

Translation in momentum space and minimal length

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We show that in the presence of the Snyder algebra the notion of translation in momentum space is modified to a formula similar to the relativistic addition of velocities. These results confirm the strict connection between Snyder algebra and the Lorentz group.

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1. Introduction

Among the possible methods to quantize gravity we can mention the introduction of a minimal length in physical theories. This idea was introduced in 1947 by Harland Snyder¹ but was later abandoned due both to the difficulty of introducing a minimal length in quantum field theory and to the success of the renormalization theory for the standard model.

The fact that gravity is the only nonrenormalizable physical theory has however left open the possibility of defining a physical theory in the presence of a minimal length, a natural cutoff for the ultraviolet divergences that afflict quantum field theory.

In general, the simplest noncommutative field theories are defined by introducing a star-product between the fields (a type of noncommutative product). However, reconciling Snyder's algebra with quantum field theory remains a subject of considerable difficulty. In this work, we try to give a new direction with which to face this long-standing problem.

Normally, the Fourier transform is used to establish a relationship between the functions defined on a Minkowski space and the operators defined on a Hilbert

space. In the case of Snyder's algebra the operator $e^{ik_\alpha \hat{x}^\alpha}$ (which is central in defining the mapping) can be considered as a deformation of the translation operator in momentum space. However, as we will calculate later, this deformation introduces some fictitious singularities and leads to poorly defined results.

To overcome this difficulty, we introduce an alternative deformation of the translation operator in momentum space which is free of singularities. We anticipate that a close relationship is obtained between this deformation and the formula for adding the velocities in special relativity. This confirms the close connection between Snyder's algebra and the Lorentz group, while the approach with the Fourier transform seems to be incompatible with the structure of Snyder's algebra.

2. Noncommutative Field Theories and the Star Product

Noncommutative field theories have been the subject of recent studies.²⁻⁴ In the case that space-time is noncommutative in the sense that

$$[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu\nu}, \tag{2.1}$$

where $\theta^{\mu\nu}$ is a constant matrix, there is a correspondence between functions f defined on the Minkowski space and operators F defined on a Hilbert space given by

$$F(\hat{x}) = \int \frac{d^4x}{(2\pi)^2} e^{ik \cdot \hat{x}} \tilde{f}(k), \tag{2.2}$$

where \tilde{f} is linked to a function f in the position space:

$$\tilde{f}(k) = \int \frac{d^4x}{(2\pi)^2} e^{-ik \cdot x} f(x). \tag{2.3}$$

In addition, the following inversion formula applies:

$$\tilde{f}(k) = \text{Tr}(e^{-ik \cdot \hat{x}} F(\hat{x})), \tag{2.4}$$

where the trace is defined by

$$\text{Tr}(A) = \lim_{\Lambda \rightarrow \infty} \frac{(2\pi)^2}{\Lambda^4} \int^\Lambda d^4q \langle q|A|q \rangle, \tag{2.5}$$

where the $|q\rangle$ are momentum eigenvectors. This inversion formula works because the following identity holds:

$$\langle h| e^{iq \cdot \hat{x}} e^{-ik \cdot \hat{x}} |h \rangle = \delta(q - k). \tag{2.6}$$

Similarly, the product of operators maps to the star-product of functions

$$F G \leftrightarrow f \star g, \tag{2.7}$$

where the star-product is defined by

$$f \star g = \lim_{x' \rightarrow x} e^{\frac{i}{2} \theta^{\alpha\beta} \partial_\alpha \partial'_\beta} f(x) g(x'). \tag{2.8}$$

However, our interest is in Snyder's algebra, defined in terms of the compact variable ρ^i as follows:

$$\hat{x}^i = i\hbar\sqrt{1-\beta\rho^2} \frac{\partial}{\partial\rho^i} \quad p^i = \frac{\rho^i}{\sqrt{1-\beta\rho^2}} \quad 0 < \rho^2 < \frac{1}{\beta}. \quad (2.9)$$

In this case, it is still possible to define a relationship between functions f and operators F through (2.2) and (2.3) ($f \rightarrow F$) but it is not possible to easily reverse this relationship ($F \rightarrow f$) (see for details Ref. 2). This fact makes it difficult to calculate the star product in the presence of Snyder's algebra.

