From pseudo-objects in dynamic explorations to proof by contradiction

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Abstract

Proof by contradiction presents various difficulties for students relating especially to the formulation and interpretation of a negation, the managing of impossible mathematical objects, and the acceptability of the validity of the statement once a contradiction has been reached from its negation. This paper discusses how a Dynamic Geometry Environment (DGE) can contribute to students' argumentation processes when trying to explain contradictions. Four cases are presented and analysed; the actors are students from high school, undergraduate and graduate students. The analyses make use of a symbolic logical chain approach and of the notion of *pseudo object*. Such analyses lead to a hypothesis, that is, experiencing a pseudo object during an exploration can foster DGE-supported processes of argumentation culminating in geometrical proofs by contradiction, while the lack of experience of a pseudo object may hinder such processes. If this hypothesis is confirmed by further studies, we foresee important didactical implications since it sheds light onto the transition from students' DGE-based argumentations to proofs by contradiction.

Keywords

Dynamic geometry, Indirect argument, Proof by contradiction, Pseudo object

1 Introduction

Previous studies have highlighted how a Dynamic Geometry Environment (DGE) can mediate students' proof processes (e.g., Laborde 2000; Mariotti 2000; De Villiers 2004; Sinclair and Robutti 2013) especially when the activities foster students' reasoning and production of conjectures (e.g., Pedemonte, 2007). Indeed, as stated by Laborde and Laborde, speaking about a particular DGE, Cabri, reasoning processes are supported by the software, which brings changes to the solving process: "the changes in the solving process brought by the dynamic possibilities of Cabri come from an active and reasoning visualisation, from what we call an interactive process between inductive and deductive reasoning." (Laborde and Laborde 1991, p. 185).

Existing literature also explains how certain argumentation processes potentially contribute to students' production of proofs by contradiction (e.g., Leung and Lopez-Real 2002; Baccaglini-Frank, Antonini, Leung and Mariotti 2013), which is what we focus on in this paper. Indeed, research centred on proof by contradiction has pointed to various difficulties it presents for students (see for example, Leron 1985; Wu Yu, Lin and Lee 2003; Antonini 2004; Antonini and Mariotti, 2006, 2007, 2008; Mariotti and Antonini 2009). These difficulties relate especially to the formulation and interpretation of a negation, the managing of impossible mathematical objects and the acceptability of the validity of the statement once a contradiction has been reached from its negation.

In managing mathematical objects, the "active and reasoning visualization" (Laborde and Laborde 1991) offered by DGEs seems to yield great potential, because it allows students to see simultaneously the consequences of all the geometrical properties according to which a figure was constructed, maintaining theoretical control (Mariotti, 2002) over the figure for the student. This should allow the student to allocate more cognitive resources to potentially conflicting properties in the case of impossible mathematical objects, that is, properties that cannot coexist in a robustly constructed[1] (Healy, 2000) dynamic figure. How students deal with the coexistence of such properties is what this paper looks into.

In particular, we illustrate the potential of a specific type of open problems, problems that ask for the construction of a geometrical object that cannot exist within the Theory of Euclidean Geometry, with respect to proof by contradiction (Baccaglini-Frank et al. 2013; Baccaglini-Frank, Antonini, Leung and Mariotti 2017). Building on previous work (Leung and Lopez-Real 2002; Baccaglini-Frank, Antonini, Leung and Mariotti, 2011), we analyse such potential through the notion of *pseudo object*, "a geometrical figure associated to another geometrical figure either by construction or by projected-perception in such a way that it contains properties that are contradictory in the Euclidean theory" (Baccaglini-Frank et al. 2013, p. 65). We further elaborate on such notion to better illustrate the DGE's potential to help students perceive and deal with contradictions as

[1] By robust construction in a DGE, we mean construction that can maintain the desired properties of a figure invariant under dragging; we use *robust* in contrast to *soft* (Healy, 2000).

they engage in the exploration of a non-constructability problem. We present analyses of four cases in which students produced argumentations after a phase of dynamic exploration. The analyses make use of a symbolic logical chain approach to illustrate the emergence of pseudo objects, as a main feature of the potential of the proposed problems in a DGE with respect to explaining contradictions and leading to proof by contradiction.

2 Conceptual Framework

2.1 Proof by contradiction and indirect argumentation

The relations between argumentation and proof constitute one of the main issues in research in mathematics education (see, for example, Boero 2007; Hanna and de Villiers 2012; Stylianides, Bieda and Morselli 2016). Many articles, based on different theoretical assumptions, have proposed different approaches and, therefore, different didactical implications. Some researchers (e.g., Duval 1992-93) have highlighted a distance between argumentation and proof, while others have focused on the analogies between argumentation and proof, seen as two processes (Boero, Garuti and Mariotti 1996; Garuti, Boero, Lemut and Mariotti 1996). In this second case, the main didactical implication is the importance for students to engage in generating conjectures, in order to promote certain processes that are relevant to developing their competences in mathematical proof. Here we focus on the relationship between processes of argumentation and of proof in the specific case of proof by contradiction in DGEs. First of all, we need to clarify terminology such as 'indirect proof', 'proof by contradiction', 'proof by contraposition', 'proof ad absurdum', etc., since it is not always used consistently by practicing mathematicians and textbooks.

2.1.1 Indirect proof: proof by contradiction and proof by contraposition

Given a statement S, a *proof by contradiction* is a direct proof of the statement $\neg S \rightarrow r \wedge \neg r$, where r is a previously proven theorem, an axiom or any proposition. If the statement S can be expressed as $p \rightarrow q$, since $p \rightarrow q$ is logically equivalent to $\neg p \vee q$, the negation of $p \rightarrow q$ can be substituted by $\neg(\neg p \vee q)$ that is equivalent to $p \wedge \neg q$: then the negation of S is $p \wedge \neg q$. In this case, a proof by contradiction of S is a direct proof of $p \wedge \neg q \rightarrow r \wedge \neg r$.

A *proof by contraposition* of S is a direct proof of $\neg q \rightarrow \neg p$.

We refer to *indirect proof* as a proof of a statement of which the premise contains the negation of the conclusion. So, both proofs by contradiction and proofs by contraposition are indirect proofs, because they refer to statements that contain a negation ($\neg q$) in their premise.

2.1.2 Indirect argumentation

Studies in mathematics education have revealed that proof by contradiction is a very complex activity for students, as mentioned above. However, some studies show that students spontaneously produce argumentations very similar to proofs by contradiction:

> The indirect proof is a very common activity ('Peter is at home since otherwise the door would not be locked'). A child who is left to himself with a problem, starts to reason spontaneously '... if it were not so, it would happen that...' (Freudenthal 1973, p. 629).

In agreement both with Freudenthal and with the characterization of indirect proof given above, we use *indirect argumentation* to refer to an argumentation stemming from assumptions that contain the negation of the statement to be argued, or the negation of part of such statement, that is an argumentation with a structure like: "*...if it were not so, it would happen that...*". (For a more articulated and refined definition see Antonini, 2010).

2.2 Open Construction Problems, Non-constructability Problems and Proof by Contradiction

Construction problems constitute the core of classic Euclidean geometry. The use of specific artefacts, i.e. ruler and compass, can be considered at the origin of the set of axioms defining the theoretical system of Euclid's *Elements*. Any geometrical construction corresponds to a theorem. This means that there is a proof that validates the construction procedure that solves the corresponding construction problem. Thus, in classic Euclidean Geometry the *theoretical nature* of a geometrical construction is clearly stated (e.g., Vinner 1999) in spite of the apparent practical objective, i.e. the accomplishment of a drawing following the construction procedure. We note that the "non-constructability" of a figure may become manifest in fundamentally two different ways: (1) a figure, though existing, may be non-constructible with certain (predefined) tools, let's say with a straightedge and compass; or (2) a figure's non-constructability may derive from the non-existence of the geometrical object per se, that is, from the contradiction that follows once its existence is assumed. Historically, there have been many examples of the first case such as the trisection of an angle, the doubling of a cube, or the squaring of a circle. Non-constructability of the second type does not depend on the tools used to accomplish the construction because it is a logical consequence within the theory of Euclidean geometry: if the figure exists, there would be a contradiction. This paper considers the second type of non-constructability.

