$u \le \frac{3}{2} aM, \quad \text{a.e. in} \quad (x,t) + \mathcal{Q}_{\rho/2}^-(M). \tag{3.40}$$

Moreover, if u is nonnegative in $\mathcal{Q}_\rho^-(M)$ we have the counterpart for supersolutions:

$$|[u \le aM] \cap (x,t) + \mathcal{Q}_\rho^-(M)| \le \nu_a |\mathcal{Q}_\rho^-(M)|, \qquad \Rightarrow \qquad \inf_{(x,t)+\mathcal{Q}_{\rho/2}^-(M)} u \ge \frac{a}{2} M. \tag{3.41}$$

Proof. If u is a local weak solution to (1.1) in such that $||u||_{L^\infty(\Omega_T)} \le M$, then $\mathcal{T}u$ is a local weak solution to (1.1) in Q_1^- such that $||\mathcal{T}u||_{L^\infty(Q_1^-)} \le M$. Furthermore, the measure theoretical information (3.39) translates into (3.28) by the simple change of variables $(y,s) = T_{\rho,M}(x,t)$

$$\begin{aligned} \nu_a \rho^N |Q_1^-| =& \nu_a |\mathcal{Q}_{\rho(M)}^-| \ge \int\int_{(x,t)+\mathcal{Q}_\rho^-(M)} \chi_{[u \ge aM]}(x)\, dx = \\ &= \int\int_{T_{\rho,M}(x,t)+Q_1^-} \chi_{[\mathcal{T}u \ge a]}(T_{\rho,M}(x))\, \rho^N d(T_{\rho,M}(x)) = \rho^N |[\mathcal{T}u(y,s) > a] \cap (y,s) + Q_1^-|. \end{aligned}$$

This is enough to infer from Lemma 3.4 that

$$\mathcal{T}u \le 3a/2 \quad \text{in} \quad Q_1^-, \qquad \Longleftrightarrow \qquad u \le 3aM/2 \quad \text{in} \quad \mathcal{Q}_\rho^-(M).$$

□

3.5 Local Clustering Revisited

First we recall a classic Lemma from [25], that we prove in detail for convenience of the reader and for the purpose of further inspection. Roughly it asserts that we can always find a point where positivity clusters.

Lemma 3.5. *Let $u \in W^{1,1}(K_\rho)$ satisfy for some constants $\gamma > 0$, $\alpha \in (0,1)$ the two properties*

$$||Du||_{L^1(K_\rho)} \le \gamma \rho^{N-1}, \quad \text{and} \quad |[u > 1] \cap K_\rho| \ge \alpha |K_\rho|. \tag{3.42}$$

Then for every choice of $\lambda, \bar{\nu} \in (0,1)$ there exist a point $y \in K_\rho$ and a number $\epsilon = \epsilon(\lambda, \bar{\nu}, \gamma, \alpha, N) \in (0,1)$ such that

$$|[u > \lambda] \cap K_{\epsilon\rho}(y)| > (1 - \bar{\nu}) |K_{\epsilon\rho}(y)| \tag{3.43}$$

Proof. Consider the function $v(x) = u(x/\rho) \in W^{1,1}(K_1)$, so that (3.42) implies, as $Dv = \rho^{-1}Du$,

$$||Dv||_{L^1(K_1)} \le \gamma, \quad \text{and} \quad |[v > 1] \cap K_1| \ge \alpha |K_1|.$$

For $n \in \mathbb{N}$ we partition K_1 into n^N cubes K_j, with pairwise disjoint interiors and edge $1/n$. We divide these cubes into two finite subcollections $\mathbb{K}^+$ and $\mathbb{K}^-$ by stating that $K \in \mathbb{K}^+$ if $|[v > 1] \cap K| > (\alpha/2)|K|$, and conversely $K \in \mathbb{K}^-$ if converse inequality holds. We denote by $\#(\mathbb{K}^+)$ the number of cubes in $\mathbb{K}^+$. Let us fix $\delta, \lambda \in (0,1)$. We claim that the integer n can be chosen big enough, depending on $\lambda, \delta, \alpha, \gamma, N$ such that for some $K \in \mathbb{K}^+$ we have

$$|[v > \lambda] \cap K] \ge (1 - \delta)|K|.$$

This would establish (3.43) when transforming back to u. So we have an alternative: or there exists a cube $K \in \mathbb{K}^+$ where this measure information on the superlevel set is satisfied, or we have

$$|[v > \lambda] \cap K] < (1 - \delta)|K|, \qquad \forall K \in \mathbb{K}^+. \tag{3.44}$$

From now on we proceed by reduction ad absurdum, to show that this hypothesis leads to a contradiction with (3.42). Assumptions (3.42)-(3.44) imply for each $K \in \mathbb{K}^+$ the following estimates

$$|[v \le \lambda] \cap K| \ge \delta |K|, \qquad \text{and} \qquad |[v > (1 + \lambda)/2] \cap K| > |[v > 1] \cap K| > (\alpha/2)|K|.$$

Let $x \in [v \le \lambda] \cap K$ and $y \in [v > (1 + \lambda)/2] \cap K$ be two Lebesgue points for v and let $\tilde{v} \in AC(l_x^y)$ be the absolutely continuous representative of v along the line l_x^y connecting x with y. So we have

$$\frac{1-\lambda}{2} = \left(\frac{1+\lambda}{2} - \lambda \right) \le \tilde{v}(y) - \tilde{v}(x) = \int_0^{|y-x|} D\tilde{v}(x + t\mathbf{n}) \cdot \mathbf{n} \, dt, \quad \text{where} \quad \mathbf{n} = \frac{y - x}{|y - x|}.$$

We integrate the previous inequality with respect to y along the set $\tilde{v} > (1 + \lambda)/2] \cap K$, estimating from below the left-hand side by using the lower bound on the measure of such a set and from above the right hand side by extending the integral over the whole K,

$$\begin{aligned} \frac{\alpha(1-\lambda)}{4}|K| \le &\left(\frac{1-\lambda}{2}\right)|[\tilde{v} > (1+\lambda)/2] \cap K| \le \\ &\int_K \int_0^{|y-x|} D\tilde{v}(x + t\mathbf{n}) \cdot \mathbf{n} \, dt dy = \int_K \int_0^{|y-x|} t^{N-1} \frac{D\tilde{v}(x + t\mathbf{n}) \cdot \mathbf{n}}{|t\mathbf{n}|^{N-1}} \, dt dy. \end{aligned}$$

Now we estimate the integral on the right in the following way. Let $R(x, \mathbf{w})$ be the polar representation of ∂K with pole at x, thus representing $y = x + s\mathbf{w}$, $dy = s^{N-1} ds d\mathbf{w}$, with normal vector $\mathbf{w} = (y - x)/|y - x| = \mathbf{n}$, and observe that $\operatorname{diam}(K) = |K|^{1/N} N^{1/2}$ to get

$$\begin{aligned} \frac{\alpha(1-\lambda)}{4}|K| &\le \int_{\mathbf{w}=1} \int_{R(x,\mathbf{w})} s^{N-1} \left(\int_0^{|s\mathbf{w}|} t^{N-1} \frac{|D\tilde{v}(x + t\mathbf{w})|}{|t\mathbf{w}|^{N-1}} \, dt \right) ds d\mathbf{w} \\ &= \int_0^{\operatorname{diam}(K)} t^{N-1} \left(\int_{\mathbf{w}=1} \int_{R(x,\mathbf{w})} s^{N-1} \frac{|D\tilde{v}(x + s\mathbf{w})|}{|s\mathbf{w}|^{N-1}} \, ds \right) d\mathbf{w} \right) dt \\ &\le \operatorname{diam}(K)^N \int_K \frac{|D\tilde{v}(z)|}{|z - x|^{N-1}} \, dz = N^{N/2}|K| \int_K \frac{|D\tilde{v}(z)|}{|z - x|^{N-1}} \, dz, \end{aligned}$$

with a touch of Fubini's Theorem on the first equality. Now we integrate with respect to x over the set $[\tilde{v} \le \lambda] \cap K$, again bounding from below the resulting left-hand side by the assumption on the measure of such a set and extending on the right hand side the integral to the whole K, to obtain

$$\begin{aligned} \frac{\alpha\delta(1-\lambda)}{4N^{N/2}}|K| \le \frac{\alpha(1-\lambda)}{4}|[\tilde{v} \le \lambda] \cap K| &\le \int_K \int_K \frac{|D\tilde{v}(z)|}{|z - x|^{N-1}} \, dz dx \\ &\le \left(\sup_{z \in K} \int_K \frac{dx}{|z - x|^{N-1}} \right) \int_K |D\tilde{v}(z)| \, dz \le \gamma C(N) |K|^{1/N}, \end{aligned}$$

using the assumption (3.42) together with $||D\tilde{v}||_{L^1(K)} = ||Dv||_{L^1(K)}$, and estimating for each $z \in K$ the Riesz potential by

$$\int_K \frac{dx}{|z-x|^{N-1}} \le w_N \int_{R(x,\mathbf{n})} \left(\int_0^{\mathrm{diam}(K)} s^{N-1-(N-1)} ds \right) d\mathbf{n} = C(N)\mathrm{diam}(K) = C(N)|K|^{1/N},$$

by enclosing K in a ball $B_{2\sqrt{N}}(z)$ of radius $2N^{1/2}$ centered at z, using polar coordinates and estimating $|z-x| = s$. Summarizing, by assumptions (3.42)-(3.44) we have

$$||Dv||_{L^1(K)} \ge C(N,\gamma,\alpha,\delta,\lambda)n^{1-N}, \qquad \forall K \in \mathbb{K}^+. \tag{3.45}$$

Moreover, the assumption (3.42) implies also

$$\alpha n^N |K| = \alpha |K_1| \le \sum_{K_j \in \mathbb{K}^+} |[v>1] \cap K_j| + \sum_{K_i \in \mathbb{K}^-} |[v>1] \cap K_i|,$$

so that by definition of families $\mathbb{K}^{\pm}$ we have

$$\alpha n^N \le \sum_{K_j \in \mathbb{K}^+} \frac{|[v>1] \cap K_j|}{K_j} + \sum_{K_i \in \mathbb{K}^-} \frac{|[v>1] \cap K_i|}{K_i} < \#(\mathbb{K}^+) + \frac{\alpha}{2}(n^N - \#(\mathbb{K}^-)).$$

This implies

$$\#(\mathbb{K}^+) > \frac{\alpha}{\alpha-2} n^N. \tag{3.46}$$

Finally, as we are supposing that our estimates are valid for all $K \in \mathbb{K}^+$, we can sum estimate (3.45) over $\mathbb{K}^+$ and use (3.46) to obtain

$$||Dv||_{L^1(K_1)} \ge \sum_{K \in \mathbb{K}^+} ||Dv||_{L^1(K)} \ge \#(\mathbb{K}^+) C(N,\gamma,\alpha,\delta,\lambda) n^{1-N} \ge C(N,\gamma,\alpha,\delta,\lambda)n.$$

This condition leads us to a contradiction when n is chosen big enough.

□

Remark 3.5. *We observe that if we have an information of the kind* (3.42) *for the truncated function* $(u-1)_+$, *the result of the above Lemma would not be at our disposal for the function* u. *This information can however be recovered when the truncation from below is at stake.*

Corollary 3.2. *Let* $u \in W^{1,1}(K_\rho)$ *satisfy for some constants* $\gamma > 0, \alpha \in (0,1)$

$$\int_{K_\rho} |D(u-1)_-|\, dx \le \gamma \rho^{N-1} \quad \text{and} \quad |[u>1] \cap K_\rho| \ge \alpha |K_\rho|. \tag{3.47}$$

Then for every choice of $\lambda, \bar{\nu} \in (0,1)$ *there exists a point* $y \in K_\rho$ *and a number* $\epsilon = \epsilon(\lambda,\bar{\nu},\gamma,\alpha,N) \in (0,1)$ *such that*

$$|[u>\lambda] \cap K_{\epsilon\rho}(y)| > (1-\bar{\nu})|K_{\epsilon\rho}(y)|. \tag{3.48}$$

Proof. If $u \in W^{1,1}(K_\rho)$, then for $a>0$ the function $u^a := \min\{u, 1+a\} \in W^{1,1}(K_\rho)$ and we have trivially[3] for each $b \in [0,1]$, $\rho \in (0,1)$ the equivalence of the two measures

$$|[u>b] \cap K_\rho| = |[u^a > b] \cap K_\rho|. \tag{3.49}$$

By monotonicity of the function $a \to \int_{K_\rho} |D(u-1-a)_-|\, dx$ we have

$$\lim_{a \downarrow 0} \int_{K_\rho} |D(u-1-a)_-|\, dx = \int_{K_\rho} |D(u-1)_-|\, dx. \tag{3.50}$$

[3]Indeed, if $x \in [u^a > b]$ then $b < \min\{u, 1+a\} \le u$, therefore implying one inequality. On the other hand, as $b<1$ there holds that if $x \in [u>b]$ then the possibilities are either $\min\{u,1+a\} = 1+a > b$ trivially or the claim itself.

This means that we can choose an $a \in (0,1)$ close enough to zero such that the energy information of (3.47) gives

$$\int_{K_\rho} |D\, u^a|\, dx = \int_{K_\rho} |D\,(u-1-a)_-|\, dx \le 2\gamma \rho^{N-1}, \quad \text{and} \quad |[u^a > 1] \cap K_\rho| = |[u>1] \cap K_\rho| \ge \alpha |K_\rho|.$$

Consequently we apply Lemma 3.5 getting that for each $\lambda, \delta \in (0,1)$ there exist a point $y \in K_\rho$ and a number $\epsilon \in (0,1)$ both not depending on a, and giving

$$|[u^a > \lambda] \cap K_{\epsilon\rho}(y)| > (1-\delta)|K_{\epsilon\delta}| \quad \Rightarrow \quad |[u>\lambda] \cap K_{\epsilon\rho}(y)| > (1-\delta)|K_{\epsilon\delta}|. \tag{3.51}$$

□

Proposition 3.6. *Let $\rho, \theta > 0$ be such that $u \ge 0$ is a solution to* (1.1) *in $\mathcal{Q}_\rho^-(\theta)(\bar{x},\bar{t}) = (\bar{x},\bar{t}) + \mathcal{Q}_\rho^-(\theta)$, and such that for $a, \nu \in (0,1)$ the following condition holds*

$$|[u > a\theta] \cap \mathcal{Q}_\rho^-(\theta)(\bar{x},\bar{t})| > \nu |\mathcal{Q}_\rho^-(\theta)|, \tag{3.52}$$

Then for every choice of $\bar{\lambda}, \bar{\nu} \in (0,1)$ there exist a point $(\bar{y},\bar{s}) \in (\bar{x},\bar{t}) + T_{\rho,\theta}(K_1 \times (-1, -\nu/4])$ and a number $\epsilon \in (0,1)$ determined only by means of $N, p_i, \nu, \bar{\nu}, a, \bar{\lambda}$ such that $K_{\epsilon\rho}(\theta)(\bar{y}) \subset K_\rho$ and

$$|[u(\cdot,\bar{s}) > \bar{\lambda} a\theta] \cap K_{\epsilon\rho}(\epsilon\theta)(\bar{y})| > \bar{\nu} |K_{\epsilon\rho}(\epsilon\theta)|. \tag{3.53}$$

Proof. We write down the energy estimates (3.19) for $(u-k)_-$, for $k = a\theta$, over the pair of anisotropic cylinders with same vertex

$$(\bar{x},\bar{t}) + \mathcal{Q}_{\rho/2}^-(\theta) \subset (\bar{x},\bar{t}) + \mathcal{Q}_\rho^-(\theta).$$

The non-negative, piece-wise smooth cut-off function $\eta = \prod_{i=1}^N \eta_i(x_i,t)$ is taken to be equal to 1 in the smallest of these cylinders, vanishing outside the largest, and satisfying

$$0 \le \partial_t \eta_t \le \frac{2^{\bar{p}}\gamma}{\theta^{2-\bar{p}}\rho^{\bar{p}}}, \quad \text{and} \quad |\partial_i \eta_i| \le \frac{2^{\bar{p}}\gamma}{\theta^{(p_i-\bar{p})/p_i}\rho^{\bar{p}/p_i}}.$$

These energy estimates give

$$\sum_{i=1}^N \int\int_{(\bar{x},\bar{t})+\mathcal{Q}_{\rho/2}^-(\theta)} |\partial_i (u-k)_-|^{p_i}\, dx d\tau \le \gamma \frac{k^{\bar{p}}}{\rho^{\bar{p}}} |\mathcal{Q}_\rho^-(\theta)|, \tag{3.54}$$

which can be rewritten for the transformed function

$$x_i \to \frac{2^{\frac{\bar{p}}{p_i}}(x_i - \bar{x}_i)}{\theta^{\frac{p_i-\bar{p}}{p_i}}\rho^{\frac{\bar{p}}{p_i}}}, \quad t \to \frac{2^{\bar{p}}(t-\bar{t})}{\theta^{2-\bar{p}}\rho^{\bar{p}}}, \quad v = \frac{1}{k} u\left(\bar{x}_i + x_i\left(\frac{k^{p_i-\bar{p}}\rho^{\bar{p}}}{2^{\bar{p}}}\right)^{\frac{1}{p_i}}, \bar{t} + \frac{tk^{2-\bar{p}}\rho^{\bar{p}}}{2^{\bar{p}}}\right), \tag{3.55}$$

transforming $(\bar{x},\bar{t}) + \mathcal{Q}_{\rho/2}^-(\theta)$ in $Q_1^- = K_1 \times (-1, 0]$ and yielding by standard calculations

$$\sum_{i=1}^N \int\int_{Q_1^-} |\partial_i (v-1)_-|^{p_i}\, dxdt \le \gamma, \quad \text{and} \quad |[v>1] \cap Q_1^-| > \nu. \tag{3.56}$$

Now we claim that this two conditions imply that there exists a time level $\bar{s} \in (-1, -\nu/4]$ such that

$$\sum_{i=1}^N \int_{K_1} |\partial_i (v-1)_-|^{p_i}\, dx \le 2\gamma/\nu, \quad \text{and} \quad |[v(\cdot,\bar{s}) > 1] \cap K_1| \ge \nu/2. \tag{3.57}$$

PROOF OF THE CLAIM. Indeed, let us define the sets

$$T_1 := \left\{ t \in (-1,0] : \sum_{i=1}^N \int_{K_1} |\partial_i (v-1)_-|^{p_i}(\cdot,t)\, dx > 4\gamma/\nu \right\}, \quad \text{and}$$

$$T_2 := \Big\{ t \in (-1,0] : |[v(\cdot,t) > 1] \cap K_1| \geq \nu/2 \Big\}.$$

From the definition of T_1 and the first assumption we have

$$\frac{4\gamma}{\nu}|T_1| \leq \sum_{i=1}^N \int_{-1}^0 \int_{K_1} |\partial_i (v-1)_-|^{p_i}(\cdot,t)\, dxdt \leq \gamma, \quad \Rightarrow \quad |T_1| < \nu/4.$$

On the other hand, by the definition of T_2 we have

$$\begin{aligned}\nu < |[v>1] \cap Q_1^-| &= \int_{-1}^0 |[v(\cdot,t) > 1] \cap K_1|\, dt \\ &= \int_{T_2} |[v(\cdot,t)>1] \cap K_1|\, dt + \int_{T_2^c} |[v(\cdot,t) > 1] \cap K_1|\, dt \leq |T_2| + \nu/2, \quad \Rightarrow |T_2| > \nu/2.\end{aligned}$$

This proves the claim, because if we consider the partition of the interval $(-1,0] = (-1,\,-\nu/4] \cup (-\nu/4, 0]$, then T_2 exceeds the second interval while T_1^c has full measure in the first one.

Subsequently, being in a unitary cube, using that $p_i > 2$ and the Hölder inequality twice we get

$$\int_{K_1} |D(v(\cdot,\bar{s})-1)_-| dx \leq \Bigg[\sum_{i=1}^N \left(\frac{2\gamma}{\nu}\right)^{\frac{2}{p_i}} \Bigg]^{\frac{1}{2}} =: \gamma_\nu, \quad \text{and} \quad |[v(\cdot,\bar{s}) > 1] \cap K_1| \geq \nu/2. \tag{3.58}$$

Finally we can invoke Corollary 3.2 and obtain that for every $\bar{\lambda}, \bar{\nu} \in (0,1)$ there exists at least a point $y \in K_1$ and a constant $\epsilon \in (0,1)$ that can be determined a priori only in terms of γ, ν above such that

$$K_\epsilon(y) \subset K_1, \quad \text{and} \quad |[(v(\cdot,s)-1)_- > \bar{\lambda}] \cap |K_\epsilon(y)|)| > \bar{\nu}|K_\epsilon(y)|.$$

Returning to the original coordinates, this implies

$$\left| \left\{ \frac{1}{k} u\left(\bar{x}_i + x_i \left(\frac{k^{p_i - \bar{p}} \rho^{\bar{p}}}{2^{\bar{p}}} \right)^{\frac{1}{p_i}}, \bar{t} + s \frac{k^{2-\bar{p}} \rho^{\bar{p}}}{2^{\bar{p}}} \right) > \bar{\lambda} \right\} \cap K_\epsilon(\bar{y}) \right| > \bar{\nu} |K_\epsilon(\bar{y})|,$$

that is, by calling $\bar{y}$ the center of $K_\epsilon(y)$ in the transformed coordinates by $\bar{y}_i = y_i \left(\frac{k^{p_i-\bar{p}}\rho^{\bar{p}}}{2^{\bar{p}}} \right)^{\frac{1}{p_i}}$, by recalling that $k = a\theta$, and defining $\bar{s} = \bar{t} + s\frac{k^{2-\bar{p}}\rho^{\bar{p}}}{2^{\bar{p}}}$, the obtained measure information is

$$|[u(\cdot,\bar{s}) > \bar{\lambda} a\theta] \cap K_{(\epsilon\rho)}(\epsilon\theta)| > \bar{\nu}|K_{(\epsilon\rho)}(\epsilon\theta)|, \tag{3.59}$$

so that the statement can be obtained by suitably redefining the constants. □

Previous Proposition 3.6 could be proven in a simpler way in the unitary cylinder. We preferred the general case this time to show for once how to get the general estimate, and then pass to the estimate in the unitary cylinder, by taking $(\bar{x},\bar{t}) = 0$, $\rho = 1$ and $\theta = 1$.

Lemma 3.6. *Let $u \geq 0$ be a local weak solution to* (1.1) *in Q_1^-, and suppose that for some $\bar{\nu} \in (0,1)$ $a > 0$ it holds*

$$|[u > a] \cap Q_1^-| > \bar{\nu}\, |Q_1^-|. \tag{3.60}$$

Then for every choice of $\lambda, \nu \in (0,1)$ there exist $\bar{y} \in K_1$, $\bar{t} \in (-1, -\bar{\nu}/4]$ and $\epsilon \in (0,1)$ determined only by means of $N, \mathbf{p}, \nu, \bar{\nu}, a$ and λ, such that $\bar{y} + K_\epsilon \subset K_1$ and

$$|[u(\cdot,\bar{t}) > \lambda\, a] \cap (\bar{y} + K_\epsilon)| > \bar{\nu}\, |K_\epsilon|. \tag{3.61}$$

3.6 Comparison Principles.

We consider in this Section the Cauchy problem for (1.1), namely

$$\begin{cases} \partial_t u = \sum_{i=1}^N \partial_i(|\partial_i u|^{p_i-2}\partial_i u) & \text{weakly in } \Omega_T, \\ u(x,0) = u_0(x) & \text{strongly in } L^2(\Omega), \end{cases} \tag{3.62}$$

and a similar one for the Fokker-Planck equation (3.10). We refer to Definition 2.2, for the meaning of $u(x,0) = u_0(x)$ in $L^2(\Omega)$. Given two local weak solutions u, v of this problem (3.62), classically we say that $u \geq v$ on the parabolic boundary of Ω_T if $(u-v)_+ \in L^{\mathbf{p}}_{loc}(0,T; W_0^{1,\mathbf{p}}(\Omega))$ and $u_0 \geq v_0$.
This condition can be rephrased by requiring $u(x,t) \geq v(x,t)$ for $\mathcal{L}^{N-1}$-almost every $(x,t) \in \Sigma_T \cup (\Omega \times \{0\})$.
We begin with a classic statement of local nature.

Proposition 3.7. *Let u, v be weak local solutions to the equation (1.1) in Ω_T, satisfying $u(x,t) \geq v(x,t)$ in the parabolic boundary ot Ω_T. Then $u \geq v$ in Ω_T.*

Proof. We write (9.12) for u and v separately, in terms of the Steklov-averages, against the test function

$$[(v-u)_h]_+(x,t) = \left[\frac{1}{h}\int_t^{t+h} (v-u)(x,\tau)d\tau\right]_+, \quad \text{for} \quad h \in (0,T),\ t \in [0, T-h].$$

This function is admissible because of the hypothesis on the lateral boundary Σ_T. Thus we have respectively

$$\int_{\Omega\times\{t\}} \partial_t(u_h)[(v-u)_h]_+(x,t)dx + \int_{\Omega\times\{t\}} \sum_{i=1}^N \Big(|\partial_i u|^{p_i-2}\partial_i u\Big)_h \cdot \partial_i[(v-u)_h]_+(x,t)dx = 0, \tag{3.63}$$

$$\int_{\Omega\times\{t\}} \partial_t(v_h)[(v-u)_h]_+(x,t)dx + \int_{\Omega\times\{t\}} \sum_{i=1}^N \Big(|\partial_i v|^{p_i-2}\partial_i v\Big)_h \cdot \partial_i[(v-u)_h]_+(x,t)dx = 0. \tag{3.64}$$

We subtract equation (3.63) from equation (3.64), obtaining

$$\int_{\Omega\times\{t\}} \partial_t(v_h - u_h)[(v-u)_h]_+(x,t)dx = -\int_{\Omega\times\{t\}} \sum_{i=1}^N \Big(|\partial_i v|^{p_i-2}\partial_i v - |\partial_i u|^{p_i-2}\partial_i u\Big)_h \cdot \partial_i[(v-u)_h]_+(x,t)dx.$$

Now we integrate between 0 and t and use the properties of differentiability of Steklov averages, to get

$$\int_\Omega \int_0^t \partial_t\left[\frac{[(v-u)_h]_+^2(x,s)}{2}\right] dxds = -\int_\Omega\int_0^t \sum_{i=1}^N \Big(|\partial_i v|^{p_i-2}\partial_i v - |\partial_i u|^{p_i-2}\partial_i u\Big)_h \cdot \partial_i[(v-u)_h]_+(x,s)\, dxds,$$

and so

$$\begin{aligned} &\int_\Omega [(v-u)_h]_+^2(x,t)\,dx - \int_\Omega [(v-u)_h]_+^2(x,0)\,dx = \\ &\qquad = -2\int_\Omega\int_0^t \sum_{i=1}^N \Big(|\partial_i v|^{p_i-2}\partial_i v - |\partial_i u|^{p_i-2}\partial_i u\Big)_h \cdot \partial_i[(v-u)_h]_+(x,s)\, dxds. \end{aligned}$$

As $h \to 0$, the second term on the left hand side tends to zero as $(v-u)_+(x,0) = 0$ by the hypothesis on initial conditions along the set $\Omega \times \{0\}$, while as $u, v \in C(0,T; L^2(\Omega))$ the property (d) of Proposition 9.6 applies. Hence the convergence of the Steklov averages and (9.1) yield for each $t \in [0,T)$ the following estimate

$$\begin{aligned} &\int_{\Omega\cap[v(\cdot,t)\geq u(\cdot,t)]} (v-u)^2(x,t)\,dx = \\ &\quad = -2\int\int_{\{\Omega\times(0,t)\}\cap[v\geq u]} \sum_{i=1}^N \Big(|\partial_i v|^{p_i-2}\partial_i v - |\partial_i u|^{p_i-2}\partial_i u\Big) \cdot \partial_i(v-u)(x,s)\,dxds \\ &\quad \leq -2\sum_{i=1}^N \gamma_i\, ||\partial_i v - \partial_i u||^{p_i}_{L^{p_i}(\Omega_T\cap[v\geq u])} \leq 0. \end{aligned} \tag{3.65}$$

□

Next we provide a comparison principle for the class of $L^{\mathbf{p}}$-solutions, that will be useful for next purposes. This time higher integrability replaces the boundary datum on the lateral boundary to arrive to the same conclusion.