3. Translation in Momentum Space

Let's analyze the following operator in details:

$$e^{ik^\alpha \cdot \hat{x}_\alpha} = e^{-k^\alpha \sqrt{1-\beta\rho^2} \frac{\partial}{\partial\rho^\alpha}}. \quad (3.1)$$

In the limit $\beta \rightarrow 0$ this operator is nothing else than the translation operator in momentum space:

$$|\rho_0 + k\rangle = e^{-k^\alpha \frac{\partial}{\partial\rho^\alpha}} |\rho_0\rangle. \quad (3.2)$$

If $\beta \neq 0$ we can exactly calculate its action on the operator ρ^α :

$$\begin{aligned} e^{-ik \cdot \hat{x}} \rho^\alpha e^{ik \cdot \hat{x}} &= \rho^\alpha + k^\alpha \\ &\times \left[\sqrt{1-\beta\rho^2} \frac{\sin(\sqrt{\beta k^2})}{\sqrt{\beta k^2}} + \beta(k \cdot \rho) \left(\frac{\cos(\sqrt{\beta k^2}) - 1}{\beta k^2} \right) \right] \end{aligned} \quad (3.3)$$

from which we get

$$\begin{aligned} |\rho_0'^\alpha\rangle &= e^{ik \cdot \hat{x}} |\rho_0^\alpha\rangle \\ &= \left| \rho_0^\alpha + k^\alpha \left[\sqrt{1-\beta\rho_0^2} \frac{\sin(\sqrt{\beta k^2})}{\sqrt{\beta k^2}} + \beta(k \cdot \rho_0) \left(\frac{\cos(\sqrt{\beta k^2}) - 1}{\beta k^2} \right) \right] \right\rangle. \end{aligned} \quad (3.4)$$

The problem we raise now is that the possible values of ρ_0^α must meet the condition

$$0 < \rho_0^2 < \frac{1}{\beta} \quad (3.5)$$

while the transformed $\rho_0'^\alpha$ does not meet this requirement. Hence, the operator (3.1) takes out of the allowed space and is poorly defined. For example in $d = 1$ a finite translation ($\sqrt{\beta k^2} = \alpha$) can bring a finite momentum ($\beta\rho_0^2 = \cos^2 \alpha$) to an infinite momentum ($\beta\rho_0'^2 = 1$), a rather unphysical behavior. Furthermore, the product of two operators of the type (3.1) $e^{ih \cdot \hat{x}} e^{ik \cdot \hat{x}}$ is very complicated.

In general one can define a generic deformation for the translation in momentum space using the following formula:

$$(x', \rho') = e^{k^\alpha f(\beta\rho^2) \frac{\partial}{\partial \rho^\alpha}} (x, \rho) e^{-k^\alpha f(\beta\rho^2) \frac{\partial}{\partial \rho^\alpha}} \quad (3.6)$$

but for a generic $f(\beta\rho^2)$ it is not warranted that the transformed ρ'^α satisfies the constraint $0 < \rho'^2 < \frac{1}{\beta}$, leading to singularities in the transformed momentum p' for some finite value of ρ and k .

In the following section, we will define a new deformation which doesn't introduce fictitious singularities in the mapping $(x, \rho) \rightarrow (x', \rho')$ and which reduces to a translation in the momentum space in the limit $\beta \rightarrow 0$.

4. One-Dimensional Case

We require that the mapping $T : \rho \rightarrow \rho'$ meets the following two requirements:

(i) if ρ belongs to the range $0 < \rho^2 < \frac{1}{\beta}$ then also the transformed ρ' does the same

$$0 < \rho'^2 < \frac{1}{\beta} \quad (4.1)$$

(ii) in the limit $\beta \rightarrow 0$ the mapping T reduces to simple translation $\rho' = \rho + k$.

These two requirements are met by the following mapping:

$$\rho \rightarrow \rho' = \frac{\rho + k}{1 + \beta k \rho}. \quad (4.2)$$

In particular it is true that

$$(1 - \beta\rho^2) \rightarrow (1 - \beta\rho'^2) = \frac{(1 - \beta k^2)(1 - \beta\rho^2)}{(1 + \beta k \rho)^2} \quad (4.3)$$

so $0 < \rho'^2 < \frac{1}{\beta}$ is valid if $0 < k^2 < \frac{1}{\beta}$ and $0 < \rho^2 < \frac{1}{\beta}$. We obtain as a consequence that the translation parameter k is also limited in the same interval.