The problems we are concerned with there are construction problems. If the construction *is* possible we speak of *constructability problems*, while if the construction is *not* possible, of *non-constructability problems*. Clearly, the solver initially does not know whether the construction problem s/he is addressing is a constructability or non-constructability problem. In this paper, the non-constructability problems we deal with are of the following type: the solver is asked whether a figure with prescribed properties is constructible or not, and in either case s/he is required to provide an argumentation supporting the answer. Usually, the solver attempts to construct the requested or hypothesized geometrical object. The solution can be provided either producing the construction procedure and its validation according the theory available (in this case Euclidean Geometry), or proving the fact that no construction procedure can be exhibited. This latter case, because of its very nature, may lead to an indirect argumentation, sowing seeds that may eventually become a proof by contradiction. As a matter of fact, a non-constructability statement expresses the fact that it is impossible to display a valid procedure for constructing a certain figure. By assuming that the negation of a certain property is always true, the solver proves that such property is not possible.

2.3 Pseudo Objects

Within the very little literature in this area, a study conducted by Leung and Lopez-Real describes a proof by contradiction produced by two students working in Cabri (a DGE), which triggered the development of a framework on theorem acquisition and justification in a DGE. The researchers used such framework to describe a "scheme for 'seeing' proof by contradiction" (Leung & Lopez-Real, 2002, p. 150). Within this framework, the idea of a *pseudo-quadrilateral* (in this case, a quadrilateral that cannot exist, unless degenerate, in a figure containing all the required properties) was first introduced to *visualize a proof by contradiction* in a DGE. Figures 1 and 2 visually summarize the idea. Building on such introductory work, Baccaglini-Frank, Antonini, Leung, and Mariotti (2013) extended this idea introducing the notion of a *pseudo object* to describe

> A geometrical figure associated to another geometrical figure either by construction or by projected-perception in such a way that it contains properties that are contradictory in the Euclidean theory (ibid., p.65).

By "projected-perception" we mean something visually perceived in the DGE that is affected with a mentally imposed condition (we discuss this in greater depth in section 3.3). Such notion has been used to suggest that a pseudo object can be conducive to a dialectic between dynamic visual reasoning in a DGE and theoretical reasoning in the Euclidean axiomatic system, eventually leading to a proof by contradiction.

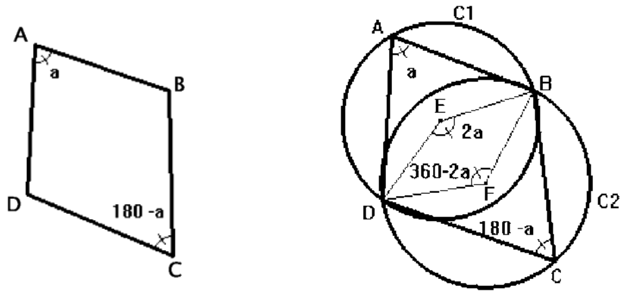


Fig 1. A learner's projected-perception on an arbitrary quadrilateral ABCD: a *pseudo-quadrilateral* EBFD associated to ABCD is constructed visually inheriting an "impossible" Euclidean property. (Leung and Lopez-Real 2002, p.155)

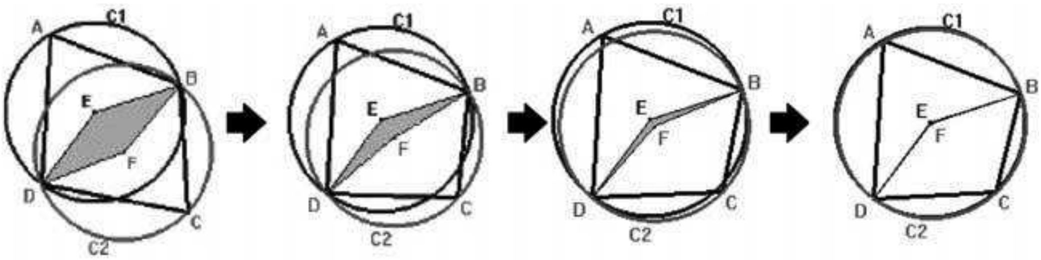


Fig 2. By dragging the vertices of ABCD, the pseudo-quadrilateral EBFD can be made to "vanish" (degenerate into a linear object), thus realizing a possible theorem associated to the imposed condition (Leung and Lopez-Real 2002, p.157).

2.4 Dragging and Invariant Properties of Figures in a DGE

Any figure in a DGE that has been constructed using specific primitives can be *acted upon* through dragging, which determines the phenomenon of *moving figures*. A Dragging Exploration Principle was proposed (Leung, Baccaglini-Frank and Mariotti 2013) to epitomize the DGE dragging phenomenon:

> During dragging, a figure maintains all the properties according to which it was constructed and all the consequences that the construction properties entail within the axiomatic world of Euclidean geometry" (ibid., p.458).

As it has been previously discussed in the literature, the perception of a *moving figure* in a DGE is the phenomenon through which something about the figure changes while something is preserved during dragging (Mariotti 2015). What is preserved during dragging (the *invariant*) becomes the identity of the object/figure in contrast with what changes which determines its *variation* and consequently its movement.

As a figure is acted upon in a DGE, there are two kinds of invariants appearing simultaneously: the invariants determined by the geometrical relations defined by the commands used to construct the figure which are called *direct invariants*, and the invariants that are derived consequently within the theory of Euclidean Geometry are called *indirect invariants* (Laborde and Sträßer 1990). The relationship of dependency between these two types of invariants constitutes a crucial point in the process of exploration in DGE. The experience of dragging allows the user to interpret what appears on the screen in terms of logical consequences between geometrical properties; in particular, indirect invariants will be interpreted in terms of consequences of the direct invariants. Discerning invariants and discerning invariant relations between invariants are cognitively different tasks (Leung et al. 2013). Simultaneous appearance of direct and indirect invariants during dragging leads to the possibility of perceiving the dependence of an indirect invariant (B) from a set of direct invariants (A_1, A_2, $\cdots$ A_n). We express this dependency relationship between (direct and indirect) invariants in the logical form: $A_1 \wedge$

$A_2 \wedge \ldots \wedge A_n \Rightarrow B$. In the following Section, we will exemplify how direct invariants, indirect invariants and pseudo objects come into dialectical play in the case of non-constructability problems.