Proposition 3.8. *Let $u, v \in \cap_{i=1}^N L^{p_i}(S_T)$ be two local weak solutions of (3.62) in S_T, satisfying $u(x,0) \geq v(x,0)$ for $x \in \mathbb{R}^N$ for initial data such that $u(x,0), v(x,0) \in L^2(S_T)$. Then $u \geq v$ in S_T.*

Proof. First notice that if u is an $L^{\mathbf{p}}$ solution to (3.62) in S_T with $u_0 \in L^2(\mathbb{R}^N)$, then $u \in L^{\mathbf{p}}(0,T; W^{1,\mathbf{p}}_{loc}(\mathbb{R}^N))$. Indeed, by the energy estimate (3.13) with a linear test function $0 \leq \eta_R = \eta_R(x) \leq 1$ vanishing on ∂B_R and such that $\eta \equiv 1$ on $B_{R/2}$, we deduce that

$$\sum_{i=1}^N \int_0^T \int_{B_{R/2}} |\partial_i u_+|^{p_i}\, dxdt \leq ||u_0||^2_{S_T} + \sum_{i=1}^N \frac{\gamma}{R^{p_i}} \int_0^T \int_{B_R} |u_+|^{p_i}\, dxdt \leq ||u_0||^2_{S_T} + \sum_{i=1}^N \frac{\gamma}{R^{p_i}} ||u_+||^{p_i}_{L^{p_i}(S_T)}\, dxdt$$

is finite, and similarly for u_- and v. Now we test the equation with

$$[(v-u)_h]_+(x,t)\zeta_R(x),$$

being ζ_R a cut-off function between the balls B_R and B_{2R}, independent of time and such that $|\partial_i \zeta_R| \leq \frac{\gamma}{R}$, $0 \leq \zeta_R \leq 1$. So we have respectively, for the Steklov averages of u, v,

$$\int_{B_R \times \{t\}} \partial_t(u_h)[(v-u)_h]_+\, dx + \sum_{i=1}^N \int_{B_{2R}\times\{t\}} \Big(|\partial_i u|^{p_i-2}\partial_i u\Big)_h \partial_i([(v-u)_h]_+\zeta_R)\, dx \leq 0,$$

and

$$\int_{B_R \times \{t\}} \partial_t(v_h)[(v-u)_h]_+\, dx + \sum_{i=1}^N \int_{B_{2R}\times\{t\}} \Big(|\partial_i v|^{p_i-2}\partial_i v\Big)_h \partial_i([(v-u)_h]_+\zeta_R)\, dx \leq 0.$$

Subtracting one equation from the other and by integrating between 0 and t, we reduce the integral to the non-zero terms and we pass to the limit on h to get

$$\begin{aligned}
&\int_{B_R \cap [v(\cdot,t)\geq u(\cdot,t)]} (v-u)^2(x,t)\, dx \\
&\quad \leq \int_{B_R \cap [v(\cdot,t)\geq u(\cdot,t)]} (v-u)^2(x,t)\, dx - \int_{B_R \cap [v(\cdot,0)\geq u(\cdot,0)]} (v-u)^2(x,0)\, dx + \\
&\quad + \gamma \sum_{i=1}^N \int_0^t \int_{B_{2R}\cap[v(\cdot,s)\geq u(\cdot,s)]} \Big(|\partial_i v|^{p_i-2}\partial_i v - |\partial_i u|^{p_i-2}\partial_i u\Big)\partial_i(v-u)\, dxds \\
&\quad \leq \sum_{i=1}^N \frac{\gamma}{R} \int_0^t \int_{B_{2R}\cap[v(\cdot,s)\geq u(\cdot,s)]} (|\partial_i v|^{p_i-1}|u| + |\partial_i v|^{p_i-1} v + |\partial_i u|^{p_i-1}|v| + |\partial_i u|^{p_i-1}|u|)\, dxds \\
&\quad \leq \frac{\gamma}{R}\sum_{i=1}^N \Big(||\partial_i v||^{p_i}_{L^{p_i}(Q_{t,2R})} + ||v||^{p_i}_{L^{p_i}(S_T)} + ||\partial_i u||^{p_i}_{L^{p_i}(Q_{t,2R})} + ||u||^{p_i}_{L^{p_i}(S_T)}\Big) \\
&\quad \leq \frac{\gamma}{R}\sum_{i=1}^N \Big(||v||^{p_i}_{L^{p_i}(S_T)} + ||u||^{p_i}_{L^{p_i}(S_T)}\Big) \to 0, \quad \text{when} \quad R \to +\infty,
\end{aligned} \tag{3.66}$$

being $Q_{t,2R} = B_{2R} \times (0,t)$ and by using monotonicity of the operator joint with the use of Young's inequality. □

As a corollary, we have the following comparison principle for solutions to the Fokker-Planck equation.

Corollary 3.3. *Let w_1, w_2 be $L^{\mathbf{p}}$-solutions to the equation (3.10) satisfying $w_2(x,0) \geq w_1(x,0)$ for $x \in \mathbb{R}^N$ and $w_1(x,0), w_2(x,0) \in L^2(S_T)$. Then $w_2 \geq w_1$ in S_T.*

4 Global and local L^∞-estimates

to semi-continuity and beyond

Even the official definitions are sometimes

of necessity very loose, corresponding to the well-known principle that,

in a formal theory, some terms must in strict logic be left undefined.

- L.J. Savage -

The Foundations of Statistics.

The following section is about the derivation of estimates on the essential supremum both of local weak solutions to (1.4) in the strip $S_T = \mathbb{R}^N \times (0,T)$ and of local weak solutions u to (1.1). Both are proved by De Giorgi iteration, using a combination of energy estimates, that provide us with a sort of reverse Sobolev-Poincaré inequality, and the parabolic Sobolev-Poincaré embedding of Section 2. Their nested use generates an iteration of integrals of a positive integrand over sub-level sets of u, eventually giving the result by showing that the limit vanishes. Finally we use the local estimates obtained for the supremum of the solution to generate a proof *per absurdum* of the lower semicontinuity of supersolutions, in a technique which is reminescent of [44]. First we start by the annouced global estimate.

Theorem 4.1. *Let $p_1 \le \cdots \le p_N$, $\bar p < N$ and $u \in \cap_{i=1}^N L^{p_i}(S_T)$ solve the Cauchy problem* (1.4) *for $u_0 \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ attained within the meaning of Definition 2.2. Then:*

1. *If $\bar p_2 > 2$, then $u \in L^\infty_{\rm loc}(0,T; L^\infty(\mathbb{R}^N))$ and for any $\tilde q \in [2, \bar p_2]$, $\theta > 0$ the following estimate holds true*

$$\sup_{t\in[\theta,T]} \|u(\cdot,t)\|_\infty \le \frac{C}{\theta^{\frac{N+\bar p}{\lambda_{\tilde q}}}} \left(\int_{\theta/2}^T \int_{\mathbb{R}^N} |u|^{\tilde q}\, dx\, dt \right)^{\frac{\bar p}{\lambda_{\tilde q}}}, \qquad \lambda_{\tilde q} = N(\bar p - 2) + \bar p\, \tilde q. \tag{4.1}$$

2. *If $\bar p_1 > 2$, the following $L^1 - L^\infty$ estimate holds true for any $\tau \in (0,T]$*

$$\|u(\cdot,\tau)\|_\infty \le \frac{C}{\tau^{\frac{N}{\lambda}}} \|u_0\|_1^{\frac{\bar p}{\lambda}}, \qquad \lambda := \lambda_1 = N(\bar p - 2) + \bar p. \tag{4.2}$$

Proof. First we observe that $\lambda_q \ge \lambda_2 > 0$ for all $q \ge 2$. The global condition $u \in \cap_{i=1}^N L^{p_i}(S_T)$, together with Corollary 2.1 and Proposition 3.4, implies that $u \in L^{\bar p_2}(S_T) \cap L^2(S_T)$. Therefore, by interpolation,

$$u \in L^q(S_T), \qquad \text{for all } q \in [\min\{2, p_1\}, \max\{\bar p_2, p_N\}].$$

Let $k>0$ to be determined, $T>\theta>0$ and define for any $n\geq 0$

$$k_n=k-\frac{k}{2^n},\qquad \theta_n=\theta-\frac{\theta}{2^{n+1}},\qquad S_n=\mathbb{R}^N\times[\theta_n,T],\qquad \psi_n(t)=\min\left\{1,\frac{2^{n+2}}{\theta}(t-\theta_n)_+\right\},$$

so that

$$\psi_n\equiv 0 \text{ on } [0,\,\theta_n],\qquad \psi_n\equiv 1 \text{ on } [\theta_{n+1},T],\qquad |\partial_t(\psi_n)|\leq 2^{n+2}/\theta.$$

It is useful to observe that θ_n are increasing with n from $\theta/2$ to θ, so that sets S_n are shrinking from $\mathbb{R}^N\times[\theta/2,\,T]$ to $\mathbb{R}^N\times[\theta,\,T]$. Now, since $\psi_n(0)=0$, the energy estimate (3.25) reads

$$\sup_{t\in[\theta_n,T]}\int_{\mathbb{R}^N}(u-k_n)_+^2\,dx+\frac{1}{C}\sum_{i=1}^N\iint_{S_n}|\partial_i(u-k_n)_+|^{p_i}\,dx\,dt\leq\frac{2^{n+2}}{\theta}\iint_{S_n}(u-k_n)_+^2\,dx\,dt. \tag{4.3}$$

Choose $\tilde q\in[2,\bar p_2]$. If $A_n:=\{(x,t)\in S_n : u(x,t)\geq k_n\}$, Tchebichev's inequality yields

$$\begin{aligned}\iint_{S_n}(u-k_{n-1})_+^{\tilde q}\,dx\,dt\geq(k_n-k_{n-1})^{\tilde q}|A_n|=\frac{k^{\tilde q}}{2^{n\,\tilde q}}|A_n|,\\ \Rightarrow|A_n|\leq\frac{2^{\tilde q n}}{k^{\tilde q}}\iint_{S_n}(u-k_{n-1})_+^{\tilde q}\,dx\,dt,\end{aligned} \tag{4.4}$$

so that, by Hölder's inequality and the monotonicity of $\{S_n\}$ we get

$$\iint_{S_n}(u-k_n)_+^2\,dx\,dt\leq\left(\iint_{S_n}(u-k_n)_+^{\tilde q}\,dx\,dt\right)^{\frac{2}{\tilde q}}|A_n|^{1-\frac{2}{\tilde q}}\leq\frac{2^{n(\tilde q-2)}}{k^{\tilde q-2}}\iint_{S_{n-1}}(u-k_{n-1})_+^{\tilde q}\,dx\,dt.$$

Therefore (4.3) becomes

$$\sup_{t\in[\theta_n,T]}\int_{\mathbb{R}^N}(u-k_n)_+^2\,dx+\frac{1}{C}\sum_{i=1}^N\iint_{S_n}|\partial_i(u-k_n)_+|^{p_i}\,dx\,dt\leq C\,\frac{2^{n\,(\tilde q-1)}}{\theta\,k^{\tilde q-2}}\iint_{S_{n-1}}(u-k_{n-1})_+^{\tilde q}\,dx\,dt \tag{4.5}$$

By Hölder inequality and (4.4)

$$\begin{aligned}\iint_{S_n}(u-k_n)_+^{\tilde q}\,dx\,dt&\leq|A_n|^{1-\frac{\tilde q}{\bar p_2}}\left(\iint_{S_n}(u-k_n)_+^{\bar p_2}\,dx\,dt\right)^{\frac{\tilde q}{\bar p_2}}\\&\leq C\left(\frac{2^{n\,\tilde q}}{k^{\tilde q}}\iint_{S_{n-1}}(u-k_{n-1})_+^{\tilde q}\,dx\,dt\right)^{1-\frac{\tilde q}{\bar p_2}}\left(\iint_{S_n}(u-k_n)_+^{\bar p_2}\,dx\,dt\right)^{\frac{\tilde q}{\bar p_2}}.\end{aligned}$$

Applying (2.16) for $\alpha_i\equiv 1$, $\sigma=2$ and $\theta=\frac{\bar p}{\bar p^*}$ gives $q=\bar p_2$ and thus by (4.5)

$$\begin{aligned}\iint_{S_n}(u-k_n)_+^{\bar p_2}\,dx\,dt&\leq C\left(\sup_{t\in[\theta_n,T]}\int(u-k_n)_+^2(x,t)\,dx\right)^{1-\frac{\bar p}{\bar p^*}}\prod_{i=1}^N\left(\iint_{S_n}|\partial_i(u-k_n)_+|^{p_i}\,dx\,dt\right)^{\frac{\bar p}{N\,p_i}}\\&\leq C\left(\frac{2^{n\,(\tilde q-1)}}{\theta\,k^{\tilde q-2}}\iint_{S_{n-1}}(u-k_{n-1})_+^{\tilde q}\,dx\,dt\right)^{1+\frac{\bar p}{N}}.\end{aligned}$$

Gathering together the previous two estimates, we obtain, for suitable $b>1$,

$$\iint_{S_n}(u-k_n)_+^{\tilde q}\,dx\,dt\leq\frac{C}{(\theta\,k^{\tilde q-2})^{\frac{\tilde q}{\bar p_2}(1+\frac{\bar p}{N})}}\frac{b^n}{k^{\tilde q\,(1-\frac{\tilde q}{\bar p_2})}}\left(\iint_{S_{n-1}}(u-k_{n-1})_+^{\tilde q}\,dx\,dt\right)^{(1+\frac{\bar p}{N})\,\frac{\tilde q}{\bar p_2}+1-\frac{\tilde q}{\bar p_2}}.$$

Setting

$$X_n=\iint_{S_n}(u-k_n)_+^{\tilde q}\,dx\,dt,\qquad\alpha=\frac{\tilde q}{N+2},\qquad\beta=\frac{\tilde q}{\bar p}\frac{N+\bar p}{N+2},\qquad\gamma=\frac{\tilde q}{\bar p}\,\frac{N\,(\bar p-2)+\bar p\,\tilde q}{N+2}$$

we thus obtained the recursive inequality

$$X_n \leq \frac{C}{\theta^\beta\, k^\gamma}\, b^n\, X_{n-1}^{1+\alpha}, \qquad n \geq 1.$$

We use Lemma 9.3 with $N = 1$, to have that $X_n \to 0$ provided $X_0 \leq C\theta^{\frac{\beta}{\alpha}} k^{\frac{\gamma}{\alpha}}$ for a suitable constant C depending only on the data. Choosing k such that equality holds we therefore get

$$\sup_{\mathbb{R}^N\times[\theta,T]} u \leq k = \frac{C}{\theta^{\frac{N+\bar p}{\lambda_{\tilde q}}}} \left(\int_{\theta/2}^T \int_{\mathbb{R}^N} u_+^{\tilde q}\, dx\, dt\right)^{\frac{\bar p}{\lambda_{\tilde q}}} \qquad \forall \theta \in\ (0,T)$$

implying (4.1) (by considering $-u$ as well). To prove (4.2), assume $\bar p_1 \geq 2$ and choose $\tilde q \in [2, \bar p_1)$. The function $v(x,t) = u(x, t+\theta)$ still solves the equation on $S_{T-\theta}$ and is bounded there by the previous estimate. Moreover $v \in \cap L^{p_i}(S_{T-\theta})$ hence (4.1) holds, reading

$$\sup_{\mathbb{R}^N\times[\tilde\theta,T-\theta]} |v| \leq \frac{C}{\tilde\theta^{\frac{N+\bar p}{\lambda_{\tilde q}}}} \left(\int_{\frac{\tilde\theta}{2}}^{T-\theta} \int_{\mathbb{R}^N} |v|^{\tilde q}\, dx\, dt\right)^{\frac{\bar p}{\lambda_{\tilde q}}} \qquad \forall \tilde\theta \in\ (0,T-\theta).$$

In the latter inequality we set

$$\tilde\theta = \theta_n = \frac{T-\theta}{2^n}, \qquad M_n = \sup_{\mathbb{R}^N\times[\theta_n,T-\theta]} |v|.$$

These time levels θ_n are decreasing from $T-\theta$ to zero, so that the sets $\mathbb{R}^N \times [\theta_n,\, T-\theta]$ are increasing in size, and as a consequence numbers M_n increase too. So we obtain

$$M_n \leq \frac{C}{\theta_n^{\frac{N+\bar p}{\lambda_{\tilde q}}}} \left(\int_{\theta_{n+1}}^{T-\theta} \int_{\mathbb{R}^N} |v|^{\tilde q}\, dx\, dt\right)^{\frac{\bar p}{\lambda_{\tilde q}}} \leq C\, \frac{2^{n\,\frac{N+\bar p}{\lambda_{\tilde q}}}}{(T-\theta)^{\frac{N+\bar p}{\lambda_{\tilde q}}}} \left(\int_0^{T-\theta} \int_{\mathbb{R}^N} |v|\, dx\, dt\right)^{\frac{\bar p}{\lambda_{\tilde q}}} M_{n+1}^{(\tilde q-1)\frac{\bar p}{\lambda_{\tilde q}}}$$

By the boundedness of v we infer the boundedness of $\{M_n\}$, while

$$(\tilde q - 1)\,\frac{\bar p}{\lambda_{\tilde q}} < 1 \quad \Leftrightarrow \quad \bar p > \frac{2N}{N+1} \quad \Leftrightarrow \quad \bar p_1 > 2.$$

So, interpolation Lemma 9.2 with $(\tilde q-1)\bar p/\lambda_{\tilde q} = 1-\alpha$ and $\alpha = 1-(\tilde q-1)\bar p/\lambda_{\tilde q}$ provides the estimate

$$M_0 \leq C\left(\frac{1}{(T-\theta)^{\frac{N+\bar p}{\lambda_{\tilde q}}}}\Big(\int_0^{T-\theta}\int_{\mathbb{R}^N} |v|\, dx\, dt\Big)^{\frac{\bar p}{\lambda_{\tilde q}}}\right)^{\frac{1}{1-(\tilde q-1)\frac{\bar p}{\lambda_{\tilde q}}}}.$$

Writing the latter in terms of u and noting that

$$\lambda_{\tilde q}\big(1-(\tilde q-1)\bar p/\lambda_{\tilde q}\big) = \lambda_1,$$

we obtain

$$\sup_{\mathbb{R}^N} |u(\cdot, T-\theta)| \leq C\, \frac{1}{(T-\theta)^{\frac{N+\bar p}{\lambda_1}}} \left(\iint_{\mathbb{R}^N\times[\theta,T]} |u|\, dx\, dt\right)^{\frac{\bar p}{\lambda_1}}, \qquad \forall \theta \in\ (0,T).$$

Using Proposition 3.5 while setting $\tau := T-\theta \in\ (0,T)$ finally gives (4.2), with $\lambda = \lambda_1$ by definition. □

Another estimate of the essential supremum is the following, which instead is purely local and requires that the p_is are very close (in harmonic mean) to the biggest one p_N. The estimate involves a sum weighted on p_is of the $\bar p_2$-norm of the sub-solution, taken in some suitable weird cylinders that we are about to describe.

Lemma 4.1. *Let*

$$2 < p_1 \le \cdots \le p_N \quad \textit{and} \quad \max\{2, p_N\} < \bar{p}_2. \tag{4.6}$$

We define for $k > 0$ the continuous and increasing functions

$$g(k) = \sum_{i=1}^{N} k^{p_i - 2}, \qquad h(k) = \left(\sum_{i=1}^{N} k^{p_i - \bar{p}_2} \right)^{-1}, \tag{4.7}$$

and for $T \in \mathbb{R}$, $M, \lambda > 0$ the cylinder

$$Q_{\lambda,M} = \prod_{i=1}^{N} \left[-\lambda^{\frac{1}{p_i}}, \lambda^{\frac{1}{p_i}} \right] \times [T - M\,\lambda, T]. \tag{4.8}$$

If u is a weak sub-solution to (1.1) *in $Q_{\lambda,M}$, then*

$$\|u_+\|_{L^\infty(Q_{\lambda/2,M})} \le g^{-1}(1/M) + h^{-1}\left(C \Big(M \fint\!\!\fint_{Q_{\lambda,M}} u_+^{\bar{p}_2}\, dx \Big)^{\frac{\bar{p}}{N+\bar{p}}} \right). \tag{4.9}$$

Proof. Define for any $k > 0$ and λ, M, T

$$k_n = k - \frac{k}{2^n}, \quad \theta_n = T - \frac{M\,\lambda}{2}\left(1 + \frac{1}{2^n}\right), \quad r_{n,i} = \frac{\lambda^{\frac{1}{p_i}}}{2^{\frac{1}{p_i}}}\left(1 + \frac{1}{2^{n+m}}\right),$$

$$Q_n = K_n \times [\theta_n, T] = \prod_{i=1}^{N} [-r_{n,i}, r_{n,i}] \times [\theta_n, T],$$

where we choose $m \in \mathbb{N}$ so that $Q_{n+1} \subseteq Q_n \subseteq Q_{\lambda,M}$ for all $n \ge 0$ and formally $Q_\infty = Q_{\lambda/2,M}$. Construct functions $\eta_n \in C^\infty(Q_n; [0,1])$ of the form (3.12) such that

$$\eta_n|_{\partial_p Q_n} \equiv 0, \qquad \eta_n|_{Q_{n+1}} \equiv 1, \qquad |\partial_i \eta_n^{\frac{1}{p_i}}| \le \frac{C\,2^n}{\lambda^{\frac{1}{p_i}}}, \qquad |\partial_t(\eta_n)| \le \frac{C\,2^n}{M\,\lambda}$$

Let us apply (3.19) to obtain

$$\sup_{t \in [\theta_n, T]} \int_{K_n} (u - k_n)_+^2\, \eta_n(x,t)\, dx + \sum_{i=1}^{N} \iint_{Q_n} \left| \partial_i\big((u - k_n)_+\, \eta_n \big) \right|^{p_i}\, dx\, dt$$
$$\le C\,2^n \left(\frac{1}{M\,\lambda} \iint_{Q_n} (u - k_n)_+^2\, dx\, dt + \frac{1}{\lambda} \sum_{i=1}^{N} \iint_{Q_n} (u - k_n)_+^{p_i}\, dx\, dt. \right).$$

Letting $p_0 = 2$, $A_n = \{(x,t) \in Q_n : u \ge k_n\}$ and recalling (8.3), we use that $\bar{p}_2 > p_N$ to get for each $i \in \{1, \dots, N\}$ the estimate

$$\iint_{Q_n} (u - k_n)_+^{p_i}\, dx\, dt \le \Big(\iint_{Q_n} (u - k_n)_+^{\bar{p}_2}\, dx\, dt \Big)^{\frac{p_i}{\bar{p}_2}} |A_n|^{1 - \frac{p_i}{\bar{p}_2}} \le \frac{C b^n}{k^{\bar{p}_2 - p_i}} \iint_{Q_{n-1}} (u - k_{n-1})_+^{\bar{p}_2}\, dx\, dt, \tag{4.10}$$

for some number $b > 1$, so that by the definition (4.7) we obtain

$$\sup_{t \in [\theta_n, T]} \int_{K_n} (u - k_n)_+^2\, \eta_n(x,t)\, dx + \sum_{i=1}^{N} \iint_{Q_n} \left| \partial_i\big((u - k_n)_+\, \eta_n \big) \right|^{p_i}\, dx\, dt$$
$$\le C\, b^n \left(\frac{1}{M\,\lambda\, k^{\bar{p}_2 - 2}} + \frac{1}{\lambda\, h(k)} \right) \iint_{Q_{n-1}} (u - k_{n-1})_+^{\bar{p}_2}\, dx\, dt.$$

As $\{k_n\}$ and $\{\theta_n\}$ are increasing and $\eta_n \equiv 1$ on $\mathrm{supp}(\eta_{n+1})$,

$$\sup_{t\in[\theta_{n+1},T]} \int_{K_{n+1}} \big((u-k_{n+1})_+\eta_{n+1}\big)^2(x,t)\,dx \le \sup_{t\in[\theta_n,T]} \int_{K_n} (u-k_n)_+^2\,\eta_n(x,t)\,dx$$
$$\le C\,b^n \left(\frac{1}{M\,\lambda\,k^{\bar p_2-2}} + \frac{1}{\lambda\,h(k)}\right) \iint_{Q_{n-1}} (u-k_{n-1})_+^{\bar p_2}\,dx\,dt.$$

Again by the monotonicity of $\{k_n\}$ and $\{Q_n\}$ it holds

$$\sum_{i=1}^N \iint_{Q_{n+1}} \big|\partial_i\big((u-k_{n+1})_+\,\eta_{n+1}\big)\big|^{p_i}\;dx\,dt \le C\,b^{n+1}\left(\frac{1}{M\,\lambda\,k^{\bar p_2-2}} + \frac{1}{\lambda\,h(k)}\right) \iint_{Q_n} (u-k_n)_+^{\bar p_2}\,dx\,dt$$
$$\le C\,b^n \left(\frac{1}{M\,\lambda\,k^{\bar p_2-2}} + \frac{1}{\lambda\,h(k)}\right) \iint_{Q_{n-1}} (u-k_{n-1})_+^{\bar p_2}\,dx\,dt$$

Therefore, applying (2.16) with $\alpha_i \equiv 1$, $\sigma = 2$, $\theta = \bar p/\bar p^*$ and thus $q = \bar p_2$, we deduce, for some other constants $C, b \ge 1$, the recursive inequality

$$\iint_{Q_{n+1}} (u-k_{n+1})_+^{\bar p_2}\,dx\,dt \le C\,b^n \left(\frac{1}{M\,\lambda\,k^{\bar p_2-2}} + \frac{1}{\lambda\,h(k)}\right)^{1+\frac{\bar p}{N}} \Big(\iint_{Q_{n-1}} (u-k_{n-1})_+^{\bar p_2}\,dx\,dt\Big)^{1+\frac{\bar p}{N}}$$

Now if k is so large that

$$\frac{1}{M\,k^{\bar p_2-2}} \le \frac{1}{h(k)}, \quad \Leftrightarrow \quad k \ge g^{-1}(1/M), \tag{4.11}$$

then the previous iterative inequality reads

$$X_{n+1} \le C\,b^n \left(\frac{1}{\lambda\,h(k)}\right)^{1+\frac{\bar p}{N}} X_n^{1+\frac{\bar p}{N}}, \qquad n \ge 0$$

where

$$X_n = \iint_{Q_{2n}} (u-k_{2n})_+^{\bar p_2}\,dx\,dt.$$

By Lemma 9.3 for $N=1$, $X_n \to 0$ whenever $X_0 \le C(\lambda\,h(k))^{\frac{N+\bar p}{\bar p}}$ and, taking account of (4.11), this in turn implies that

$$\sup_{Q_{\lambda/2,M}} u_+ \le \max\left\{g^{-1}(1/M), h^{-1}\left(\frac{C}{\lambda}\Big(\iint_{Q_0} u_+^{\bar p_2}\,dx\,dt\Big)^{\frac{\bar p}{N+\bar p}}\right)\right\}.$$

Note that $Q_0 \subseteq Q_{\lambda,M}$ and

$$|Q_{\lambda,M}| = M\,\lambda \prod_{i=1}^N \lambda^{\frac{1}{p_i}} = M\,\lambda^{\frac{N+\bar p}{\bar p}}, \tag{4.12}$$

so that being h monotone increasing h^{-1} is monotone increasing too and we obtain

$$\sup_{Q_{\lambda/2,M}} u_+ \le \max\left\{g^{-1}(1/M), h^{-1}\left(C\Big(M ⨌_{Q_{\lambda,M}} u_+^{\bar p_2}\,dx\,dt\Big)^{\frac{\bar p}{N+\bar p}}\right)\right\}.$$

□

Using the estimate on the supremum given in previous Lemma 4.6, we can show that each weak local supersolution to the equation when $\bar p > Np_N/(N+2)$ has a lower-semicontinuous representative. We will use the notion of Lebesgue points of u under $\bar p_2$ norm, that has to be considered within the suitable weird cylinders of Lemma 4.6. Instead of repeating the proof of Lebesgue's differentiation Theorem, using Besicovitch covering result on balls, to define what is a Lebesgue point for u by approximating it in $L^{\bar p_2}$-norm on weird cylinders $Q_{\lambda,M}$, we use an abstract framework. For each $M \in \mathbb{N}$ we define a proper distance dist_M so that the space $(\Omega_T, \mathcal{L}, \mathrm{dist}_M)$ is a doubling measure space, then we take the intersection of all Lebesgue points for u through all such spaces, and finally choose $M > 0$ such that the estimate (4.9) is small enough to ensure semicontinuity of u for dist_1-topology. Being semicontinuity a topological property, the representative u is lower semicontinuous also in the classical topology.