To obtain a symmetry of the algebra

$$[x, \rho] = i\hbar\sqrt{1 - \beta\rho^2} \quad (4.4)$$

we also have to change $x \rightarrow x'$:

$$x \rightarrow x' = \frac{1 + \beta k \rho}{\sqrt{1 - \beta k^2}} x. \quad (4.5)$$

Thus it is ensured that

$$[x, \rho] = i\hbar\sqrt{1 - \beta\rho^2} \rightarrow [x', \rho'] = i\hbar\sqrt{1 - \beta\rho'^2}. \quad (4.6)$$

Going from the reduced variable ρ to the momentum variable p we obtain

$$p \rightarrow p' = \frac{1}{\sqrt{1 - \beta k^2}} [p + k \sqrt{1 + \beta p^2}], \quad (4.7)$$

$$x \rightarrow x' = \frac{1}{\sqrt{1 - \beta k^2}} \left[1 + \frac{\beta k p}{\sqrt{1 + \beta p^2}} \right] x.$$

The composition of two transformations of this type is simple:

$$\rho \rightarrow \rho' = \frac{\rho + k}{1 + \beta k \rho} \rightarrow \rho'' = \frac{\rho' + \tilde{h}}{1 + \beta h \rho'} = \frac{\rho + \tilde{h}}{1 + \beta \tilde{h} \rho}, \quad (4.8)$$

where

$$\tilde{h} = \frac{h + k}{1 + \beta h k}. \quad (4.9)$$

Obviously also $\tilde{h}^2 < \frac{1}{\beta}$ since $h^2 < \frac{1}{\beta}$ and $k^2 < \frac{1}{\beta}$.

5. Generalization to the Snyder Algebra

The generalization to Snyder's algebra is trivial if one remembers how the addition of velocities is done in special relativity;

$$\rho'^\alpha = \frac{1}{(1 + \beta \rho \cdot k)} \left\{ \rho^\alpha \left[1 + (1 - \sqrt{1 - \beta \rho^2}) \frac{\rho \cdot k}{\rho^2} \right] + \sqrt{1 - \beta \rho^2} k^\alpha \right\}. \quad (5.1)$$

The following properties are valid:

$$\begin{aligned} \rho' \cdot \rho &= \frac{\rho^2 + \rho \cdot k}{(1 + \beta \rho \cdot k)}, \\ \sqrt{1 - \beta \rho^2} &\rightarrow \sqrt{1 - \beta \rho'^2} = \frac{\sqrt{1 - \beta k^2}}{(1 + \beta \rho \cdot k)} \sqrt{1 - \beta \rho^2}. \end{aligned} \quad (5.2)$$

To obtain a symmetry of Snyder's algebra

$$[x^\alpha, \rho_\beta] = i\hbar \sqrt{1 - \beta \rho^2} \delta_{\alpha\beta} \rightarrow [x'^\alpha, \rho'_\beta] = i\hbar \sqrt{1 - \beta \rho'^2} \delta_{\alpha\beta}, \quad (5.3)$$

we must transform $x^\alpha \rightarrow x'^\alpha$ as follows:

$$x'^\alpha = i\hbar \sqrt{1 - \beta \rho'^2} \frac{\partial}{\partial \rho'_\alpha} = \frac{\sqrt{1 - \beta k^2}}{(1 + \beta \rho \cdot k)} \sum_\beta \frac{\partial \rho_\beta}{\partial \rho'_\alpha} x^\beta. \quad (5.4)$$

We note that this definition allows a simple composition of these transformations:

$$\begin{aligned} x''^\alpha &= i\hbar \sqrt{1 - \beta \rho''^2} \frac{\partial}{\partial \rho''_\alpha} \\ &= \frac{\sqrt{1 - \beta \rho''^2}}{\sqrt{1 - \beta \rho'^2}} \sum_\beta \frac{\partial \rho'_\beta}{\partial \rho''_\alpha} x'^\beta = \frac{\sqrt{1 - \beta \rho''^2}}{\sqrt{1 - \beta \rho^2}} \sum_\beta \frac{\partial \rho_\beta}{\partial \rho''_\alpha} x^\beta. \end{aligned} \quad (5.5)$$