3 Research Hypothesis and Methodology

Over the last decade we have conducted a number of studies to investigate processes of conjecture generation, argumentation and proof, within a DGE; in particular, we have collected data in Italy and in Hong Kong on students' solution processes when working on non-constructability problems (results have been published in: Leung and Lopez-Real 2002; Baccaglini-Frank 2010; Baccaglini-Frank and Mariotti 2010; Baccaglini-Frank et al. 2011; Baccaglini-Frank et al. 2013; Leung et al. 2013; Baccaglini-Frank et al. 2017). The research presented in this paper stems from a revisitation of the part of such data regarding students' solution processes mobilized to solve non-constructability problems, one of which we will introduce in section 3.2. The practice of data revisitation has been suggested and used insightfully, for example, by Nachlieli and Tabach (2012) who revisited data collected during a project that took place over 15 years before. From the revisitation of our data in light of the notion of pseudo object the following new hypothesis emerged, which is at the heart of this paper:

> *Experiencing a pseudo object during an exploration can foster DGE-supported processes of argumentation culminating in geometrical proofs by contradiction, while the lack of experience of a pseudo object can hinder such processes.*

3.1 The Revisited Data

We collected our data in the following forms: audio and video recordings and transcriptions of the introductory lessons; Cabri and GeoGebra files worked on by the instructor and by the students during the classroom activities; audio and video recordings, screenshots of the students' explorations, transcriptions of the task-based interviews, and the students' work on paper that was produced during the interviews. For the revisitation in light of the notion of pseudo object, we singled out the interviews of students working alone or in pairs on two non-constructability problems proposed in the context of a DGE.

A total of 12 interviews were found in the data and revisited in light of the notion of pseudo object, 6 of which dealt with the same non-constructability problem presented in the following section. All participants had been working with dynamic geometry for at least two months prior to the time that the task-based interviews were carried out. The four cases chosen for this paper were chosen because of the heterogeneity of the students involved: two pairs of students are from Italian high schools in different regions (ages 15-16), an Italian graduate student enrolled in a Mathematics Education Course, part of his master's programme in a Math Department (age 21), and a pair of Hong Kong undergraduate students (age 21) in a joint Mathematics and Mathematics Education bachelor's programme. The high school students had been introduced to different forms of dragging in a DGE in the context of conjecture-generation by the first author of the paper; they had never worked with her before on non-constructability problems, but they had been using a DGE with their teacher for a year prior to the interview. The Italian graduate student had been introduced to dynamic geometry as part of the Mathematics Education Course two months prior to the interview, and had not used such software before during his education. The Hong Kong Undergraduates had been introduced to dynamic geometry during their bachelor's degree at least one year prior to the interview. In terms of their previous exposure to Euclidean Geometry, all students had worked on proof in this domain for about one year in high school. The Hong Kong students had also taken an undergraduate class on this topic.

The methodology we will use to explore our research hypothesis consists in: (1) presenting an a priori analysis of the problem assigned in the selected interviews, through which we highlight the potential of perceiving pseudo objects with respect to transitioning from argumentation to proof by contradiction. This is done by developing a *symbolic logical chain approach* that allows us to identify and describe pseudo objects and their potential role in argumentation processes. Then (2) we use such frame to analyse the four selected cases and explore our research hypothesis.

3.2 A Priori Analysis of the Problem

The problem we introduce here is an open non-constructability problem, upon which the selected student interviews that we will discuss in later sections are all based. It is formulated as follows:

> Is it possible to construct a triangle with two angle bisectors that are mutually perpendicular? If so, provide steps for a construction. If not, explain why not.

To simplify the reading of our analyses using such approach we add a table summarising the properties and abbreviations we will be using (Table 1).

Geometrical property	*Abbreviation used*	*Appearance in Analyses*
$\angle CPA$ is right	A_1	a priori, all cases
CP is bisector of $\angle BCA$	A_2	a priori, all cases
AP is bisector of $\angle CAB$	A_3	a priori, all cases
ABC is a triangle	A_4	a priori, all cases
$m\angle BCA + m\angle BAC = 180°$	B_1	a priori, case 2
sides BC and BA coincide	C_1	a priori, case 2
sides BC and BA are non-coincident and parallel	C_2	a priori
B, the intersection point of the triangle's sides that are not common to the two angle bisectors, does not exist	D_1	a priori, case 3
A, B, and C lie on the same line	D_2	a priori, case 2, 3, 4
all sides of the figure collapse into a point	D_3	case 3
the figure is a rhombus	E_1	case 1
the figure is a square	E_2	case 1
EF≅E'F'	F_1	case 3
the figure is a parallelogram	F_2	case 3

Table 1: abbreviations of the geometrical properties considered in the analyses. The letters in the first column from the left refer to the labels in the figures corresponding to the cases indicated in the third column.

The cases in which the DGE gives the solver feedback in the form of an indirect invariant (e.g., $A_1 \wedge A_2 \wedge A_3 \Rightarrow \neg A_4$) that conflicts with what s/he expected (e.g., $A_1 \wedge A_2 \wedge A_3 \Rightarrow \ldots \Rightarrow A_4$) are situations with a particularly high potential of fostering reasoning by contradiction in continuity with the exploration.

The answer to the question posed by the problem is "No. A triangle with two angle bisectors that are mutually perpendicular cannot be constructed". A proof by contradiction might go along the following lines (Fig. 3).

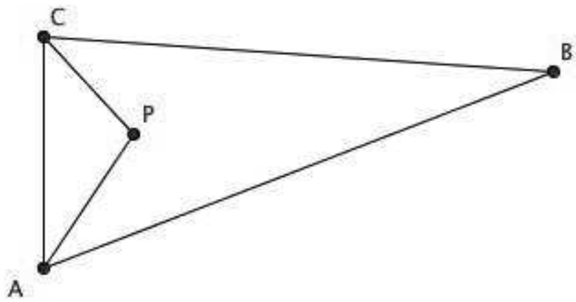


Fig 3. Possible attempt at constructing a suitable triangle.

Let $\angle CPA$ be a right angle (property A_1) and CP bisector of $\angle BCA$ (A_2) and AP bisector of $\angle CAB$ (A_3). Then, using the angle measures, $\frac{1}{2} m\angle BCA + \frac{1}{2} m\angle BAC = 90^\circ$, so $m\angle BCA + m\angle BAC = 180^\circ$ (property B_1). On the other hand, $m\angle BCA + m\angle BAC < 180^\circ$ then $m\angle BCA + m\angle BAC \neq 180^\circ$ (we can write this as $\neg B_1$) because $\angle BCA$ and $\angle BAC$ are two angles of a triangle ABC (property A_4). Therefore, we have a contradiction, that is, the co-existence of a proposition and its negation ($B_1 \wedge \neg B_1$). We proved that $A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow B_1 \wedge \neg B_1$.

As we have shown in previous studies (e.g., Mariotti and Antonini 2009), from a cognitive point of view, we could have the necessity to visualize the consequences of the property $m\angle BCA + m\angle BAC = 180^\circ$ (B_1): sides BC and BA either

$$\left.\begin{array}{l}\text{coincide } (C_1)\\ \text{or}\\ \text{are non-coincident and parallel } (C_2)\end{array}\right\} \quad C_1 \vee C_2$$

and continuing the chain of derived properties, either

vertex B does not exist hence the initial triangle does not exist (D_1, essentially $\neg A_4$),

or

A, B, and C must lie on the same line (D_2, a different way of obtaining $\neg A_4$) hence the triangle cannot exist in a non-degenerate form.

A determining difference of how this situation may be seen is how the figure degenerates. In one case the triangle can be seen to degenerate, breaking into an open figure when CB and BA are seen as becoming parallel (see Mariotti and Antonini 2009), or in the other case it can be perceived as turning into a single line (for example, BC and BA are seen as collapsing onto the same line). So, to explain the impossibility we might need to envision a logical sequence, like

$A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow B_1 \Rightarrow C_2 \Rightarrow D_1$ (recognised as $\neg A_4$)

or

$A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow B_1 \Rightarrow C_1 \Rightarrow D_2$ (recognised as $\neg A_4$)

and capture a conflict between initially assumed properties and those derived (e.g. $A_4 \wedge D_1$, that is $A_4 \wedge \neg A_4$). We will argue that recognizing such a deductive chain and capturing conflicting properties within it constitute an important step towards explaining the impossibility, and eventually constructing a proof by contradiction.