Corollary 4.1. *Let* (4.6) *hold and* u *be a weak super-solution to* (1.1) *in* Ω_T*. Then* u *has a lower semicontinuous representative.*

Proof. We recall that all the infima and suprema are taken in the essential sense. For any $M \in \mathbb{N}$ define a metric in Ω_T as

$$\mathrm{dist}_M\big((x,t),(x',t')\big) = \max\{M^{-1}|t-t'|, |x_1 - x_1'|^{p_1}, \dots, |x_N - x_N'|^{p_N}\},$$

with corresponding balls $B_{r,M}$. Observe that $B_{r,M_1} \subset B_{r,M_2}$ if $M_2 > M_1$. We will prove that for some $M \in \mathbb{N}$ there is a dist_M-metric essential lower-semicontinuous representative of u. Let us fix an arbitrary representative, which we will still denote by u. By (4.12), Ω_T with the induced metric and the Lebesgue measure is a locally doubling measure space. Therefore the set E_M of Lebesgue points for u has full measure, as well as $E = \cap_{M \in \mathbb{N}} E_M$. We can therefore suppose that for any $(x_0, t_0) \in E$ and for every $M \in \mathbb{N}$

$$\lim_{r \downarrow 0} \fint\!\!\fint_{B_{r,M}} |u(x,t) - u(x_0,t_0)|^{\bar{p}_2}\, dx\, dt = 0.$$

We claim that for any $(x_0,t_0) \in E$

$$u(x_0,t_0) \leq \lim_{r \to 0} \inf_{B_{r,1}(x_0,t_0)} u. \tag{4.13}$$

Suppose by contradiction that for some $\epsilon > 0$

$$u(x_0,t_0) - \inf_{B_{r,1}(x_0,t_0)} u \geq \epsilon > 0 \qquad \forall r < r_0 \tag{4.14}$$

and consider the sub-solution $v = u(x_0,t_0) - u$ to (1.1). Since $g(0) = h(0) = 0$, g and h are continuous and increasing, we can choose $M \geq 1$ such that

$$g^{-1}(1/M) + h^{-1}(C/M) < \epsilon/2,$$

(C being the constant in (4.9)) and, being $(x_0,t_0) \in E_M$, choose $r(M) < r_0$ such that

$$B_{2r,M}(x_0,t_0) \subseteq \Omega_T, \qquad \fint\!\!\fint_{B_{2r,M}(x_0,t_0)} |u - u(x_0,t_0)|^{\bar{p}_2}\, dx\, dt \leq \frac{1}{M^{\frac{N+\bar{p}}{\bar{p}}}}.$$

The previous Theorem (applied to $v(x - x_0, t)$) then assures that

$$\begin{aligned}
\sup_{B_{r,1}(x_0,t_0)} u(x_0,t_0) - u &\leq \sup_{B_{r,M}(x_0,t_0)} u(x_0,t_0) - u \\
&\leq g^{-1}(1/M) + h^{-1}\left(C\Big(M \fint\!\!\fint_{B_{2r,M}(x_0,t_0)} (u - u(x_0,t_0))_+^{\bar{p}_2}\, dx\, dt \Big)^{\frac{\bar{p}}{N+\bar{p}}} \right) \\
&\leq g^{-1}(1/M) + h^{-1}(C/M) < \epsilon/2
\end{aligned}$$

contradicting (4.14). Finally, for $(x_0,t_0) \in \Omega_T \setminus E$ we modify the representative forcing the equality in (4.13). □

Remark 4.1. *The existence of a lower-semicontinuous representative can be proven also by use of Lemma 3.4, see for instance [47]. The use of Lemma 3.4 seems preferable for these situations when adding a constant to a solution does not necessarily give another solution to the same equation.*

5 Properties and existence of $L^{\mathbf{p}}$-solutions

the Cauchy problem with square-integrable data

Là, tout n'est qu'ordre et beauté,

Luxe, calme et volupté.

- Charles Baudelaire -

L'Invitation au Voyage

The present chapter is entirely devoted to the study of well-posedness and finite speed of propagation for the class of $L^{\mathbf{p}}(S_T)$ solutions, defined in Definition (2.1) to be local weak solutions which are globally integrable, together with their directional derivatives. The interesting feature of anisotropic equation (1.1) relatively to the isotropic counterpart (1.2) is that the support evolves compactly on each direction x_i with a different (finite) speed, dictated by the quantity $(N(\bar{p}-p_i)+\bar{p})$. This exponent is always positive in the range of p_is that is under interest for the Harnack inequality to hold, which is $2 < p_N < \bar{p}_1$. In order to show finite speed of propagation (see for instance [33]) we use the general energy estimates of Section 3 with a proper choice of test functions that eliminate the contribute of initial datum. These are then combined with the parabolic embedding of Section 2, which allows us to perform the De Giorgi integral iteration when estimating terms as $\partial_i(|u|^{\alpha_i})$. The initial step of iteration can be recovered thanks to the use of conservation of mass (3.27) and global boundedness (4.1).
Further we study the existence and uniqueness of solutions to the Cauchy problem (1.4) in S_T with initial datum taken in $L^2(\mathbb{R}^N)$, through a limit process along a sequence of expanding problems ([23] Chap VI Sect.12, or [19]). As we look for local weak solutions, we need to recover the convergence in L^2 of initial data that are not prescribed by the assumption. This can be done by a precise use of Aubin-Lions theorem, and the identification of the energy term is then achieved by means of Minty's trick ([56]).

5.1 Finite speed of propagation

In this Section we suppose $u \in \cap_{i=1}^N L^{p_i}(0,T;L^{p_i}_{\rm loc}(\mathbb{R}^N))$. As long as $\max\{p_1,\dots,p_N\} < \bar{p}_2$, this directly follows from the condition of being a weak solution together with Theorem 2.2, as by Hölder's inequality

$$L^{p_i}(0,T;L^{p_i}_{\rm loc}(\mathbb{R}^N)) \subseteq L^\infty(0,T;L^2_{\rm loc}(\mathbb{R}^N)) \cap L^{\bar{p}_2}(0,T;L^{\bar{p}_2}_{\rm loc}(\mathbb{R}^N)), \quad \forall i=1,\dots,N.$$

Theorem 5.1. *Let us suppose that for all $i \in \{1, \ldots, N\}$ we have the condition*

$$2 < p_i \le p_N < \bar{p}_1 = \bar{p}\left(1 + \frac{1}{N}\right) < N + 1. \tag{5.1}$$

Let $u \in \cap_{i=1}^N L^{p_i}(S_T)$ be a local weak solution to Cauchy problem (1.4) in S_T with

$$u_0 \in L^2(\mathbb{R}^N), \qquad \emptyset \ne \operatorname{supp}(u_0) \subseteq [-R_0, R_0]^N = K_{R_0}\,.$$

Then the support of u evolves with the law

$$\operatorname{supp}(u(\cdot, t)) \subseteq \prod_{j=1}^N [-R_j(t), R_j(t)], \tag{5.2}$$

for any $t < T$, where

$$R_j(t) = 2\,R_0 + Ct^{\frac{N\,(\bar{p}-p_j)+\bar{p}}{\lambda\,p_j}}\,\|u_0\|_1^{\frac{\bar{p}(p_j-2)}{\lambda p_j}}, \qquad \lambda = N(\bar{p}-2)+\bar{p}. \tag{5.3}$$

Proof. Since $2 < p_N < \bar{p}_1$, second point of Theorem 4.1 applies, ensuring (4.2). Choose $\mu \in\ (0,1)$ and for any $\epsilon > 0$ apply (3.22) with

$$F_\epsilon(s) = \int_0^s \tau\,(\tau^2+\epsilon^2)^{\frac{\mu-1}{2}}\,d\tau, \qquad F_\epsilon''(s) = \frac{\mu\,s^2+\epsilon^2}{(s^2+\epsilon^2)^{\frac{3-\mu}{2}}} > 0.$$

All the assumptions of Lemma 3.3 hold true, except the boundedness of F', which however is not necessary being u bounded on $S_T \setminus S_t$, $t > 0$ by (4.2). Using

$$\mu(s^2+\epsilon^2)^{\frac{\mu-1}{2}} \le F_\epsilon''(s), \qquad |F_\epsilon'(u)|^{p_i}|F_\epsilon''(u)|^{1-p_i} \le |u|^{p_i}(u^2+\epsilon^2)^{\frac{\mu-1}{2}}$$

for $i = 1, \ldots, N$, we get for all $0 < t_1 < t_2 < T$ and η of the form (3.12)

$$\begin{aligned}
&\int_{\mathbb{R}^N} F_\epsilon(u(x,t))\,\eta(x)\,dx\Big|_{t_1}^{t_2} + \mu\sum_{i=1}^N \int_{t_1}^{t_2}\int_{\mathbb{R}^N\cap[u\ne0]} (u^2+\epsilon^2)^{\frac{\mu-1}{2}}\,\eta\,|\partial_i u|^{p_i}\,dx\,dt \\
&\le \int_{\mathbb{R}^N} F_\epsilon(u(x,t))\,\eta(x)\,dx\Big|_{t_1}^{t_2} + \mu\sum_{i=1}^N \int_{t_1}^{t_2}\int_{\mathbb{R}^N} (u^2+\epsilon^2)^{\frac{\mu-1}{2}}\,\eta\,|\partial_i u|^{p_i}\,dx\,dt \\
&\qquad \le \gamma\sum_{i=1}^N \int_{t_1}^{t_2}\int_{\mathbb{R}^N} |u|^{p_i}(u^2+\epsilon^2)^{\frac{\mu-1}{2}}\,|\partial_i \eta^{\frac{1}{p_i}}|^{p_i}\,dx\,dt.
\end{aligned}$$

Being $\mu \in\ (0,1)$ and $\partial_i u(x) = 0$ for almost every $x \in [u = 0]$, we define the function

$$\eta\,|u|^{\mu-1}\,|\partial_i u|^{p_i} =:= \begin{cases} \eta\,|u|^{\mu-1}\,|\partial_i u|^{p_i}, & \text{when} \quad [u \ne 0], \\ 0, & \text{in} \quad [u = 0]. \end{cases}$$

Therefore, by monotone convergence on all the terms we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u(x,t)|^{1+\mu}\,\eta(x)\,dx\Big|_{t_1}^{t_2} + \frac{1}{C}\sum_{i=1}^N \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \eta\,|u|^{\mu-1}\,|\partial_i u|^{p_i}\,dx\,dt \\
&\qquad \le C\sum_{i=1}^N \int_{t_1}^{t_2}\int_{\mathbb{R}^N} |u|^{p_i+\mu-1}\,|\partial_i \eta^{\frac{1}{p_i}}|^{p_i}\,dx\,dt
\end{aligned} \tag{5.4}$$

for some constant $C = C(\mu) > 0$. By assumption u solves the Cauchy problem with datum taken in $L^{1+\mu}_{loc}(\mathbb{R}^N) \subset L^2_{loc}(\mathbb{R}^N)$, so that estimate (5.4) above is valid for $t_1 = 0$ by a dominated convergence argument. Let $j \in \{1, \ldots, N\}$ and choose $\tilde{\eta}, \psi \in C_c^\infty(\mathbb{R}; [0,1])$

$$\eta(x) = \tilde{\eta}(x_j)^{p_j}\prod_{i\ne j}\psi(x_i)^{p_i}$$

with the properties

$$\tilde\eta\lfloor_{\{|s|\le R_0\}}\equiv 0, \qquad \psi\lfloor_{\{|s|\le R\}}\equiv 1, \qquad |\psi'|\le C/R.$$

With this test function, we let $R\to+\infty$ in (5.4): all the terms on the right hand side except the j-th one vanish, since $u\in\cap_{i=1}^N L^{p_i}(S_T)$ and $u\in L^1(S_T)$ by Proposition (3.5), therefore by interpolation $u\in L^{p_i+\mu-1}(S_T)$. On the left-hand side the term for $t_1=0$ vanishes since $\mathrm{supp}(u_0)\subseteq\{|x_j|\le R_0\}$ and on the other we apply Fatou's lemma, to obtain

$$\int_{\mathbb{R}^N}|u(x,t_2)|^{1+\mu}\,\tilde\eta\,dx+\frac{1}{C}\sum_{i=1}^N\int_0^{t_2}\int_{\mathbb{R}^N}\tilde\eta^{p_j}\,|u|^{\mu-1}\,|\partial_i u|^{p_i}\,dx\,dt\le C\int_0^{t_2}\int_{\mathbb{R}^N}|u|^{p_j+\mu-1}\,|\partial_j\tilde\eta|^{p_j}\,dx\,dt \tag{5.5}$$

where we set for brevity $\tilde\eta(x)=\tilde\eta(x_j)$. We define for $r>2R_0$, $n\in\mathbb{N}$ the sequence of sets

$$r_n=2\,r+\frac{r}{2^n}, \qquad s_n=r-\frac{r}{2^{n+1}}, \qquad E_n=\{x\in\mathbb{R}^N: s_n\le|x_j|\le r_n\},$$

and specify for each E_n the function $\tilde\eta=\tilde\eta_n$, to be constructed from a suitable $\tilde\eta_n\in C^\infty(\mathbb{R};[0,1])$ satisfying

$$\tilde\eta_n\equiv 1 \text{ on } E_{n+1}, \qquad |\tilde\eta_n'|\le C\,\frac{2^n}{r}, \qquad \mathrm{supp}(\tilde\eta_n)\subseteq\{s_n\le|s|\le r_n\}. \tag{5.6}$$

Let finally

$$\beta_i=\frac{p_i+\mu-1}{p_i}<1, \qquad \beta=\min\{\beta_i: j=1,\dots,N\}, \qquad \eta_n(x):=\tilde\eta_n^{1/\beta}(x_j).$$

Clearly η_n still satisfies (5.6), while being $0\le\tilde\eta_n\le1$ we have for each $i\in\{1,\dots,N\}$

$$\left|\partial_j|\eta_n\,u|^{\beta_j}\right|^{p_j}\le|\partial_j|\eta_n|^{\beta_i}|^{p_j}u^{\beta_j p_j}+\beta_j^{p_j}\,\eta_n^{\beta_j\,p_j}\,|u|^{\mu-1}\,|\partial_j u|^{p_j}\le\frac{C\,2^{p_j n}}{r^{p_j}}|u|^{p_j+\mu-1}+\beta_j^{p_j}\,\tilde\eta_n^{p_j}\,|u|^{\mu-1}\,|\partial_j u|^{p_j},$$

where we used $\beta_i\ge\beta$ in the last inequality. Considering the chain rule above, it is possible because the function $z\to z^{\beta_j}$ maps zero-measure sets in zero-measure sets (Theorem 3.44 in [46]) so that we can apply the chain rule to the function of single variable x_i given by $|\eta_n u|^{\beta_i}(\hat x,x_i)$. When $i\neq j$ in this estimate the first term on the right vanishes, because η_n is independent of x_i, and we have the simpler estimate

$$\left|\partial_i|\eta_n u|^{\beta_i}\right|^{p_i}=\eta_n^{\beta_i\,p_i}\,\left|\partial_i|u|^{\beta_i}\right|^{p_i}\le\beta_i^{p_i}\,\tilde\eta_n^{p_i}\,|u|^{\mu-1}\,|\partial_i u|^{p_i}.$$

Therefore (5.5) for $\tilde\eta=\tilde\eta_n$ provides for all $i\in\{1,\dots,N\}$ the inequality

$$\int_0^{t_2}\int_{\mathbb{R}^N}\left|\partial_i|\eta_n\,u|^{\beta_i}\right|^{p_i}\,dx\,dt\le\frac{C\,2^n}{r^{p_j}}\int_0^{t_2}\int_{E_{n-1}}|u|^{p_j+\mu-1}\,dx\,dt, \tag{5.7}$$

where we used the properties in (5.6) and the monotonicity of E_n. Since (5.5) also implies

$$\int_{\mathbb{R}^N}|\tilde\eta_n\,u|^{1+\mu}(x,t)\,dx\le\int_{E_n}\tilde\eta_{n-1}\,|u|^{1+\mu}(x,t)\,dx\le\frac{C\,2^n}{r^{p_j}}\int_0^{t_2}\int_{E_{n-1}}|u|^{p_j+\mu-1}\,dx\,dt$$

for any $t\in\ (0,t_2]$, we can apply Theorem 2.1 in the compact set $E_n\subset\mathbb{R}^N$, with parameters

$$\sigma=1+\mu, \quad \alpha_i=\beta_i=\frac{p_i+\mu-1}{p_i}, \quad \theta=\frac{p_j-2}{p_\alpha^*-\mu-1}, \quad q=p_j+\mu-1.$$

Substitution gives

$$\tilde\alpha=N\left(\frac{1}{\bar p'}+\frac{\mu}{\bar p}\right), \quad \theta=(p_j-2)\frac{N-\bar p}{\lambda_{1+\mu}}, \qquad \text{with} \quad \frac{1}{\bar p'}+\frac{1}{\bar p}=1,$$

where we recall that

$$\lambda_{1+\mu}=N\,(\bar p-2)+(1+\mu)\,\bar p.$$

The necessary condition $\theta \in\ [0,\, \bar p/\bar p^*]$ reads, after some algebraic manipulations, as

$$p_j \le \bar p\left(1+\frac{\mu+1}{N}\right) \quad\Leftrightarrow\quad \mu \ge N\left(\frac{p_j}{\bar p}-1\right)-1,$$

and the latter quantity is always negative under assumption (5.1). Therefore for any $\mu \in\ (0,1)$ (2.16) gives, through the previous estimates and some algebra

$$\int_0^{t_2}\int_{\mathbb{R}^N} |\tilde\eta_n\, u|^{p_j+\mu-1}\,dx\,dt \le C\,2^n\,T^{1-(p_j-2)\,\frac{N}{\lambda_{1+\mu}}}\left(\frac{2}{r^{p_j}}\int_0^{t_2}\int_{E_{n-1}}|u|^{p_j+\mu-1}\,dx\,dt\right)^{1+(p_j-2)\,\frac{\bar p}{\lambda_{1+\mu}}}$$

which, being $\tilde\eta_n \equiv 1$ on E_{n+1}, implies

$$\int_0^{t_2}\int_{E_{n+1}} |u|^{p_j+\mu-1}\,dx\,dt \le C\,2^n\,T^{1-(p_j-2)\,\frac{N}{\lambda_{1+\mu}}}\left(\frac{2}{r^{p_j}}\int_0^{t_2}\int_{E_{n-1}}|u|^{p_j+\mu-1}\,dx\,dt\right)^{1+(p_j-2)\,\frac{\bar p}{\lambda_{1+\mu}}}.$$

Applying the classical form of Lemma 9.3 for $N=1$ gives that the condition

$$\int_0^{t_2}\int_{E_0}|u|^{p_j+\mu-1}\,dx\,dt \le C\,r^{p_j\left(1+\frac{\lambda_{1+\mu}}{(p_j-2)\,\bar p}\right)}\,T^{\frac{N}{\bar p}-\frac{\lambda_{1+\mu}}{(p_j-2)\,\bar p}}, \tag{5.8}$$

with $E_0 = B_{3r}\setminus B_{r/2}$, implies the following convergence when $n\to\infty$,

$$\int_0^{t_2}\int_{E_{2n}}|u|^{p_j+\mu-1}\,dx\,dt \to 0,$$

and hence, since $t_2\in(0,T)$, we have

$$\operatorname{supp}(u(\cdot,t)) \subseteq \mathbb{R}^N\setminus E_\infty = \mathbb{R}^N\setminus\{r\le |x_j|\le 2\,r\}\quad \forall t\in[0,T). \tag{5.9}$$

To obtain (5.8), we estimate from above with the integral in the whole $\mathbb{R}^N$ and we employ Proposition 3.5 and Theorem 4.1 as follows:

$$\begin{aligned}\int_0^T\int_{E_0}|u|^{p_j+\mu-1}\,dx\,dt &\le \int_0^T \|u(\cdot,t)\|_1\|u(\cdot,t)\|_{L^\infty}^{p_j+\mu-2}\,dt\\ &\le C\|u_0\|_1\int_0^T\frac{\|u_0\|_1^{\frac{\bar p}{\lambda}\,(p_j+\mu-2)}}{t^{\frac{N}{\lambda}\,(p_j+\mu-2)}}\,dt\\ &\le C\|u_0\|_1^{1+\frac{\bar p}{\lambda}(p_j+\mu-2)}T^{1-\frac{N}{\lambda}\,(p_j+\mu-2)},\end{aligned}$$

where we recall that $\lambda=\lambda_1 = N\,(\bar p-2)+\bar p$ and integrating in time at the last inequality, we assumed

$$\frac{N}{\lambda}\,(p_j+\mu-2)<1 \quad\Leftrightarrow\quad \mu<\bar p_1-p_j,$$

the latter being positive due to (5.1). The previous discussion shows that if r and T obey

$$\|u_0\|_1^{1+\frac{\bar p}{\lambda}(p_j+\mu-2)}T^{1-\frac{N}{\lambda}\,(p_j+\mu-2)} \le C\,r^{p_j\left(1+\frac{\lambda_{1+\mu}}{(p_j-2)\,\bar p}\right)}\,T^{\frac{N}{\bar p}-\frac{\lambda_{1+\mu}}{(p_j-2)\,\bar p}}$$

for some constant C depending only on the data and on μ, then (5.9) holds. This inequality can be rewritten through some algebra as

$$r\ge CT^{\frac{N\,(\bar p-p_j)+\bar p}{\lambda\,p_j}}\,\|u_0\|_1^{\frac{\bar p}{p_j}\,\frac{p_j-2}{\lambda}}.$$

Thus (5.9) holds for any $r\ge 2\,R_0$ satisfying the previous one-sided inequality, concluding the proof. □

5.2 Existence of $L^{\mathbf{p}}$ solutions for square integrable data

Here we consider the Cauchy problem (1.4), attained in L^2 with bounded and a compactly supported initial data. This problem can be read in formulas as

$$\begin{cases} \partial_t u = \sum_{i=1}^N \partial_i\big(|\partial_i u|^{p_i-2}\partial_i u\big) & \text{in } S_T, \qquad p_i > 2 \quad \forall i = 1,\dots,N, \\ u_0 = g \in L^2(\mathbb{R}^N), & \operatorname{supp} g \subset \bar{B}_{R_0}, \qquad g \in L^\infty(B_{R_0}). \end{cases} \tag{5.10}$$

We show in this Section that this problem has a unique $L^{\mathbf{p}}$-solution, by a standard approximation technique relying on the monotonicity of the operator.

Proposition 5.1. *Problem* (5.10) *has a unique $L^{\mathbf{p}}$-solution which takes g as initial datum in $L^2(\mathbb{R}^N)$.*

Proof. We divide the proof into three steps: existence, uniqueness and attainment of initial data.

STEP 1. Existence.
We let, for $n \geq \operatorname{diam}(\operatorname{supp} g)$, $B_n = \{|x| < n\}$ and consider the boundary value problems

$$\begin{cases} v_n \in C(0,T;L^2(B_n)) \cap L^{\mathbf{p}}(0,T;W_0^{1,\mathbf{p}}(B_n)) \\ \partial_t v_n - \sum_{i=1}^N \partial_i(|\partial_i v_n|^{p_i-2}\partial_i v_n) = 0, \quad \text{in} \quad B_n \times (0,T), \\ v_n(\cdot,0) = g_{|B_n}. \end{cases} \tag{5.11}$$

We regard the solutions to these problems as defined in the whole S_T by extending them to be zero on $|x| \geq n$. The problems (5.11) can be uniquely solved by a monotonicity method (see for instance [49, Example 1.7.1]), and give solutions v_n satisfying

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^N} |v_n(x,t)|^2\,dx + 2\sum_{i=1}^N \int\!\!\int_{S_T} |\partial_i v_n|^{p_i}\,dx\,dt = \|g\|_2^2, \quad \forall n \in \mathbb{N}, \tag{5.12}$$

and thus $v_n \in L^\infty(0,T;L^2(\mathbb{R}^N))$ and $\partial_i v_n \in L^{p_i}(S_T)$ uniformly. Considering the nature of the equation, the function g can be considered a stationary solution to (5.11), so that by the local comparison principle Proposition 3.7 we have $\|v_n\|_\infty \leq \|g\|_\infty$. In the weak formulation of (5.11) we take (modulo a Steklov averaging process) the test function $|v_n|^{p_j-2}v_n$, $j = 1,\dots,N$, obtaining $\forall t \in (0,T)$

$$\int_{\mathbb{R}^N} \frac{|v_n|^{p_j}}{p_j}(x,t)\,dx + (p_j-1)\sum_{i=1}^N \int\!\!\int_{S_t} |\partial_i v_n|^{p_i}\,|v_n|^{p_j-2}\,dxd\tau = \int_{\mathbb{R}^N} \frac{|g|^{p_j}}{p_j}\,dx.$$

implying

$$v_n \in \cap_{i=1}^N L^\infty(0,T;L^{p_i}(\mathbb{R}^N)), \quad \text{with a uniform bound.} \tag{5.13}$$

This estimate, together with (5.12), provides an uniform bound for v_n in

$$L^{\mathbf{p}}(0,T;W^{1,\mathbf{p}}(\mathbb{R}^N)) \cap L^\infty(0,T;L^2(\mathbb{R}^N)).$$

This bound implies that a (not relabelled) subsequence v_n converges weakly* to a function v in these spaces. Moreover, the weak formulation of the equation implies that the right hand side of

$$\partial_t v_n = \sum_{i=1}^N \partial_i(|\partial_i v_n|^{p_i-2}\partial_i v_n),$$

is uniformly bounded in

$$\big(L^{\mathbf{p}}\big(0,T;W_0^{1,\mathbf{p}}(B_m)\big)\big)' =: L^{\mathbf{p}'}(0,T;W^{-1,\mathbf{p}'}(B_m))$$

by Hölder inequality, for any $m \in \mathbb{N}$. By Aubin-Lions theorem [62, Chap. III Proposition 1.3], applied to the triple

$$W_0^{1,\mathbf{p}}(B_m) \hookrightarrow L^2(B_m) \to W^{-1,\mathbf{p}'}(B_m),$$

we can select for each m a subsequence v_n that converges to a function v in $L^2(0,T;L^2(B_m))$. A diagonal argument provides a subsequence (still not relabeled) converging in $L^2(0,T;L^2_{loc}(\mathbb{R}^N))$ to the weak* limit v and such that

1. $\displaystyle\int_{\mathbb{R}^N} v_n(x,t)\,\varphi(x,t)\,dx \to \int_{\mathbb{R}^N} v(x,t)\,\varphi(x,t)\,dx$ for a.e. t and all $\varphi \in C^\infty_{\rm loc}(0,T;C^\infty_c(\mathbb{R}^N))$,

2. $\partial_i(|\partial_i v_n|^{p_i-2}\partial_i v_n) \rightharpoonup \eta_i$, weakly in $L^{\mathbf{p}'}(0,T;W^{-1,\mathbf{p}'}(\mathbb{R}^N))$ for some η_i, $\forall i = 1,\ldots,N$.