The calculation of this matrix of partial derivatives

$$f_{\alpha\beta} = \frac{\partial \rho_\beta}{\partial \rho'_\alpha} \quad (5.6)$$

is complicated. Let's first calculate:

$$\frac{\partial \rho'_\beta}{\partial \rho_\alpha} = \eta_{\alpha\beta} A_1 + \rho_\alpha \rho_\beta A_2 + k_\alpha k_\beta A_3 + \rho_\alpha k_\beta A_4 + k_\alpha \rho_\beta A_5, \quad (5.7)$$

where

$$\begin{aligned}
 A_1 &= \frac{1}{(1 + \beta\rho \cdot k)} \left[1 + (1 - \sqrt{1 - \beta\rho^2}) \frac{\rho \cdot k}{\rho^2} \right], \\
 A_2 &= \frac{1}{(1 + \beta\rho \cdot k)} \left[\frac{\beta\rho \cdot k}{\sqrt{1 - \beta\rho^2} \rho^2} - 2(1 - \sqrt{1 - \beta\rho^2}) \frac{\rho \cdot k}{\rho^4} \right], \\
 A_3 &= -\frac{\beta\sqrt{1 - \beta\rho^2}}{(1 + \beta\rho \cdot k)^2}, \\
 A_4 &= -\frac{\beta}{(1 + \beta\rho \cdot k) \sqrt{1 - \beta\rho^2}}, \\
 A_5 &= -\frac{\beta}{(1 + \beta\rho \cdot k)^2} + \frac{(1 - \sqrt{1 - \beta\rho^2})}{\rho^2(1 + \beta\rho \cdot k)^2}.
 \end{aligned} \tag{5.8}$$

Let us define

$$f_{\alpha\beta} = \eta_{\alpha\beta} B_1 + \rho_\alpha \rho_\beta B_2 + k_\alpha k_\beta B_3 + \rho_\alpha k_\beta B_4 + k_\alpha \rho_\beta B_5. \tag{5.9}$$

The coefficients B_i can be obtained from A_i . Finally, we get the following expressions:

$$\begin{aligned}
 B_1 &= \frac{1}{A_1}, \\
 B_2 &= -\frac{A_2}{\Delta} - \frac{k^2}{A_1 \Delta} (A_2 A_3 - A_4 A_5), \\
 B_3 &= -\frac{A_3}{\Delta} - \frac{\rho^2}{A_1 \Delta} (A_2 A_3 - A_4 A_5), \\
 B_4 &= -\frac{A_4}{\Delta} + \frac{\rho \cdot k}{A_1 \Delta} (A_2 A_3 - A_4 A_5), \\
 B_5 &= -\frac{A_5}{\Delta} + \frac{\rho \cdot k}{A_1 \Delta} (A_2 A_3 - A_4 A_5),
 \end{aligned} \tag{5.10}$$

where the denominator is

$$\begin{aligned}
 \Delta &= (A_1 + \rho^2 A_2 + (\rho \cdot k) A_5)(A_1 + k^2 A_3 + (\rho \cdot k) A_4) \\
 &\quad - (k^2 A_5 + (\rho \cdot k) A_2)(\rho^2 A_4 + (\rho \cdot k) A_3).
 \end{aligned} \tag{5.11}$$

6. Connection with β -Canonical Transformations

It is possible to show that the transformation (5.4) also satisfies the condition of β -canonical transformation (see Ref. 5), therefore it is well defined. In the one-dimensional (1D) case, we simply have to prove that

$$\left(\frac{\partial x'}{\partial x} \frac{\partial \rho'}{\partial \rho} - \frac{\partial \rho'}{\partial x} \frac{\partial x'}{\partial \rho} \right) \sqrt{1 - \beta\rho^2} = \sqrt{1 - \beta\rho'^2}, \tag{6.1}$$

where

$$\rho' = \frac{\rho + k}{1 + \beta k \rho} \quad x' = \frac{1 + \beta k \rho}{\sqrt{1 - \beta k^2}} x. \quad (6.2)$$

Verification reduces to the following identity:

$$\frac{\partial \rho'}{\partial \rho} = \frac{1 - \beta k^2}{(1 + \beta k \rho)^2}, \quad (6.3)$$

which is true.

