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Title: Gödel's theorems and Zermelo's axioms—a firm foundation of mathematics.

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The book under review is a compact and rich manual (surprisingly rich, given its compactness I would say), that would well serve the purpose declared by the authors themselves, namely to be suitable for a one- or two-semester introductory course into logic and set theory for undergraduate mathematics students. I am inclined to think that it may also do even better and provide with a good reference also young researchers who are looking for a self-contained introduction to first-order logic and more advanced topics such as Gödel's completeness and incompleteness theorems, set theory and model theory. The book is divided into four parts, each of which is made out of chapters. Useful excercises are offered at the end of each of them, which often offers the occasion of nicely completing the information of the text.

Part 1 contains three chapters devoted to an introduction to various aspects of first-order logic, formal theories and their relation to 'natural' (semi-formal) proofs. Chapter 1 presents formal languages and their syntax, explains the way in which a formal theory is built and introduces the concept of formal proof and tautology (a list of tautologies is given as an appendix to the volume). It also treats the concept of logical equivalence as a way to simplify the alphabet. Chapter 2 revolves around the concept of proof, and goes from presenting basic properties of it (the deduction theorem) and the alternative approach to the concept offered by natural deduction, to an overview of methods of proofs, an analysis of the properties of consistency and compactness, and ends with relating formal to semi-formal proofs and to proof-strategies like backward and forward reasoning. Chapter 3 is devoted to the semantics of formal languages and to some basic notions of model theory. It also offers an explanation of the concept of soundness and completeness for a formal theory (*via* the notion of "completion" of it), which are the bridge to the parts covering Gödel's contributions.

The theorem generalizing the result that Gödel notoriously presented in his doctoral dissertation from 1929, namely the generalized completeness theorem for formal theories, is the main topic of Part 2. It is proved, as it is nowadays standard, by Henkin's construction from [The completeness of first-order functional calculus, *The Journ. of Symb. Logic*, 14, 1949; MR0033781] based on maximally consistent theories of axioms. Like the previous one, also this part of the book is made out of three chapters. Chapter 4 sets the basis and proves Lindembaum's lemma about the existence of a maximally consistent extension of any consistent theory. Chapter 5 shows how to use this result to prove the generalized completeness theorem for theories based upon languages with a countable signature, and explores some consequences and some equivalent formulations of it. Finally, Chapter 6 is devoted to extending a formal language by newly defined symbols, and how to prove the completeness theorems to theories based on such extended languages.

Part III of the book presents Gödel's first incompleteness theorem, formulated as the statement that Peano Arithmetic PA is not syntactically complete, and a proof of Gödel's second incompleteness theorem for PA by means of Löb's derivability conditions strategy of [Solution of a problem of Leon Henkin, *The Journ. of Symb. Logic*, 20, 1955; MR0070596]. Everyone who is acquainted with the topic knows that these results are full of insidious technical details, and that is particularly helpful to rely on a steady guide for avoiding traps related to those. This is what the book provides the reader with. The part starts with a couple of introductory chapters on PA: Chapter 7 is a short, preliminary introduction to standard and non-standard countable models of the theory, while Chapter 8 illustrates how doing arithmetic inside it looks like. Chapter 9 presents the so-called arithmetization of the syntax *à la* Gödel (i.e., with the use of the β-function). Chapter 10 contains the proof of the first incompleteness theorem *via* the diagonalization lemma, and a useful discussion about refinements (by presenting Robinson Arithmetic as an attempt to restrict the schema of complete induction), and generalizations (i.e., the theorems by Rosser and Tarski). Finally, Chapter 12 presents Presburger arithmetic PrA from [On the completeness of a certain system of arithmetic of whole numbers in which addition occurs as the only operation, Hist. Philos. Logic 12, 1991; MR1111343]. This is a sort of dual of Skolem arithmetic (that is obtained from PA by retaining multiplication as the only operation available), as it contains only the addition operation, for which the completeness theorem is proved as a way to reflect about the cost of escaping Gödel's result.

The last part of the book is an equally compact, and equally rich introduction to the axioms of set theory, that starts in Chapter 13 with a detailed presentation of the theory ZFC and some basic results on ordinal and cardinal arithmetic. The remaining chapters of this part of the book deal with model-theoretic topics. Chapter 14 is about models of ZFC, with a presentation of the cumulative hierarchy of sets and Gödel's constructible model. Chapter 15 deals with the construction that leads to the upward and to the downward Löwenheim-Skolem theorems. Chapter 16 is a short note about standard and non-standard models of PA within ZFC. Finally, Chapter 17 deals with models of the real numbers, and compares the well-known construction based on Cauchy sequences with N. A'Campo's simpler one [A natural construction for the real numbers, Elem. Math. 76, 2021; MR4292704], as the latter uses only the integers and not the rational numbers, by proving that they lead to isomorphic structures. The chapter ends with a (very brief) note on non-standard analysis and non-standard models of the reals.