We can thus pass to the limit in the weak formulation of the equation, identifying $\eta_i = |\partial_i v|^{p_i-2}\partial_i v$ through Minty's trick, that we explain here below.
The idea is to use the equation to identify the last term. Indeed, let $K \subset\subset \mathbb{R}^N$ and let n be so large that $K \subset B_n$. Each v_n satisfies for every $\varphi \in C^\infty_{loc}(0,T;C^\infty_0(K))$ and for almost every $0 \le s < t \le T$, the equation

$$\int_K v_n\varphi\,dx\Big|_s^t - \int_s^t\int_K v_n\varphi_\tau\,dxd\tau + \sum_{i=1}^N\int_s^t\int_K |\partial_i v_n|^{p_i-2}\partial_i v_n\partial_i\varphi\,dxd\tau = 0,$$

and

$$\begin{aligned}
&v_n(\cdot,t) \to v(\cdot,t), \quad \text{weakly in } L^2(\mathbb{R}^N), \quad \text{for almost every } t \in (0,T)\\
&v_n \to v, \quad \text{weakly* in } L^\infty(0,T;L^2(\mathbb{R}^N)),\\
&v_n \to v, \quad \text{weakly in } L^2((0,T);L^2_{loc}(\mathbb{R}^N)),\\
&|\partial_i v_n|^{p_i-2}\partial_i v_n \rightharpoonup \eta_i, \quad \text{weakly in } L^{\mathbf{p}'}(0,T;W^{-1,\mathbf{p}'}(\mathbb{R}^N)).
\end{aligned} \tag{5.14}$$

Indeed, for each i the monotone operator $A_i(v_n) = \partial_i(|\partial_i v_n|^{p_i-2}\partial_i v_n)$ lives in a bounded subset of $L^{\mathbf{p}'}(0,T;W^{-1,\mathbf{p}'}(\mathbb{R}^N))$, thus for $i = 1,..,N$ there exists $\eta_i \in L^{p_i'}(0,T;W^{-1,p_i'}(\mathbb{R}^N))$ such that A_i converges weakly to η_i and the limit equation is now, for almost every $0 \le s < t \le T$, $K \subset\subset \mathbb{R}^N$,

$$\begin{aligned}
0 &= \lim_{n\to\infty}\left\{\int_K v_n\varphi\,dx\Big|_s^t - \int_s^t\int_K v_n\varphi_\tau\,dxd\tau + \sum_{i=1}^N\int_s^t\int_K |\partial_i v_n|^{p_i-2}\partial_i v_n\partial_i\varphi\,dxd\tau\right\}\\
&= \int_K v\varphi\,dx\Big|_s^t - \int_s^t\int_K v\varphi_\tau\,dxd\tau + \sum_{i=1}^N\int_s^t\int_K \eta_i\partial_i\varphi\,dxd\tau.
\end{aligned} \tag{5.15}$$

As each v_n takes initial value g with strong L^2 meaning, we can let $s \to 0$. The game is over if we show that each component is $\eta_i = |\partial_i v|^{p_i-2}\partial_i v$. From the monotonicity properties of the operator A_i we obtain

$$X_{n,i} = \int_0^t\int_{\mathbb{R}^N}\langle A_i(v_n(\tau)) - A_i(z(\tau)), v_n(\tau) - z(\tau)\rangle\,dxd\tau \ge 0,$$

$$\Rightarrow \quad X_n := \sum_{i=1}^N X_{n,i} \ge 0, \quad \forall z \in L^{\mathbf{p}}(0,T,W^{1,\mathbf{p}}_{loc}(\mathbb{R}^N)).$$

Next, from (5.11) and n big enough it holds that

$$\int_0^t\int_{\mathbb{R}^N}\sum_{i=1}^N\langle A_i(v_n), v_n\rangle\,dxd\tau = \frac{1}{2}\Big[||g||^2_{L^2(\mathbb{R}^N)} - ||v_n(t)||^2_{L^2(\mathbb{R}^N)}\Big].$$

Therefore, as $\liminf ||v_n(t)||_2^2 \ge ||v(t)||_2^2$, we have

$$0 \le \limsup X_n \le \frac{||g||_2^2}{2} - \frac{||v(t)||_2^2}{2} - \int_0^t\int_{\mathbb{R}^N}\sum_{i=1}^N\langle\eta_i, z\rangle\,dxd\tau - \int_0^t\int_{R^N}\sum_{i=1}^N\langle A_i(z), v - z\rangle\,dxd\tau.$$

But from (5.15), up to a Steklov averaging, we have

$$\frac{||g||_2^2}{2} - \frac{||v(t)||_2^2}{2} = \sum_{j=1}^N\int_0^t\int_{\mathbb{R}^N}\langle\eta_j, v\rangle\,dxd\tau,$$

and so by passing to the limit $n \to \infty$ we get

$$\int_0^t \int_{\mathbb{R}^N} \sum_{i=1}^N \langle \eta_i - A_i(z), v - z \rangle \, dx d\tau \geq 0, \quad \forall z \in L^{\mathbf{p}}(0, T, W^{1,\mathbf{p}}_{loc}(\mathbb{R}^N)). \tag{5.16}$$

We end Minty's trick by hemicontinuity of operators A_is. Indeed, at this point it is enough to consider a function $w \in L^{\mathbf{p}}(0, T, W^{1,\mathbf{p}}_{loc}(\mathbb{R}^N))$, $\epsilon \in (0,1)$, to define $v_\epsilon = v - \epsilon w$ and write (5.16) against $z = v_\epsilon$, so to get

$$0 \leq \lim_{\epsilon \to 0} \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^N} \langle \eta_i, w \rangle - \langle A_i(v_\epsilon), w \rangle dx d\tau = \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^N} \langle \eta_i - A_i(v), w \rangle \, dx d\tau, \quad \forall w \in L^{\mathbf{p}}(0, T; W^{1,\mathbf{p}}_{loc}(\mathbb{R}^N)).$$

Henceforth $\eta_i = A_i(v) = \partial_i(|\partial_i v|^{p_i - 2} \partial_i v) \in L^{\mathbf{p}'}(0, T; W^{-1,\mathbf{p}'}(\mathbb{R}^N))$, and we end up with a function $v \in L^2((0,T); L^2_{loc}(\mathbb{R}^N))$ solving (5.10). To show the continuity of the law $v : [0,T] \to L^2_{loc}(\mathbb{R}^N)$, it is sufficient to proceed as in Proposition 2.1, and the bound (5.13) confirms that v is an $L^{\mathbf{p}}$-solution.

STEP 2. Uniqueness.
Let v_1, v_2 be two possibly distinct solutions originating from the same initial datum w_0. The function $w = v_1 - v_2$ satisfies

$$\begin{cases} w \in C([0,T]; L^2_{loc}(\mathbb{R}^N)) \cap L^{\mathbf{p}}(0, T; W^{1,\mathbf{p}}_{loc}(\mathbb{R}^N)), \\ w_t - \sum_{i=1}^N \left(|\partial_i v_1|^{p_i-2} \partial_i v_1 - |\partial_i v_2|^{p_i-2} \partial_i v_2 \right) = 0, \quad \text{in} \quad S_T, \\ w(\cdot, 0) = 0, \quad \text{taken in } L^2_{loc}(\mathbb{R}^N). \end{cases} \tag{5.17}$$

In the weak formulation of (5.17) we take the test function $w\zeta$, modulo a Steklov averaging, where the function $x \to \zeta(x)$ is a nonnegative piecewise smooth cutoff function in the ball B_{2R}, that equals one in B_R and such that $|\partial_i \zeta| \leq 1/R, \quad \forall i = 1, .., N$. This gives, together with the monotonicity properties of the operator A_i, that for all $0 < t < T$ we have

$$\begin{aligned} &\frac{1}{2} \int_{B_R} |w|^2(t) \, dx \\ &\qquad \leq \frac{1}{2} \int_{B_R} |w|^2(t) \, dx + \int_0^t \int_{B_{2R}} \sum_{i=1}^N \langle |\partial_i v_1|^{p_i-2} \partial_i v_1 - |\partial_i v_2|^{p_i-2} \partial_i v_2, \partial_i v_1 - \partial_i v_2 \rangle \zeta \, dx d\tau = \\ &\qquad\qquad = - \int_0^t \int_{B_{2R}} \sum_{i=1}^N \langle |\partial_i v_1|^{p_i-2} \partial_i v_1 - |\partial_i v_2|^{p_i-2} \partial_i v_2, \partial_i \zeta \rangle w \, dx d\tau. \end{aligned} \tag{5.18}$$

Therefore as R grows to infinity the L^2-norm of w vanishes,

$$\int_{B_R} |w|^2(t) \, dx \leq \frac{\gamma}{R} \sum_{i=1}^N ||w||_{L^{p_i}(S_T)} (||\partial_i v_1||_{L^{p_i}(S_T)} + ||\partial_i v_2||_{L^{p_i}(S_T)}) \to 0, \tag{5.19}$$

and we are done.

STEP3. Initial datum in L^2.
Next we check that the unique solution v takes the initial datum g in $L^2(\mathbb{R}^N)$. Let $\eta \in (0,1)$ be arbitrary and let g_η be a mollification of g such that

$$||g - g_\eta||_{L^2(\mathbb{R}^N)} \to 0, \quad \text{as} \quad \eta \downarrow 0.$$

In the weak formulation of (5.11) for v_n, we consider the testing function $v_n - g_\eta$, modulo a Steklov averaging process, for n big enough to let supp $(g_\eta) \subset B_n$. We first split the derived terms in the energy term and then use Young inequality to get the power p_i on the term $\partial_i g_\eta$ and a small contribute to the power p_i on the energy term which can be reabsorbed on the left hand side of the equation, and finally discarded (because it is positive), to get

$$\int_{B_n} |v_n - g_\eta|^2(t)\,dx \le ||g - g_\eta||^2_{L^2(\mathbb{R}^N)} + \gamma \int_0^t \int_{\mathbb{R}^N} \sum_{i=1}^N |\partial_i g_\eta|^{p_i}\,dxd\tau, \quad \forall 0 < t < T,$$

for a constant $\gamma > 0$ depending only on p_i's. Then we apply the triangle inequality and we let $n \to \infty$ and use the convergence properties of v_n to obtain the inequality

$$||v(\cdot,t) - g||^2_{L^2(\mathbb{R}^N)} \le 2||g - g_\eta||^2_{L^2(\mathbb{R}^N)} + \gamma \int_0^t \int_{\mathbb{R}^N} \sum_{i=1}^N |\partial_i g_\eta|^{p_i}\,dxd\tau.$$

From this inequality we can take the limit $t \downarrow 0$,

$$\lim_{t\downarrow 0} ||v(\cdot,t) - g||^2_{2,K} \le 2||g - g_\eta||^2_{L^2(\mathbb{R}^N)}, \qquad \forall \eta \in (0,1),$$

and we conclude letting $\eta \downarrow 0$. $\square$

6 Barenblatt Fundamental Solutions

construction and properties

They that have power to hurt and will do none,
That do not do the thing they most do show,
Who moving others are themselves as stone,
Unmoved, cold and to temptation slow;
They rightly do inherit heaven's graces
And husband nature's riches from expense;
They are the lords and owners of their faces,
Others but stewards of their excellence.
The summer's flower is to the summer sweet,
Though to itself it only live and die,
But if that flower with base infection meet
The basest weed outbraves his dignity.
For sweetest things turn sourest by their deeds:
Lilies that fester smell far worse than weeds.

- William J. Shakespeare -

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Throughout this Section we construct a self-similar solution $\mathcal{B}$ to (1.1), i.e., by the discussion in Section 3.1, a stationary solution to the Fokker-Planck equation (3.10). To do this, we show that if the bounded compactly supported initial datum g has L^1-norm small enough, then the support of the solution to the Fokker-Planck equation stays in a cube of side one, and then we turn the problem into a fixed-point one. Once the theorem of Schauder is applied and the existence of $\mathcal{B}$ is proved, we study the positivity properties of such self-similar solution, which, together with the comparison principle, will be the main tool to expand the positivity set of non-negative solutions of (1.1) itself.

We begin this section by unveiling the correspondence between the two fundamental Cauchy problems.

6.1 The resolvent operator

By the results of Section 5.2 we can define, at least for bounded compactly supported initial data g, for each time $t \in [0,T]$ the *resolvent* operator $\mathcal{S}_t : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ such that

$$\mathcal{S}_t g := u(\cdot,t), \qquad t \geq 1,$$

where u is the unique $L^{\mathbf{p}}$-solution to

$$\begin{cases} \partial_t u = \sum_{i=1}^N \partial_i\big(|\partial_i u|^{p_i-2}\partial_i u\big) & \text{in } S_{1,\infty} := \mathbb{R}^N \times (1,\infty), \\ u(\cdot,1) = g, \quad \text{taken in } L^2(\mathbb{R}^N). \end{cases} \tag{6.1}$$

In terms of the Fokker-Planck equation, this also defines through (3.8) an operator $\tilde{\mathcal{S}} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by

$$\tilde{\mathcal{S}}_s g := (\Phi u)(\cdot,s) \qquad s \geq 0, \tag{6.2}$$

which assigns to each initial datum $g \in L^2(\mathbb{R}^N)$ the solution at the time $s \in \mathbb{R}_+$, of the problem

$$\begin{cases} \partial_s w = \sum_{i=1}^N \partial_i[(\partial_i w)^{p_i-1} + \alpha_i y_i w] & \text{in } S_\infty := \mathbb{R}^N \times (0,\infty), \\ w_0 = g, \quad \text{taken in } L^2(\mathbb{R}^N). \end{cases} \tag{6.3}$$

The relation (6.2) implies that

$$\tilde{\mathcal{S}}_s g = \Phi_{e^{s/\lambda}} \mathcal{S}_{e^s} g, \qquad \text{for} \qquad \lambda = N(\bar{p}-2) + \bar{p}, \tag{6.4}$$

and where Φ_ρ is given in (3.11), allowing us to prove properties for $\tilde{\mathcal{S}}_s$ by proving them for $\mathcal{S}_t$.

6.2 Construction of a Barenblatt solution

In order to state some basic properties of the operator $\tilde{\mathcal{S}}_s$ we define the following spaces:

$$X_{R,M} = \{g \in L^\infty(\mathbb{R}^N) : 0 \leq g \leq M,\ \operatorname{supp}(g) \subseteq K_R\}, \qquad X = \bigcup_{R,M>0} X_{R,M}. \tag{6.5}$$

Lemma 6.1. *If* (5.1) *holds true, the operator* $\tilde{\mathcal{S}}_s$*,* $s \geq 0$ *defined in* (6.2) *has the following properties.*

1. *If* $g \in L^2(\mathbb{R}^N)$ *and* $\operatorname{supp}(g) \subseteq K_{R_0}$ *then for some* $c = c(N, \mathbf{p})$ *it holds*

$$\operatorname{supp}(\tilde{\mathcal{S}}_s g) \subseteq \prod_{i=1}^N [-\tilde{R}_i(s), \tilde{R}_i(s)], \qquad \tilde{R}_i(s) = 2\, e^{-s\alpha_i} R_0 + c\, \|g\|_1^{\bar{p}(p_i-2)/(p_i\lambda)}. \tag{6.6}$$

2. *If* $g \in X$*, then* $\|\tilde{\mathcal{S}}_s g\|_1 = \|g\|_1$ *and* $0 \leq \tilde{\mathcal{S}}_s g \leq \|g\|_\infty$*. In particular* $\tilde{\mathcal{S}}_s : X \to X$ *for all* $s \geq 0$*.*
3. *For any* $R, M > 0$ *and* $s \geq 0$*,* $\tilde{\mathcal{S}}_s : X_{R,M} \to X$ *is continuous when* $X_{R,M}$ *and* X *are equipped with the weak-*L^2 *topology.*

Proof. Consider the corresponding problem (6.1) and the therein defined operator $\mathcal{S}_t$. By Theorem 5.1 we know that if $\operatorname{supp}(g) \subseteq K_{R_0}$, then for $\lambda = N(\bar{p}-2) + \bar{p}$, the support of $L^{\mathbf{p}}$-solutions evolves with the law

$$\operatorname{supp}(\mathcal{S}_t g) \subseteq \prod_{i=1}^N [-R_i(t), R_i(t)], \qquad R_i(t) = 2\, R_0 + c\, (t-1)^{\alpha_i}\, \|g\|_1^{\bar{p}(p_i-2)/(p_i\lambda)}. \tag{6.7}$$

Letting $t = e^s$ and using (6.4) we get the first assertion, since

$$\operatorname{supp} \tilde{\mathcal{S}}_s g \subseteq \prod_{i=1}^N [-\tilde{R}_i(s), \tilde{R}_i(s)], \qquad \tilde{R}_i(s) = e^{-s\alpha_i} R_i(e^s) \leq 2\, e^{-s\alpha_i} R_0 + c\, \|g\|_1^{\bar{p}(p_i-2)/(p_i\lambda)}.$$

Also the second statement follows from its counterpart on the corresponding solution u of (1.1): to prove conservation of mass we take advantage of the compactness of the supports of u dictated by (6.7) and test (1.1) with $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ on $\cup_{t<T}\mathrm{supp}(u(\cdot,t))$, $T>0$ arbitrary. Let $\mathrm{supp}(g) \subset K_{R(t)}$, fix any time $t \in \mathbb{R}$ and choose hence the test functions

$$C_c^\infty(\mathbb{R}^N) \ni \varphi = \begin{cases} 1, & \text{for } x \in K_{R(t)+1}, \\ \psi \in C_c^\infty(\mathbb{R}^N \setminus K_{R(t)+1}), & \text{otherwise.} \end{cases} \tag{6.8}$$

So we have, modulo a Steklov averaging process,

$$\begin{aligned} &||u(\cdot,t)||_{L^1(\mathbb{R}^N)} - ||u(\cdot,0)||_{L^1(\mathbb{R}^N)} = \int_0^t \int_{\mathbb{R}^N} \partial_\tau u \, dx d\tau = \\ &= -\sum_{i=1}^N \int_0^t \int_{K_{R(t)+1}} |\partial_i u|^{p_i-2}\partial_i u \cdot \partial_i \varphi \, dx d\tau - \int_0^t \int_{\mathbb{R}^N \setminus K_{R(t)+1}} |\partial_i u|^{p_i-2}\partial_i u \cdot \partial_i \psi \, dx d\tau = 0, \end{aligned} \tag{6.9}$$

owing last equality respectively to $\partial_i \varphi = 0$ and $\partial_i u = 0$ in the respective sets of integration.

The point-wise bounds follow from the local comparison principle, Proposition 3.7, again taking advantage of the compactness of the support and comparing u with the solutions $v \equiv 0$ and $v \equiv \|g\|_\infty$, respectively.

There remains to prove the continuity of $\tilde{\mathcal{S}}_s : X_{R,M} \to X$ within the weak L^2 topologies from departure to arrival, which by (6.4) is equivalent to prove the same statement for $\mathcal{S}_t$. Fix $T > t \geq 1$ and let

$$\bar{R} = \max\left\{2\,R + C\,(T-1)^{\alpha_i}\,(|K_R|M)^{\bar{p}(p_i-2)/(p_i\lambda)} : i = 1, \dots, N\right\}.$$

Assume $g_n \to g$ weakly in L^2 with $g_n \in X_{R,M}$ and let u_n be the $L^{\mathbf{p}}$ solution to (6.1) with initial data g_n. we observe that thanks to (6.7), it holds $\mathrm{supp}(u_n(\cdot,\tau)) \subseteq K_{\bar{R}}$ for every $\tau \in [0,T]$, $n \geq 1$. The boundedness of $\|g_n\|_2$ and standard energy estimates then give a uniform bound for u_n in $L^{\mathbf{p}}(1,T;W_0^{1,\mathbf{p}}(K_{\bar{R}})) \cap L^\infty(1,T;L^2(\mathbb{R}^N))$ and for $\partial_\tau u_n$ in $L^{\mathbf{p}'}(0,T;W^{-1,\mathbf{p}'}(K_{\bar{R}}))$, similarly to the proof of existence for Proposition 5.1. Applying Aubin-Lions theorem as in the proof of Proposition 5.1, we can extract a subsequence converging weakly* to some u in those spaces and such that

$$u_n(\cdot,\tau) \to u(\cdot,\tau) \qquad \text{in } L^2(K_{\bar{R}}), \text{ for a.e. } \tau \in [1,T].$$

Therefore we can pass to the limit in the weak form of the equation to get in $S_{1,\tau} = \mathbb{R}^N \times (1,\tau)$ the equation

$$\int_{\mathbb{R}^N} u(x,\tau)\,\varphi(x,\tau)\,dx - \int_{\mathbb{R}^N} g(x,1)\,\varphi(x,1)\,dx - \int_{S_{1,\tau}} u\,\partial_\tau\varphi\,dx\,dt + \int\int_{S_{1,\tau}} \sum_{i=1}^N \eta_i\,\partial_i\varphi\,dx\,dt = 0$$

for almost every $1 < \tau < T$, so that it only remains to show that $\eta_i = |\partial_i u|^{p_i-2}\partial_i u$. We cannot directly employ Minty's trick, since this time we are missing the strong convergence of the initial data. However, for any τ such that $u_n(\cdot,\tau) \to u(\cdot,\tau)$ in $L^2(K_{\bar{R}})$, we look at $\{u_n\}$ as a sequence of solutions to (6.1) on $[\tau,T]$ with strongly convergent initial data and now Minty's trick allows to deduce $\eta_i = |\partial_i u|^{p_i-2}\partial_i u$ on $S_{\tau,T}$. Since τ can be chosen arbitrarily close to 1 we obtain that u is a $L^{\mathbf{p}}$ solution to (6.1) with initial datum g and from uniqueness we infer that $u(\cdot,t) = \mathcal{S}_t g$ for any $t \geq 1$. A standard sub-subsequence argument concludes the proof of the third statement. □

Theorem 6.1. *Under assumption* (5.1)*, there exists a nontrivial stationary solution* $w \in X_{1,1}$ *to* (6.3)*, and therefore a Barenblatt Fundamental solution.*

Proof. Let us consider the convex set

$$C_\epsilon := \left\{g \in L^2(\mathbb{R}^N) : \mathrm{supp}\,g \subset K_1,\, 0 \leq g \leq 1, \|g\|_{L^1(\mathbb{R}^N)} = \epsilon\right\} \subseteq X_{1,1}.$$

If c is given in (6.6), for $\bar{s}$ sufficiently large and $\bar{\epsilon}$ sufficiently small it holds

$$\tilde{R}_i(\bar{s}) = 2\,e^{-\bar{s}\alpha_i} + c\,\bar{\epsilon}^{\bar{p}(p_i-2)/(p_i\lambda)} \leq 1 \qquad \forall i = 1, \dots, N,$$

so that (6.6) implies that $\mathrm{supp}(\tilde{\mathcal{S}}_{\bar{s}}g) \subseteq K_1$ for all $g \in C_{\bar{\epsilon}}$. Using also point (2) of the previous lemma we have that $\tilde{\mathcal{S}}_{\bar{s}} C_{\bar{\epsilon}} \subseteq C_{\bar{\epsilon}}$. Moreover, $C_{\bar{\epsilon}}$ with the weak L^2 topology is compact, and by point (3) of the previous lemma, $\tilde{\mathcal{S}}_{\bar{s}} : C_{\bar{\epsilon}} \to C_{\bar{\epsilon}}$ is continuous, so that Schauder's theorem ensures the existence of a fixed point $\bar{g} \in C_{\bar{\epsilon}}$ for $\tilde{\mathcal{S}}_{\bar{s}}$. Therefore the function $\bar{w}(\cdot, s) = \tilde{\mathcal{S}}_s \bar{g}$ is a times-periodic, bounded and compactly supported solution to (6.3), which can therefore be extended to $\mathbb{R}^{N+1}$ as an aeternal solution. Consider the bounded, compactly supported function

$$g(y) = \sup_{s \in \mathbb{R}} \bar{w}(y, s), \quad g \in X_{1,1},$$

for which $\|g\|_1 \geq \bar{\epsilon}$. Then $\tilde{\mathcal{S}}_0 g = g \geq \bar{w}(\cdot, \tau)$ for every $\tau \in \mathbb{R}$, so that by Corollary 3.3 it holds $\tilde{\mathcal{S}}_s g \geq \bar{w}(\cdot, \tau + s)$ for any $s \geq 0$. Taking the supremum in $\tau \in \mathbb{R}$ gives $\tilde{\mathcal{S}}_s g \geq g$, but since $\|\tilde{\mathcal{S}}_s g\|_1 = \|g\|_1$, this implies $\tilde{\mathcal{S}}_s g = g$ for every $s \geq 0$, i.e. g is a stationary solution to (6.3). □

Remark 6.1. *It is worth underlying that the same proof does not work if we use just $S_s g$, because in this case $\mathrm{supp}(S_s g) \not\subset K_1$. The resolvent operator $\tilde{S}_s$ has the property to contract the support of the initial datum. We further observe that the whole problem can be solved, in particular that this contraction property holds (Lemma 6.1, point 1) is possible, because problems* (6.1) *and* (6.3) *have both the same initial datum $u(x, 1) = g = w(x, 0) \in L^2(\mathbb{R}^N)$.*

6.3 Properties of the Barenblatt solutions

Our next aim is to prove that Barenblatt Fundamental solutions (see Definition 3.3) are positive in a quantitative way, which is, their positivity set spreads in time in a way controlled by scaling. This amounts in proving that stationary non-negative solutions of the Fokker Planck equation are bounded from below near the origin, which is the content of the next theorem.

Theorem 6.2. *Let us suppose that* (5.1) *holds, let $w \in X_{1,1}$ (see* (6.5)*) be a nontrivial stationary solution to the Fokker-Planck equation* (6.3) *and $\mathcal{B}$ the corresponding Barenblatt solution to* (6.1) *with initial datum $g \in L^2(\mathbb{R}^N)$. Then there exists $\bar{\eta} > 0$, depending on g and $N, \mathbf{p}$, such that*

$$\mathcal{B}(x,t) \geq \bar{\eta}\, t^{-\alpha} \qquad \text{if} \quad |x_i| < \bar{\eta}\, t^{\alpha_i} \quad \forall i = 1, \dots, N; \qquad \alpha, \alpha_i \quad \text{specified in (3.9).}$$

Proof. Suppose that $\mathcal{B}$ is given by

$$\mathcal{B}(x,t) = t^{-\alpha}\, w(x_i\, t^{-\alpha_i}), \qquad t \geq 1. \tag{6.10}$$

By Corollary 4.1 we can fix a lower-semicontinuous representative of $\mathcal{B}$ and thus of w. Since $w > 0$ somewhere, we can pick a point $x^{(0)}$ and numbers $\delta_0, \eta_0 > 0$ such that

$$\inf_{K_{\delta_0}(x^{(0)})} w(y) > \eta_0. \tag{6.11}$$

By (6.10), the latter implies for any $t \geq 1$

$$\mathcal{B}(x,t) \geq \eta_0\, t^{-\alpha}, \quad \text{when} \quad \big\{|x_i - x_i^{(0)}\, t^{\alpha_i}| < \delta_0/2\, t^{\alpha_i}\big\}.$$

Consider now for $\sigma > 0$ the function

$$\mathcal{B}_\sigma(x,t) = \sigma\, t^{-\alpha}\, w\big(\sigma^{(2-p_i)/p_i}\, t^{-\alpha_i}\, (x_i^{(0)} - x_i)\big),$$

which solves (6.1) by translation invariance and Proposition 3.1. Notice that, since $w \in X_{1,1}$ we have

$$\|\mathcal{B}_\sigma(\cdot,t)\|_\infty = \sigma\, t^{-\alpha} \qquad \text{and} \qquad \mathrm{supp}(\mathcal{B}_\sigma(\cdot,t)) \subseteq \big\{2\,|x_i^{(0)} - x_i| \leq t^{\alpha_i}\, \sigma^{(p_i-2)/p_i}\big\}.$$

We seek for $\sigma > 0$ such that the comparison principle can be applied between $\mathcal{B}_\sigma$ and $\mathcal{B}$ with starting time $t = 1$. We need

$$\begin{cases} \|\mathcal{B}_\sigma(\cdot,1)\|_\infty \leq \eta_0, \\ \mathrm{supp}(\mathcal{B}_\sigma(\cdot,1)) \subseteq K_{\delta_0}(x^{(0)}), \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \sigma \leq \eta_0, \\ \sigma^{(p_i-2)/p_i} \leq \delta_0/2, \end{cases}$$

which, being $p_1 > 2$, can be solved for some $\sigma = \sigma_1 \in (0,1)$. Consequently, by comparison and (6.11), there holds

$$\mathcal{B}(x,t) \geq \mathcal{B}_{\sigma_1}(x,t) > \sigma_1\, t^{-\alpha}\, \eta_0, \qquad \text{for} \qquad \big|x_i^{(0)} - \sigma_1^{(2-p_i)/p_i}\, t^{-\alpha_i}\, (x_i^{(0)} - x_i)\big| < \delta_0/2.$$

We let $t_1^{\alpha_1} = \sigma_1^{(2-p_1)/p_1} \geq 1$ and, consequently,

$$\eta_1 = \sigma_1\, t_1^{-\alpha}\, \eta_0, \quad x_i^{(1)} := x_i^{(0)}\big(1 - t_1^{\alpha_i}\, \sigma_1^{(p_i-2)/p_i}\big), \quad \delta_1 := (\delta_0/2) \min\big\{t_1^{\alpha_i}\, \sigma_1^{(p_i-2)/p_i} : i = 1, \ldots, N\big\}$$

(observe that, by the choice of t_1, it holds $x_1^{(1)} = 0$), to get

$$\inf_{K_{\delta_1}(x^{(1)})} \mathcal{B}(\cdot, t_1) \geq \eta_1$$

Proceeding by induction, we will find sequences $t_n, \eta_n, \delta_n, x^{(n)}$ with the properties

$$\inf_{K_{\delta_n}(x^{(n)})} \mathcal{B}(\cdot, t_n) \geq \eta_n, \qquad x_i^{(n)} = 0 \quad \text{for } i = 1, \ldots, n,$$

so that after N steps $x^{(N)} = 0$ and we find

$$\inf_{K_{\delta_N}} \mathcal{B}(\cdot, t_N) \geq \eta_N.$$

By (6.10), this implies $w(x) \geq \eta_N\, t_N^{\alpha}$ when $|x_i| < t_N^{\alpha_i}\, \delta_N/2$ for $i = 1, \ldots, N$. We set $\bar\eta = \min\{\eta_N, \delta_N/2\}$ and scale back to $\mathcal{B}$ through (6.10) again, to get the desired property of $\mathcal{B}$. $\square$

We suppose that w is a fixed stationary solution in $X_{1,1}$ of (6.3). For future purposes we summarise some properties derived from a scaling argument for a large family of corresponding Barenblatt solutions.