In general we have to show that

$$\{x'_i, \rho'_j\}_{\{x_i, \rho_j\}} = \sqrt{1 - \beta \rho'^2} \delta_{ij}, \quad (6.4)$$

$$\{x'_i, x'_j\}_{\{x_i, \rho_j\}} = \beta \left(\frac{x'_i \rho'_j - x'_j \rho'_i}{\sqrt{1 - \beta \rho'^2}} \right), \quad (6.5)$$

where the bracket is modified:

$$\begin{aligned} \{u_i, v_j\}_{\{q_i, w_j\}} &= \sqrt{1 - \beta w^2} \sum_{k=1}^n \left(\frac{\partial u_i}{\partial q_k} \frac{\partial v_j}{\partial w_k} - \frac{\partial u_j}{\partial q_k} \frac{\partial v_i}{\partial w_k} \right) \\ &+ \beta \sum_{l,m=1}^n \left(\frac{q_l w_m - q_m w_l}{\sqrt{1 - \beta w^2}} \right) \frac{\partial u_i}{\partial q_l} \frac{\partial v_j}{\partial q_m} \end{aligned} \quad (6.6)$$

and

$$x'_i = \frac{\sqrt{1 - \beta \rho'^2}}{\sqrt{1 - \beta \rho^2}} \sum_k \frac{\partial \rho_k}{\partial \rho'_i} x_k. \quad (6.7)$$

We first prove Eq. (6.4). We obtain as an intermediate step

$$\sum_{k=1}^n \left(\frac{\partial x'_i}{\partial x_k} \frac{\partial \rho'_j}{\partial \rho_k} \right) = \frac{\sqrt{1 - \beta \rho'^2}}{\sqrt{1 - \beta \rho^2}} \delta_{ij}, \quad (6.8)$$

which is true.

Let us prove Eq. (6.5). We can rewrite the β -canonical bracket as

$$\begin{aligned} \{x'_i, x'_j\} &= \alpha_1 + \alpha_2, \\ \alpha_1 &= \sqrt{1 - \beta \rho^2} \sum_{k=1}^n \left(\frac{\partial x'_i}{\partial x_k} \frac{\partial x'_j}{\partial \rho_k} - \frac{\partial x'_j}{\partial \rho_k} \frac{\partial x'_i}{\partial x_k} \right), \\ \alpha_2 &= \beta \sum_{l,m=1}^n \left(\frac{x_l \rho_m - x_m \rho_l}{\sqrt{1 - \beta \rho^2}} \right) \frac{\partial x'_i}{\partial x_l} \frac{\partial x'_j}{\partial x_m}. \end{aligned} \quad (6.9)$$

Calculating the derivatives is easy. We obtain

$$\frac{\partial x'_i}{\partial x_k} = \frac{\sqrt{1 - \beta k^2}}{(1 + \beta \rho \cdot k)} \frac{\partial \rho_k}{\partial \rho'_i}, \quad (6.10)$$

$$\frac{\partial x'_i}{\partial \rho_k} = -\frac{\beta k_k x'_i}{(1 + \beta \rho \cdot k)} + \frac{\sqrt{1 - \beta k^2}}{(1 + \beta \rho \cdot k)} \sum_{l=1}^n \frac{\partial}{\partial \rho_k} \left(\frac{\partial \rho_l}{\partial \rho'_i} \right) x_l.$$

The first term α_1 gives rise to

$$\alpha_1 = \frac{\sqrt{1 - \beta \rho'^2}}{(1 + \beta \rho \cdot k)} \left[-x'_j \frac{\partial(\beta \rho \cdot k)}{\partial \rho'_i} - (i \leftrightarrow j) \right]. \quad (6.11)$$

The second term proportional to β gives rise to

$$\alpha_2 = \frac{\beta(1 - \beta k^2)}{2(1 + \beta \rho \cdot k)^2 \sqrt{1 - \beta \rho^2}} \left[x_l \frac{\partial \rho_l}{\partial \rho'_i} \frac{\partial \rho^2}{\partial \rho'_j} - (i \leftrightarrow j) \right]. \quad (6.12)$$

Using this identity

$$(1 - \beta \rho^2) = \frac{(1 + \beta \rho \cdot k)^2}{(1 - \beta k^2)} (1 - \beta \rho'^2), \quad (6.13)$$

we obtain

$$\alpha_2 = \beta \left(\frac{x'_i \rho'_j - x'_j \rho'_i}{\sqrt{1 - \beta \rho'^2}} \right) + \frac{\sqrt{1 - \beta \rho'^2}}{(1 + \beta \rho \cdot k)} \left[-x'_i \frac{\partial(\beta \rho \cdot k)}{\partial \rho'_j} - (i \leftrightarrow j) \right]. \quad (6.14)$$

Adding the two contributions equation (6.5) is verified.