What is the potential of a DGE in supporting the development of this process? When working with paper and pencil, slight inaccuracies in the drawing allow the figure to represent properties, which a *proper* construction would not permit. For example, on paper, with no trouble one can assume to have drawn a triangle, of which two angle bisectors intersect at a right angle. In this case one may easily be perceptually unaware of the presence of contradictory properties; awareness of a contradiction depends on the solver's conceptual control on the figure, that allows him/her to construct a deductive chain such as $A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow B_1 \Rightarrow C_2 \Rightarrow D_1$ and perceive conflicting properties. In a DGE solvers can make different choices on which properties to use to construct the figure robustly (Healy 2000). The choice determines the type of guidance that the DGE can provide to reasoning, as shown by the authors previously (Baccaglini-Frank et al. 2013). For example, a similar situation to that described in paper and pencil occurs when the solver constructs a figure with robust properties while mentally imposing on it a soft contradictory property. We could start with a robust triangle (A_4) and robust bisectors ($A_2 \wedge A_3$), and try to obtain bisectors that are perpendicular (A_1) through dragging. This allows the solver to use the DGE (only) as a sort of "amplified paper-and-pencil drawing" in that it allows the exploration of many cases without having to redraw the figure. An exploration in this situation might lead to experiencing that *when* $A_2 \wedge A_3 \wedge A_4$, imposing A_1 leads *every time* to D_2. To correctly solve the problem, the observation "Given $A_2 \wedge A_3$ a conflict between A_1 and A_4 may be perceived" needs to be transformed into a chain of deductive chain (e.g., $A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow B_1 \Rightarrow C_1 \Rightarrow D_2$ or $A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow \ldots \Rightarrow \neg A_4$), exposing a contradiction (e.g., $\ldots A_4 \Rightarrow \neg A_4$). The fact that a DGE can potentially "show" the solver conflicting soft properties simultaneously, is a first aspect of its potential with respect to transitioning from argumentation to proof by contradiction. What happens if, instead, the solver attempts to construct the three properties $A_1 \wedge A_2 \wedge A_3$ robustly, expecting to find the third vertex of the triangle and thus also obtain A_4 robustly? Since a DGE generates immediately all properties that are logical consequences of the construction properties, the figure obtained imposing $A_1 \wedge A_2 \wedge A_3$ will also show the robust property D_1 (and so $\neg A_4$). That is, the solver can face a surprising situation in which s/he expected A_4, but robustly obtains $\neg A_4$. The unexpected feedback may generate uncertainty about the possibility of the construction and require an interpretation. The crucial point for the solver is to realize that impossibility emerging as a feedback can be related to the properties constructed robustly and to properties deriving from them. In other words, in this case the DGE gives the solver feedback in the form of an indirect invariant $A_1 \wedge A_2 \wedge A_3 \Rightarrow \neg A_4$ that conflicts with the expected $A_1 \wedge A_2 \wedge A_3 \Rightarrow \ldots \Rightarrow A_4$. We see this situation as having a particularly high potential with respect to transitioning from argumentation to proof by contradiction.

3.3 Perception of Pseudo Objects

In the case we examined where we had a robust construction of the triangle and of the angle bisectors (properties: A_4, A_2, A_3) the notion of pseudo object introduced in Section 2.1 (Baccaglini-Frank et al. 2013, p.65) can be used to describe the non-degenerate triangle (A_4) along with the property $m\angle BCA + m\angle BAC = 180°$ (B_1) derived from the projected perception $\angle CPA$ is right (A_1). By *projected perception*, we mean what is visually perceived in the DGE is affected with a mentally imposed condition. Analogously, in the case of robust construction of the bisectors that are perpendicular (A_1), together with two bisectors (properties: A_2, A_3), a pseudo object could be perceived in the projected perception of the figure being a triangle (A_4) together with the property of having two parallel sides (C_2). Table 2 presents a summary of conditions to perceive possible pseudo objects.

Robust Construction (Direct Invariant)	Projected Perception or Imposed Condition	Indirect Invariant	Contradiction
A_2, A_3, A_4	A_1	D_2 (as $\neg A_4$)	$A_4 \wedge \neg A_4$
A_1, A_2, A_3	A_4	D_1 (as $\neg A_4$)	$A_4 \wedge \neg A_4$

Table 2: Conditions to perceive possible pseudo objects.

In the process of dragging the pseudo object (mentally or within the DGE), contradiction can arise or disappear (when the pseudo object is made to degenerate) to obtain configurations in which the imposed condition is visually verified. The arising and disappearing of contradictions may guide the argumentation supporting the claim of an impossibility in the case of a non-constructability problem. In the perception of a pseudo object it is not a single property that is important, but a *conflicting relationship* between a certain geometrical property and other geometrical properties of the figure. We can interpret a *pseudo object* (for a solver) as a geometrical figure in which the solver simultaneously perceives a *conflict* between types of invariant (direct or indirect). The main reason we are interested in using the construct of pseudo object is because its presence can give rise to contradiction in the process of argumentation, and we seek to explore its educational potential with respect to proof by contradiction.

4 Student Cases

In this section, we first present analyses of two cases in which we argue that the students did encounter pseudo objects during their explorations, and, indeed, successfully transformed their argumentations into geometrical proofs by contradiction. Afterwards, we present two cases in which we argue that the students fail to encounter pseudo objects, and no proof by contradiction stemming from the exploration is reached.

4.1 Case 1: Matteo

Matteo, an Italian graduate student, who had become quite fluent in using DGE during a course of mathematics education. Matteo seemed to have extremely strong conceptual control over the figure he was thinking about. He claimed to be thinking about the figure "as if he had constructed it in a dynamic file", and drew the following figure (Fig. 4[2]).

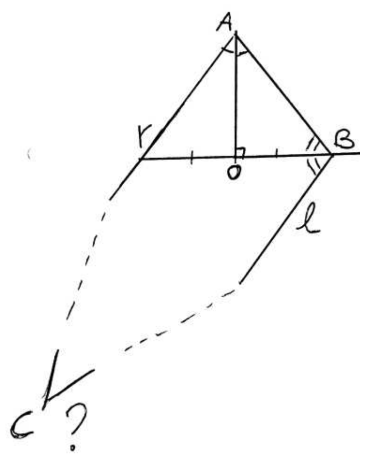


Fig. 4. Matteo's drawing of a triangle with two perpendicular angle bisectors

Matteo seemed to visualize a figure with properties $A_1 \wedge A_2 \wedge A_3 \wedge A_4$ in which he questions the existence of the point C. He seemed to be expecting a contradiction because

[2] This figure was redrawn by the authors because the scan of the original was too light.

he drew r and l (initially) parallel, then deviating towards C. Matteo then opened GeoGebra and constructed a figure like the one he drew: starting from a right triangle AOB he extended OB past O of a segment of the length of OB, then constructed the ray through this point and A, and the ray through the symmetric image of AB over OB. The two rays appeared to be parallel. Matteo looked at the figure and said: "the triangle's sides r and l just pop open." We can hypothesize the appearance of a pseudo object for Matteo in ABC (the figure he drew and the "open triangle" that appeared on the screen) because Matteo seemed to perceive $A_1 \wedge A_2 \wedge A_3 \Rightarrow \neg A_4$ and, simultaneously $A_1 \wedge A_2 \wedge A_3 \Rightarrow A_4$.