Corollary 6.1. *Let $\mathcal{B}(x,t) = t^{-\alpha} w(x_i t^{-\alpha_i})$ be a fixed Barenblatt Fundamental solution to* (6.1) *with $w \in X_{1,1}$ and initial datum $g \in L^2(\mathbb{R}^N)$. There exists $\bar\eta > 0$ such that the family of Barenblatt solutions*

$$\mathcal{B}_\sigma(x,t) = \mathcal{T}_{1,\sigma^{-\lambda/\bar p}}\mathcal{B}\,(x,t) = \sigma\, t^{-\alpha}\, w(\sigma^{(2-p_i)/p_i}\, x_i\, t^{-\alpha_i}), \qquad \sigma > 0, \quad \textit{and} \quad \alpha, \alpha_i \textit{ as in } (3.9),$$

has the following properties

1. $\|\mathcal{B}_\sigma(\cdot,t)\|_\infty = \sigma\, t^{-\alpha}$;
2. $\mathrm{supp}(\mathcal{B}_\sigma(\cdot,t)) \subseteq \displaystyle\prod_{i=1}^N \big\{|x_i| \leq \sigma^{(p_i-2)/p_i}\, t^{\alpha_i}\big\}$;
3. $\{\mathcal{B}_\sigma(\cdot,t) \geq \bar\eta\, \sigma\, t^{-\alpha}\} \supseteq \displaystyle\prod_{i=1}^N \big\{|x_i| \leq \bar\eta\, \sigma^{(p_i-2)/p_i}\, t^{\alpha_i}\big\} =: P_t$.

We will refer in the following to P_t as the set of *positivity* of $\mathcal{B}_\sigma$, when $\sigma > 0$ will be clear from the context.

7 Intrinsic Harnack Inequality

A mathematician, like a painter or a poet, is a maker of patterns.

If his patterns are more permanent than theirs,

it is because they are made with ideas.

-G.H. Hardy-

A Mathematician's Apology

In this Section we prove the main result of the present work. It is an Harnack estimate holding in the appropriate geometry dictated by the equation itself. The way we prove it is similar to [22]: first we use a trick called of Krylov-Safonov and a local clustering argument, in order to recover an estimate from below at a certain time $\bar{t}$, concentrated in some unknown spatial neighborhood inside the domain Ω. Then we consider a Barenblatt solution $\mathcal{B}$ and we scale it back so that it has a smaller value than u at the time $\bar{t}$ and support contained in the former neighborhood. This permits to use the comparison principle (Proposition 3.7) and expand the positivity of u according to the law of evolution of $\mathcal{B}$. We will therefore prove the following.

Theorem 7.1. *Let $u \geq 0$ be a local weak solution to* (1.1) *in $\Omega \times [-T, T]$, suppose valid* (5.1) *and, assume without loss of generality that the origin is a Lebesgue point for u in $\Omega \times [-T, T]$ with $u(0, 0) > 0$. Then, there exist constants $C_1 \geq 0, C_3 \geq C_2 \geq 1$ depending only on N and the p_i's such that, letting $M = u(0, 0)/C_1$ it holds*

$$\frac{1}{C_3} \sup_{\mathcal{K}_\rho(M)} u(\,\cdot\,, -M^{2-\bar{p}}\,(C_2\,\rho)^{\bar{p}}) \leq u(0, 0) \leq C_3 \inf_{\mathcal{K}_\rho(M)} u(\,\cdot\,, M^{2-\bar{p}}\,(C_2\,\rho)^{\bar{p}}) \tag{7.1}$$

whenever

$$M^{2-\bar{p}}\,(C_3\,\rho)^{\bar{p}} < T, \qquad \textit{and} \qquad \mathcal{K}_{C_3\,\rho}(M) \subseteq \Omega. \tag{7.2}$$

In order to provide a lower bound for the solution, we cannot use the same procedure of [22] for the trick of Krylov and Safonov, because the intrinsic geometry that we are considering is quite weird. Instead, we give a generalisation of ([14], Lemma 4.7 pag.34) that is valid for a topological space equipped with a less stringent notion of distance. To this end, we make the following observations: for $\rho \in [0, 1]$ the translates of the cylinders $\mathcal{Q}_\rho^-(\rho^{-N})$ arise naturally from the *quasi-metric*[4]

$$\mathrm{d}((x,t),(y,s)) = \max\left\{ |2^{-1}(x_i - y_i)|^{p_i/(\bar{p}+N(\bar{p}-p_i))}, |t-s|^{1/(\bar{p}+N(\bar{p}-2))} \right\}. \tag{7.3}$$

Indeed, all the exponents appearing in the previous definition are positive thanks to condition (5.1) on the spareness of p_i's, therefore the *quasi-triangle inequality*

$$\mathrm{d}(z_1, z_3) \leq \gamma\,(\mathrm{d}(z_1, z_2) + \mathrm{d}(z_2, z_3)), \qquad \forall z_1, z_2, z_3 \in \mathbb{R}^{N+1},$$

holds true for a constant $\gamma = \gamma(N, \mathbf{p}) \geq 1$ which is the *quasi-metric constant.* We denote the balls of center z and radius $\rho > 0$ in this *quasi-metric* with the symbol $\mathbb{B}_\rho(z)$. Finally, notice that the cylinder $\bar{z} + \mathcal{Q}_\rho^-(\rho^{-N})$ is the bottom half part of the ball $\mathbb{B}_\rho(\bar{z})$ with respect to this distance.

[4]This terminology is borrowed from Grafakos, but it appears there's no general consensus on the term "quasi": sometimes -pseudo-metric- is used instead.

Lemma 7.1. *Let (X,d) be a quasi-metric space with quasi-metric constant γ and $x_0 \in X$. For any $\beta > 0$ there exists a constant $\omega = \omega(\gamma,\beta) > 1$ such that for any bounded function $u : \mathbb{B}_1(x_0) \to \mathbb{R}$ with $u(x_0) \geq 1$ there exist $x \in \mathbb{B}_1(x_0)$ and $r > 0$ such that*

$$\mathbb{B}_r(x) \subseteq \mathbb{B}_1(x_0), \qquad r^\beta \sup_{\mathbb{B}_r(x)} u \leq \omega, \qquad r^\beta u(x) \geq 1/\omega. \tag{7.4}$$

Proof. Extend u as 0 outside $\mathbb{B}_1(x_0)$ and suppose that the claim is false. For ω a parameter to be determined depending only on β and γ, we will construct a sequence of points contradicting the boundedness of u. Set $r_0 = 1/(2\gamma)$ and choose $\omega > (2\gamma)^\beta$. By hypothesis $u(x_0) \geq 1$ and our choice of r_0 it follows $r_0^\beta u(x_0) \geq 1/\omega$ and as also $\mathbb{B}_{r_0}(x_0) \subset \mathbb{B}_1(x_0)$, it must hold by contradiction

$$r_0^\beta \sup_{\mathbb{B}_{r_0}(x_0)} u > \omega.$$

Choose $x_1 \in \mathbb{B}_{r_0}(x_0)$ such that $r_0^\beta\, u(x_1) \geq \omega$ and set $r_1 = r_0\, \omega^{-2/\beta}$, so that

$$r_1^\beta\, u(x_1) \geq 1/\omega.$$

If $\mathbb{B}_{r_1}(x_1) \subseteq \mathbb{B}_1(x_0)$, we can similarly construct by contradiction $x_2 \in \mathbb{B}_{r_1}(x_1)$ such that

$$r_2^\beta\, u(x_2) \geq 1/\omega, \qquad r_2 = r_1\, \omega^{-2/\beta}.$$

Proceed by induction to get a sequence of points and radii such that, if $\mathbb{B}_{r_n}(x_n) \subseteq \mathbb{B}_1(x_0)$,

$$r_n^\beta\, u(x_n) \geq 1/\omega, \qquad r_n = r_{n-1}\, \omega^{-2/\beta}.$$

As $\omega > 1$, the first condition contradicts the boundedness of u if all the balls $\mathbb{B}_{r_n}(x_n)$ are contained in $\mathbb{B}_1(x_0)$. This can be achieved if for any $n \geq 0$

$$\mathrm{d}(x_0, x_n) \leq \gamma \sum_{i=0}^{n-1} \gamma^i\, \mathrm{d}(x_i, x_{i+1}) \leq \gamma r_0 \sum_{i=0}^{+\infty} \gamma^i\, \omega^{-2i/\beta} < 1,$$

which holds for $\gamma\, \omega^{-2/\beta} < 1/2$. □

7.1 Proof of Theorem 7.1, first step. Looking for a bound from below.

We look in the first place for a bound from below, somewhere. More in detail, this is the purpose of next Lemma: it states that there exists a time $\bar t$ and an intrinsic cube, where the sole function of space $u(\cdot,\bar t)$ is above a multiple of the volume of the intrinsic cube itself. To begin with, we suppose that a semicontinuous representative for the solution has been chosen, through Corollary 4.1; hence $u(0,0)$ makes sense as a number.

Lemma 7.2. *Let $u \geq 0$ be a bounded solution to* (1.1) *in Q_1^-. There exist $C_1 > 0$ such that if $u(0,0) \geq C_1$, then there exist $\varepsilon > 0$ and a point $(\bar x, \bar t) \in Q_1^-$, both depending only on N and* **p** *such that*

$$\inf_{\bar x + \mathcal{K}_\rho(\varepsilon \rho^{-N})} u(\cdot,\bar t) \geq \varepsilon\, \rho^{-N} \qquad \textit{for } \rho > 0 \textit{ with } \bar x + \mathcal{K}_\rho(\varepsilon\, \rho^{-N}) \subseteq K_1. \tag{7.5}$$

Proof. Let $\beta = N$ in Lemma 7.1, with the quasi-metric (7.3) and let $C_1 = 1/\omega$, where $\omega(N)$ is given in Lemma 7.1. We apply the Lemma 7.1 to u/C_1 and extend u as 0 in the upper half-space. Then, (7.4) implies the existence of a point $z_1 \in Q_1^-$ and $r \in (0,1)$ such that

$$z_1 + \mathcal{Q}_r^-(r^{-N}) \subseteq \mathcal{Q}_1^-, \qquad \sup_{z_1 + \mathcal{Q}_r^-(r^{-N})} u \leq r^{-N}, \qquad u(z_1) \geq C_1^2 r^{-N}.$$

The solution $v = \mathcal{T}_{r,r^{-N}} u(\cdot + z_1)$ in Q_1^-, (with $\mathcal{T}$ given in (3.6)) obeys

$$\sup_{Q_1^-} v \leq 1, \qquad v(0) \geq C_1^2. \tag{7.6}$$

We prove that (3.60) holds for $\bar\nu = \nu_a$ given in Lemma 3.4 when $a = C_1^2/3$ (thus $\bar\nu$ depends only on N and $\mathbf{p}$). Indeed, if, by contradiction, we have

$$|[v \geq a] \cap Q_1^-| \leq \nu_a\, |Q_1^-|,$$

then since $0 \leq v \leq 1$ in Q_1^-, Lemma 3.4 and lower semi-continuity of u give the point-wise estimate

$$v(0) \leq \sup_{Q_{1/2}^-} v \leq \frac{3}{2}\, a = \frac{C_1^2}{2},$$

contradicting the second condition in (7.6). Therefore the thesis of Proposition 3.6 holds true for any ν, λ to be chosen and for the corresponding point $z_2 = (\bar x, \bar t) \in Q_1^-$, the following measure estimate holds at the time $\bar t$

$$|[v(\cdot, \bar t) \leq \lambda\, a] \cap (\bar x + K_\epsilon)| \leq \nu\, |K_\epsilon|, \qquad \bar t \in (-1, -\bar\nu/4].$$

Recall that this measure estimate is valid for any $\nu, \lambda > 0$ to be chosen, which in turn determine an arbitrarily small ϵ, so we can also suppose

$$\nu < \bar\nu, \qquad \epsilon^{-2}\bar\nu > 1, \qquad \epsilon^{-1} a > 2,$$

where $a = C_1^2/3$. We choose $\lambda = 1/2$ and scale again considering $w = \mathcal{T}_{\epsilon/2, \epsilon/2} v(\cdot + z_2)$. Since v solves (1.1) in Q_1^- and by (3.5) it holds $K_\epsilon = \mathcal{K}_\epsilon(\epsilon) = T_{\epsilon/2,\epsilon/2}(K_2)$, w solves (1.1) in $K_2 \times (0, \epsilon^{-2}\,\bar\nu]$ and it satisfies

$$|[w(\cdot, 0) \leq 2] \cap K_2| \leq |[w(\cdot, 0) \leq \epsilon^{-1} a] \cap K_2| \leq \nu. \tag{7.7}$$

We propagate forward in time the information in (7.7) as follows. Fix a time $0 < \tau < \nu < 1$, so that we can write down the energy inequality for $(w-2)_-$ in the subcylinder $K_2 \times (0, \tau^2]$ with $0 \leq \eta \leq 1$ independent of time and such that $\eta = 1$ in K_1, $\eta = 0$ outside of K_2 and $|\partial_i \eta| \leq C$ for a constant $C > 0$, to get

$$\int_{K_1} (w(\cdot, t) - 2)_-^2\, dx \leq \int_{K_2} (w_0 - 2)_-^2\, dx + C \sum_{i=1}^N \int_0^t \int_{K_2} (w(\cdot, s) - 2)_-^{p_i}\, dx\, ds,$$

for all $t \in (0, \tau^2]$. The second term on the right hand side is bounded by $C\, 2^{N + p_{\max}}\, \nu$, while the first one is smaller than $4\,\nu$ due (7.7). The term on the left hand side bounds $|[w(\cdot, t) \leq 1] \cap K_1|$, hence we get

$$|[w(\cdot, t) \leq 1] \cap K_1| \leq C\, \nu \qquad \forall t \in (0, \tau^2],$$

which implies by integration

$$|[w \leq 1] \cap Q^+| \leq C\, \nu\, |Q^+|, \qquad Q^+ := K_1 \times (0, \tau^2].$$

We need now to recover the correct intrinsic geometry, to apply a De Giorgi-type Lemma. Let $\tau = 2^{-n}$ for some $n \in \mathbb{N}$ to be determined. We partition K_1 in 2^{Nn} dyadic cubes $x_i + K_{2^{-n}} = x_i + K_\tau$ and consider the corresponding cylinders $\mathcal{Q}_i^+ := (x_i + K_\tau) \times (0, \tau^2]$. Notice that for any such τ, the latter are intrinsically scaled, since $\mathcal{Q}_i^+ = (x_i, \tau^2) + \mathcal{Q}_\tau^-(\tau)$. On at least one of these cylinders it must hold

$$|[w \leq \tau] \cap \mathcal{Q}_i^+| \leq |[w \leq 1] \cap \mathcal{Q}_i^+| \leq C\, \nu\, |\mathcal{Q}_i^+|.$$

Now the levels in consideration are scaled according to the intrinsic geometry, and we can apply (3.41), choosing ν such that $C\,\nu \leq \nu_1$, (determining ϵ, τ and n in the process, depending only on N and $\mathbf{p}$). This implies

$$w \geq \tau/2 \qquad \text{in} \qquad (x_i, \tau^2) + T_{\tau,\tau} Q_{1/2}^- = (x_i, \tau^2) + \mathcal{Q}_{\tau/2}^-(\tau/2)$$

and in particular

$$w \geq \tau/2 \qquad \text{in} \quad z_3 + \mathcal{K}_{\tau/2}(\tau/2)$$

for some z_3. Scaling back to $u = \mathcal{T}^{-1}_{\epsilon r/2, \epsilon r^{-N}/2} w$ we get for some $z_0 \in \mathcal{Q}_1^-$ the estimate

$$u \geq \tau\, \epsilon\, r^{-N}/4 \qquad \text{in} \quad z_0 + \mathcal{K}_{\tau \epsilon r/4}(\tau\, \epsilon\, r^{-N}/4),$$

To conclude the proof of (7.5), it suffices to set

$$\rho = \frac{\tau\, \epsilon\, r}{4}, \qquad \varepsilon\, \rho^{-N} = \frac{\tau\, \epsilon\, r^{-N}}{4},$$

so that $\varepsilon = (\tau\, \epsilon/4)^{N+1}$ depends only on N and $\mathbf{p}$. $\square$

7.2 Proof of Theorem 7.1, final step. Expansion of positivity.

We begin setting C_1 as given in Lemma 7.2. To define C_2 and C_3, we begin by considering the inequality

$$u(0,0) \le C_3 \inf_{\mathcal{K}_\rho(M)} u(\cdot,\, M^{2-\bar p}\,(C_2\,\rho)^{\bar p}), \qquad M = u(0,0)/C_1. \tag{7.8}$$

We claim that there exist $\bar D > 0$ and functions $\bar A(\cdot) > 0$, $\bar B(\cdot) > 0$ all depending only on N and $\mathbf{p}$ such that, whenever

$$D \ge \bar D, \qquad A \ge \bar A(D), \qquad B \ge \bar B(D), \tag{7.9}$$

then it holds

$$\inf_{\mathcal{K}_r(M)} u(\cdot, D\,M^{2-\bar p}\,r^{\bar p}) \ge u(0,0)/B \quad \text{if} \quad \mathcal{K}_{Ar}(M) \times [-M^{2-\bar p}\,(A\,r)^{\bar p}, M^{2-\bar p}\,(A\,r)^{\bar p}] \subseteq \Omega_T. \tag{7.10}$$

Taking $C_2 \ge \bar D$ and, accordingly, $C_3 \ge \max\{\bar A(C_2), \bar B(C_2)\}$ will then give (7.8) as long as

$$\mathcal{K}_{C_3 r}(M) \times [-M^{2-\bar p}\,(C_3\,r)^{\bar p}, M^{2-\bar p}\,(C_3\,r)^{\bar p}] \subseteq \Omega_T.$$

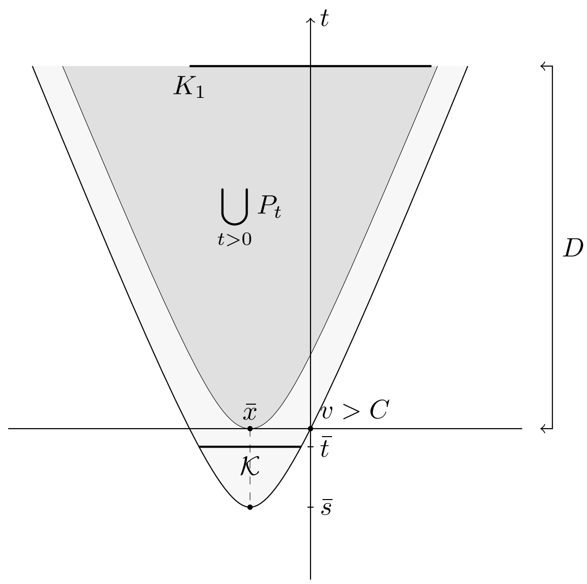

Figure 1: Scheme of proof of (7.10). The light-gray part is the support of the Barenblatt starting at $(\bar x, \bar s)$, while $\mathcal{K}$ is $\mathcal{K}_\rho(\epsilon\,\rho^{-N})$.

PROOF OF THE CLAIM (7.10).
In order for (7.10) to make sense we start by prescribing $\bar A(D)^{\bar p} \ge \max\{D, 1\}$. We suppose (this will determine (7.2)) that the function $v = \mathcal{T}_{r,M} u$ solves the equation in $\mathcal{Q}_A := \mathcal{K}_A \times [-A^{\bar p}, A^{\bar p}]$ and $v(0,0) = C_1$. Then (7.5) holds, namely there exists $(\bar x, \bar t) \in Q_1^-$, $\rho \in (0,1)$ and $\epsilon = \epsilon(N, \mathbf{p})$ such that

$$\inf_{\bar x + \mathcal{K}_\rho(\varepsilon\,\rho^{-N})} v(\cdot, \bar t) \ge \varepsilon\,\rho^{-N} \qquad \text{for} \quad (\bar x, \bar t) + \mathcal{K}_\rho(\varepsilon\,\rho^{-N}) \subseteq K_1.$$

We choose $\sigma > 0$, $-2 < s < 0$ so that the Barenblatt solution centered at $(\bar x, s)$ defined as

$$b_{\sigma,s}(x,t) = \mathcal{B}_\sigma(x - \bar x, t - s)$$

is below v in $\mathcal{K}_A$, which is implied by

$$\begin{cases} \operatorname{supp} b_{\sigma,s}(\cdot, \bar t) \subseteq \bar x + \mathcal{K}_\rho(\varepsilon\,\rho^{-N}), \\ \|b_{\sigma,s}(\cdot, \bar t)\|_\infty \le \varepsilon\,\rho^{-N}. \end{cases}$$

By Corollary 6.1, with α, α_i as specified in (3.9), this amounts to require

$$\begin{cases} \sigma^{(p_i-2)/p_i}\,(\bar t - s)^{\alpha_i} \le \frac{1}{2}(\varepsilon\,\rho^{-N})^{(p_i-\bar p)/p_i}\,\rho^{\bar p/p_i} = \frac{1}{2}\varepsilon^{(p_i-\bar p)/p_i}\,\rho^{\lambda\alpha_i}, \\ \sigma\,(\bar t - s)^{-\alpha} \le \varepsilon\,\rho^{-N}, \end{cases}$$

which holds true for $s = \bar s$ obeying $\bar s = \bar t - \rho^{\lambda}$ with $\rho < 1$, $\lambda = N(\bar p - 2) + \bar p = N/\alpha$, and $\bar\sigma = \sigma(N, \mathbf{p})$ sufficiently small. Since $\bar s > -2$, by Corollary 6.1 it holds

$$b_{\bar\sigma,\bar s}(x,t) \ge \bar\sigma\,\bar\eta\,(t-\bar s)^{-\alpha} \ge \bar\sigma\,\bar\eta\,(t+2)^{-\alpha}$$

for all

$$t>0, \qquad x \in \prod_{i=1}^{N}\{|\bar x_i - x_i| < \bar\eta\,\bar\sigma^{(p_i-2)/p_i}\,(t-\bar s)^{\alpha_i}\} \supseteq P_t(\bar x) := \prod_{i=1}^{N}\{|\bar x_i - x_i| < \bar\eta\,\bar\sigma^{(p_i-2)/p_i}\,t^{\alpha_i}\}.$$

We then choose $\bar\tau > 0$ sufficiently large so that $P_{\bar\tau}(\bar x) \supseteq K_1$ and set $\bar D = \bar\tau$ (this is possible by (5.1), which ensures $\alpha_i > 0$ for each $i = 1, \dots, N$). Then, for any $D \ge \bar D$ we additionally prescribe

$$\bar A(D)^{\bar p} \ge D + 2 \quad \text{and} \quad \bigcup_{\bar x\in K_1} \operatorname{supp}\mathcal{B}_{\bar\sigma}(\cdot - \bar x, D+2) \subseteq \mathcal{K}_{\bar A(D)}. \tag{7.11}$$

Notice that this choice can be made depending only on the parameters $N, \mathbf{p}$ and D and that if the latter conditions holds for $\bar A$ then they hold for any $A \ge \bar A$. The prescribed conditions on A permits the use of the comparison principle (Proposition 3.7) between v and $b_{\bar\sigma,\bar s}$ in $K_A \times [\bar t, D]$ (since on the lateral part of its boundary $b_{\bar\sigma,\bar s}$ vanishes), which then yields

$$v(\cdot, D) \ge b_{\bar\sigma,\bar s}(\cdot, D) \ge \bar\sigma\,\bar\eta\,(D+2)^{-\alpha} \qquad \text{in } K_1$$

for any $D \ge \bar D$. Defining $\bar B(D) = C(D+2)^{\alpha}/(\bar\eta\bar\sigma)$ and scaling back gives (7.10), and the claim is proved.

We next deal with the left inequality in (7.1), sketching its proof as some arguments are identical to the previous one. The constant C_1 is the same C as before and we claim that the inequality

$$\sup_{\mathcal{K}_r(M)} u(\cdot, -D\,M^{2-\bar p}\,r^{\bar p}) \le B\,u(0,0) \quad \text{if} \quad \mathcal{K}_{Ar}(M) \times [-M^{2-\bar p}(A\,r)^{\bar p}, M^{2-\bar p}(A\,r)^{\bar p}] \subseteq \Omega_T \tag{7.12}$$

(with $M = u(0,0)/C$) holds true for any A, B, D as in (7.9), for a possibly different choice of $\bar D$ and of the functions $\bar A, \bar B$.
To prove (7.12), we fix $\gamma > N/\bar p$ and start by prescribing

$$\bar A(D)^{\bar p} \ge D, \qquad \bar B(D) \ge D^{\gamma}.$$

Next, consider A, B, D fulfilling (7.9) together with $\mathcal{K}_{Ar}(M) \times [-M^{2-\bar p}(A\,r)^{\bar p}, M^{2-\bar p}(A\,r)^{\bar p}] \subseteq \Omega_T$, but such that

$$\sup_{\mathcal{K}_r(M)} u(\cdot, -D\,M^{2-\bar p} r^{\bar p}) > B\,u(0,0). \tag{7.13}$$

We rewrite the latter in terms of $v = \mathcal{T}_{r,MD^{\gamma}}u$, which is a solution in $\mathcal{Q}_A(D^{-\gamma})$: the resulting information is

$$v(0,0) = C\,D^{-\gamma}, \qquad \sup_{\mathcal{K}_1(D^{-\gamma})} v(\cdot, -D^{1+\gamma(\bar p-2)}) > B\,v(0,0) \ge C, \tag{7.14}$$

where we used $B \ge \bar B(D) \ge D^{\gamma}$ in the last inequality. We fix a point $x_0 \in \mathcal{K}_1(D^{-\gamma})$ such that

$$v(x_0, -D^{1+\gamma(\bar p-2)}) > C,$$

and suppose that $\bar A(D)$ is additionally large enough so that v is a solution in $(x_0, -D^{1+\gamma(\bar p-2)}) + Q_1$. We can then apply Lemma 7.2 and, proceeding exactly as in the first part of the proof, we find

$$\bar x \in x_0 + K_1, \qquad -D^{1+\gamma(\bar p-2)} - 2 \le \bar s < \bar t \le -D^{1+\gamma(\bar p-2)}$$

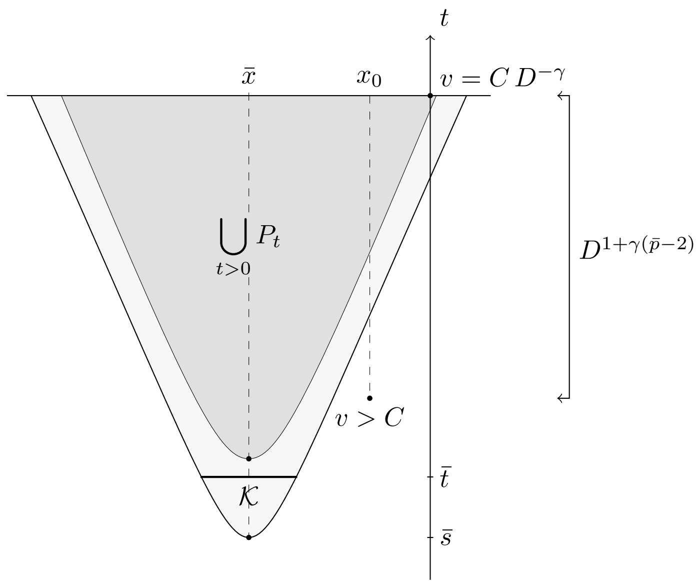

Figure 2: Scheme of proof of (7.12). The light-gray part is the support of the Barenblatt starting at $(\bar{x}, \bar{s})$ while $\mathcal{K}$ is $\mathcal{K}_\rho(\epsilon\, \rho^{-N})$.

and $\bar{\sigma}(N, \mathbf{p}) > 0$ such that the Barenblatt solution $b_{\bar{\sigma},\bar{s}}$ centered at $(\bar{x}, \bar{s})$ is below v at the time $\bar{t}$. As before, for some $\bar{\eta}(N, \mathbf{p})$ it holds

$$b_{\bar{\sigma},\bar{s}}(\cdot, t) \geq \bar{\sigma}\, \bar{\eta}\, (t + D^{1+\gamma(\bar{p}-2)} + 2)^{-\alpha} \quad \text{in} \quad P_{t+D^{1+\gamma(\bar{p}-2)}}(\bar{x}), \quad \forall t > -D^{1+\gamma(\bar{p}-2)}.$$

If needed, we further increase $\bar{A}(D)$ so that v solves the equation in a rectangle containing the support of any possible $b_{\bar{\sigma},\bar{s}}$ so constructed, up to the time $t = 0$ (through a condition of the type (7.11)).