7. A Solvable Example

To define a mapping between functions and operators in the case of the deformation described in this paper, we limit ourselves to the soluble case in 1D. We must first find the explicit representation:

$$\begin{aligned} \rho \rightarrow \rho' &= \frac{\rho + k}{1 + \beta k \rho} = e^H \rho e^{-H}, \\ \frac{\partial}{\partial \rho} \rightarrow \frac{\partial}{\partial \rho'} &= e^H \frac{\partial}{\partial \rho} e^{-H}, \end{aligned} \quad (7.1)$$

where $H = f(\rho) \frac{\partial}{\partial \rho}$ is a linear operator.

The H operator must satisfy the condition:

$$f(\rho') \frac{\partial}{\partial \rho'} = f(\rho) \frac{\partial}{\partial \rho} \quad (7.2)$$

from which we obtain

$$f(\rho') = \frac{(1 - \beta k^2)}{(1 + \beta k \rho)^2} f(\rho). \quad (7.3)$$

The general solution of this equation is

$$f(\rho) = c(k)(1 - \beta\rho^2), \quad (7.4)$$

where c is a constant dependent on k .

The constant $c(k)$ can be obtained with a perturbative calculation

$$c(k) = k(1 + \frac{1}{3}\beta k^2 + \frac{1}{5}\beta^2 k^4 + \dots) = \frac{1}{\sqrt{\beta}} \tanh^{-1}(\sqrt{\beta k^2}) \quad (7.5)$$

or from the condition that the composition of two transformations

$$e^{H(h)}e^{H(k)} = e^{H(\tilde{h})}, \quad (7.6)$$

where \tilde{h} is defined by Eq. (4.9).

Let us notice that $c(k) = \eta(k)$ is the rapidity function of special relativity:

$$\eta(k) + \eta(h) = \eta(\tilde{h}). \quad (7.7)$$

At this point we notice the substantial difference between our solution

$$H = \eta(k)(1 - \beta\rho^2) \frac{\partial}{\partial \rho} \quad (7.8)$$

and the operator (3.1).

Furthermore, the rapidity variable η can extend to infinity while the variable k is bounded $0 < k^2 < \frac{1}{\beta}$.

Also the bounded variable ρ can be replaced with the variable

$$\rho \rightarrow y = \frac{1}{\sqrt{\beta}} \tanh^{-1}(\sqrt{\beta\rho^2}), \quad (7.9)$$

which is unbounded. Then the translation operator defined in this work takes the standard form

$$e^{i\eta\hat{x}}, \quad (7.10)$$

where the rapidity η takes the role of momentum variable in the Fourier transform and $\hat{x} = -i\frac{\partial}{\partial y}$.

So, we can try to define a mapping between functions and operators of the form:

$$f(x) = \int d\eta e^{i\eta x} \tilde{f}(\eta), \quad (7.11)$$

$$F(\hat{x}) = \int d\eta e^{i\eta\hat{x}} \tilde{f}(\eta),$$

which is the basis for defining a field theory in noncommutative geometry.

In the general case there is certainly an operator H such that

$$\rho^a \rightarrow \rho'^a = e^H \rho^a e^{-H} \quad (7.12)$$

but it is difficult to find a closed form for the linear operator H as we did in case 1D. We leave this discussion for future work.

8. Conclusions

Normally noncommutative field theories are defined in terms of a star-product that modifies fields in interaction. In the case of Snyder algebra we have criticized this method based on the Fourier transform, because applying it we obtain a deformation of the translation operator in momentum space that introduces a fictitious singularity for some finite value of k and ρ .

In this work, we have introduced a correct deformation of the translation operator in momentum space constructed starting from the formula of the addition of velocities in special relativity.

This deformation is a true symmetry of Snyder's algebra. Confirmation of this is the verification that this deformation has the property of being a β -canonical transformation (introduced as symmetry of the path-integral in our previous work⁵).

Therefore we find a close connection between Snyder algebra and the Lorentz group, while the Fourier transform method is more suitable for noncommutative theories in which the translation in momentum space is the standard one.

We hope that the knowledge of this deformation can stimulate the construction of a new method to define quantum field theory in the presence of Snyder's algebra.

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