Matteo wrote:

- Construct two angle bisectors that are perpendicular
- Visualise ABO right triangle $\Rightarrow \angle OAB + \angle OBA = 90°$
- Construct sides of the "triangle"
- From theory:
  Alternate interior angles supplementary $\Rightarrow r \parallel l \Rightarrow$ ABC is NOT a triangle

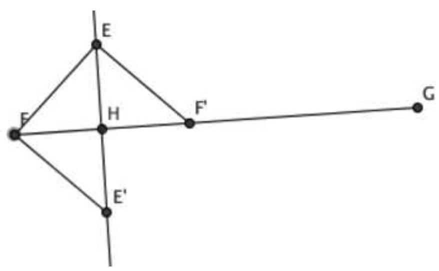
In the written argumentation Matteo seemed to be logically working out the relationship between the given properties of the triangle AOB and the existence of C and $\angle$AOB, r and l. He correctly derived the property $r \parallel l$ (D_1) which can "resolve" the contradiction, eliminating the existence of the triangle ABC.

Matteo's high conceptual control on the figure allowed him to construct and explore his figure, even before acting in the DGE, indeed he did not seem surprised by the feedback he received, since he seemed to already have imagined it and envisioned the contradiction. This way of reasoning allowed him to perceive a pseudo object through which he could directly elaborate a proof by contradiction. We see this example as strongly supporting the part of our working hypothesis that states that experiencing a pseudo object during an exploration can foster processes of indirect argumentation.

4.2 Case 2: Gille and Bernard

The two Hong Kong students, Gille (G.) and Bernard (B.), in the following excerpts robustly constructed two bisectors that are perpendicular, EE' and FF' (properties $A_2 \wedge A_3 \wedge A_1$) as follows: they constructed segment EF, and a second segment FG; they constructed a robust perpendicular line to FG through E, that intersects FG in H; they reflected points E and F on the bisectors, obtaining E' and F', respectively, and connected E with F' and F with E', to obtain segments along which the sides of the triangle-to-be should lie (Fig. 5).

Excerpt 1

What was said and done	***Our interpretation***
 Fig 5. Initial construction by G. and B. 1 G (exclaims): There are 3 sides in total, but they do not stick together[3] .	G seems to have the expectation of seeing sides EF' and FE' converge. That is, she seems to have projected the following indirect invariant onto the figure: $A_2 \wedge A_3 \wedge A_1$ (in her construction) $\Rightarrow A_4$. However, the software's feedback is guiding her to perceive the invariant D_2, seen as $\neg A_4$, (or possibly an indirect invariant: $A_2 \wedge A_3 \wedge A_1 \Rightarrow D_2$), which she will discover to be robust and to cause the quadrilateral to degenerate into a point whenever she tries to "close" the triangle bringing E' and F' together. These conflicting indirect invariants ($A_2 \wedge A_3 \wedge A_1 \Rightarrow A_4$ and $A_2 \wedge A_3 \wedge A_1 \Rightarrow D_2$) seem to be perceived simultaneously, giving rise to a pseudo object for G.

[3] This is a translation from Chinese. Another possible translation might be "they do not close".

The rest of the exploration was dedicated to explaining this surprising feedback. The exploration proceeded as follows (the text is translated from Chinese).

Excerpt 2

What was said and done	***Our interpretation***
2 G: [She drags E so that the "vertexes" converge]. We need to have E' and F' stick together. They stick together when they become one point.	G seems to be looking for a case in which her construction also has the property A_4. As she attempts to impose the soft property A_4 on the figure, she experiences the indirect invariant: $A_2 \wedge A_3 \wedge A_1 \wedge A_4 \Rightarrow$ all sides of the triangle-to-be collapse into a point (D_3).
3 G: If E' and F' stick together, E' and F' will have to move towards the intersection point [H].	
4 G: …and E will tend to F then. So all points will stick together [Fig. 6].  Fig. 6 Degeneration of the triangle-to-be	
5 B: The distance EF is equal to the distance E'F'. Therefore, if we want E' and F' to become one point E and F need to become one point also.	B tries to explain the figure's behaviour, identifying a new invariant property: EF≅E'F' (property F_1), which implies degeneration of the triangle-to-be in the desired case.
6 G: There should be a line right there [she points to a line that seems to connect E' and F'] ...	Possibly G is remembering the construction accomplished using symmetry, and relates it to a known property of parallelograms.
7 B: Because this is [pointing to EF'E'F] is a parallelogram.	
8 G: Because here FF' and here EE' … are all mirrors.	
9 B: Yes, this is a parallelogram. So there is no triangle.	What becomes crucial is the perception of a robust

10 G: Not possible to make the triangle with angle bisectors perpendicular to each other.	parallelogram (F_2) in place of the triangle-to-be, that conflicts with A_4.

The exploration seemed to lead the students to identify a chain of invariants:

$$A_2 \wedge A_3 \wedge A_1 \Rightarrow F_1 \Rightarrow \ldots \Rightarrow F_2 \Rightarrow D_2 \text{ (seen as } \neg A_4\text{)}$$

exposing the impossibility of obtaining A_4 from the constructed figure. Moreover, this explains the unexpected behaviour experienced when the students perceive $A_2 \wedge A_3 \wedge A_1 \wedge A_4 \Rightarrow D_3$ as all sides of the triangle-to-be collapsed into a point when trying to softly impose A_4. In fact, this "explanation" is what seemed to eliminate the contradiction overcoming the pseudo object: in lines 9 and 10 the students did not seem to be projecting A_4 onto the figure any more or perceiving it as an invariant relationship between invariants. The triangle simply did not exist and it had been replaced by the parallelogram.

The proof produced by the students went as follows and is accompanied by a figure (Fig. 7[4]). The written text (originally in English) is transcribed literally.

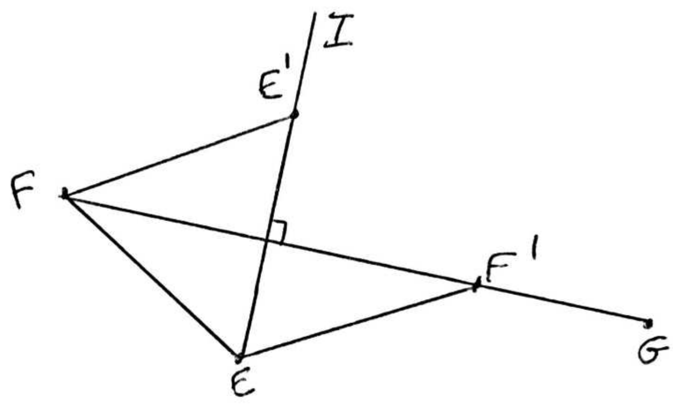


Fig. 7. The figure G. and B. drew for the proof

[4] This figure was redrawn by the authors because the scan of the original was too light.

1. Construct a point E and a line segment FG. [omitted in written text: "construct line segment EI so that"] $EI \perp FG$.
2. Reflect E along FG to E' such that FG bisects $\angle E'FE$.
3. Reflect F along EI to F' such that EI bisects $\angle F'EF$.
4. Drag E on F to make E' and F' be the same point.

Result: All the points E, F, E', F' intersect [they seem to use this as a synonym of "coincide"] together at the same point.

Reason: EFF'E' is a parallelogram (diag. $\perp$)

$\therefore$ EF = F'E' (prop. of parallelogram)

$\therefore$ E' and F' won't intersect [i.e. coincide]

$\therefore$ Contradict with the statement, no triangles can be formed under this situation.