So far, the definition of the functions $\bar{A}(D)$ and $\bar{B}(D)$ is concluded, and we now look for all the values of D such that $0 \in P_{\bar{D}^{1+\gamma(\bar{p}-2)}}(\bar{x})$. Since $x_0 \in \mathcal{K}_1(D^{-\gamma})$ and $\bar{x} \in x_0 + K_1$, this is true if

$$1 + D^{-\gamma(p_i - \bar{p})/p_i} \leq \bar{\eta}\, \bar{\sigma}^{(p_i-2)/p_i}\, D^{(1+\gamma(\bar{p}-2))\alpha_i}, \qquad \forall i = 1, \dots, N. \tag{7.15}$$

We claim that the exponent of D on the left is less than the one on the right. Indeed, from the definition of α_i, the claim reduces through elementary algebraic manipulations to

$$\gamma\, \bar{p}\, (2 - p_i) < N\, (\bar{p} - p_i) + \bar{p},$$

which is always true since the left hand side is negative by $p_i > 2$ and the right hand side is positive by (5.1). It follows that (7.15) holds true for any $D \geq \bar{D}_1$, and in this case we get by comparison

$$v(0, 0) \geq b_{\bar{\sigma},\bar{s}}(0, 0) \geq \bar{\sigma}\, \bar{\eta}\, (D^{1+\gamma(\bar{p}-2)} + 2)^{-\alpha}. \tag{7.16}$$

Next, we claim that there exists $\bar{D}_2$ such that if $D \geq \bar{D}_2$, then

$$\bar{\sigma}\, \bar{\eta}\, (D^{1+\gamma(\bar{p}-2)} + 2)^{-\alpha} > C\, D^{-\gamma}. \tag{7.17}$$

Indeed, it suffices to show that the exponent on the left is greater than the one on the right, which, recalling that $\alpha = N/(N(\bar{p} - 2) + \bar{p})$, amounts to

$$\gamma - \alpha\, (1 + \gamma\, (\bar{p} - 2)) = \frac{\gamma\, \bar{p} - N}{N\, (\bar{p} - 2) + \bar{p}} > 0 \quad \Leftrightarrow \quad \gamma > \frac{N}{\bar{p}}$$

as we assumed. Thus (7.17) is proved, which in turn contradicts the first condition in (7.14) via the lower bound in (7.16). All in all, letting $\bar{D} = \max\{\bar{D}_1, \bar{D}_2\}$ shows that if A, B, D obey (7.9), then (7.13) cannot hold, completing the proof of (7.12). We conclude choosing the constants C_2 and C_3 as in the previous step, and finally pick the largest between the so defined constants and previous ones.

Corollary 7.1. *Let u be a non-negative local weak solution to the equation*

$$-\sum_{i=1}^{N}\partial_i\Big(|\partial_i u|^{p_i-2}\partial_i u\Big)=0,\quad in\quad \Omega\subset\subset\mathbb{R}^N, \tag{7.18}$$

and let condition (5.1) be satisfied. Suppose furthermore for a point $x_o\in\Omega$ that $u(x_o)>0$. Then there exist constants $C_1>0$ and $C_3\geq 1$ depending only on N and the p_i's such that

$$\frac{1}{C_3}\sup_{\mathcal{K}_\rho(M)} u\leq u(x_o)\leq C_3\inf_{\mathcal{K}_\rho(M)} u) \tag{7.19}$$

whenever $x_o+\mathcal{K}_{C_3\,\rho}(M)\subseteq\Omega$ and being $M=u(x_o)/C_1$.

Although most arguments in our proofs are local, previous Theorem 7.1 expresses a global information. Let $(x_o,t_o)\in\Omega_T$ be a point where the local weak solution u is positive. We will distinguish the sets where the point-wise control (7.1) holds true.

Proposition 7.1. *Suppose satisfied the assumptions of Theorem 7.1 for $(x_0,t_0)\in\Omega\times[-T,T]$. Then,*

$$\inf_{\mathcal{P}_u^+(x_0,t_0)} u\geq u(x_0,t_0)/C_3,\qquad and\qquad \sup_{\mathcal{P}_u^-(x_0,t_0)} u\leq C_3u(x_0,t_0), \tag{7.20}$$

where $\mathcal{P}_u^+(x_o,t_o)$ and $\mathcal{P}_u^-(x_o,t_o)$ are defined, up to side-condition (7.2), by

$$\mathcal{P}_u^+(x_0,t_0)=\Big\{(x,t)\in\Omega_T:\quad C_2^{\bar p}|x_i-x_{0,i}|^{p_i}\theta^{2-p_i}\leq(t-t_0),\quad\forall i=1,..N\Big\},$$

and

$$\mathcal{P}_u^-(x_0,t_0)=\Big\{(x,t)\in\Omega_T:\quad (t-t_0)\leq -C_2^{\bar p}|x_i-x_{0,i}|^{p_i}\theta^{2-p_i},\quad\forall i=1,..N\Big\}.$$

The union of these two sets through the point (x_o,t_o) is the whole anisotropic intrinsic paraboloid

$$\mathcal{P}_u(x_0,t_0)=\Big\{(x,t)\in\Omega_T:\quad |x_i-x_{0,i}|^{p_i}\leq C_2^{-\bar p}\theta^{p_i-2}|t-t_0|<(\rho^+)^{\bar p}\theta^{p_i-\bar p},\quad\forall i=1,..N\Big\}, \tag{7.21}$$

where ρ^+ depends on u, Ω_T and (x_0,t_0) with the following expression

$$\rho^+=C_3^{-\bar p}\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{\bar p-2}\min_{i=1,\dots,N}\Big\{(T-|t_0|),\ \Big(\frac{\mathrm{dist}(x_0,\partial\Omega)}{2}\Big)^{p_i}\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{2-p_i}\Big\}. \tag{7.22}$$

Proof of (7.20). Estimates of Harnack inequality (7.1) and (7.2) give a condition that can be expressed for different radii $\rho\in(0,\rho^+)$, where ρ^+ is identified by means of (7.2) as

$$\begin{cases}(T-|t_0|)\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{\bar p-2}C_3^{-\bar p}>\rho^{\bar p},\\ (C_3\rho)^{\frac{\bar p}{p_i}}\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{\frac{p_i-\bar p}{p_i}}\leq\Big(\frac{\mathrm{dist}(x_0,\partial\Omega)}{2}\Big)\end{cases}\Rightarrow\begin{cases}\rho^{\bar p}\leq C_3^{-\bar p}(T-|t_0|)\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{\bar p-2},\\ \rho^{\bar p}\leq C_3^{-\bar p}\Big(\frac{\mathrm{dist}(x_o,\partial\Omega)}{2}\Big)^{p_i}\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{\bar p-p_i},\end{cases}$$

therefore we may take ρ^+ as the minimum as in (7.22). Finally, to show estimates (7.20) we use (7.1) to get a precise description of radii ρ in terms of the fixed times $t_0\pm(C_2\rho)^{\bar p}(u(x_0,t_0)/C_1)^{2-\bar p}$, which amounts in the case of $\mathcal{P}_u^+(x_0,t_0)$ to

$$\begin{cases}|x_i-x_{0,i}|^{p_i}<\rho^{\bar p}\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{p_i-\bar p},\\ t=t_0+(C_2\rho)^{\bar p}\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{2-\bar p}\end{cases}\Rightarrow\begin{cases}|x_i-x_{0,i}|^{p_i}\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{\bar p-p_i}<\rho^{\bar p},\\ \rho^{\bar p}=C_2^{-\bar p}\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{\bar p-2}|t-t_0|,\end{cases}$$

$$\Rightarrow\quad |x_i-x_{0,i}|^{p_i}<C_2^{-\bar p}\Big(\frac{u(x_0,t_0)}{C_1}\Big)^{p_i-2}(t-t_0).$$

In case $t<t_0$ we obtain a similar statement for $\mathcal{P}_u^-(x_0,t_0)$. If the equation is solved in all $\mathbb{R}^N$, then we set $\rho^+=+\infty$. $\square$

8 Consequences of Harnack's inequality

Future perspectives and open problems

Quanta energia racchiusa in una frase,
quanta struttura imprigionata in due simboli.
Libertà di scrivere, voi direte, è questa.
Saltano però i tiranti delle vele maestre
allo soccar delle implicazioni funeste
che al paradosso scontrano muro a testa.
Lá dove fallisca rappresentazione,
leggera astrazione liberasi amichevole
tra voce e parole, tra nascita e lutto,
con messaggio ch'evapora il messaggero.
Al formalismo questa passione deve tutto,
giacché, se non è falso e non è vero,
da questa prigione v'è natural evasione
se alla fantasia è posta alta recinzione.

- S.C. -

Formalismo e Libertà

In this last Section we show that some classical consequences of Harnack inequality hold true, for local weak solutions to (1.1). Following the lines of ([27], Section 10), we show that Harnack estimates (7.1) imply a reduction of the essential oscillation. From this point to Hölder continuity, we use a different strategy than the one we found in literature ([23], Lemma 3.1 Chap. III, or [65]); because if $\theta_0 > \theta_n$ the inclusion of $\mathcal{Q}_\rho(\theta_0)$ in $\mathcal{Q}_\rho(\theta_n)$ is no more valid for anisotropic intrinsic cylinders. Moreover, we show some interesting rigidity results of Liouville-type. On a third step we show a version of the Harnack inequality that frees the time variable of the intrinsic geometry. To achieve this estimate, we use the Harnack inequality first to recover a lower bound, and then we use comparison principle with $\mathcal{B}$. As soon as side-condition (7.2) is satisfied, this particular Harnack estimate is equivalent to the previous (7.1) for small radii.

Finally we comment some open problems; some of them easier, some of them harder and some of them very general and challenging. We wish so to attract the attention of the research community on the present research topic which, as we have already discussed in detail along the Introduction, has deep roots into the study of singular and degenerate equations. It is nowadays evident that new techniques are required.

8.1 Hölder Continuity

We use the intrinsic Harnack inequality (7.1) to establish locally quantitative Hölder estimates for local, weak solutions u of (1.1), conditioned to the assumption $2 < p_i < \bar{p}(1 + 1/N)$ for each $i = 1, .., N$. This condition is part of the hypothesis of Theorem 7.1, and it is stronger than the sole boundedness condition (4.6). It may happen that in this range of p_is the diffusion is so slow that no expansion of positivity occurs.

Theorem 8.1. *Local weak solutions to* (1.1) *are locally Hölder continuous under condition* (5.1). *More precisely, there exist constants* $\gamma > 1$, $\chi \in (0, 1)$ *depending only upon* p_i, N *with the following property. For each compact set* $K \subset\subset \Omega_T$ *there exist an open set* $K \subset \Lambda \subseteq \Omega_T$ *and a constant* $\omega_o = \omega_o(K, \|u\|_{\infty,K})$ *as defined in* (8.3)-(8.4)*, such that for every pair of points* (x, t), $(y, s) \in K$

$$|u(x,t) - u(y,s)| \leq \gamma\omega_o\bigg(\frac{\sum_{i=1}^{N}|x_i - y_i|^{\frac{p_i}{\bar{p}}}\omega_o^{\frac{\bar{p}-p_i}{\bar{p}}} + |t-s|^{\frac{1}{\bar{p}}}\omega_o^{\frac{\bar{p}-2}{\bar{p}}}}{\mathbf{p}\text{-}dist(K, \partial\Lambda)}\bigg)^{\chi}, \tag{8.1}$$

with

$$\begin{aligned}
&\mathbf{p}\text{-}dist(K, \partial\Lambda) := \inf\{\mathbf{p}_x, \mathbf{p}_t\}, \quad \textit{being}\\
&\mathbf{p}_x = \inf\bigg\{|x_i - y_i|^{\frac{p_i}{\bar{p}}}(\omega_o/C_1)^{\frac{\bar{p}-p_i}{\bar{p}}} \,:\, (x,t) \in K, (y,s) \in \partial\Lambda,\, i = 1, .., N\bigg\},\\
&\mathbf{p}_t = \inf\bigg\{|t-s|^{\frac{1}{\bar{p}}}(\omega_o/C_1)^{\frac{\bar{p}-2}{\bar{p}}} \,:\, (x,t) \in K, (y,s) \in \partial\Lambda\bigg\},
\end{aligned} \tag{8.2}$$

Furthermore, if u is bounded in Ω_T then (8.1) *holds with* $\Lambda = \Omega_T$.

We will prove Theorem 8.1 in five steps.

Proof. Let us fix a compact set $K \subset\subset \Omega$ and two points $(y, s), (x, t) \in K$.

STEP 1-*A global bound for the solution in K.*

By compactness of K it is possible to find a number $m \in \mathbb{N}$ of points $(x_i, t_i) \in K$ and $\lambda_i, H_i \in \mathbb{R}_+$ for $i = 1, \ldots, m$ such that

$$K \subset \Lambda := \bigcup_{j=1}^{m}\{(x_j, t_j) + Q_{\lambda_j,H_j}\} \subseteq \bigcup_{j=1}^{m}\{(x_j, t_j) + Q_{2\lambda_j,H_j}\} \subseteq \Omega_T,$$

being $Q_{\lambda,M}$ as in (4.8). This is because $D = \text{dist}(K, \partial\Omega_T) > 0$ and we may consider $\lambda_j, H_j < \min\{(D/2)^{p_i}, \sqrt{D/2}\}$. Then by (4.9) we have for each $Q_{\lambda_j,M_j} = (x_j, t_j) + Q_{\lambda_j,M_j}$, $j = 1, \ldots, m$, the estimate

$$\begin{aligned}
\|u\|_{L^\infty(Q_{\lambda_j,M_j})} &\leq g^{-1}(1/M_j) + h^{-1}\bigg(C\bigg(M_j \fint\!\!\fint_{Q_{2\lambda_j,M_j}} |u|^{\bar{p}_2}\, dxdt\bigg)^{\frac{\bar{p}}{N+\bar{p}}}\bigg)\\
&\leq g^{-1}(1/\min_j M_j) + h^{-1}\bigg(C \max_{j=1,\ldots,m}\bigg(M_j \fint\!\!\fint_{Q_{2\lambda_j,M_j}} |u|^{\bar{p}_2}\, dxdt\bigg)^{\frac{\bar{p}}{N+\bar{p}}}\bigg) =: \mathcal{I},
\end{aligned} \tag{8.3}$$

because h,g, are monotone increasing. Observe that $\mathcal{I}$ does not depend anymore on $j \in \{1, \ldots, m\}$. Finally, we define

$$\omega_0(K) = 2m\mathcal{I}, \tag{8.4}$$

so that

$$K \subset \bigcup_{j=1}^{m} Q_{\lambda_j,M_j}(x_j, t_j) = \Lambda, \qquad \& \qquad 2\|u\|_{L^\infty(K)} \leq \omega_o(K).$$

Let us call $\omega_o = \omega_o(K)$ for the sake of brevity.

STEP 2-*Accommodation of degeneracy.*

We recall (8.2) and we define $R := [\mathbf{p}\text{-dist}(K,\partial\Lambda)]/(2C_3)$. We claim that the intrinsic cylinder centered in $(y,s)\in K$ and constructed with R and ω_o is contained inside Λ,

$$(y,s)+\mathcal{Q}_R(\omega_o/C_1)\subseteq\Lambda.$$

Let us check it component-wise. We denote with π_k the projection on the x_k-th component of $\mathbb{R}^{N+1}$, and we have for all $i=1,\dots,N$

$$\pi_i\bigg((y,s)+\mathcal{Q}_R(\omega_o/C_1)\bigg)=\bigg\{|y_i-x_i|<(\omega_o/C_1)^{\frac{p_i-\bar{p}}{p_i}}\bigg(\frac{\inf\{|w_i-z_i|^{\frac{p_i}{\bar{p}}}(\omega_o/C_1)^{\frac{\bar{p}-p_i}{\bar{p}}}\}}{2C_3}\bigg)^{\frac{\bar{p}}{p_i}}\bigg\},$$

for $w\in K, z\in\partial\Lambda$, by definition of R. Hence for $x_i\in\pi_i((y,s)+\mathcal{Q}_R(\omega_o/C_1)))$ we have

$$|y_i-x_i|\le(\omega_o/C_1)^{\frac{p_i-\bar{p}}{p_i}}\bigg(\frac{\inf\{|y_i-z_i|^{\frac{p_i}{\bar{p}}}(\omega_o/C_1)^{\frac{\bar{p}-p_i}{\bar{p}}}\}}{2C_3}\bigg)^{\frac{\bar{p}}{p_i}}\le\text{dist}(y_i,\pi_i(\partial\Lambda)),$$

where the infimum is taken over all $z_i\in\pi_i(\partial K)$ and those terms involving ω_o cancel out. Moreover,

$$\pi_t\bigg((y,s)+\mathcal{Q}_R(\omega_o(K)/C_1)\bigg)=\bigg\{|s-t|<(\omega_o/C_1)^{2-\bar{p}}\bigg(\frac{\inf\{|h-l|^{\frac{1}{\bar{p}}}(\omega_o/C_1)^{\frac{\bar{p}-2}{\bar{p}}}\}}{2C_3}\bigg)^{\bar{p}}\bigg\},$$

for $h\in\pi_t(K), l\in\pi_t(\Lambda)$ using again the definition of R, is included in Λ. Indeed for all times t in such a set it holds

$$|s-t|\le(\omega_o/C_1)^{2-\bar{p}}\bigg(\frac{\inf\{|s-l|^{\frac{1}{\bar{p}}}(\omega_o/C_1)^{\frac{\bar{p}-2}{\bar{p}}}\}}{2C_3}\bigg)^{\bar{p}}\le\text{dist}(s,\pi_t(\partial K)),$$

where the infimum is take over all $l\in\pi_t(K)$. Our claim is proved.

STEP 3-*Alternatives.*

Consider now any other point $(x,t)\in K$. We show that we can formulate the problem for $(y,s)+Q_R(\omega_o/C_1)$, having otherwise the Lipschitz continuity of u. If $|s-t|\ge(\omega_o/C_1)^{2-\bar{p}}R^{\bar{p}}$, we have

$$|u(y,s)-u(x,t)|\le|u(y,s)|+|u(x,t)|\le2\omega_o\le4\omega_o\bigg(\frac{(\omega_o/C_1)^{\frac{\bar{p}-2}{\bar{p}}}|s-t|^{\frac{1}{\bar{p}}}}{\mathbf{p}\text{-dist}(K,\partial\Lambda)}\bigg).$$

Indeed by assumption

$$1\le2\bigg(\frac{(\omega_o/C_1)^{\frac{\bar{p}-2}{\bar{p}}}|s-t|^{\frac{1}{\bar{p}}}}{\mathbf{p}\text{-dist}(K,\partial\Lambda)}\bigg)\qquad\Leftarrow\qquad\mathbf{p}\text{-dist}(K,\partial\Lambda)\le(\omega_o/C_1)^{\frac{\bar{p}-2}{\bar{p}}}|s-t|^{\frac{1}{\bar{p}}}.$$

Same conclusion pertains the case $|y_i-x_i|\ge(\omega_o/C_1)^{\frac{p_i-\bar{p}}{\bar{p}}}R^{\frac{\bar{p}}{p_i}}$ for some $i\in\{1,\dots,N\}$, arriving again to

$$|u(y,s)-u(x,t)|\le|u(y,s)|+|u(x,t)|\le2\omega_o\le4\omega_o\bigg(\frac{(\omega_o/C_1)^{\frac{\bar{p}-p_i}{\bar{p}}}|y_i-x_i|^{\frac{p_i}{\bar{p}}}}{\mathbf{p}\text{-dist}(K,\partial\Lambda)}\bigg).$$

In conclusion we are left with the following possibility

$$\bigg\{|s-t|<(\omega_0/C_1)^{2-\bar{p}}R^{\bar{p}}\bigg\}\quad\&\quad\bigg\{|y_i-x_i|<(\omega_o/C_1)^{\frac{p_i-\bar{p}}{p_i}}R^{\frac{\bar{p}}{p_i}}\bigg\}\quad\forall i=1,\dots,N.\tag{8.5}$$

which means that if we pick any point (x,t) in K we can assume that

$$(x,t)\in(y,s)+\mathcal{Q}_R(\omega_o/C_1).$$

We take this cylinder as the first element $\mathcal{Q}_0:=\mathcal{Q}_R(\omega_o/C_1)$ of a net of particular cylinders $\mathcal{Q}_n$ shrinking to their center (y,s) in such a way that the oscillation in each of them is controlled uniformly.

STEP 4- *Controlled reduction of oscillation*

Proposition 8.1 (Reduction of oscillation in shrinking cylinders)**.** *Let the hypothesis of Theorem 8.1 be satisfied and assume also* (8.5)*. Let* C_1, C_2, C_3 *the constants of Theorem 7.1. Then we can define numbers* $\delta, \varepsilon \in (0,1)$ *such that if we set*

$$\begin{cases} \omega_0 = \omega_o(K), \\ \omega_n = \delta\omega_{n-1},\ 1 < n \in \mathbb{N} \end{cases}, \begin{cases} \theta_n = (\omega_n/C_1), \\ \rho_n = \varepsilon\rho_{n-1},\ 1 < n \in \mathbb{N}, \\ \rho_0 = R \end{cases}, \begin{cases} \delta = 4C_3/(1+4C_3), \\ \varepsilon = \delta^{\frac{\bar p - 2}{\bar p}}/A \end{cases}, \quad A > 1,$$

we have both the inclusions

$$\mathcal{Q}_n \subset \mathcal{Q}_{n-1}, \quad with \quad \mathcal{Q}_n = (y,s) + \mathcal{Q}_{\rho_n}(\theta_n) = \prod_{i=1}^N \left\{ |y_i - x_i| < \theta_n^{\frac{p_i - \bar p}{p_i}} \rho_n^{\frac{\bar p}{p_i}} \right\} \times \left(s - \theta_n^{2-\bar p}(C_2\rho_n)^{\bar p},\, s \right]$$

and the controlled inequalities

$$\operatorname*{osc}_{\mathcal{Q}_n} u \le \omega_n \le \delta^n \omega_o. \tag{8.6}$$

Proof of Proposition 8.1. We proceed by induction. The first step is $\operatorname{osc}_{\mathcal{Q}_0} u = \operatorname{osc}_{Q_R(\omega_o/C_1)} \le \omega_o$, that holds true because $\omega_o \ge 2 \operatorname{osc}_{Q_0}$ having the inclusion $(y,s) + Q_R(\omega_o/C_1) \subset \Lambda$ by the previous accommodation of degeneracy. Moreover, we postpone the proof of $\mathcal{Q}_1 \subseteq \mathcal{Q}_0 = (y,s) + Q_R(\omega_o/C_1)$ for the following general fact. By direct computation,

$$\theta_{n+1}^{\bar p - 2}(C_2\rho_{n+1})^{\bar p} = \left(\frac{C_1}{\omega_{n+1}}\right)^{\bar p - 2}\left((C_2\rho_n/A)^{\bar p}\delta^{\bar p - 2}\right) = \left(\frac{C_1}{\delta\omega_n}\right)^{\bar p - 2}\left((C_2\rho_n/A)^{\bar p}\delta^{\bar p - 2}\right) = \theta_n^{\bar p - 2}(C_2\rho_n/A)^{\bar p},$$

precisely, and for each $i \in \{1,..,N\}$ as $p_i > 2$ and $\delta \in (0,1)$ it holds

$$\theta_{n+1}^{\frac{\bar p - p_i}{p_i}} \rho_{n+1}^{\frac{\bar p}{p_i}} = \left(\frac{C_1}{\omega_{n+1}}\right)^{\frac{\bar p - p_i}{p_i}} \left(\frac{\rho_n}{A}\delta^{\frac{\bar p - 2}{\bar p}}\right)^{\frac{\bar p}{p_i}} = \left(\frac{C_1}{\delta\omega_n}\right)^{\frac{\bar p - p_i}{p_i}} \delta^{\frac{\bar p - 2}{p_i}} \left(\frac{\rho_n}{A}\right)^{\frac{\bar p}{p_i}}$$

$$= \left(\frac{C_1}{\omega_n}\right)^{\frac{\bar p - p_i}{p_i}} \left(\frac{\rho_n}{A}\right)^{\frac{\bar p}{p_i}} \delta^{\frac{p_i - \bar p}{p_i} + \frac{\bar p - 2}{p_i}} = \theta_n^{\frac{\bar p - p_i}{p_i}} \left(\frac{\rho_n}{A}\right)^{\frac{\bar p}{p_i}} \delta^{\frac{p_i - 2}{p_i}} \le \theta_n^{\frac{\bar p - p_i}{p_i}} \left(\frac{\rho_n}{A}\right)^{\frac{\bar p}{p_i}}.$$

This computation show a little more, by allowing indeed $\mathcal{Q}_{n+1} \subset \mathcal{Q}_{\rho_n/A}(\omega_n/C_1) \subset \mathcal{Q}_n$. These computations holding also for $\mathcal{Q}_1 \subset \mathcal{Q}_0$, the first and $(n+1)$-th steps are checked at once.

Proof of the inductive step (8.6)

We assume now that the statement (8.6) is true until step n and we show it for $n+1$. This will determine number A. More precisely, we assume that $\operatorname{osc}_{\mathcal{Q}_n} u \le \omega_n$ and by contradiction that $\operatorname{osc}_{\mathcal{Q}_{n+1}} u > \omega_{n+1}$. With numbers ω_n we construct the following

$$M_n = \sup_{\mathcal{Q}_n} u, \qquad m_n = \inf_{\mathcal{Q}_n} u, \qquad P_n = (y,\, s - (C_2\rho_n)^{\bar p}\theta_n^{2-\bar p}).$$

Point P_n is the opposite point in $\mathcal{Q}_n$ to the vertex (y,s). Now we observe that one of the following two inequalities must hold

$$M_n - u(P_n) > \omega_{n+1}/4, \quad \text{or} \qquad u(P_n) - m_n > \omega_{n+1}/4.$$

Indeed if both alternatives are violated then by adding their opposites we obtain $\operatorname{osc}_{\mathcal{Q}_n} u \le \omega_{n+1}/2 < \operatorname{osc}_{\mathcal{Q}_{n+1}}$ generating an absurd by monotonicity of the oscillation and the proven fact that $\mathcal{Q}_{n+1} \subseteq \mathcal{Q}_n$. Let us suppose $M_n - u(P_n) \ge \omega_{n+1}/4$, the other case being similar. In particular we have the double bound

$$\omega_{n+1}/4 \le M_n - u(P_n) \le \omega_n.$$

We work within the half-paraboloid $\mathcal{P}_n^+ = \mathcal{P}^+_{M_n - u(P_n)}(P_n)$ for times limited to the ones of $\mathcal{Q}_n$, which is

$$\mathcal{P}_n^+ = \left\{ |x_i - y_i|^{p_i} < [(M_n - u(P_n)/C_1)]^{p_i - 2} \rho_n^{\bar{p}} (\omega_n/C_1)^{2-\bar{p}},\ s - (C_2\rho_n)^{\bar{p}}(\omega_n/C_1)^{2-\bar{p}}) \le t \le s \right\}.$$

The condition $\rho_n^{\bar{p}}\theta_n^{2-\bar{p}} < (\rho^+)^{\bar{p}}\theta_n^{2-\bar{p}}$ specified by right-hand inequality of (7.21) is naturally satisfied, being $\theta_n = (M_n - u(P_n)/C_1)$. Indeed, by definition of R, $\omega_o > \omega_n > M_n - u(P_n)$ and recalling that $\mathcal{P}_n^+$ is limited to the times of $\mathcal{Q}_n$, we have the following estimate along the time variable

$$\rho_n^{\bar{p}} \left(\frac{\omega_n}{C_1} \right)^{2-\bar{p}} \le R^{\bar{p}} \left(\frac{\omega_o}{C_1} \right)^{2-\bar{p}} \le (\rho^+)^{\bar{p}} \left(\frac{M_n - u(P_n)}{C_1} \right)^{2-\bar{p}}.$$

It is worth to observe that in the previous estimate the number ρ^+ depends on the function $M_n - u(P_n)$ and the point P_n. On the other hand, by definition of ω_n, we recover an estimate for the space variables

$$C_2^{\bar{p}} |x_i - y_i|^{p_i} < \left(\frac{M_n - u(P_n)}{C_1} \right)^{p_i - 2} \rho_n^{\bar{p}} \left(\frac{\omega_n}{C_1} \right)^{2-\bar{p}} = \left(\frac{M_n - u(P_n)}{C_1} \right)^{p_i - 2} \left(\frac{R}{A^n} \right)^{\bar{p}} \left(\frac{\omega_o}{C_1} \right)^{2-\bar{p}}.$$

First inequality of last estimate shows also that paraboloid $\mathcal{P}_n^+$ is contained in $\mathcal{Q}_n$. Now we show that after a certain time $\bar{t}$, the whole cylinder $\mathcal{Q}_{n+1}$ is contained in the paraboloid $\mathcal{P}_n^+$, see Figure 3 for a representation. For times $t > s - (C_2\rho_n)^{\bar{p}}(\omega_n/C_1)^{2-\bar{p}}$ we denote by $\mathcal{P}_n^+(t)$ the time-section of $\mathcal{P}_n^+$ at time t:

$$\mathcal{P}_n^+(t) = \left\{ x \in \mathbb{R}^N :\ |x_i - y_i|^{p_i} < (C_2)^{-\bar{p}} [(M_n - u(P_n)/C_1)]^{p_i - 2} (t - s + (C_2\rho_n)^{\bar{p}} (\omega_n/C_1)^{2-\bar{p}}) \right\}.$$

Let us set

$$\bar{t} = s - (C_2\rho_{n+1})^{\bar{p}} (\omega_{n+1}/C_1)^{2-\bar{p}}, \tag{8.7}$$

and let us prove that at time $\bar{t}$ we have the inclusion

$$\pi_x(\mathcal{Q}_{\rho_{n+1}}) \subset \mathcal{P}_n^+(\bar{t}) = \left\{ x \in \mathbb{R}^N :\ |x_i - y_i|^{p_i} < \rho_n^{\bar{p}} [(M_n - u(P_n)/C_1)]^{p_i - 2} (1 - A^{-\bar{p}}) (\omega_n/C_1)^{2-\bar{p}} \right\}.$$

This hence reduces to show that

$$\begin{aligned} (\rho_{n+1})^{\bar{p}} (\omega_{n+1}/C_1)^{p_i - \bar{p}} &\le (1 - A^{-\bar{p}}) \rho_n^{\bar{p}} [(M_n - u(P_n)/C_1)]^{p_i - 2} (\omega_n/C_1)^{2-\bar{p}} = \\ &= (1 - A^{-\bar{p}}) [(M_n - u(P_n)/C_1)]^{p_i - 2} A^{\bar{p}} \rho_{n+1}^{\bar{p}} (\omega_{n+1}/C_1)^{2-\bar{p}}, \\ &\iff (\omega_{n+1})^{p_i - 2} \le (A^{\bar{p}} - 1)(M_n - u(P_n))^{p_i - 2}. \end{aligned}$$

If we choose $A > 1$ such that $4^{p_N - 1} < A^{\bar{p}} - 1$ as for instance $A = 4^{p_N}$, then

$$(\omega_{n+1})^{p_i - 2} = (\omega_{n+1}/4)^{p_i - 2} 4^{p_i - 2} \le (A^{\bar{p}} - 1)(M_n - u(P_n))^{p_i - 2}.$$

Therefore we can estimate by Harnack inequality (7.1) the infimum of $M_n - u$ in $\mathcal{Q}_{n+1}$ as

$$\inf_{\mathcal{Q}_{n+1}} (M_n - u) \ge \frac{M_n - u(P_n)}{C_3} \ge\ \omega_{n+1}/(4C_3), \tag{8.8}$$

again referring to Figure 3. Now we use that $\inf(-u) = -\sup u$ to estimate

$$M_n \ge \sup_{\mathcal{Q}_{n+1}} u + \omega_{n+1}/(4C_3),$$

and we add $-\inf_{\mathcal{Q}_n} u \ge -\inf_{\mathcal{Q}_{n+1}} u$ to both sides, to get by assumption $\operatorname{osc}_{\mathcal{Q}_{n+1}} u > \omega_{n+1}$ that

$$\omega_n \ge \sup_{\mathcal{Q}_{n+1}} u + \omega_{n+1}/(4C_3) - \inf_{\mathcal{Q}_{n+1}} u = \operatorname*{osc}_{\mathcal{Q}_{n+1}} u + \omega_{n+1}/(4C_3) > \left(1 + \frac{1}{4C_3} \right) \omega_{n+1}\,.$$

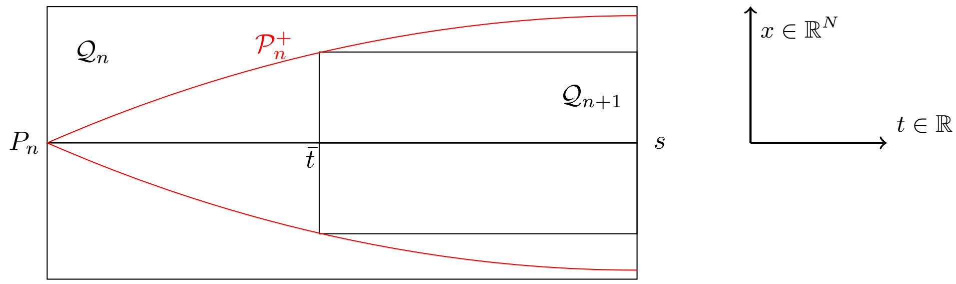

Figure 3: Scheme of the proof of (8.8), by use of the anisotropic paraboloid $\mathcal{P}_n^+$ (in red), that is centered in $P_n = (\,y,\, s - (C_2\rho_n)^{\bar{p}}(\omega_n/C_1)^{2-\bar{p}})$ and evolves in a time $(C_2\rho_n)^{\bar{p}}(\omega_n/C_1)^{2-\bar{p}}$ to cover completely $\mathcal{Q}_{n+1}$.

This leads to the contradiction by definition of δ, as

$$\omega_n > \Big(1 + \frac{1}{4C_3}\Big)\delta\omega_n = \Big(\frac{4C_3}{1+4C_3}\Big)\Big(1 + \frac{1}{4C_3}\Big)\omega_n = \omega_n,$$

and Proposition 8.1 is proved true.

□

STEP 5.*Conclusion of the proof of Theorem 8.1*

If we consider a point $(x,t) \in (y,s) + \mathcal{Q}_R(\omega_o/C_1)$, let $n \in \mathbb{N}$ be the last number such that we have $(x,t) \in \mathcal{Q}_n$, so that $(x,t) \notin \mathcal{Q}_{n+1}$. From the first condition we have

$$|u(x,t) - u(y,s)| \le \operatorname*{osc}_{\mathcal{Q}_n} u \le \delta^n \omega_o.$$

The rest of the job is to determine from condition $(x,t) \notin (y,s) + \mathcal{Q}_{n+1}$ an upper bound for δ^n. Indeed let $\beta > 0$ be such that $\delta^{\frac{\bar{p}-2}{\bar{p}}}/A = \delta^\beta$ and as the point is not contained, there must be an index $i \in \{1, \dots, N\}$ such that

$$|x_i - y_i| > \rho_{n+1}^{\frac{\bar{p}}{p_i}}(\omega_{n+1}/C_1)^{\frac{p_i-\bar{p}}{p_i}} \ge \gamma(\delta, A)\Big(\delta^n\Big)^{\frac{\bar{p}(\beta-1)+p_i}{p_i}} R^{\frac{\bar{p}}{p_i}}(\delta^n\omega_o/C_1)^{\frac{p_i-\bar{p}}{p_i}}, \tag{8.9}$$

that gives us for $\chi_i = \bar{p}/(\bar{p}(\beta-1)+p_i)$ the following estimate of δ^n,

$$\delta^n \le \gamma\Big(\frac{|x_i-y_i|^{\frac{p_i}{\bar{p}}}(\omega_o/C_1)^{\frac{\bar{p}-p_i}{\bar{p}}}}{R}\Big)^{\frac{\bar{p}}{\bar{p}(\beta-1)+p_i}} \le \gamma\Big(\frac{\sum_{i=1}^N |x_i-y_i|^{\frac{p_i}{\bar{p}}}\omega_o^{\frac{\bar{p}-p_i}{\bar{p}}} + |t-s|^{\frac{1}{\bar{p}}}\omega_o^{\frac{\bar{p}-2}{\bar{p}}}}{\mathbf{p}\text{-dist}(K,\partial\Lambda)}\Big)^{\chi_i}. \tag{8.10}$$

Now we can repeat the same reasoning in case that $x \in \mathcal{K}_{\rho_{n+1}}$ but $t \notin (-\rho_{n+1}^{\bar{p}}(\omega_{n+1})^{2-\bar{p}}, 0]$, getting an estimate of δ^n by condition

$$|t-s| > \rho_{n+1}^{\bar{p}}(\omega_{n+1/C_1})^{2-\bar{p}} > \gamma(\delta,A)(\delta^n)^{\bar{p}(\beta-1)+2}R^{\bar{p}}(\omega_o/C_1)^{2-\bar{p}}$$

$$\Rightarrow \quad \delta^n < \gamma\Big(\frac{|t-s|^{\frac{1}{\bar{p}}}(\omega_o/C_1)^{\frac{\bar{p}-2}{\bar{p}}}}{R}\Big)^{\frac{\bar{p}}{(\bar{p}(\beta-1)+2)}} \le \gamma\Big(\frac{\sum_{i=1}^N |x_i-y_i|^{\frac{p_i}{\bar{p}}}\omega_o^{\frac{\bar{p}-p_i}{\bar{p}}} + |t-s|^{\frac{1}{\bar{p}}}\omega_o^{\frac{\bar{p}-2}{\bar{p}}}}{\mathbf{p}\text{-dist}(K,\partial\Lambda)}\Big)^{\chi_t},$$

this time with $\chi_t = \bar{p}/(\bar{p}(\beta-1)+2)$. Finally we put all estimates together to get

$$|u(x,t)-u(y,s)| \le \operatorname*{osc}_{\mathcal{Q}_n} u \le \delta^n\omega_o \le \gamma\omega_o\Big(\frac{\sum_{i=1}^N |x_i-y_i|^{\frac{p_i}{\bar{p}}}\omega_o^{\frac{\bar{p}-p_i}{\bar{p}}} + |t-s|^{\frac{1}{\bar{p}}}\omega_o^{\frac{\bar{p}-2}{\bar{p}}}}{\mathbf{p}\text{-dist}(K,\partial\Lambda)}\Big)^{\chi},$$

with the following choice of $\chi \in (0,1)$, motivated again by (8.5), and therefore we choose as Hölder constant

$$\chi = \min\{\chi_i, \chi_t, \quad i = 1, \dots, N\} = \frac{\bar{p}}{\bar{p}(\beta-1)+p_N}, \quad \text{being} \quad p_N > 2. \tag{8.11}$$

□

8.2 Liouville-type results

Harmonic functions in $\mathbb{R}^N$ with one-sided bound are constant. This fact, known as Liouville's property, is a sole consequence of the Harnack inequality and as such it extends to solutions to homogeneous, quasi-linear, elliptic partial differential equations in $\mathbb{R}^N$ with one-sided bound. This statement has a correspondent one in case of elliptic p-Laplacean equations, and also in case of the elliptic anisotropic $\mathbf{p}$-Laplacean one.

Theorem 8.2. *Let u be a nonnegative weak solution in the whole $\mathbb{R}^N$ to the equation* (7.18)*. If u is bounded from below and condition* (5.1) *is satisfied, then it is constant.*

Proof. We suppose that $\sup_{\mathbb{R}^N} u > \inf_{\mathbb{R}^N} u$, admitting $\sup_{\mathbb{R}^N} u \in \bar{\mathbb{R}}$, and let $\epsilon \in (0, \sup_{\mathbb{R}^N} u - \inf_{\mathbb{R}^N} u)$. Consider the non-negative solution $v_\epsilon = u - \inf_{\mathbb{R}^N} u + \epsilon/2$ to (7.18). By continuity, we can pick a point x_ϵ such that $v_\epsilon(x_\epsilon) = \epsilon$. Up to translations, the elliptic Harnack inequality (7.19) implies that $v_\epsilon \le C_3\, \epsilon$ in $x_\epsilon + \mathcal{K}_\rho(\epsilon/C_1)$, for all $\rho > 0$. Letting $\rho \to +\infty$, we get $v_\epsilon \le C_3\, \epsilon$ in the whole $\mathbb{R}^N$, i.e.

$$u \le \inf_{\mathbb{R}^N} u + (C_3 - 1/2)\, \epsilon$$

in $\mathbb{R}^N$ and letting $\epsilon \to 0$ we get the claim. □

On the other hand the parabolic theory is much different, because of the general principle that "diffusion needs some time to pass". As a matter of fact, this property does not extend to the heat equation in $\mathbb{R}^N \times \mathbb{R}$: indeed the function $0 \le u(x,t) = e^{x+t}$ is caloric and has a left-sided bound while not being constant. A one sided-bound is therefore not anymore sufficient to imply that solutions are constant. The Liouville's property continues to be false, stated as it is, also for non-negative solutions to degenerate p-Laplacean equations ($p > 2$). Indeed, the one-parameter family of non-negative functions

$$\mathbb{R} \times \mathbb{R} \ni (x,t) \to u(x,t;c) = c^{\frac{1}{p-2}} \left(\frac{p-2}{p-1} \right)^{\frac{p-1}{p-2}} (1 - x + ct)_+^{\frac{p-1}{p-2}},$$

is a family of non-negative, non-constant weak solutions to $u_t = \Delta_p u$ in $\mathbb{R}^2$. This naturally provides a counterexample also in case of equation (1.1) in one spatial dimension. On the other hand, if a left-sided bound is coupled with a right-sided bound at some time level, it is possible to prove the following result.

Theorem 8.3. *Let $T \in \mathbb{R}$, $S_T = \mathbb{R}^N \times (-\infty, T)$ and u be a solution to* (1.1) *with condition* (5.1)*, which is bounded below in S_T. Assume moreover that for some time $s < T$*

$$\sup_{\mathbb{R}^N} u(\cdot, s) = M_s < \infty. \tag{8.12}$$

Then u is constant in $S_s = \mathbb{R}^N \times (-\infty, s)$.

Corollary 8.1. *Let $T \in \mathbb{R}$, $S_T = \mathbb{R}^N \times (-\infty, T)$ and u be a solution to* (1.1) *with condition* (5.1)*, which is bounded above and below in S_T. Then, u is constant.*

Proof. For u is bounded below in S_T, let us set

$$m = \inf_{S_T} u,$$

and for points $(y,s) \in S_T$ for which $u(y,s) > m$ let us construct the intrinsic backward $\mathbf{p}$-paraboloid

$$\mathcal{P}_m^-(y,s) := \mathcal{P}_{\frac{u(y,s)-m}{C_1}}(y,s) = \left\{ (x,t) \in S_T | \quad \forall i = 1,..N, \quad (t-s) \le -C_2^{\bar{p}} (2|x_i - y_i|)^{p_i} \left(\frac{u(y,s) - m}{C_1} \right)^{2-p_i} \right\},$$

where the constants C_1, C_2 are the constants of the intrinsic Harnack inequality (7.1). First of all we prove the following fact which is interesting per se: for all $x \in \mathbb{R}^N$ fixed,

$$\lim_{t \to -\infty} u(x,t) = \inf_{S_T} u. \tag{8.13}$$

To prove (8.13) we unfold the definition of infimum: for every chosen $\varepsilon > 0$ there exists a point $(y_\varepsilon, s_\varepsilon) \in S_T$ such that $u(y_\varepsilon, s_\varepsilon) - m \le \varepsilon/C_3$, being C_3 the constant appearing in (7.1). Now we apply left-hand side of inequality (7.1) to the solution $(u - m)$, to obtain

$$m \le u(y,s) \le m + \varepsilon, \quad \text{for all} \quad (y,s) \in P_m^-(y_\varepsilon, s_\varepsilon).$$

Now, we consider that the intrinsic **p**-paraboloid $\mathcal{P}_m^-(y_\varepsilon, s_\varepsilon)$ expands until it spreads all over the space when time diverges to $-\infty$. For a fixed point $x \in \mathbb{R}^N$, the half line $[t < T] \times \{x\}$ intersects the intrinsic **p**-paraboloid for some first time $t_{\varepsilon,x}$. To sum up, for each chosen $x \in \mathbb{R}^N$ and $\varepsilon > 0$ there exists a time $t_{\varepsilon,x} \in \mathbb{R}$ such that $\forall s < t_{\varepsilon,x}$ the function $u(x,s)$ gets arbitrarily close to m.

A similar argument shows that

$$\sup_{S_T} u = M < \infty \quad \Rightarrow \quad \lim_{t \to -\infty} u = M.$$

This implies that if u is a non-negative solution to (1.1) bounded from both above and below in the whole S_T, it is necessarily constant. Indeed, by (8.13) we would have $\sup_{S_T} u = \inf_{S_T} u$.

To end the proof of Theorem 8.3, we use the assumption that there exists a time $\bar{s} \in (-\infty, T]$ such that the function of the sole space $u(\cdot, \bar{s})$ is bounded from above by $M_s \in \mathbb{R}$ in the whole $\mathbb{R}^N$. Indeed in this case for each $x \in \mathbb{R}^N$ we achieve again, by the intrinsic backward Harnack inequality, the uniform bound

$$u(y,s) \le C_3 u(x,\bar{s}) \le C_3 M_{\bar{s}}, \quad \text{for all} \quad (y,s) \in \mathcal{P}_{u(x,\bar{s})/C_1}(x,\bar{s}),$$

and these intrinsic paraboloids $\mathcal{P}^-_{u(x,\bar{s})/C_1}(x,\bar{s})$ invade all the strip $S_{\bar{s}}$ when x ranges all over $\mathbb{R}^N$. By previous considerations, being u bounded from both above and below, it is necessarily constant in $S_{\bar{s}}$. □

Moreover, when the equation is solved in $\mathbb{R}^N \times \mathbb{R}$, it suffices to check the asymptotic (in time) two-sided boundedness of the solution at a single point $y \in \mathbb{R}^N$ to conclude its constancy.

Theorem 8.4. *Let u be a local weak solution to* (1.1) *in $\mathbb{R}^N \times \mathbb{R}$, bounded from below. Let condition* (5.1) *be satisfied. If in addition there exists $y \in \mathbb{R}^N$ and a sequence of points $\mathbb{N} \ni \{s_n\}_{n\in\mathbb{N}} \to +\infty$ such that $u(y, s_n)$ is bounded for large $n \in \mathbb{N}$, then u is constant.*

Remark 8.1. *It is an easy application to see that if we replace the previous assumption with the request that for some point $y \in \mathbb{R}^N$ the following limit is finite*

$$\liminf_{t \to +\infty} u(y,t) = \alpha \in \mathbb{R}, \tag{8.14}$$

then u is constant.

Proof. We consider the solution $\tilde{u} = u + 1$ of equation (1.1). By assumption there exists some divergent sequence in time $\{s_n\}_{n\in\mathbb{N}} \to \infty$ and a number $\bar{n} \in \mathbb{N}$ such that

$$\tilde{u}(y, s_n) < M_{\bar{n}}, \qquad \forall n \ge \bar{n}.$$

Let us fix $\bar{s} = s_{\bar{n}}$, and consider the following sequence of radii $\{\rho_n\}_{n\in\mathbb{N}}$, defined by being equidistant to $\bar{s}$ in the following sense

$$s_n - \left(\frac{C_1}{\tilde{u}(y,s_n)}\right)^{\bar{p}-2} (C_2 \rho_n)^{\bar{p}} = \bar{s} \qquad \Rightarrow \qquad \rho_n = \frac{1}{C_2}\left[(s_n - \bar{s})\left(\frac{\tilde{u}(y,s_n)}{C_1}\right)^{\bar{p}-2}\right]^{1/\bar{p}}.$$

As our aim is to apply Harnack inequality and invade all the space, we need to check that the intrinsic anisotropic cubes $\mathcal{K}_{\rho_n}(\tilde{u}(y,s_n)/C_1)$ expand. Indeed by explicit computation

$$\mathcal{K}_{\rho_n}(\tilde{u}(y,s_n)/C_1) = \prod_{i=1}^{N} \left\{ \left(\frac{\tilde{u}(y,s_n)}{C_1}\right)^{\frac{p_i-2}{p_i}} \frac{1}{C_2^{\bar{p}/p_i}} (s_n - \bar{s})^{\frac{1}{p_i}} \right\} \quad \Rightarrow_{n\to\infty} \quad \mathbb{R}^N,$$

being $\tilde u > 1$, $p_i > 2$ and $\{s_n\}$ divergent. Now by the intrinsic backward Harnack inequality (7.1) we have

$$\sup_{\mathcal{K}_{\rho_n}} \tilde u\bigg(\ \cdot\ , s_n - \Big(\frac{C_1}{\tilde u(y,s_n)}\Big)^{\bar p-2}\rho_n^{\bar p}\bigg) \le \gamma\, \tilde u(y,s_n) \le \gamma M_{\bar n},$$

which gives for all $n \ge \bar n$, by exploiting the definition of $\{\rho_n\}$, the uniform estimate

$$\sup_{\mathcal{K}_{\rho_n}} \tilde u(\cdot,\bar s) \le \gamma M_{\bar n} \qquad \Rightarrow^{\bar n \text{ fixed}}_{n\to\infty} \qquad \sup_{\mathbb{R}^N} \tilde u(\cdot,\bar s) \le \gamma\, M_{\bar n}.$$

This is precisely the hypothesis of Theorem 8.3, and under its application the proof is concluded. □

8.3 An alternative formulation of Harnack inequality

In this small Section we show how it is possible to free the Harnack inequality from its intrinsic geometry in time. More precisely, we can give a formulation where the function can be calculated at any chosen time level, independently of the anisotropic geometry but always provided that there is enough room for the anisotropic evolution inside Ω_T. Unlike the isotropic case, it looks here more difficult to get rid of the intrinsic geometry along space variables.

Theorem 8.5. *Let u be a nonnegative local weak solution to (1.1) in Ω_T, and suppose (5.1) valid. Then, there exist constants $\tilde\eta > 0$, $\gamma > 1$ depending only on N, p_is such that for all points $(x_o,t_o) \in \Omega_T$ and all $\rho,\theta > 0$ such that*

$$(x_o, t_o + \theta) + \mathcal{Q}_{C_3\rho}(u(x_o,t_o)/C_1) \subseteq \Omega_T,$$

we have

$$u(x_o,t_o) \le \gamma\bigg\{ \Big(\frac{\rho^{\bar p}}{\theta}\Big)^{\frac{1}{\bar p-2}} + \Big(\frac{\theta}{\rho^{\bar p}}\Big)^{N/\bar p}\bigg[\inf_{x_o+\mathcal{K}_{\tilde\eta\rho}(\tilde\eta u(x_o,t_o))} u(\cdot,\, t_o+\theta)\bigg]^{\lambda/\bar p}\bigg\}, \tag{8.15}$$

where $\lambda = N(\bar p-2)+\bar p$ and $C_1, C_3 > 1$ are the constants of Theorem 7.1 while $\eta > 0$ is the constant of Corollary 6.1.