In the students' written argumentation, they described the construction created in the DGE, reproduced it on paper, and argued that E' and F' cannot coincide unless the whole figure degenerates, because "EFF'E' is a parallelogram" (F_2). Though a correct derivation of "EFF'E' is a parallelogram" is not actually given (perpendicular diagonals is not a sufficient condition, and the students did not prove that E' belongs to EI and F' belongs to FG), in the proof we can find an attempt at directly deriving property F_2 and expressing the conflict between this condition and the coincidence of points E' and F', which would give rise to a triangle. Although the argumentation is not a proper proof by contradiction and the derivations are not completely correct, it explains the perceived contradiction in the figure assumed to be a triangle but turning out necessarily to be a parallelogram. We suggest that the argumentation could have become a proof by contradiction if the property A_4 had been explicitly added to the students' initial assumptions.

According to this analysis the case of Gille and Bernard supports the part of our working hypothesis that states that experiencing a pseudo object during an exploration can foster DGE-supported processes of argumentation culminating in a proof by contradiction.

We now introduce two cases supporting the part of our working hypothesis in which we claim that lack of experience of a pseudo object may hinder processes culminating in the production of a proof by contradiction. In these cases, we argue, the students fail to encounter pseudo objects and they do not construct indirect argumentations or proofs by contradiction as a culmination of the dynamic exploration process.

4.3 Case 3: Simone

Simone was a 16 year-old Italian high school student. He proceeded by constructing a proper triangle (A_4) and two of its angle bisectors ($A_2 \wedge A_3$). Then he marked an angle formed by the bisectors and started dragging one vertex of the triangle in the attempt to get the measure to say "90°" (A_1) (Fig. 8). So, in the situation of having constructed the robust properties $A_4 \wedge A_2 \wedge A_3$ he was trying to induce the soft property A_1, and possibly investigate the conditions under which A_1 was visually verified.

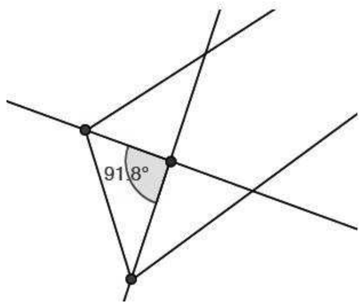


Fig. 8. Simone's attempt at obtaining a right angle at the intersection of the two bisectors. The third vertex of the triangle is too "far away" to be shown here.

Excerpt 3

What was said and done	***Our interpretation***
1 *Sim*: It's endless! [dragging "up" the third vertex of the triangle]	Sim seems to be convinced that it is possible to obtain the (soft) property A_1.
2 *Sim*: 91.2 [reading the measure of the angle between the bisectors.]	
3 *Sim*: Well, yes, in any case it will come out!	
4 *Sim:* Well, of course! It's not like it can go on forever! At the end it will make it to be 90!	
…	
5 *Sim:* Eh, it is impossible to construct it!	

Because… I only have these two bisectors.	
6 *Interviewer:* Hmm.	
7 *Sim:* How can I…	
8 *Sim:* Since… the bisectors are perpendicular … it means here there is a rhombus…or a square.	Although he cannot obtain the desired property visually, Sim projects onto the figure the property of bisectors being perpendicular (A_1), and expresses two implied properties: rhombus (E_1) or square (E_2). A relationship between invariants seems to be perceived: $A_4 \wedge A_2 \wedge A_3 \wedge A_1 \Rightarrow E_1 \vee E_2$.
9 *Sim:* If like here… [he draws a segment]… Here… there were… a rhombus…this would be 90, 90… or a square. And therefore… then… Eh, I mean, if this is like a rhombus, no? here there is 90 and here there is 90, and these are the bisectors.	
10 *Sim:* And then… and then I bring these up [pointing to the "open" looking sides of the triangle (see Fig. 8)] and I find their point of… of intersection.	His new argumentation seems to lead to the conclusion that it is in fact possible to construct the triangle with the desired properties. At this point Sim seems to have forgotten that A_4 is already a property imposed on the figure, that is, the point of intersection of the seemingly open sides exists robustly. No conflicting invariants seem to be perceived.

Although in line 5 Simone claimed that it is not possible to construct the triangle, the behaviour that followed was not consistent with this claim. On the contrary, he behaved as if it were possible to construct the triangle, and his exclamation can be interpreted as a sign of distress in that he may be having trouble conceiving a set of steps leading to the desired construction. There seem to be three possible indirect invariants that Simone perceived (of course this is our interpretation):

$$A_4 \wedge A_2 \wedge A_3 \wedge A_1 \Rightarrow E_1 \vee E_2$$

$$A_2 \wedge A_3 \wedge A_1 \Rightarrow E_1 \vee E_2$$

$$A_4 \wedge A_2 \wedge A_3 \wedge (E_1 \vee E_2) \Rightarrow A_1$$

Consistently with this interpretation, he proceeded to construct a new figure in which the triangle is not closed (Fig. 9).

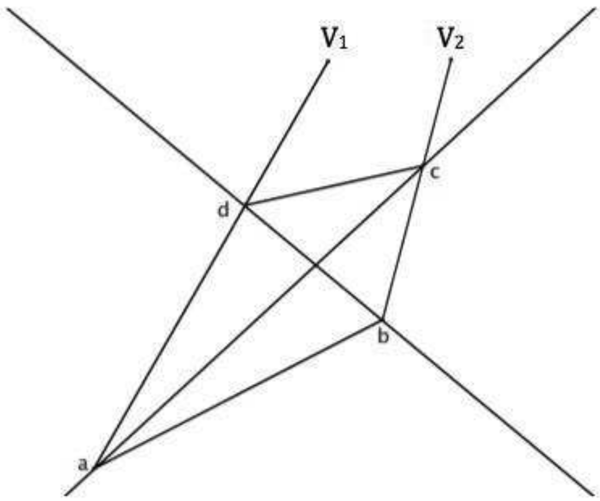


Fig. 9. Simone's new construction in which A_4 was no longer robust. That is, the points V_1 and V_2 could be dragged to merge (the labels V_1 and V_2 were inserted by the authors).

However, he did not seem to be aware of any conflict between his perceived invariant and other invariants he may have perceived simultaneously. Therefore, we can claim that Simone had not perceived any pseudo object. He was not able to continue his argumentation, nor did he discover the impossibility of the construction within the DGE. This case and its analysis support the part of our working hypothesis claiming that the lack of experience of a pseudo object can hinder DGE-supported processes of indirect argumentation.

4.4 Case 4: Emiliana and Ilaria

We conclude with the case of two Italian high school students, Emiliana and Ilaria (age 16); they started by constructing the triangle and the two bisectors robustly ($A_4 \wedge A_2 \wedge A_3$) (Fig. 10).

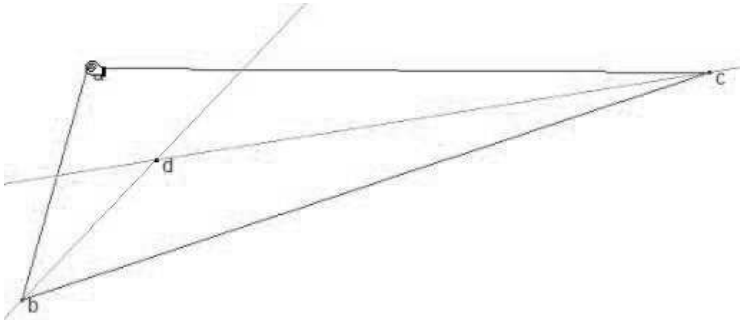


Fig.10. The constructed figure upon which Ila was acting.