Proof. Let $\rho,\theta > 0$ be two numbers such that the anisotropic intrinsic cylinder $(x_o,t_o) + \mathcal{Q}_\rho(\theta)$ is contained in Ω_T, let C_i be the constants of Theorem 7.1 for $i = 1,2,3$ and $\bar\eta$ the number specified in the third point of Corollary 6.1. Now we consider the time level

$$t^* := \Big(\frac{C_1}{u(x_o,t_o)}\Big)^{\bar p-2}(C_2\rho)^{\bar p}, \quad \text{such that} \quad t^* \le \theta/2, \tag{8.16}$$

indeed otherwise $u(x_o,t_o) \le \gamma(\rho^{\bar p}/\theta)^{1/(\bar p-2)}$ for $\gamma = \gamma(C_1,C_2,\bar\eta)$ and (8.15) is valid. We observe that inequality (8.16) above with the hypothesis on the domain implies also

$$\Big(\frac{C_1}{u(x_o,t_o)}\Big)^{\bar p-2}(C_2\rho)^{\bar p} < \theta/2 \le T, \quad \text{and} \quad x_o + \mathcal{K}_{C_3\rho}(u(x_o,t_o)/C_1) \subset \Omega.$$

By the Harnack inequality (7.1) and (8.16) we have a precise bound from above at the point (x_o,t_o), so that

$$u(x_o,t_o) \le C_3 u(x,\, t_o+t^*), \qquad \forall x \in \mathcal{K}_\rho(u(x_o,t_o)/C_1).$$

This initial value can be taken for a comparison principle with a Barenblatt solution $\mathcal{B}_\sigma(x-x_o, t-s)$ centered in (x_o,s), where $s \in (-\theta,0)$ and $\sigma > 0$ must be chosen to let $B_\sigma(x-x_o,t-s)$ be below u in $x_o + \mathcal{K}_\rho(u(x_o,t_o)/C_1)$. This request is written as

$$\begin{cases} \operatorname{supp} B_\sigma(\cdot, t_o+t^*-s) \subseteq x_o + \mathcal{K}_\rho(u(x_o,t_o)/C_1), \\ ||B_\sigma(\cdot,t_o+t^*-s)||_\infty \le u(x_o,t_o)/C_3, \end{cases} \quad \Leftarrow \begin{cases} \sigma^{\frac{p_i-2}{p_i}}(t_o+t^*-s)^{\alpha_i} \le \rho^{\bar p/p_i}(u(x_o,t_o)/C_1)^{\frac{p_i-\bar p}{p_i}} \\ \sigma(t_o+t^*-s)^{-\alpha} \le u(x_o,t_o)/C_3. \end{cases} \tag{8.17}$$

This can be done by choosing

$$\sigma = |t_o + t^* - s|^{N/\lambda} u(x_o,t_o)/C_3, \quad \text{and} \quad s = t_o + t^* - \Big(\frac{\rho^{\bar p}}{u(x_o,t_o)^{\bar p -2}}\Big)\gamma_1, \qquad \gamma_1 = \min\{C_3^{p_i-2} \,|\, i=1,\dots,N\}.$$

Therefore, the comparison principle implies that at the time $t_o+\theta$ we have

$$\begin{aligned} u(x, t_o+\theta) &\geq \bar\eta \sigma |t_o + t^* - (t_o+\theta)|^{-\alpha} \geq \bar\eta \Big(\frac{u(x_o,t_o)}{C_3}\Big) |t_o+t^* - s|^{N/\lambda} |\theta - t^*|^{-N/\lambda} \\ &\geq \bar\eta \Big(\frac{u(x_o,t_o)}{C_3}\Big)\Big(\frac{\gamma_1 \rho^{\bar p}}{u(x_o,t_o)^{\bar p-2}}\Big)^{N/\lambda}\Big(\frac{\theta}{2}\Big)^{-N/\lambda} \geq \gamma u(x_o,t_o)^{\bar p/\lambda}\Big(\frac{\rho^{\bar p}}{\theta}\Big)^{N/\lambda}, \end{aligned}$$

recalling that $\alpha = N/\lambda$ and being $\gamma = \gamma(\gamma_1, \bar\eta)$, for every x in the set of positivity of $\mathcal{B}_\sigma$

$$P_{t_o+\theta}(x_o) = \prod_{i=1}^N \{|x_i - x_{o,i}| < \tilde\eta \rho^{\bar p/p_i} u(x_o,t_o)^{(p_i-\bar p)/p_i}\} = x_o + \mathcal{K}_{\tilde\eta\rho}(\tilde\eta u(x_o,t_o)),$$

with $\tilde\eta = \tilde\eta(\bar\eta, C_1, C_2, C_3) = \min\{\bar\eta C_3^{(2-p_i)/p_i} C_2^{\alpha_i} C_1^{(\bar p-2)\alpha_i} \,|\, i=1,\dots,N\}$. This implies, by passing to the minimum of u on such a set, the right-hand member of (8.15). □

Remark 8.2. *In Theorem 8.5 it is not required that $u(x_o,t_o) > 0$ and $\theta > 0$ is arbitrary, between those numbers that preserve the inclusion of the intrinsic cylinder translated to time θ into Ω_T. Actually, Theorems 7.1 and 8.5 are equivalent locally.*

Indeed, we proved that Theorem 7.1 implies 8.5. Now we show that the converse statement can be obtained by a simple choice of θ. Indeed, let us pick

$$\theta = \frac{(2\gamma)^{\bar p-2}\rho^{\bar p}}{u(x_o,t_o)^{\bar p-2}},$$

and suppose that $(x_o, t_o+\theta) + \mathcal{Q}_{C_3\rho}(u(x_o,t_o)/C_1) \subset \Omega_T$. The weak Harnack inequality (8.15) leads us to

$$u(x_o,t_o) \leq \gamma \Big\{ \frac{u(x_o,t_o)}{2\gamma} + \Big(\frac{2\gamma}{u(x_o,t_o)}\Big)^{\frac{N(\bar p-2}{\bar p}} \Big[\inf_{x_o+\mathcal{K}_{\tilde\eta\rho}(\tilde\eta u(x_o,t_o))} u(\cdot,\, t_o + \Big(\frac{u(x_o,t_o)}{2\gamma}\Big)^{2-\bar p}\rho^{\bar p}\Big]^{\frac{\lambda}{\bar p}}\Big\}$$

$$\Rightarrow \quad u(x_o,t_o) \leq \tilde C_3 \inf_{x_o+\mathcal{K}_{\tilde\rho}(M)} u(\cdot,\, t_o + \tilde C_2 M^{2-\bar p}\tilde\rho^{\bar p}), \quad M = u(x_o,t_o)/\tilde C_1,$$

for all $\tilde\rho \leq \tilde\eta\rho$ and positive constants

$$\tilde{C_3} = \frac{(2\gamma)^{\frac{N(\bar p-2)}{\bar p}}}{2}, \qquad \tilde{C_1} = 1/\tilde\eta, \qquad \tilde{C_2} = \frac{(2\gamma)^{\bar p-2}}{\tilde\eta^2}.$$

8.4 Future Perspectives and Open Problems

In the isotropic case, Theorem 8.5 can be stated with a weighted space integral on the left. Indeed, considering the isotropic version of (8.15) by taking $p_i \equiv p > 2\, \forall i = 1,\dots,N$, then for at least one point $\bar x \in K_\rho$ we have

$$\fint_{\mathcal{K}_\rho(u(x_o,t_o)/C_1)} u(x,t_o)\,dx = u(\bar x, t_o) \leq \Big\{\Big(\frac{\rho^{\bar p}}{\theta}\Big)^{\frac{1}{\bar p-2}} + \Big(\frac{\theta}{\rho^{\bar p}}\Big)^{N/\bar p}\Big[\inf_{\bar x+\mathcal{K}_{\tilde\eta\rho}(\tilde\eta u(\bar x,t_o))} u(\cdot,\, t_o+\theta)\Big]^{\lambda/\bar p}\Big\},$$

that, in the isotropic geometry would have $\tilde\eta = 1$ and would not be anymore intrinsic in space, because $x_o \in \mathcal{K}_\rho(u(x_o,t_o)/C_1) = K_\rho$. That is why one can estimate from above the right-hand side with the value $u(x_o\,,\, t_o+\theta)$. Here the situation is markedly different, because the inequality (8.15) is still intrinsic in space. Therefore a natural question would be to find an estimate as in ([23], Cor. 2.1, Chap VI), that would be of the following type.

Open Problem 8.6. *Let the assumptions of Theorem 8.5 be satisfied. Then there exist constants* $\tilde{C}_1, \gamma > 1$ *depending only on* N*,* p_i*s such that for all points* $(x_o, t_o) \in \Omega_T$ *and all* $\rho, \theta > 0$ *such that*

$$(x_o, t_o + \theta) + \mathcal{Q}_{C_3\rho}(u(x_o, t_o)/C_1) \subseteq \Omega_T,$$

defining $M = u(x_o, t_o)/\tilde{C}_1$ *we have*

$$\fint_{\mathcal{K}_\rho(M)} u(x, t_o)\, dx \leq \gamma \bigg\{ \bigg(\frac{\rho^{\bar{p}}}{\theta} \bigg)^{\frac{1}{\bar{p}-2}} + \bigg(\frac{\theta}{\rho^{\bar{p}}} \bigg)^{N/\bar{p}} \Big[u(x_o,\, t_o + \theta) \Big]^{\lambda/\bar{p}} \bigg\}, \tag{8.18}$$

where $\lambda = N(\bar{p} - 2) + \bar{p}$ *and* $C_3 > 1$ *is the number of Theorem 7.1.*

Furthermore, the estimate (8.15) is usually referred to as a weak Harnack inequality, when it is stated for supersolutions. This is another interesting question, that will be the object of future investigation.

Open Problem 8.7. *Let* u *be a nonnegative local weak supersolution to* (1.1) *in* Ω_T *and let assumption* (5.1) *be at stake. Then, there exist constants* $\tilde{\eta} > 0$*,* $\gamma > 1$ *depending only on* N*,* p_i*s such that for all points* $(x_o, t_o) \in \Omega_T$ *and all* $\rho, \theta > 0$ *such that*

$$(x_o, t_o + \theta) + \mathcal{Q}_{C_3\rho}(u(x_o, t_o)/C_1) \subseteq \Omega_T,$$

we have

$$\fint_{\mathcal{K}_\rho(M)} u(x, t_o)\, dx \leq \gamma \bigg\{ \bigg(\frac{\rho^{\bar{p}}}{\theta} \bigg)^{\frac{1}{\bar{p}-2}} + \bigg(\frac{\theta}{\rho^{\bar{p}}} \bigg)^{N/\bar{p}} \Big[\inf_{x_o + \mathcal{K}_{\tilde{\eta}\rho}(\tilde{\eta} u(x_o, t_o))} u(\cdot,\, t_o + \theta) \Big]^{\lambda/\bar{p}} \bigg\}, \tag{8.19}$$

where $\lambda = N(\bar{p} - 2) + \bar{p}$ *and* $C_1, C_3 > 1$ *are the constants of Theorem 7.1 and* $\eta > 0$ *of Corollary 6.1.*

In the present work we chose to prove the results and introduce the theory for solutions to (1.1), while it is a classically known fact that many of the results shown are still valid if we replace *solution* by *super* or *sub-solution*. For example, to the aim of expanding positivity, it would be sufficient to have at hand a subsolution $\mathcal{B}^-$ whose support evolves as the one of $\mathcal{B}$. Moreover, an intrinsic Harnack inequality as (7.1) for the singular range $2N/(N+1) < p_i < 2$ is to be expected, with some suitable additional condition on the sparseness of the p_is. Reasoning again by comparison, the existence of $\mathcal{B}$ may be the consequence of extinction in finite time, instead of finite speed of propagation (see [34] for another interesting approach).

Finally, a much more difficult task (but for the same reason far more intriguing) would be the identification of a suitable class of functions $\mathcal{A}$ that enhances the properties that we already know for local weak solutions to (1.1). The class $\mathcal{A}$ may be defined with assumption similar to the ones in [27], [30]; according to the general fact explained above that the equation roughly speaking evolves both in a singular and degenerate way. Hence it is reasonable to suspect that it may include both the energy estimates (3.13) and some kind of logarithmic estimates. This question does not have an answer yet for isotropic parabolic p-Laplacean equations. We refer to [36] for a partial result and more references; there, a class of functions $\mathcal{A}$ is identified just by means of the sole energy estimates, but for a very special homogeneous case.

Last problem introduced above may look very general. It can be clarified, if we think about the classical isotropic elliptic p-Laplacean equations with measurable and bounded coefficients (see for instance [24], Chap. 10). More in detail, it is possible to show that if a function u belongs to a certain space of functions DG_p, called De Giorgi classes, then it automatically has the desired properties of Hölder continuity, Harnack inequality, et cetera. It would be therefore already a simpler but excellent starting point to identify a suitable class of functions $\mathcal{G}$ that satisfies the elliptic version of previous general parabolic problem.

Open Problem 8.8. *Let and* $\mathcal{G}$ *be the class of functions* $u \in W^{1,\mathbf{p}}_{loc}(K_1) \cap L^\infty(K_1)$ *such that for each function of the form*

$$C_o^\infty(K_1) \ni \eta = \prod_{i=1}^N \eta_i^{p_i}(x_i) \quad \textit{for} \quad \eta_i \in C_o^\infty(\pi_i(K_1)), \tag{8.20}$$

where as usual π_i *is the projection to the* i*-th coordinate, satisfy the inequality*

$$\sum_{i=1}^N \int_{K_1} |\partial_i(\eta(u-k)_\pm)|^{p_i}\, dx \leq \gamma \sum_{i=1}^N \int_{K_1} |(u-k)_\pm|^{p_i} \hat{\eta}_i |\partial_i \eta_i|^{p_i}\, dx, \quad \textit{being} \quad \hat{\eta}_i = \frac{\eta}{\eta_i^{p_i}}. \tag{8.21}$$

Then, $u \in \mathcal{G}$ *is locally Hölder continuous and satisfies an analogous version of* (7.19)*.*

9 Appendix and References

Technicalities and known facts

You'd better believe it.

- Baloo -

The Jungle Book

9.1 Some algebraic and iteration Lemmata

Lemma 9.1 ([23] Chap. I, Lemma 4.4)**.** *Let $p \geq 2$. Then for each $a, b \in \mathbb{R}^N$, there exists a constant γ_o depending only on p, N such that*

$$\langle |a|^{p-2}a - |b|^{p-2}b,\, a-b\rangle \geq \gamma_o |a-b|^p. \tag{9.1}$$

Lemma 9.2 ([23] Chap. I, Sec. IV)**.** *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of equibounded positive numbers which satisfies the recursive inequality*

$$Y_n \leq C b^n Y_{n+1}^{1-\alpha}, \qquad \alpha \in (0,1), \quad C,\, b > 1. \tag{9.2}$$

Then

$$Y_0 \leq \left(\frac{2C}{b^{1-1/\alpha}}\right)^{\frac{1}{\alpha}}. \tag{9.3}$$

Remark 9.1. *If we just have a sequence of equibounded numbers $\{Y_n\}$ such that*

$$Y_n \leq \epsilon Y_{n+1} + C b^n, \quad C, b > 1, \quad \epsilon \in (0,1), \tag{9.4}$$

then by a simple iteration, setting $(\epsilon b) = 1/2$ and letting $n \to \infty$ gives (9.3) *with $\alpha = 1$.*

The following is a simple generalization of ([23], Lemma 4.1 Chap. I), useful for the anisotropic growth.

Lemma 9.3. *Let $\beta_i > 0$ for $i = 1, \dots, N$ and suppose $Z_n > 0$ satisfies*

$$Z_{n+1} \leq C\, B^n \,\frac{1}{N}\sum_{i=1}^N Z_n^{1+\beta_i} \tag{9.5}$$

for some $C > 0$ and $B > 1$. Then letting $\beta = \min\{\beta_1, \dots, \beta_N\}$, the following logical implications hold

$$Z_0 \leq \frac{1}{C^{\frac{1}{\beta}}\, B^{\frac{1}{\beta^2}}} \quad \textit{and} \quad C B^{\frac{1}{\beta}} \geq 1 \quad \Rightarrow \quad Z_n \leq B^{-\frac{n}{\beta}} Z_0 \quad \Rightarrow \quad \lim_{n\uparrow\infty} Z_n = 0.$$

Proof. That the limit $\{Z_n\} \to_{n\to\infty} 0$ follows from the first implication is trivial. We show by induction the first implication.
For $n = 0$ it is trivially satisfied, so we suppose the validity of the formula for the n-th step and we verify it for the $(n+1)$-th one. Let us apply (9.5) to evaluate

$$Z_{n+1} \leq \frac{CB^n}{N}\sum_{i=1}^N Z_n^{1+\beta_i} \leq \frac{CB^n}{N}\sum_{i=1}^N \left(B^{-\frac{n}{\beta}} Z_0\right)^{1+\beta_i} \leq \sum_{i=1}^N \left(\frac{CB^{\frac{n(\beta-\beta_i)+1}{\beta}} Z_0^{\beta_i}}{N}\right) B^{-\frac{(n+1)}{\beta}} Z_0.$$

Last term in parenthesis is smaller than one if

$$CB^{\frac{n(\beta-\beta_i)+1}{\beta}} Z_0^{\beta_i} \le CB^{\frac{1}{\beta}} Z_0^{\beta_i} \le 1 \qquad \Longleftrightarrow \qquad Z_o^{\beta_i} \le (CB^{\frac{1}{\beta}})^{-1}.$$

As $Z_o \le 1$ by assumptions, our choice of β determines the condition

$$Z_o^{\beta_i} \le Z_o^{\beta} \le (CB^{\frac{1}{\beta}})^{-1}, \qquad \forall i = 1, \dots, N.$$

□

Remark 9.2. *Condition $CB^{1/\beta} \ge 1$ is always satisfied when $C \ge 1$, as in ([33], Lemma 2.7). The proof for the isotropic version of this simple fact, which is when $\beta_i \equiv 1$, can be found in ([45], Lemma 4.7 page 66).*

9.2 Useful inequalities

In this subsection we suppose for ease of notation that $E \subseteq \mathbb{R}^N$ is an open set. It is well known that these inequalities can be given a far more general fashion, that nonetheless is beyond the aim of the present work.

Proposition 9.1 (Young's Inequality)**.** *Let a, b, ϵ be positive numbers, and $1 < p < \infty$, $p' = 1/(1-1/p)$ its Hölder's conjugate. Then*

$$ab \le \epsilon a^p + C(\epsilon) b^{p/(p-1)}, \qquad \textit{with} \quad C(\epsilon) = \left(\frac{p-1}{p^{1/(p-1)}p}\right)\left(\frac{1}{\epsilon}\right)^{\frac{1}{p-1}}. \tag{9.6}$$

Proposition 9.2 (Generalised Hölder's Inequality)**.** *Let $m \in \mathbb{N}$ and $p_1, \dots, p_m, r > 0$ be real numbers such that $\sum_{i=1}^m 1/p_i = 1/r$. Let $f_i \in L^{p_i}(E)$ for $i = 1, \dots, m$. Then $\prod_{i=1}^m f_i \in L^r(E)$ and*

$$\|\prod_{i=1}^m |f_i|\|_{L^r(E)} \le \prod_{i=1}^m ||f_i||_{L^{p_i}(E)}. \tag{9.7}$$

Next proposition is a classic interpolation result, that heuristically states that if $f \in L^p \cap L^r$ for $0 < p < r$, then $f \in L^q$ for every $p < q < r$.

Proposition 9.3 (Interpolation)**.** *Let us suppose that $0 < p < q < r \le \infty$. Then $L^p(E) \cap L^r(E) \subset L^q(E)$ and*

$$\|f\|_{L^q(E)} \le \|f\|^{\lambda}_{L^p(E)} \|f\|^{1-\lambda}_{L^r(E)}, \tag{9.8}$$

where $\lambda \in (0, 1)$ is to be determined so that

$$\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}, \qquad \textit{and} \quad \lambda = p/q \quad \textit{when} \quad r = \infty.$$

9.3 Directional Poincaré's Inequality and Absolute Continuity along lines.

Along this subsection we recall some classic properties of functions $u \in W_o^{1,\mathbf{p}}(\Omega)$, which is, that are limits of sequences in $C_o^\infty(\Omega)$ under the norm $\|u\|_{W_o^{1,\mathbf{p}}(\Omega)} = \sum_{i=1}^N ||\partial_i u||_{L^{p_i}(\Omega)}$. We will focus on the possible inequalities that can be derived by the different summability on the weak partial derivatives. First we recall the following directional Poincaré-type inequality.

Proposition 9.4 ([35], Theorem 1)**.** *Let $\{e_i\}_{i=1,..,N}$ be a basis of $\mathbb{R}^N$ and $\Omega \subset\subset \mathbb{R}^N$ a bounded domain with Lipschitz boundary such that is contained in a slab, i.e. $\sup_{x,y\in\Omega}\langle x-y, e_j\rangle =: d_j < \infty$ for some $j \in \{1, \dots, N\}$. Then a function $u \in W_o^{1,\mathbf{p}}(\Omega)$ satisfies the inequality*

$$\|u\|_{L^{p_j}(\Omega)} \le \frac{p_j d_j}{2}\|\partial_j u\|_{L^{p_j}(\Omega)}. \tag{9.9}$$

Next we consider the following result of absolute continuity on lines of functions which have merely summable weak derivatives. We state it for the preferred direction e_1 but it is clear that every direction e_i, $i \in \{1, \dots, N\}$ can be considered.

Proposition 9.5 ([13], Theorem 8.27)**.** *Let $u \in W^{1,1}(E)$, where $E \subset \mathbb{R}^N$ is an open set having the form*

$$E = \left\{ x \in \mathbb{R}^N \,|\, \alpha(x_2, \dots, x_N) < x_1 < \beta(x_2, \dots, x_N) \right\}, \quad \textit{being possibly} \quad \alpha = -\infty,\, \beta = \infty. \tag{9.10}$$

Then there exists a function $\tilde{u}$ coinciding almost everywhere on E with u and such that the map

$$x_1 \mapsto \tilde{u}(x_1, x_2, \dots, x_N)$$

is absolutely continuous for $\mathcal{L}^{N-1}$-a.e. $(x_2, \dots, x_N) \in \mathbb{R}^{N-1}$ such that $(x_1, x_2, \dots, x_N) \in E$. Moreover, the derivative $\partial_{x_1}\tilde{u}$ coincides almost everywhere with the weak derivative $\partial_1 u$.

9.4 Steklov-averaged solutions.

Let $u \in L^1(\Omega_T)$ and for $0 < h < T$ let us define for all $0 < t < T$ the Steklov averages $v_h(\cdot, t)$ by

$$v_h(\cdot,\, t) \equiv \begin{cases} h^{-1} \int_t^{t+h} v(\cdot, \tau)\, d\tau, & t \in (0,\, T-h], \\ 0, & t > T-h. \end{cases} \tag{9.11}$$

We present the following properties of Steklov averages, that are useful for our purposes.

Proposition 9.6. *Let $v \in L^1(\Omega_T)$. Then*

(a) If $u \in L^r(0,T;L^p(\Omega))$ for some $r, p > 1$ then $u_h \in L^p(\Omega)$ for every $t \in [0,T)$ and
  - *(a1) it holds $||u_h(\cdot,\, t)||_{L^p(\Omega)} \leq h^{-1/r}||u(\cdot,\, t)||_{L^p(\Omega)}$,*
  - *(a2) $u_h \in C([0,T);L^p(\Omega)) \cap L^\infty(0,T;L^p(\Omega))$;*
  - *(a3) $||u_h||_{L^r(0,T;L^p(\Omega))} \leq ||u||_{L^r(0,T;L^p(\Omega))}$.*

(b) If $u \in L^r(0,T;L^p(\Omega))$ then $\lim_{h\downarrow 0} ||u_h - u||_{L^r(0,T;L^p(\Omega))} = 0$.

(c) Operators ∂_i and $(\cdot)_h$ commute, i.e. if for $i \in \{1, \dots, N\}$ we have $\partial_i u \in L^{r_i}(0,T;L^{p_i}(\Omega))$ then

$$\partial_i(u_h(\cdot,\, t)) = (\partial_i u)_h(\cdot,\, t), \qquad \forall t \in I_h = [0, T-h).$$

(d) If $v \in C(0,T;L^p(\Omega))$, then for every $\varepsilon \in (0,T)$, $t \in (0,\, T-\varepsilon)$ it holds $\lim_{h\downarrow 0} ||v_h(\cdot,t) - v(\cdot,t)||_{L^p(\Omega)} = 0$.

(e) If $u \in C(0,T;L^p(\Omega))$, then u_h is strongly differentiable in the time variable and

$$\partial_t(u_h)(\cdot, s) = \left(\frac{u(\cdot, s+h) - u(\cdot, s)}{h} \right) \quad (\in L^p(\Omega)), \qquad \forall t \in \mathring{I}_h.$$

Now we define what is a local weak solution to (1.1) in terms of its Steklov average and prove that this definition is equivalent to Definition 2.1.

Definition 9.1. *A function $u \in L^{\mathbf{p}}_{loc}(0,T;W^{1,\mathbf{p}}_{loc}(\Omega))$ is a Steklov averaged sub-(super)-solution to (1.1) if for all $s \in (0,T)$, $h \in (0, T-s)$, for all compact sets $K \subset\subset \Omega$ and every test function $\phi \in W^{1,\mathbf{p}}_o(K)$ it holds*

$$\int_K (\partial_t(u_h)(x,s))\phi(x)\, dx + \sum_{i=1}^N \int_K \left(|\partial_i u|^{p_i-2}\partial_i u \right)_h (x,s)\partial_i\phi(x)\, dx \leq (\geq) 0. \tag{9.12}$$

Equivalence between Definition 2.1 and Definition 9.1

[(2.2) $\Rightarrow$ (9.12)]
Fix $t \in (0,T)$ and let $h > 0$ to be so small that $0 < t < t+h < T$. In formulation (2.2) we take $\varphi = \phi(x)$ depending only on the space variables and $t_2 = t+h$, $t_1 = t$. So we get for each compact set $K \subset\subset \Omega$

$$\int_K \left[\frac{u(\cdot,t+h) - u(\cdot,t)}{h}\right]\phi\, dx + \sum_{i=1}^N \int_K \left[\frac{1}{h}\int_t^{t+h} -0 + |\partial_i u(\cdot,t)|^{p_i-2}\partial_i u(\cdot,t)\right]\partial_i\phi\, dx \leq (\geq) 0,$$

by dividing the integral inequality by h, observing that $(u\partial_\tau\phi) = 0$ and using Fubini-Tonelli on the last term. Now we use property (e) of previous Proposition 9.6 inside the first integral term to get that for each time $t \in (0,T)$, $h \in (0, T-h)$, each compact set $K \subset\subset \Omega$ and each test function $\phi \in W_o^{1,\mathbf{p}}(K)$ the integral inequality (9.12) holds true.

[(9.12) $\Rightarrow$ (2.2)]
For a compact set $K \subset\subset \Omega$, consider a function $0 \leq \phi \in W^{1,2}_{loc}(0,T; W_o^{1,\mathbf{p}}(K))$: for each fixed time $\tau \in (0,T)$ it holds (9.12). We integrate such an estimate along whatever interval $[t_1,t_2]$, to get for h sufficiently small

$$\int_{t_1}^{t_2}\int_K (\partial_t u_h(x,\tau))\phi(x,\tau)\, dxd\tau + \sum_{i=1}^N \int_{t_1}^{t_2}\int_K \Big(|\partial_i u|^{p_i-2}\partial_i u\Big)_h (x,\tau)\, \partial_i\phi(x,\tau)\, dxd\tau \leq (\geq) 0\,. \tag{9.13}$$

As $u \in C(0,T; W_o^{1,\mathbf{p}}(K))$ then by the above properties $u_h \in AC(0,T; W_o^{1,\mathbf{p}}(K))$ and we are allowed to use integration by parts to get

$$\int_K u_h(x,t_2)\phi(x,t_2)\, dx + \int_{t_1}^{t_2}\int_K \Big[-u_h(x,\tau)\partial_t\phi(x,\tau)\Big] +$$
$$+ \sum_{i=1}^N \int_{t_1}^{t_2}\int_K \Big(|\partial_i u|^{p_i-2}\partial_i u\Big)_h (x,\tau)\, \partial_i\phi(x,\tau)\, dxd\tau \leq (\geq) \int_K u_h(x,t_1)\phi(x,t_1)\, dx\,.$$

Now we use the Dominated Convergence Theorem on each term to get, as

- $u_h(\cdot,\tau) \to u(\cdot,\tau)$ in $L^2(K)$ when $h \downarrow 0$ for every $\tau \in [t_1, t_2-h] \to [t_1,t_2]$,
- $(|\partial_i u|^{p_i-2}|\partial_i u)_h(\cdot,\tau) \to |\partial_i u(\cdot,\tau)|^{p_i-2}\partial_i u(\cdot,\tau)$ in $L^{(p_i-1)/p_i}(K)$ for almost every $\tau \in [t_1, t_2-h]$,

that (2.2) is verified on the limit $h \downarrow 0$ by summability of the functions $(u_h)\phi, (u_h)\partial_t\phi, [(|\partial_i u|^{p_i-1})_h]\partial_i\phi$. Finally, the estimate (9.13) shows us that we can use this formulation involving the time-derivative of the function u_h for the purposes of deriving energy estimates (and other tools).

9.5 Miscellaneous

We recall in this last section two classic abstract theorems that find their application along some important steps of the present work. The first one is useful in both Sections 5 and 6.

Theorem 9.1 (Aubin-Lions, [62] Chap. III Prop. 1.3)**.** *Consider B_0, B_1, B_2 three Banach spaces with $B_0 \subset B_1 \subset B_2$ and B_0, B_2 reflexive. Let B_0 be compactly contained in B_1 and B_1 be continuously contained in B_2. Let $1 < p, q < \infty$ and define*

$$\mathcal{D}_p^q = \{u \in L^p(0,T;B_0)|\ \partial_t u \in L^q(0,T;B_2)\}.$$

Then, the inclusion $\mathcal{D}_p^q \subset L^p(0,T;B_1)$ is compact.

Finally, we state a general Lebesgue differentiation theorem that we use to prove semicontinuity in Section 4. Let $(X,\, d_X,\, \mu)$ be a measure space endowed with a metric d_x, i.e. *a metric measure space*. Let us denote its balls centered in $x \in X$ of radius $r > 0$ with $\mathbb{B}_r(x)$. The measure μ is said to be *doubling* with respect to the distance d_X, if there exists a positive constant $C = C(\mu)$ such that

$$\mu(\mathbb{B}_2) \leq C(\mu)\mu(\mathbb{B}_1),$$

and in this case the metric measure space $(X,\, d_X,\, \mu)$ is called a *doubling metric measure space.*

Theorem 9.2 ([42], Theorem 1.8). *Let f be a nonnegative, locally integrable function on a doubling metric measure space (X, d_X, μ), then*

$$\lim_{r\downarrow 0} \fint_{\mathbb{B}_r(x)} f\, d\mu = f(x), \qquad \text{for } \mu\text{-almost every } x \in X.$$

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