Excerpt 4

<table>
<tr><th>What was said and done</th><th>Our interpretation</th></tr>
<tr><td>1 Ila: Do the angle bisectors and try to move the two angles until we get a right angle and put ‘mark the angle’. [she takes control of the mouse]</td><td>Ila is proposing to explore what happens trying to impose the soft condition A_1 on the figure.</td></tr>
<tr><td>2 Ila: We have to move ‘a’[5]…</td><td rowspan="6">The students realise that when the vertex ‘a’ is collinear with vertexes ‘b’ and ‘c’ the bisectors appear to be perpendicular. The students initially reject this case, because ‘abc’ is no longer a triangle.</td></tr>
<tr><td>3 Emi: Why ‘a’?</td></tr>
<tr><td>4 Ila: Uh, all, uh, we have to try all, …uh…[she stops in a degenerate situation, with a, b, c on a same line. The bisectors appear to be perpendicular]</td></tr>
<tr><td>5 Emi: No, that’s a line [they giggle]</td></tr>
<tr><td>6 Ila: So [moving away from the degenerate configuration]…</td></tr>
<tr><td>…</td></tr>
<tr><td>7 Emi: It is not useful to move ‘a’ because then, uh, then…move it…the angle keeps the same measure, and we have to try to make it so that this one [pointing to ‘d’[6]] is 90.</td><td>However ‘a’ is dragged the students do not seem to be happy with what</td></tr>
</table>

[5] This point corresponds to vertex B in Fig. 3

<table>
<tr><td>8 Ila: Right.</td><td rowspan="2">they find, so they decide to drag a different vertex.</td></tr>
<tr><td>9 Emi: So move ‘b’.</td></tr>
<tr><td>…</td><td></td></tr>
<tr><td>10 Emi: Measure the angle, and do ‘b’, ‘d’, ‘c’…Ok…see that it changes? (Fig. 11)

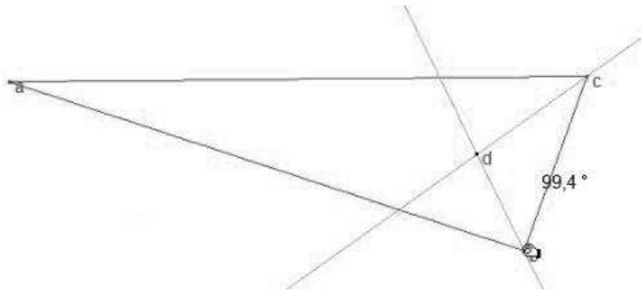


Fig. 11 The measure of the angle in ‘d’ changes as Ila drags ‘b’.</td><td rowspan="2">Again, the students experience the necessity of the degeneration that occurs whenever they obtain the desired property. This would correspond to an indirect invariant such as: $A_4 \wedge A_2 \wedge A_3 \wedge A_1 \Rightarrow D_2$ (which they seem to recognise as $\neg A_4$) or like $A_4 \wedge A_2 \wedge A_3 \wedge D_2 \Leftrightarrow A_1$ (see line 13).</td></tr>
<tr><td>11 Ila: Yes, but, excuse me, only when it is a line [referring to Fig. 12].

Fig. 12 Ila makes the triangle degenerate on purpose.

…then it's impossible!</td></tr>
<tr><td>12 Emi: Ok, it’s impossible. Then I guess it is impossible.</td><td rowspan="2">The students seem to conceive the impossibility of the construction, but limit their attention to the invariant relationship $A_4 \wedge A_2 \wedge A_3 \wedge D_2 \Leftrightarrow A_1$.</td></tr>
<tr><td>13 Ila: See? [she shows Emi other cases in which the three vertices are collinear and the measure of the angle between the bisectors is 90°]…It’s impossible.</td></tr>
<tr><td>14 Emi: No, because, if…[she starts writing what will be an</td><td></td></tr>
</table>

[6] This point corresponds to point P in Fig. 3

algebraic proof by contradiction and no longer pays attention to the screen]	

The perceived indirect invariant did contain a contradiction (if D_2 is interpreted as $\neg A_4$), however simultaneous perception of the contradictory invariants (A_4 and D_2) seemed to be lacking. Indeed, what Ila seemed to be perceiving was: "$A_4 \wedge A_2 \wedge A_3 \Leftrightarrow \neg A_1$" or (strictly) "$A_2 \wedge A_3 \wedge A_1 \Leftrightarrow D_2$" seen as "$A_2 \wedge A_3 \wedge A_1 \Leftrightarrow \neg A_4$"). She seemed unable to see these invariants simultaneously projected onto the figure, thus, according to this interpretation a pseudo object was not perceived. The students' argumentation processes in this exploration did not culminate in a geometrical proof by contradiction, so this case supports the part of our working hypothesis claiming that the lack of experience of a pseudo object can hinder DGE-supported processes of argumentation culminating in proof by contradiction.

5 Conclusion

This paper presents the revisitation of some data through the lens of a new tool of analysis: the notion of pseudo object. The analysis leads to the emergence of a research hypothesis concerning the relationship between argumentation and proof, specifically proof by contradiction. We highlighted key elements in the argumentation and reasoning process through which the solvers seem to be trying to find harmony between conflicting dynamic phenomena experienced through dragging (what is seen via dragging) and geometric properties of figures (what is expected to be seen happening according to the Theory of Euclidean Geometry). We suggest that perceiving a pseudo object can be a key to reaching such harmony: perceivable conflicts are related to one or more indirect invariant(s) (invariant relationships between properties). Furthermore, we developed a symbolic logical chain approach to identify and describe the emergence of pseudo objects and their role in argumentation processes.

The notion of pseudo object allowed us to advance the hypothesis that experiencing a pseudo object during an exploration can foster DGE-supported processes of argumentation culminating in proof by contradiction, while the lack of experience of a pseudo object may hinder such processes. In the analyses of the four cases presented we identified the invariants that we could recognise in the argumentation processes and wrote them as deductive chains, highlighting dependency relationships between perceived invariants. We note that in the two excerpts in which we inferred the presence of pseudo objects in the students' argumentations such pseudo objects emerged as a conflict between two indirect invariants, one expected and one perceived as feedback from the DGE. Indeed, as noted in the a priori analysis, the cases in which the appearance of an indirect invariant (e.g., $A_1 \wedge A_2 \wedge A_3 \Rightarrow \neg A_4$) conflicts with what is expected (e.g., $A_1 \wedge A_2 \wedge A_3 \Rightarrow \ldots \Rightarrow A_4$) are situations with a particularly high potential of fostering reasoning by contradiction in continuity with the exploration. In the third and fourth cases presented it seems that neither the phenomenon of the emergence of a pseudo object occurred nor did indirect argumentations stem from the explorations.

The theoretical tool offered by the notion of pseudo objects and the symbolic logical chain approach allowed us to notice the missing steps in each argumentation that could have led to perceiving contradiction. It seems that the appearance of the phenomenon of the pseudo object occurring during an exploration could be related to elaborating arguments supporting the answer in terms of existence of the requested figure. Moreover, the conflicting nature of the pseudo object may make a contradiction evident and in this way support the production of proof culminating with a contradiction.

A pseudo object, as discussed in this paper, can be thought of as a virtual object that exists in the interface between our cognitive world and the DGE micro-world. It withholds the *potential* that when actualized visually in a DGE (a projected perception) it can lead to realizing Euclidean possibility or impossibility. Therefore, even though one cannot construct "impossible" figures in a DGE, coupled with their mental world, DGEs have the potential to create uncertain dynamic geometrical phenomena that can be perceived (visually) leading to argumentation and proof. This Learner-DGE coupling expands the epistemic dimension of a DGE where dragging, in particular dragging a pseudo object, becomes simultaneously a mental and physical activity. We show that perceiving and resolving conflict, uncertainty, surprise and contradiction motivate students to engage in meaningful (successful or not) mathematical reasoning and argumentation. Thus, pseudo objects can be invested upon in the design of DGE-tasks with the pedagogical aim of fostering geometrical proof by contradiction. Such goal might be eventually achieved with a transitional phase in which students produce DGE-based argumentations as in the case of Gille and Bernard. Figure 2 and Table 2 show two types of pseudo object representations that are conducive to mathematical reasoning: one is visual and dynamic while the other is logical and linguistic. Their combination in pedagogical settings and task design opens a new direction in DGE and geometrical proof research.

References

Antonini, S. (2004), A statement, the contrapositive and the inverse: intuition and argumentation, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, vol. 2 (pp. 47-54). Bergen, Norway.

Antonini, S. (2010). A model to analyse argumentations supporting impossibilities in mathematics. In M.F. Pinto & T.F. Kawasaki. (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, vol. 2 (pp. 153-160). Belo Horizonte, Brazil.

Antonini, S. & Mariotti, M.A., (2006), Reasoning in an absurd world: difficulties with proof by contradiction. In J. Novotna, H. Moraova, M. Kratka, & N. Sthlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education,* vol. 2 (pp. 65-72). Prague, Czech Republic.

Antonini, S., & Mariotti, M. A. (2007). Indirect proof: an interpreting model. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 541-550). Larnaca, Cyprus: University of Cyprus.

Antonini, S., & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving? *Zentralblatt für Didaktik der Mathematik, 40*(3), 401-412.

Baccaglini-Frank, A. (2010). Conjecturing in Dynamic Geometry: A Model for Conjecture-generation through Maintaining Dragging. *Doctoral dissertation*, University of New Hampshire, Durham, NH: ProQuest.

Baccaglini-Frank, A., & Mariotti, M. A. (2010). Generating conjectures through dragging in dynamic geometry: the Maintaining Dragging Model. *International Journal of Computers for Mathematical Learning, 15*(3), 225-253.

Baccaglini-Frank, A., Antonini, S., Leung, A., & Mariotti, M.A. (2011) Reasoning by Contradiction in Dynamic Geometry. In *Proceedings of the 35rd Conference of the International Group for the Psychology of Mathematics Education*, vol. 2 (pp. 81-88). Ankara, Turkey.

Baccaglini-Frank, A., Antonini, S., Leung, A., & Mariotti, M. A. (2013). Reasoning by contradiction in dynamic geometry. *PNA, 7*(2), 63-73.

Baccaglini-Frank, A., Antonini, S., Leung, A., Mariotti, M.A. (2017). Designing non-constructability tasks in a dynamic geometry environment. In A. Leung & A. Baccaglini-Frank (Eds.), *Digital Technologies in Designing Mathematics Education Tasks - Potential and pitfalls*, (pp. 99-120). Springer.

Bartolini Bussi, M.G. & Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, and D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education, second revised edition*, (pp. 746-805). Mahwah, NJ.: Lawrence Erlbaum.

Boero, P. (Ed.) (2007). *Theorems in school: From history, epistemology and cognition to classroom practice*, pp. 249-264. Rotterdam, The Netherlands: Sense Publishers.

Boero, P., Garuti, R., & Mariotti, M. A. (1996). Some dynamic mental process underlying producing and proving conjectures. In *Proceedings of 20th PME Conference*, vol. 2, (pp. 121–128). Valencia, Spain.

De Villiers, M. (2004). Using dynamic geometry to expand mathematics teachers' understanding of proof. *International Journal of Mathematical Education in Science and Technology, 35*(5), 703-724.

Duval, R. (1992–93). Argumenter, demontrer, expliquer: coninuité ou rupture cognitive? *Petit* x, 31, 37–61.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, Holland: Reidel Publishing Company.

Garuti, R., Boero,P., Lemut, E.& Mariotti, M.A. (1996), Challenging the traditional school approach to theorems: a hypothesis about the cognitive unity of theorems, in *Proceedings of the 20th PME Conference*, Valencia, vol. 2 pp. 113-120.

Hanna, G., & de Villiers, M. (Eds.), (2012). *Proof and proving in mathematics education. The 19th ICMI study*. Dordrecht: Springer.

Healy, L. (2000). Identifying and explaining geometric relationship: interactions with robust and soft Cabri constructions. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education,* vol. 1 (pp. 103-117). Hiroshima, Japan.

Laborde, C. (2000). Dynamic geometry environments as a source of rich learning contexts for the complex activity of proving. *Educational Studies in Mathematics, 44*, 151–161.

Laborde, C., & Laborde, J. (1991). Problem solving in geometry: From microworlds to intelligent computer environments. In J. P. Ponte et al. (Eds.), *Mathematical problem solving and new information technologies*, NATO ASI Series F, vol 89 (pp. 117–192). Springer-Verlag.

Laborde, J.-M., & Sträßer, R. (1990). Cabri-Geomètre: A microworld of geometry for guided discovery learning. *Zentralblatt fur Didaktik der Mathematik, 90*(5), 171-190.

Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies in Mathematics, 16*(3), 321-325.

Leung, A., & Lopez-Real, F. (2002). Theorem justification and acquisition in dynamic geometry: a case of proof by contradiction. *International Journal of Computers for Mathematical Learning, 7*(2), 145-165.

Leung, A., Baccaglini-Frank, A. & Mariotti, M.A. (2013). Discernment in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460.

Mariotti, M. A. (2000). Introduction to proof: the mediation of a dynamic software environment. *Educational Studies in Mathematics. Special issue 44*, 25-53.

Mariotti, M. A. (2002). The influence of technological advances on students' mathematical learning. In L. D. English (Ed.), *Handbook of international research in mathematics education*, (pp. 695-723). Lawrence Erlbaum Associates.

Mariotti M. A . (2015). Transforming Images in a DGS: The Semiotic Potential of the Dragging Tool for Introducing the Notion of Conditional Statement. In S. Rezat, Sebastian, M. Hattermann & A. Peter-Koop (Eds.), *Transformation—A Fundamental Idea of Mathematics Education*, (pp.156-175). Springer.

Mariotti, M. A., & Antonini, S. (2009). Breakdown and reconstruction of figural concepts in proofs by contradiction in geometry. In F. L. Lin, F. J. Hsieh, G. Hanna, & M. de Villers (Eds.), *Proof and proving in mathematics education, ICMI Study 19 Conference Proceedings*, vol. 2 (pp. 82-87). Taipei, Taiwan: National Taiwan Normal University.

Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom – the case of function. *International Journal of Educational Research, 51-52*, 10–27.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics, 66*(1), 23–41.

Sinclair, N., & Robutti, O. (2013). Technology and the Role of Proof: The Case of Dynamic Geometry. In A. J. Bishop, M. A. Clements, C. Keitel & F. Leung (Eds.), *Third international handbook of mathematics education*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Stylianides, A., Bieda, K., & Morselli, F. (2016). Proof and argumentation in mathematics education research. In A. Gutiérrez, G. Leder, & P. Boero (Eds.), *2nd handbook on the psychology of mathematics education,* pp. 315–351. Rotterdam: Sense Publishers.

Vinner, S. (1999). The possible and the impossible. *ZDM*, *99*(2), 77.

Wu Yu, J., Lin, F., & Lee, Y. (2003). Students' understanding of proof by contradiction. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA,* vol. 4 (pp. 443-449). Honolulu, HI: College of Education, University of Hawaii.

Yerushalmy, M., Chazan, D. & Gordon, M. (1993). Posing problems: One aspect of bringing inquiry into classrooms. In J. Schwartz, M. Yerushalmy & B. Wilson (Eds.), *The Geometric Supposer, what is it a case of?* (pp. 117-142). Hillsdale, NJ: Lawrence Erlbaum Associates.